

Pluri-cotangent maps of surfaces of general type

Francesco Polizzi, Xavier Roulleau

Abstract

Let X be a compact, complex surface of general type whose cotangent bundle Ω_X is strongly semi-ample. We study the pluri-cotangent maps of X , namely the morphisms $\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n\Omega_X))$ defined by the vector space of global sections $H^0(X, S^n\Omega_X)$.

0 Introduction

Let X be a complex surface of general type and assume that its cotangent bundle Ω_X is strongly semi-ample. This means that for some integer $n \geq 1$ the symmetric power $S^n\Omega_X$ is globally generated, namely, the evaluation map

$$H^0(X, S^n\Omega_X) \otimes \mathcal{O}_X \rightarrow S^n\Omega_X \quad (1)$$

is surjective (this condition implies in particular that X is minimal and K_X is ample). Recalling that we have a natural identification between $H^0(X, S^n\Omega_X)$ and $H^0(\mathbb{P}(\Omega_X), \mathcal{O}_{\mathbb{P}(\Omega_X)}(n))$, from the surjectivity of (1) we infer that the induced evaluation map

$$H^0(X, S^n\Omega_X) \otimes \mathcal{O}_{\mathbb{P}(\Omega_X)} \rightarrow \mathcal{O}_{\mathbb{P}(\Omega_X)}(n)$$

is also surjective, and so it defines a morphism

$$\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n\Omega_X)),$$

that we call the n th pluri-cotangent map of X . The case $n = 1$ was studied by the second author in [Rou09]: it turns out that, as soon as Ω_X is globally generated and $q(X) > 3$, the cotangent map $\psi_1: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, \Omega_X)) \simeq \mathbb{P}^{q(X)-1}$ is a generically finite morphism onto its image.

In the present note we generalize this result to the case $n \geq 2$. Let us remark that, if Ω_X is strongly semi-ample, then ψ_n is a generically finite morphism onto its image for n sufficiently large if and only if the second Segre number $c_1(X)^2 - c_2(X)$ is strictly positive, see Proposition 2.2. Our aim is to give effective versions of this statement. The first result we show is

2010 Mathematics Subject Classification: 14J29

Keywords: surfaces of general type, pluri-cotangent maps

Theorem A (see Theorem 2.19). *Let $n \geq 2$ be such that $S^n\Omega_X$ is globally generated. If $h^0(X, S^n\Omega_X) > \frac{1}{2}(n+1)(n+2)$ then ψ_n is generically finite onto its image. In this case, the exceptional locus $\text{exc}(\psi_n)$ is a Zariski-closed, possibly empty subset of $\mathbb{P}(H^0(X, S^n\Omega_X))$ of dimension at most 1.*

The proof of Theorem A is obtained by generalizing the geometrical arguments used in [Rou09]. We also exploit some results contained in the recent paper [MU19] by Mistretta and Urbinati, allowing us to prove the finiteness of the n th Gauss map of X (Proposition 2.9), together with the classical description of surfaces 2-covered by curves of degree n , first obtained by Bompiani [Bom21] and later rediscovered by Pirio and Russo [PR13], see Proposition 2.17.

Furthermore, we are also able to provide explicit bounds for the generic finiteness of the pluri-cotangent maps.

Theorem B (see Theorem 3.3). *Let $n \geq 3$ be an integer such that $S^n\Omega_X$ is globally generated. If*

$$c_1^2 > \left(1 + \frac{6n-2}{2n^2-2n+1}\right)c_2 + \frac{6n+12}{2n^2-2n+1} \quad (2)$$

then the n th pluri-cotangent map ψ_n is generically finite onto its image.

The proof of Theorem B uses the explicit computation of $\chi(X, S^n\Omega_X)$, see Lemma 3.1, together with Bogomolov's cohomological vanishing (Proposition 1.3), in order to show that (2) implies the lower bound $h^0(X, S^n\Omega_X) > \frac{1}{2}(n+1)(n+2)$. Then the claim follows from Theorem A. As a consequence, we get

Corollary C (see Corollary 3.4). *Let X be a minimal surface of general type with $c_1^2 - c_2 > 0$ and such that $S^m\Omega_X$ is globally generated. Then the n th pluri-cotangent map ψ_n is generically finite onto its image for all multiples n of m such that*

$$n > \frac{\beta + \sqrt{\beta^2 - \alpha\gamma}}{\alpha},$$

where

$$\alpha := 2(c_1^2 - c_2), \quad \beta := c_1^2 + 2c_2 + 3, \quad \gamma := c_1^2 + c_2 - 12.$$

Corollary C is proved by rewriting (2) as a quadratic inequality in n with strictly positive leading coefficient, namely

$$2(c_1^2 - c_2)n^2 - 2(c_1^2 + 2c_2 + 3)n + (c_1^2 + c_2 - 12) > 0.$$

Section 4 deals with some examples and counterexamples. In Subsections 4.1 and 4.2 we consider the following two constructions giving surfaces X such that Ω_X is neither ample nor globally generated:

- (1) X is a suitable symmetric complete intersection in an abelian fourfold of the form $A \times E$, where A is an abelian threefold and E is an elliptic curve;
- (2) X is of the form $X = (C \times F)/G$, where C is a smooth hyperelliptic curve of genus 3, F is a smooth curve of odd genus and $G = \mathbb{Z}_2$ acts with four fixed points on C , freely on F and diagonally on the product.

In both situations, the vector bundle $S^2\Omega_X$ turns out to be globally generated, hence Ω_X is strongly semi-ample, and moreover the pluri-cotangent map ψ_n is generically finite onto its image for all even n (see Propositions 4.1, 4.2, 4.5, 4.6).

In Subsection 4.3 we exhibit some counterexamples to Corollary C when $c_1^2 - c_2 = 0$, namely, smooth ample divisors in abelian threefolds. In fact, if X is such a divisor, for all $n \geq 1$ we have

$$H^0(X, S^n\Omega_X) = S^n H^0(X, \Omega_X) \simeq \mathbb{C}^{\binom{(n+1)(n+2)}{2}},$$

and the image X_n of $\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n\Omega_X))$ is projectively equivalent to the n th Veronese surface $\nu_n(\mathbb{P}^2) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$. Thus, no pluri-cotangent map of X is generically finite onto its image. Under the additional assumption that the image of the Albanese map is smooth, we also show that these counterexamples are the only ones up to finite, étale covers (Proposition 4.10).

By using finite cyclic covers of a surface X as above, we are also able to construct surfaces of general type all of whose Gauss maps have arbitrarily large degree, see Remark 4.11.

Finally, in the last section of the paper we state a couple of open problems.

Acknowledgments. This work started in May 2017, when the first author visited the Institut de Mathématiques de Marseille. He is grateful to the members of the équipe Analyse, Géométrie et Topologie for the invitation and the hospitality, and to GNSAGA-INdAM for the financial support. He also thanks all the MathOverflow users that generously answered his questions in several threads, see

[MO397682](#), [MO412306](#), [MO412888](#), [MO413988](#), [MO414382](#), [MO414452](#),
[MO417972](#), [MO418607](#), [MO430933](#), [MO431327](#).

Both authors are grateful to Erwan Rousseau for providing useful references and to Antonio Rapagnetta and Igor Reider for suggestions and remarks.

Notation and conventions. We work over the field \mathbb{C} of complex numbers. By *surface* we mean a smooth, compact complex surface X , and for such a surface Ω_X denotes the holomorphic cotangent bundle, T_X the holomorphic tangent bundle, $\omega_X = \mathcal{O}_X(K_X)$ the canonical bundle, $p_g(X) = h^0(X, K_X)$ is the geometric genus, $q(X) = h^1(X, K_X)$ is the irregularity and $\chi(\mathcal{O}_X) = 1 - q(X) + p_g(X)$ is the holomorphic Euler-Poincaré characteristic. We also set $c_1 := c_1(T_X) = -c_1(\Omega_X)$ and $c_2 := c_2(T_X) = c_2(\Omega_X)$. The second Segre number of X is the integer $c_1^2 - c_2$.

For projective spaces and projective bundles we use the same conventions as in [Laz04, Chapter 6]. More specifically, if V is a vector space, $\mathbb{P}(V)$ stands by the projective space of 1-dimensional quotients of V ; we denote by $\mathbb{G}(n, \mathbb{P}(V))$ the Grassmannian of n -dimensional subspaces of $\mathbb{P}(V)$, and by $\mathbb{G}(\mathbb{P}(V), n)$ the Grassmannian of n -dimensional quotients of $\mathbb{P}(V)$. Moreover, we write $\nu_n: \mathbb{P}(V) \rightarrow \mathbb{P}(S^n V)$ for the n th Veronese embedding of $\mathbb{P}(V)$, and we call its image $\nu_n(\mathbb{P}(V))$ the n th Veronese variety of $\mathbb{P}(V)$.

If \mathcal{E} is a vector bundle on X and $x \in \mathcal{E}$, we write $\mathcal{E}(x)$ for the fibre of \mathcal{E} over x . Furthermore, we denote by $\pi: \mathbb{P}(\mathcal{E}) \rightarrow X$ the projective bundle of 1-dimensional quotients of \mathcal{E} , so that $\pi_* \mathcal{O}_{\mathbb{P}(\mathcal{E})}(n) = S^n \mathcal{E}$ for all $n \geq 1$.

1 Pluri-cotangent maps and Gauss maps

Let X be a compact, complex surface of general type. We say that its cotangent bundle Ω_X is *strongly semi-ample* if the n th symmetric power $S^n \Omega_X$ is globally generated for some $n \geq 1$, namely, if the evaluation map

$$H^0(X, S^n \Omega_X) \otimes \mathcal{O}_X \rightarrow S^n \Omega_X \quad (3)$$

is surjective. Note that this implies

$$h^0(X, S^n \Omega_X) \geq \text{rank } S^n \Omega_X = n + 1 \quad (4)$$

and if equality holds then $S^n \Omega_X \simeq \mathcal{O}_X^{n+1}$. In particular, since we are assuming that X is of general type, the strict inequality holds in (4). Moreover, the strong semi-amplicity of Ω_X implies that $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$ is semi-ample, see [MU19, Section 3.1], and so Ω_X is nef.

Lemma 1.1. *If Ω_X is strongly semi-ample, then X does not contain any smooth rational curve. In particular, X is minimal and K_X is ample.*

Proof. If C is a smooth curve contained in X , then Ω_C is a quotient of the restricted bundle $\Omega_X|_C$. Since passing to the n th symmetric product preserves epimorphisms, it follows that $S^n \Omega_C$ is a quotient of $S^n \Omega_X|_C$. This implies that Ω_C is strongly semi-ample, hence $g(C) \geq 1$. \square

Remark 1.2. It is not hard to construct examples where K_X is ample and Ω_X is not strongly semi-ample. For instance, take a smooth quintic surface $X \subset \mathbb{P}^3$ containing a line L . Since L is a smooth rational curve (with $L^2 = -3$) we see that Ω_X is not strongly semi-ample. On the other hand, by using adjunction formula we can check that there are no (-1) -curves or (-2) -curves on X , so X is a minimal model and K_X is ample.

Lemma 1.1 allows us to apply to our situation the next result, based on Bogomolov's work [Bog78], see [Kob80, Corollary A.1] and [RouRous13, p. 1341].

Proposition 1.3. *Let X be a surface of general type with ample canonical class. Then for all $n \geq 1$ the vector bundle $S^n \Omega_X$ is semi-stable with respect to the polarization K_X , and moreover*

$$H^0(X, S^n T_X \otimes \omega_X^k) = 0 \quad \text{for } n - 2k > 0. \quad (5)$$

Setting $k = 1$ in (5) and applying Serre duality, we get Bogomolov's vanishing

$$H^2(X, S^n \Omega_X) = 0 \quad \text{for } n \geq 3. \quad (6)$$

Corollary 1.4. *Let X be a surface of general type with ample canonical class. Then for all $n \geq 3$ we have $h^0(X, S^n \Omega_X) \geq \chi(X, S^n \Omega_X)$.*

Proof. Immediate consequence of (6). □

Remark 1.5. The extremal case $n = 2, k = 1$ in Proposition 1.3 is characterized as follows, cf. [Kob80, Theorem B and Corollaries B.1 and B.2]. If X is a minimal surface of general type with ample canonical bundle, then we have

$$H^0(X, S^2 T_X \otimes \omega_X) = H^0(X, S^2 \Omega_X \otimes \omega_X^{-1}) = 0$$

if and only if Ω_X is an indecomposable rank 2 vector bundle. One direction is clear: if $\Omega_X = L_1 \oplus L_2$ is the direct sum of two line bundles, then a straightforward computation shows that $S^2 \Omega_X \otimes \omega_X^{-1}$ has a direct summand isomorphic to \mathcal{O}_X , hence $H^0(X, S^2 \Omega_X \otimes \omega_X^{-1}) \neq 0$. Conversely, let us assume that $S^2 \Omega_X \otimes \omega_X^{-1}$ has a non-zero global section and let us show that Ω_X is decomposable; to this purpose, we will use the following argument suggested to us by Igor Reider. Identifying $S^2 \Omega_X \otimes \omega_X^{-1}$ with the sheaf $End_0(\Omega_X)$ of trace-zero endomorphisms of Ω_X ,¹ a non-zero global section corresponds to an endomorphism $f: \Omega_X \rightarrow \Omega_X$ whose trace is zero at every point. Now we have two cases:

- (i) There is a point $x \in X$ such that $f_x: \Omega_X(x) \rightarrow \Omega_X(x)$ has two non-zero eigenvalues $\pm \lambda$; then

$$\Omega_X = \ker(f - \lambda I) \oplus \ker(f + \lambda I)$$

is the desired splitting.²

¹This is a consequence of the following linear algebra facts. Consider a rank 2 vector space V over a field \mathbb{K} of characteristic different from 2. Since every square matrix can be written in a unique way as the sum of a symmetric matrix and a skew-symmetric one, we have the direct sum decomposition $V \otimes V = S^2 V \oplus \wedge^2 V$. Taking the tensor product with $\wedge^2 V^*$, and using the identification $V \otimes \wedge^2 V^* = V^*$ (coming from the bilinear pairing on V induced by the wedge product, namely $v \otimes w \mapsto v \wedge w$), we get an identification $V^* \otimes V = (S^2 V \otimes \wedge^2 V^*) \oplus \mathbb{K}$. On the other hand, $V^* \otimes V = \text{Hom}(V, V)$, so we get a further identification $\text{Hom}(V, V) = (S^2 V \otimes \wedge^2 V^*) \oplus \mathbb{K}$. Under this identification, an endomorphism $f: V \rightarrow V$ satisfies $\text{Trace}(f) = 0$ if and only if it lies in the first summand $S^2 V \otimes \wedge^2 V^*$. This is a straightforward computation based on the interpretation of the trace as the functional $V^* \otimes V \rightarrow \mathbb{K}$ given by the natural evaluation on decomposable tensors, namely $\text{Trace}(f \otimes v) = f(v)$. Therefore $S^2 V \otimes \wedge^2 V^*$ is naturally identified with the vector space $\text{Hom}_0(V, V)$ of trace-zero endomorphisms of V .

²If we have an endomorphism $g: E \rightarrow E$ of a vector bundle E , then its determinant $\det g: \det E \rightarrow \det E$ is a scalar multiple of the identity and so, if it vanishes at one point, it vanishes everywhere. Taking $E = \Omega_X$ and $g = f - \lambda I, g = f + \lambda I$, we get

$$\Omega_X(x) = \ker(f - \lambda I)(x) \oplus \ker(f + \lambda I)(x)$$

for all $x \in X$.

(ii) The endomorphism $f: \Omega_X \rightarrow \Omega_X$ is nilpotent everywhere, hence $f^2 = 0$. We will rule out this case, by exploiting the ampleness of K_X . In fact, by the nilpotency condition, the sheaf $\text{im}(f)$ injects into $\ker(f)$; then, setting

$$c_1(\ker(f)) = L, \quad c_1(\text{im}(f)) = L'$$

the divisor $L - L'$ is effective. The semi-stability of Ω_X with respect to the polarization K_X now gives

$$K_X L = \mu(L) \leq \mu(\Omega_X) = K_X^2/2, \quad K_X L' = \mu(L') \geq \mu(\Omega_X) = K_X^2/2,$$

and so

$$0 \leq K_X(L - L') \leq K_X^2/2 - K_X^2/2 = 0,$$

that implies $L = L'$. Thus $K_X = c_1(\Omega_X) = 2L$, hence L is ample. But this is impossible because, by a result of Bogomolov, Ω_X cannot have ample subsheaves of rank 1, see [Reid77, Theorem 2].

Recalling that we have a natural identification between $H^0(X, S^n \Omega_X)$ and $H^0(\mathbb{P}(\Omega_X), \mathcal{O}_{\mathbb{P}(\Omega_X)}(n))$, from the surjectivity of (3) we infer that the induced evaluation map

$$H^0(X, S^n \Omega_X) \otimes \mathcal{O}_{\mathbb{P}(\Omega_X)} \rightarrow \mathcal{O}_{\mathbb{P}(\Omega_X)}(n)$$

is also surjective, and so defines a morphism

$$\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n \Omega_X)).$$

Definition 1.6. We call ψ_n the n th cotangent map of X , and we denote its image by $X_n := \psi_n(\mathbb{P}(\Omega_X)) \subset \mathbb{P}(H^0(X, S^n \Omega_X))$.

By [Laz04, Appendix A] there is a relative n th Veronese embedding $\nu_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(S^n \Omega_X)$ such that, for every $x \in X$, the fibre $\mathbb{P}(\Omega_X(x))$ of $\mathbb{P}(\Omega_X)$ over x is sent to a rational normal curve of degree n inside the n -dimensional projective space $\mathbb{P}(S^n \Omega_X(x))$. Moreover, passing to projective bundles in the evaluation map (3), we get a morphism $e_n: \mathbb{P}(S^n \Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n \Omega_X))$ and a factorization of ψ_n of the form

$$\begin{array}{ccc} \mathbb{P}(\Omega_X) & \xrightarrow{\psi_n} & \mathbb{P}(H^0(X, S^n \Omega_X)) \\ & \searrow \nu_n & \nearrow e_n \\ & \mathbb{P}(S^n \Omega_X) & \end{array} \quad (7)$$

Let us now consider two important examples: the case where Ω_X is ample and the case where Ω_X is globally generated.

Example 1.7. The case where Ω_X is ample. If Ω_X is ample, then it is automatically strongly semi-ample, see [Laz04, Theorem 6.1.10], and the Chern numbers

of X satisfy the inequality $c_1^2 - c_2 > 0$, see [Kl69]. Furthermore, by [Laz04, Example 6.1.5 and Theorem 6.1.15], the ampleness of Ω_X is equivalent to the fact that e_n is finite onto its image. Summing up, we can state what follows:

Assume that $S^n\Omega_X$ is globally generated. Then the n th cotangent map ψ_n is a finite morphism onto its image X_n if and only if Ω_X is ample.

For the sake of completeness, let us shortly explain how to construct surfaces for which Ω_X is strongly semi-ample but not ample. Let A be an abelian 3-fold containing an elliptic curve E , and let $X \subset A$ be a sufficiently positive, smooth divisor containing E . Then X is a surface of general type whose Albanese morphism $a_X: X \rightarrow \text{Alb}(X)$ coincides with the inclusion $X \rightarrow A$. Since A contains no rational curves, the same is true for X , which is therefore a minimal model with ample K_X . Furthermore, since Ω_X is a quotient of $\Omega_A|_X = \mathcal{O}_X^{\oplus 3}$, it follows that Ω_X is globally generated, hence strongly semi-ample. However, Ω_X is not ample: in fact, varieties with ample cotangent bundle are Kobayashi hyperbolic ([Laz04, Theorem 6.3.26]), in particular, they do not contain any elliptic curves. For a detailed analysis of a similar construction in codimension 2, see Subsection 4.1.

Example 1.8. The case where Ω_X itself is globally generated. Since the evaluation map (3) for $n = 1$ is the co-differential of the Albanese morphism

$$a_X: X \rightarrow \text{Alb}(X),$$

the cotangent bundle Ω_X is globally generated if and only if a_X is a local immersion. In this case, $S^n\Omega_X$ is globally generated for all $n \geq 1$, and we have a natural symmetrization homomorphism $\sigma_n: S^n H^0(X, \Omega_X) \rightarrow H^0(X, S^n\Omega_X)$, that fits into a commutative diagram

$$\begin{array}{ccc} \mathbb{P}(\Omega_X) & \xrightarrow{\psi_n} & \mathbb{P}(H^0(X, S^n\Omega_X)) \\ \psi_1 \downarrow & & \downarrow \mathbb{P}(\sigma_n) \\ \mathbb{P}(H^0(X, \Omega_X)) & \xrightarrow{\nu_n} & \mathbb{P}(S^n H^0(X, \Omega_X)). \end{array} \quad (8)$$

Here ν_n stands for n th Veronese embedding and the rational map $\mathbb{P}(\sigma_n)$ is an embedding of projective spaces if σ_n is surjective, and a linear projection otherwise.³ By [Rou09, Proposition 2.14] it follows that if $q(X) = h^0(X, \Omega_X) > 3$ then ψ_1 is generically finite, hence we can draw the following conclusion.

Assume that Ω_X is globally generated and $q(X) > 3$. Then the n th cotangent map ψ_n is generically finite onto its image for all $n \geq 1$.

Again for the sake of completeness, let us provide examples where Ω_X is strongly semi-ample but not globally generated. If X is a fake projective plane (see [PY07]), then Ω_X is ample (this is true for every smooth compact complex

³According to our understanding, not much is known about the behaviour of σ_n in general. A result in this direction would provide a higher-dimensional generalization of the celebrated Max Noether's Theorem for curves, see [ACGH85, p. 117] and the MathOverflow thread [MO273557].

variety uniformized by the ball $\mathbb{B}^n \subset \mathbb{C}^n$, see [Laz04, Construction 6.3.36]) and thus strongly semi-ample. However, $h^0(X, \Omega_X) = h^1(X, \mathcal{O}_X) = 0$, namely, Ω_X has no global sections at all.

2 Finiteness of the Gauss map and dimension of the pluri-cotangent image

Assumption 2.1. From now on, X will denote a surface of general type with strongly semi-ample cotangent bundle Ω_X . Note that we are *neither* assuming that Ω_X is ample *nor* that Ω_X is globally generated, having already analyzed these cases before.

Proposition 2.2. *Let X be a surface that satisfies Assumption 2.1. Then ψ_n is generically finite onto its image X_n for n sufficiently large if and only if $c_1^2 - c_2 > 0$. In this case, we have $\deg X_n = n^3(c_1^2 - c_2) / \deg \psi_n$.*

Proof. Assume $n \geq 3$. Using the asymptotic form of Riemann-Roch theorem for vector bundles together with the vanishing (6), we get

$$h^0(X, S^n \Omega_X) \geq \chi(X, S^n \Omega_X) = \frac{n^3}{6}(c_1^2 - c_2) + O(n^2).$$

Thus, the positivity of the second Segre number $c_1^2 - c_2$ implies that Ω_X is big, hence ψ_n is generically finite onto its image for n sufficiently large. Conversely, suppose that ψ_n is generically finite onto X_n for some n . Then, if $\xi \in |\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)|$, the same argument used in the proof of [Rou09, Proposition 2.15] shows that

$$0 < \deg X_n \cdot \deg \psi_n = (n\xi)^3 = n^3(c_1^2 - c_2). \quad (9)$$

□

Remark 2.3. In Section 3 we will provide a quantitative version of Proposition 2.2, see in particular Corollary 3.4.

Remark 2.4. The statement of Proposition 2.2 boils down to the fact that, if we assume that the line bundle $\mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$ is nef, then it is big if and only if its top self-intersection is strictly positive.

Remark 2.5. Subsection 4.3 contains a detailed analysis of some examples where Ω_X is globally generated, $c_1^2 - c_2 = 0$ and $\dim X_n = 2$ for all $n \geq 1$. This shows that the assumption about the positivity of the second Segre number in Proposition 2.2 cannot be removed.

Remark 2.6. There exist examples of surfaces X of general type with big cotangent bundle and $c_1^2 - c_2 \leq 0$, see [RouRous13]. They are obtained by taking the minimal resolution of some singular models with rational double points, hence they contain smooth rational curves and so Ω_X is not strongly semi-ample (Lemma 1.1). However, the bigness of Ω_X implies that ψ_n is a birational

map onto its image for n large enough. This shows that the “only if” part in the statement of Proposition 2.2 does not hold if one drops strongly semi-ampleness in Assumption 2.1.

In this paper we focus on finding explicit lower bounds on n such that ψ_n is generically finite onto its image. Our arguments are geometric in nature, and generalize the ones used in [Rou09, Section 2]; furthermore, we use in an essential way some results from [PR13] and [MU19].

Let $\pi: \mathbb{P}(\Omega_X) \rightarrow X$ be the structure projection and let us look at the restriction of ψ_n to the fibre $\pi^{-1}(x)$ over a point $x \in X$. Such a fibre is the curve $\mathbb{P}(\Omega_X(x)) \simeq \mathbb{P}^1$, and the restriction of $|\mathcal{O}_{\mathbb{P}(\Omega_X)}(n)|$ to it is the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(n)|$, that embeds $\pi^{-1}(x)$ as a rational normal curve C_x of degree n in $\mathbb{P}(H^0(X, S^n\Omega_X))$. There is a unique n -dimensional linear subspace $L_x \subset \mathbb{P}(H^0(X, S^n\Omega_X))$ containing C_x , so we have a morphism

$$g_n: X \rightarrow \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X))), \quad x \mapsto L_x.$$

Definition 2.7. We call g_n the n th Gauss map of X , and we denote its image by $Y_n := g_n(X) \subset \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X)))$.

Let us now provide an alternative description of the Gauss map. Being Ω_X globally generated, for every point $x \in X$ there is a surjection

$$H^0(X, S^n\Omega_X) \rightarrow S^n\Omega_X(x) \rightarrow 0.$$

Since the fibre $S^n\Omega(x)$ of $S^n\Omega$ over x is a vector space of dimension $n + 1$, after passing to projective spaces and dualizing we obtain a quotient of dimension n of $\mathbb{P}(H^0(X, S^n\Omega_X))^*$, hence an element

$$s_x \in \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X))) \simeq \mathbb{G}(\mathbb{P}(H^0(X, S^n\Omega_X))^*, n).$$

Thus, we get a morphism

$$k_n: X \rightarrow \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X))), \quad x \mapsto s_x.$$

Following [MU19, p. 2230] we call k_n the n th Kodaira map of X .

Proposition 2.8. The two morphisms g_n and k_n do actually coincide.

Proof. Diagram (7) shows that the n -dimensional linear space L_x containing C_x coincides with the image of $\mathbb{P}(S^n\Omega_X(x))$ via the map e_n . By construction, this image is precisely s_x . \square

Proposition 2.9. The n th Gauss map g_n is a finite morphism onto its image Y_n . In particular, we have $\dim Y_n = 2$ for all $n \geq 1$.

Proof. By Proposition 2.8, it is equivalent to prove the result for the Kodaira map k_n . Let $\tilde{Y}_n \rightarrow Y_n$ be the normalization map of Y_n , and let $\tilde{k}_n: X \rightarrow \tilde{Y}_n$ be the corresponding lifting of k_n (which exists since X is smooth, hence normal). If the result is true for \tilde{k}_n then it is true for k_n as well, because the normalization is a finite map. Thus, we may assume that Y_n is normal.

Next, the assumption that $S^n\Omega_X$ is globally generated implies that $S^{tn}\Omega_X$ is globally generated for all positive integers t ; according to [MU19, Lemma 3.3], we have a factorization

$$\begin{array}{ccc} X & \xrightarrow{k_n} & Y_n \\ & \searrow k_{tn} & \nearrow u \\ & & Y_{tn} \end{array}$$

where $u: Y_{tn} \rightarrow Y_n$ is a finite map. Furthermore, by [MU19, Theorem 3.4], there exists a diagram

$$\begin{array}{ccc} X & \xrightarrow{k_{\det}} & Y_\infty \\ & \searrow k_G & \nearrow v \\ & & Y_G \end{array}$$

where k_{\det} is the litaka fibration induced by $K_X = \det \Omega_X$ and v is a finite map, such that for $t \gg 0$ we have $Y_{tn} = Y_G$ and $k_{tn} = k_G$. So, for t sufficiently large, we get

$$k_n = u \circ k_{tn} = u \circ v \circ k_{\det}.$$

Since K_X is ample (Lemma 1.1), it follows that k_{\det} is an isomorphism onto its image; therefore k_n is a composition of finite maps, and the proof is complete. \square

Lemma 2.10. *For all $p \in X_n$ (the image of ψ_n in $\mathbb{P}(H^0(X, S^n\Omega_X))$), the restriction of $\pi: \mathbb{P}(\Omega_X) \rightarrow X$ to $\psi_n^{-1}(p)$ is injective.*

Proof. Given $x \in X$, the intersection $\pi^{-1}(x) \cap \psi_n^{-1}(p)$ is either empty or consists of a single point, because the restriction $\psi_n: \pi^{-1}(x) \rightarrow C_x \subset \mathbb{P}(H^0(X, S^n\Omega_X))$ is an embedding. \square

Remark 2.11. The degree of the Gauss map divides the degree of the pluricotangent map. Denoting by U_n the universal vector bundle over the affine Grassmannian of $(n+1)$ -dimensional subspaces of $H^0(X, S^n\Omega_X)$ and by $\mathbb{P}(U_n)$ the corresponding projectivization, we have a commutative diagram

$$\begin{array}{ccccc} & & \psi_n & & \\ & & \curvearrowright & & \\ \mathbb{P}(\Omega_X) & \xrightarrow{\tilde{\psi}_n} & \mathbb{P}(U_n) & \xrightarrow{\pi_1} & \mathbb{P}(H^0(X, S^n\Omega_X)) \\ \pi \downarrow & & \downarrow \pi_2 & & \\ X & \xrightarrow{g_n} & \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X))) & & \end{array}$$

where π_1, π_2 are the natural projections and $\tilde{\psi}_n$ is such that $\pi_1 \circ \tilde{\psi}_n = \psi_n$. Since the restriction of ψ_n to the fibres of π is an embedding, it follows that $\tilde{\psi}_n$ is a finite map onto its image, whose degree is the same as the degree of the Gauss map g_n . So, assuming that the pluricotangent map ψ_n is generically finite onto

its image X_n , and denoting by $\deg \psi_n$ the degree of $\psi_n: \mathbb{P}(\Omega_X) \rightarrow X_n$, we obtain

$$\deg \psi_n = \deg g_n \cdot \deg \pi'_1,$$

where π'_1 is the restriction of π_1 to the image of $\tilde{\psi}_n$. As a consequence, if ψ_n is generically finite onto its image, then $\deg g_n$ divides $\deg \psi_n$.

If $p \in X_n$, let us denote by D'_p the subvariety $\pi(\psi_n^{-1}(p)) \subset X$. By Lemma 2.10, we have

$$\dim D'_p = \dim \psi_n^{-1}(p). \quad (10)$$

Proposition 2.12. *Let $x \in X$. Then $x \in D'_p$ if and only if $p \in C_x$.*

Proof. If $x \in D'_p$ then $x = \pi(z)$, with $z \in \psi_n^{-1}(p)$. Thus $z \in \pi^{-1}(x)$ and so $p = \psi_n(z) \in \psi_n(\pi^{-1}(x)) = C_x$. Conversely, if $p \in C_x$ then $p = \psi_n(w)$, with $w \in \pi^{-1}(x)$. Then $x = \pi(w) \in \pi(\psi_n^{-1}(p)) = D'_p$. \square

Proposition 2.13. *For all $p \in X_n$, the subvariety $\psi_n^{-1}(p) \subset \mathbb{P}(\Omega_X)$ has dimension at most 1. Hence $D'_p \subset X$ has dimension at most 1, too.*

Proof. Write $Z := \psi_n^{-1}(p)$ and $N := h^0(X, S^n \Omega_X)$. Since ψ_n is not a constant map, we have $\dim Z \leq 2$. By contradiction, assume $\dim Z = 2$. Then the restriction $\pi|_Z: Z \rightarrow X$ is a bijective morphism by Lemma 2.10 and so, since X is normal, it is an isomorphism by Zariski's Main Theorem, see [Mum99, Chapter III]. Thus we get a regular section $t: X \rightarrow \mathbb{P}(\Omega_X)$ of the projective bundle $\pi: \mathbb{P}(\Omega_X) \rightarrow X$, that in turn corresponds to a rank 1 quotient $\Omega_X \rightarrow \mathcal{L}$, where $\mathcal{L} = t^* \mathcal{O}_{\mathbb{P}(\Omega_X)}(1)$, see [Ha77, Chapter II, Proposition 7.12]. Since $\psi_n \circ t$ contracts X to the point $p \in \mathbb{P}(H^0(X, S^n \Omega_X)) \simeq \mathbb{P}^{N-1}$, it follows that $(\psi_n \circ t)^* \mathcal{O}_{\mathbb{P}^{N-1}}(1)$ is the trivial line bundle on X , and so

$$\mathcal{L}^n = t^* \mathcal{O}_{\mathbb{P}(\Omega_X)}(n) = t^* \psi_n^* \mathcal{O}_{\mathbb{P}^{N-1}}(1) = \mathcal{O}_X.$$

So, taking the n th symmetric product of $\Omega_X \rightarrow \mathcal{L}$, we get a quotient $S^n \Omega_X \rightarrow \mathcal{O}_X$, contradicting the semi-stability of $S^n \Omega_X$ with respect to K_X , see Proposition 1.3. Hence the only possibility is $\dim Z \leq 1$, and this proves the first statement. The second statement follows from (10). \square

Definition 2.14. *A point $p \in X_n$ is called exceptional for ψ_n if the fibre $\psi_n^{-1}(p)$ has dimension 1. The set of such exceptional points will be denoted by $\text{exc}(\psi_n)$.*

Let p be an exceptional point for ψ_n , so that $\dim D'_p = 1$, and let $D_p \subseteq D'_p$ be an irreducible component of dimension 1. If we set $\Sigma := \psi_n(\pi^{-1}(D_p))$, then we have

$$\Sigma = \bigcup_{x \in D_p} C_x. \quad (11)$$

Proposition 2.15. *The variety Σ is a surface in $\mathbb{P}(H^0(X, S^n \Omega_X))$ containing a 1-dimensional family of rational normal curves of degree n passing through p . More precisely, for every point $q \in \Sigma$, different from p , there exists a rational normal curve of degree n contained in Σ and joining p and q .*

Proof. The fact that Σ has dimension at least 1 is an immediate consequence of (11). By contradiction, assume $\dim \Sigma = 1$; then there is a rational normal curve C of degree n such that $C_x = C$ for all $x \in D_p$. This in turn implies that the n -plane $L_x \subset \mathbb{P}(H^0(X, S^n \Omega_X))$ is constant on D_p , and so the n th Gauss map $g_n: X \rightarrow \mathbb{G}(n, \mathbb{P}(H^0(X, S^n \Omega_X)))$ contracts D_p to a point, against Proposition 2.9. It follows that $\Sigma \subset \mathbb{P}(H^0(X, S^n \Omega_X))$ is a surface. If $q \in \Sigma$, then $q \in C_x$ for some $x \in D_p$; thus we have $p \in C_x$ by Proposition 2.12, hence the rational normal curve C_x joins p and q . \square

Definition 2.16. Let $N > 3$ be a positive integer. An irreducible variety $V \subset \mathbb{P}^{N-1}$ is said to be 2-covered by curves of degree n if a general pair of points of V can be joined by an irreducible curve of degree n .

The case of surfaces 2-covered by curves was classically considered in [Bom21]; a modern treatment can be found in [PR13, pp. 718-722], see also [Ion05, Theorem 2.8] and [PT13, Théorème 1.5]. It turns out that those maximizing the dimension of the ambient space \mathbb{P}^{N-1} are precisely the Veronese embeddings of \mathbb{P}^2 :

Proposition 2.17. If $\Sigma \subset \mathbb{P}^{N-1}$ is a non-degenerate surface which is 2-covered by curves of degree n , then $N \leq \frac{1}{2}(n+1)(n+2)$. Moreover, equality holds if and only if Σ is projectively equivalent to the n th Veronese surface $\nu_n(\mathbb{P}^2)$ and, in this case, every curve in the 2-covering family is a rational normal curve of degree n and there exists a unique such a curve passing through two distinct points of Σ .

Proof. See [PR13, Theorem 2.2]. \square

Let us study now the image of the n th cotangent map and the geometry the locus $\text{exc}(\psi_n)$; this generalizes the analysis of the case $n = 1$ that was carried out in [Rou09, Proposition 2.14].

Proposition 2.18. If $n \geq 2$ and the map $\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n \Omega_X))$ is not generically finite, then its image $X_n \subset \mathbb{P}(H^0(X, S^n \Omega_X))$ is a non-degenerate, linearly normal surface 2-covered by rational normal curves of degree n . Therefore, by the last result, it is projectively equivalent to $\nu_n(\mathbb{P}^2)$.

Proof. By Proposition 2.13 we have $2 \leq \dim X_n \leq 3$. If $\dim X_n = 2$ then every fibre of $\psi_n: \mathbb{P}(\Omega_X) \rightarrow X_n$ has dimension 1, i.e. every point $p \in X_n$ is an exceptional point for ψ_n . Hence D_p has dimension 1, whereas $\Sigma = \psi_n(\pi^{-1}(D_p))$ has dimension 2 and is contained in the irreducible surface X_n . Therefore $\Sigma = X_n$ and so, by using Proposition 2.15 and the fact that $p \in X_n$ is arbitrary, we infer that X_n is 2-covered by curves of degree n . Finally, X_n is non-degenerate and linearly normal, being the image of the morphism induced by a complete linear system in $\mathbb{P}(\Omega_X)$. \square

Theorem 2.19. Let $n \geq 2$ be an integer such that $S^n \Omega_X$ is globally generated. If $h^0(X, S^n \Omega_X) > \frac{1}{2}(n+1)(n+2)$ then ψ_n is generically finite onto its image, namely, $\dim X_n = 3$, and we have

$$\deg \psi_n \leq \frac{n^3(c_1^2 - c_2)}{h^0(X, S^n \Omega_X) - 3}. \quad (12)$$

In this case, $\text{exc}(\psi_n)$ is a Zariski-closed, possibly empty subset of $\mathbb{P}(H^0(X, S^n\Omega_X))$ of dimension at most 1.

Proof. The first statement immediately follows from Propositions 2.17 and 2.18, setting $N = h^0(X, S^n\Omega_X)$. Since X_n is a non-degenerate threefold in $\mathbb{P}(H^0(X, S^n\Omega_X))$, we have $\deg X_n \geq h^0(X, S^n\Omega_X) - 3$ and so (12) is a consequence of (9). Regarding the last statement, if ψ_n is generically finite onto its image then we have $\dim \psi_n^{-1}(\text{exc}(\psi_n)) \leq 2$, hence the exceptional locus $\text{exc}(\psi_n)$ has dimension at most 1. Such a locus is a (possibly empty) Zariski-closed subset of $\mathbb{P}(H^0(X, S^n\Omega_X))$ because ψ_n is a proper morphism, see [EGAIV, Corollaire 13.1.4]. \square

Note that, as explained in Example 1.7, the exceptional locus $\text{exc}(\psi_n)$ is empty if and only if Ω_X is ample.

3 An explicit bound for the generic finiteness of the pluri-cotangent maps

We start this section by a straightforward calculation, whose details are included because we could not find a suitable reference.

Lemma 3.1. *Let X be a compact, complex surface. We have*

$$\chi(X, S^n\Omega_X) = \frac{1}{12}(n+1)((2n^2 - 2n + 1)c_1^2 - (2n^2 + 4n - 1)c_2). \quad (13)$$

Proof. This is a standard application of the splitting principle, as stated in [Fried98, p. 28]: every universal formula on Chern classes which holds for direct sum of line bundles holds in general. Let $\mathcal{E} = L_1 \oplus L_2$ be a decomposable rank 2 vector bundle on X ; then

$$c_1(\mathcal{E}) = c_1(L_1) + c_1(L_2), \quad c_2(\mathcal{E}) = c_1(L_1)c_1(L_2). \quad (14)$$

We can compute $c_1(S^n\mathcal{E})$ as follows:

$$\begin{aligned} c_1(S^n\mathcal{E}) &= c_1(S^n(L_1 \oplus L_2)) = c_1\left(\bigoplus_{i=0}^n L_1^i \otimes L_2^{n-i}\right) \\ &= \sum_{i=0}^n (ic_1(L_1) + (n-i)c_1(L_2)) = \frac{n(n+1)}{2}(c_1(L_1) + c_1(L_2)) \\ &= \frac{n(n+1)}{2}c_1(\mathcal{E}). \end{aligned} \quad (15)$$

Let us now compute $c_2(S^n \mathcal{E})$. We have

$$\begin{aligned}
c_2(S^n \mathcal{E}) &= c_2(S^n(L_1 \oplus L_2)) = c_2\left(\bigoplus_{i=0}^n L_1^i \otimes L_2^{n-i}\right) \\
&= \sum_{0 \leq i < j \leq n} c_1(L_1^i \otimes L_2^{n-i}) c_1(L_1^j \otimes L_2^{n-j}) \\
&= \sum_{0 \leq i < j \leq n} (i c_1(L_1) + (n-i) c_1(L_2)) (j c_1(L_1) + (n-j) c_1(L_2)) \\
&= A c_1(L_1)^2 + B c_1(L_1) c_1(L_2) + C c_1(L_2)^2,
\end{aligned} \tag{16}$$

where

$$\begin{aligned}
A &= \sum_{0 \leq i < j \leq n} ij, \\
B &= \sum_{0 \leq i < j \leq n} (i(n-j) + (n-i)j), \\
C &= \sum_{0 \leq i < j \leq n} (n-i)(n-j).
\end{aligned} \tag{17}$$

The quantities in (17) can be calculated by means of the standard formulas for the sum of integers and squares, obtaining

$$\begin{aligned}
A = C &= \frac{1}{24} (n-1)n(n+1)(3n+2) \\
B &= \frac{1}{12} n(n+1)(3n^2+n+2).
\end{aligned} \tag{18}$$

Plugging (18) into (16), and taking into account (14), we get

$$c_2(S^n \mathcal{E}) = \frac{1}{24} (n-1)n(n+1)(3n+2) c_1(\mathcal{E})^2 + \frac{1}{6} n(n+1)(n+2) c_2(\mathcal{E}). \tag{19}$$

Now, the Riemann-Roch theorem for vector bundles on surfaces, see [Fried98, p. 31], implies

$$\chi(X, S^n \mathcal{E}) = \frac{c_1(S^n \mathcal{E})(c_1(S^n \mathcal{E}) - K_X)}{2} - c_2(S^n \mathcal{E}) + (n+1) \frac{c_1^2 + c_2}{12}. \tag{20}$$

Setting $\mathcal{E} = \Omega_X$ in (20) and using (15) and (19), by standard computations we obtain (13). \square

Corollary 3.2. *Let X be surface of general type with ample canonical class. Then for all $n \geq 3$ we have*

$$h^0(X, S^n \Omega_X) \geq \frac{1}{12} (n+1) ((2n^2 - 2n + 1) c_1^2 - (2n^2 + 4n - 1) c_2).$$

Proof. Combine Lemma 3.1 with Corollary 1.4. \square

Theorem 3.3. *Let X be a surface of general type satisfying Assumption 2.1 and let $n \geq 3$ be an integer such that $S^n\Omega_X$ is globally generated. If*

$$c_1^2 > \left(1 + \frac{6n-2}{2n^2-2n+1}\right)c_2 + \frac{6n+12}{2n^2-2n+1} \quad (21)$$

then the n th pluri-cotangent map ψ_n is generically finite onto its image X_n .

Proof. Using (13) one checks that (21) is equivalent to $\chi(X, S^n\Omega_X) > \frac{1}{2}(n+1)(n+2)$. Using Corollary 1.4 we infer $h^0(X, S^n\Omega_X) > \frac{1}{2}(n+1)(n+2)$, so the claim follows from Theorem 2.19. \square

We can now state the following quantitative version of Proposition 2.2.

Corollary 3.4. *Let X be a minimal surface of general type with $c_1^2 - c_2 > 0$ and such that $S^m\Omega_X$ is globally generated. Then the n th pluri-cotangent map ψ_n is generically finite onto its image for all multiples n of m such that*

$$n > \frac{\beta + \sqrt{\beta^2 - \alpha\gamma}}{\alpha},$$

where

$$\alpha := 2(c_1^2 - c_2), \quad \beta := c_1^2 + 2c_2 + 3, \quad \gamma := c_1^2 + c_2 - 12.$$

Proof. We can rewrite (21) as a quadratic inequality in n with strictly positive leading coefficient, namely

$$Q(n) = 2(c_1^2 - c_2)n^2 - 2(c_1^2 + 2c_2 + 3)n + (c_1^2 + c_2 - 12) > 0.$$

Let us define $u = c_1^2/c_2 > 1$. By the Bogomolov-Miyaoka-Yau inequality, one has $u \leq 3$. The reduced discriminant $\delta = \beta^2 - \alpha\gamma$ of $Q(n)$ is

$$\delta = (-u^2 + 4u + 6)c_2^2 + (30u - 12)c_2 + 9.$$

We see δ as a quadratic function of c_2 , depending on the parameter $u \in (1, 3]$. The reduced discriminant of δ is the function

$$\bar{\delta}(u) = 18(u-1)(13u+1),$$

which satisfies $\bar{\delta}(u) > 0$ for $u \in (1, 3]$. Since all the coefficients of δ are positive for $u \in (1, 3]$, it follows that both zeros of δ are real and negative for u in the same range. As a consequence, δ is positive for all $c_2 > 0$ and $u \in (1, 3]$. Summing up, when n is greater than the root $\frac{\beta + \sqrt{\delta}}{\alpha}$ of $Q(n)$, one has $Q(n) > 0$. \square

4 Examples and counterexamples

4.1 Example: symmetric complete intersections in abelian fourfolds of product type

Let us consider an abelian fourfold of the form $A \times E$, where A is an abelian threefold and E is an elliptic curve. Let M be a polarization on $A \times E$ and let $x \in A$ be a point which is not 2-torsion. Up to replacing the polarization M on $A \times E$ with a suitable positive multiple, by using parameter counting and Bertini-type arguments we can find two smooth hypersurfaces $Y_1, Y_2 \in |M|$ such that

- Y_1 and Y_2 are both symmetric, i.e. invariant with respect to the involution $-1: A \times E \rightarrow A \times E$;
- Y_1 and Y_2 both contain the elliptic curve $\{x\} \times E$;
- Y_1 and Y_2 do not contain any 2-torsion points of $A \times E$;
- the intersection $Y = Y_1 \cap Y_2$ is smooth.

The conditions above imply that Y is a smooth surface on which -1 acts freely; then the quotient $f: Y \rightarrow X$ provide a smooth surface X , containing an elliptic curve E' isomorphic to E .

Proposition 4.1. *The surface X is of general type with*

$$c_1(X)^2 = 2M^4, \quad c_2(X) = \frac{3}{2}M^4, \quad (22)$$

hence $c_1(X)^2 - c_2(X) = \frac{1}{2}M^4 > 0$. Moreover $H^0(X, \Omega_X) = 0$, in particular Ω_X is not globally generated. Finally, $S^n \Omega_X$ is globally generated for all even n , hence Ω_X is strongly semi-ample; however, Ω_X is not ample.

Proof. Using the short exact sequence of tangent bundles

$$0 \rightarrow T_Y \rightarrow T_{A \times E}|_Y \rightarrow N_Y \rightarrow 0,$$

and recalling that $T_{A \times E} = \mathcal{O}_{A \times E}^{\oplus 4}$ and $N_Y = \mathcal{O}_Y(M)^{\oplus 2}$, we get the equality of total Chern classes

$$c(T_Y) = c(\mathcal{O}_Y(M))^{-2}$$

that in turn yields $c_1(Y) = \mathcal{O}_Y(-2M)$ and $c_2(Y) = 3M^4$. Since $f: Y \rightarrow X$ is an étale double cover, we deduce (22). Moreover, we have $\Omega_Y = f^* \Omega_X$, hence the vector space $H^0(X, \Omega_X)$ is isomorphic to the involution-invariant subspace of $H^0(Y, \Omega_Y)$. Now, by Lefschetz theorem for Hodge groups, see [Laz04, Example 3.1.24], the global holomorphic 1-forms on Y are precisely the restrictions of those on $A \times E$, so none of them is invariant and we get $H^0(X, \Omega_X) = 0$. On the other hand, if n is even then all global sections of $S^n \Omega_{A \times E}$ are invariant and thus, when restricted to Y , they descend to X . These sections generate $S^n \Omega_{A \times E}$, hence they also generate $S^n \Omega_Y$ and $S^n \Omega_X$. Finally, Ω_X is not ample because X contains the elliptic curve E' , cf. Example 1.7. \square

Proposition 4.2. *Let X be a surface as above. Then the pluri-cotangent map $\psi_n: \mathbb{P}(\Omega_X) \rightarrow H^0(X, S^n\Omega_X)$ is generically finite onto its image for all even $n \geq 2$.*

Proof. From $\Omega_Y = f^*\Omega_X$ we infer $S^n\Omega_Y = f^*(S^n\Omega_X)$ for all $n \geq 1$. This implies that $H^0(X, S^n\Omega_X)$ is isomorphic to the involution-invariant subspace $H^0(Y, S^n\Omega_Y)^+ \subseteq H^0(Y, S^n\Omega_Y)$. On the other hand, [Deb05, Proposition 13] implies that the restriction map

$$H^0(A \times E, S^n\Omega_{A \times E}) \rightarrow H^0(Y, S^n\Omega_Y)$$

is injective for all n ; moreover, if n is even then every global section of $S^n\Omega_{A \times E}$ is invariant, and so its restriction belongs to $H^0(Y, S^n\Omega_Y)^+$. Summing up, for all even $n \geq 2$ we have

$$\begin{aligned} h^0(X, S^n\Omega_X) &= \dim H^0(Y, S^n\Omega_Y)^+ \geq \dim H^0(A \times E, S^n\Omega_{A \times E}) \\ &= \frac{1}{6}(n+1)(n+2)(n+3) > \frac{1}{2}(n+1)(n+2). \end{aligned}$$

The claim now follows from Theorem 2.19. □

4.2 Example: some product-quotient surfaces

We start by considering a hyperelliptic curve C of genus 3, endowed with an action of the cyclic group $G = \mathbb{Z}_2$ as follows. The curve C has affine equation of the form

$$y^2 = a_8x^8 + a_6x^6 + a_4x^4 + a_2x^2 + a_0,$$

where the coefficients a_i are such that the zeros of the polynomial at the right side are distinct. If g is the generator of G , we define the action of G on C as

$$g(x, y) = (-x, y).$$

One checks that g has the four fixed points

$$(0, \sqrt{a_0}), \quad (0, -\sqrt{a_0}), \quad (\infty, \sqrt{a_8}), \quad (\infty, -\sqrt{a_8}),$$

hence the quotient map $C \rightarrow C/G$ is branched at four points and, by the Hurwitz formula, the curve C/G has genus 1.

Lemma 4.3. *Let $H^0(C, \omega_C) = V^+ \oplus V^-$ be the decomposition of $H^0(C, \omega_C)$ into the G -invariant subspace V^+ and the G -antiinvariant subspace V^- . Then there exists a basis $\{\xi_1, \xi_2, \xi_3\}$ of $H^0(C, \omega_C)$ such that*

$$V^+ = \langle \xi_1 \rangle, \quad V^- = \langle \xi_2, \xi_3 \rangle$$

and the three canonical divisors $\text{div}(\xi_1), \text{div}(\xi_2), \text{div}(\xi_3)$ have pairwise disjoint supports.

Proof. The vector space $H^0(C, \omega_C)$ is generated by the holomorphic 1-forms which, in affine coordinates, can be written as

$$\omega_1 := \frac{dx}{y}, \quad \omega_2 := x \frac{dx}{y}, \quad \omega_3 := x^2 \frac{dx}{y}.$$

Note that ω_2 is G -invariant, whereas ω_1 and ω_3 are G -antiinvariant. Now set $\xi_1 := \omega_2$ and take as ξ_2 and ξ_3 two general elements in the G -antiinvariant subspace $\langle \omega_1, \omega_3 \rangle$. \square

Let us consider now another curve F with a G -action.

Lemma 4.4. *Let $g = 2k + 1 \geq 5$ be an odd integer. Then there exists a curve F of genus g , endowed with a free G -action having the following property. Denoting by $H^0(F, \omega_F) = W^+ \oplus W^-$ the decomposition of $H^0(F, \omega_F)$ into G -invariant and G -antiinvariant subspace, we can find a basis $\{\tau_1, \dots, \tau_g\}$ of $H^0(F, \omega_F)$ such that*

$$W^+ = \langle \tau_1, \dots, \tau_{k+1} \rangle, \quad W^- = \langle \tau_{k+2}, \dots, \tau_g \rangle$$

and the g canonical divisors $\text{div}(\tau_1), \dots, \text{div}(\tau_g)$ have pairwise disjoint supports.

Proof. Let D be a curve of genus $k + 1$ and let \mathcal{L} be a non-trivial line bundle on D such that $\mathcal{L}^2 = \mathcal{O}_D$. Then there exists an étale double cover $f: F \rightarrow D$, with F of genus $2k + 1$ and $f_*\mathcal{O}_F = \mathcal{O}_D \oplus \mathcal{L}^{-1}$; the curve F comes with a free G -action, corresponding to the automorphism exchanging the two sheets of the cover. Furthermore, since $f_*\omega_F = \omega_D \oplus (\omega_D \otimes \mathcal{L})$, we deduce

$$W^+ = f^*H^0(D, \omega_D), \quad W^- = f^*H^0(D, \omega_D \otimes \mathcal{L}).$$

Therefore the desired result follows if both ω_D and $\omega_D \otimes \mathcal{L}$ are globally generated. In fact, for a base-point free line bundle, a general section avoids any given finite set of points, so we can choose recursively a basis where each section avoids the base loci of the previous ones; moreover, this property is preserved by étale pullbacks. It is well known that ω_D is base-point free, see for instance [Ha77, Lemma 5.1 p. 341]. Regarding $\omega_D \otimes \mathcal{L}$, a point p is in its base locus if and only if

$$H^0(D, \omega_D \otimes \mathcal{L}(-p)) = H^0(D, \omega_D \otimes \mathcal{L}),$$

namely, if and only if $h^1(D, \omega_D \otimes \mathcal{L}(-p)) = 1$. By Serre duality, this is equivalent to the fact that there exists $q \in D$ such that the divisor class of \mathcal{L} is of the form $q - p$. But then $\mathcal{O}_D(2q - 2p) = \mathcal{O}_D$, hence the linear system spanned by $2p$ and $2q$ is a g_2^1 on D and so D is hyperelliptic. Summing up, $\omega_D \otimes \mathcal{L}$ is globally generated if and only if one of these conditions (both implying $g(D) \geq 3$, and so $g = g(F) \geq 5$) hold:

- D is non-hyperelliptic;
- D is hyperelliptic and the divisor class of \mathcal{L} is not the difference of two Weierstrass points.

This concludes the proof. \square

Taking C and F as above, for all $k \geq 2$ we can now define a product-quotient surface $X = (C \times F)/G$, where G acts diagonally on the product. Such action is free (because the action of G on F is so), hence X is a smooth surface of general type, whose invariants are

$$p_g(X) = 3k + 1, \quad q(X) = k + 2, \quad K_X^2 = 16k.$$

Thus $h^0(X, \Omega_X) = k + 2$ and $c_1(X)^2 - c_2(X) = 8k > 0$. The natural projections of $C \times F$ induce two isotrivial fibrations $X \rightarrow F/G$ and $X \rightarrow C/G$, whose general fibres are isomorphic to C and F , respectively. In particular, the cotangent bundle Ω_X is not ample, see [Rou09, Corollaire 3.8].

Let us now show that Ω_X is not globally generated, either, but that it is nevertheless strongly semi-ample.

Proposition 4.5. *Let C and F be curves with a G -action as above, and $X = (C \times F)/G$. Then Ω_X is not globally generated, whereas its second symmetric power $S^2\Omega_X$ is globally generated (and so $S^n\Omega_X$ is globally generated for all even n).*

Proof. Denoting by $\pi_C: C \times F \rightarrow C$, $\pi_F: C \times F \rightarrow F$ the two natural projections, we have

$$\Omega_{C \times F} = L \oplus M,$$

where $L = \pi_C^*\omega_C$, $M = \pi_F^*\omega_F$. Moreover, the covering map $C \times F \rightarrow X$ being étale, for all $n \geq 1$ we have $H^0(X, S^n\Omega_X) = H^0(C \times F, S^n\Omega_{C \times F})^G$. Thus, since the action of G on $C \times F$ does not exchange the two factors, in order to show that $S^n\Omega_X$ is globally generated it suffices to show that it is possible to generate every summand $L^k \otimes M^{n-k}$ of $S^n\Omega_{C \times F}$ by using G -invariant global sections.

In the case $n = 1$, the space of invariant global sections of $\Omega_{C \times F}$ is

$$V^+ \oplus W^+ = \langle \xi_1 \rangle \oplus \langle \tau_1, \dots, \tau_{k+1} \rangle.$$

This shows that Ω_X is not globally generated, since V^+ is 1-dimensional and so it is not possible to generate the summand L by means of G -invariant global sections.

In the case $n = 2$, we have

$$S^2\Omega_{C \times F} = L^2 \oplus (L \otimes M) \oplus M^2.$$

We recall that $\text{div}(\xi_i)$ are pairwise disjoint divisors, and the same is true for $\text{div}(\tau_j)$, see Lemmas 4.3 and 4.4; thus, using the notation $\alpha \boxtimes \beta$ for $\pi_C^*\alpha \otimes \pi_F^*\beta$, we can say that

- the G -invariant sections $(\xi_1)^2, (\xi_2)^2$ generate L^2 ;
- the G -invariant sections $\xi_1 \boxtimes \tau_1, \xi_2 \boxtimes \tau_{k+2}, \xi_3 \boxtimes \tau_{k+3}$ generate $L \otimes M$;
- the G -invariant sections $(\tau_1)^2, (\tau_2)^2$ generate M^2 .

This shows that $S^2\Omega_X$ is globally generated. \square

Proposition 4.6. *Let $k \geq 2$ and X be the surface constructed above. Then the pluricotangent map $\psi_n: \mathbb{P}(\Omega_X) \rightarrow H^0(X, S^n \Omega_X)$ is generically finite onto its image for all even n .*

Proof. We first analyze the case $n = 2$. Using the G -invariant global sections of $S^2 \Omega_X$ produced at the end of the proof of Proposition 4.5, we get

$$\begin{aligned} h^0(X, S^2 \Omega_X) &= \dim [H^0(X, L^2)^G \oplus H^0(X, L \otimes M)^G \oplus H^0(X, M^2)^G] \\ &\geq 2 + 3 + 2 = 7. \end{aligned}$$

Thus, for all $k \geq 2$ we obtain $h^0(X, S^2 \Omega_X) > 6$, hence ψ_2 is generically finite onto its image by Theorem 2.19. Now, let us consider the general case. We have $c_1(X)^2 = 16k$ and $c_2(X) = 8k$ and so, by Corollary 3.4, the n th pluricotangent map ψ_n is generically finite onto its image for all even n such that

$$n > \frac{32k + 3 + \sqrt{640k^2 + 384k + 9}}{16k}.$$

Straightforward calculations show that function $h(k) = \frac{32k+3+\sqrt{640k^2+384k+9}}{16k}$ is strictly decreasing in the interval $[2, +\infty)$ and that $h(2) \simeq 3.9$. Thus, for k in the same interval, we have $n > h(k)$ as soon as $n \geq 4$, and this completes the proof. \square

4.3 Counterexamples to the generic finiteness of ψ_n : smooth ample divisors in abelian threefolds

We will now show that the assumption $h^0(X, S^n \Omega_X) > \frac{1}{2}(n+1)(n+2)$ in Theorem 2.19 cannot be dropped. In fact, we will provide examples of surfaces X of general type, with Ω_X globally generated, such that $h^0(X, S^n \Omega_X) = \frac{1}{2}(n+1)(n+2)$ and X_n is the n th Veronese surface for all $n \geq 1$. Thus, no pluricotangent map of X is generically finite onto its image. All these surfaces satisfy $c_1^2 - c_2 = 0$.

Let (A, M) be a polarized abelian threefold, with polarization M of type (d_1, d_2, d_3) , such that there exists a smooth element $X \in |M|$. By the Riemann-Roch Theorem and the ampleness of M , we have

$$h^0(A, M) = \chi(A, M) = \frac{1}{6}M^3 = d_1 d_2 d_3,$$

see [BL04, Chapter 3]. Moreover, by adjunction we get $\omega_X = \mathcal{O}_X(X)$ and so

$$K_X^2 = M^3 = 6h^0(A, M).$$

From the short exact sequence

$$0 \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}_A(X) \rightarrow \omega_X \rightarrow 0$$

we infer

$$p_g(X) = h^0(A, M) + 2, \quad q(X) = 3,$$

hence $\chi(\mathcal{O}_X) = h^0(A, M)$. Summing up, X is a minimal surface of general type with $K_X^2 = 6\chi(\mathcal{O}_X)$, namely $c_1(X)^2 - c_2(X) = 0$; moreover the canonical bundle Ω_X is globally generated (cf. Example 1.8) and so K_X is ample (Lemma 1.1).

Proposition 4.7. *For all $n \geq 1$, we have*

$$H^0(X, S^n \Omega_X) = S^n H^0(X, \Omega_X) \simeq \mathbb{C}^{\frac{(n+1)(n+2)}{2}}. \quad (23)$$

Furthermore, the image X_n of $\psi_n: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, S^n \Omega_X))$ is projectively equivalent to the n th Veronese surface $\nu_n(\mathbb{P}^2) \subset \mathbb{P}^{\frac{n(n+3)}{2}}$.

Proof. Since Ω_X is globally generated and $h^0(X, \Omega_X) = 3$, we have a short exact sequence

$$0 \rightarrow \mathcal{O}_X(-K_X) \rightarrow H^0(X, \Omega_X) \otimes \mathcal{O}_X \rightarrow \Omega_X \rightarrow 0. \quad (24)$$

From [Eis94, p. 577] it follows that (24) gives rise to a short exact sequence

$$0 \rightarrow S^{n-1} H^0(X, \Omega_X) \otimes \mathcal{O}_X(-K_X) \rightarrow S^n H^0(X, \Omega_X) \otimes \mathcal{O}_X \rightarrow S^n \Omega_X \rightarrow 0, \quad (25)$$

where the exactness on the left follows by comparing ranks. The canonical divisor K_X is effective and ample, so $h^0(X, -K_X) = h^1(X, -K_X) = 0$; taking cohomology in (25), we obtain (23). As a consequence, the map $\mathbb{P}(\sigma_n)$ in diagram (8) is an isomorphism for all n . Since the 1-cotangent map $\psi_1: \mathbb{P}(\Omega_X) \rightarrow \mathbb{P}(H^0(X, \Omega_X)) \simeq \mathbb{P}^2$ is surjective, the image of ψ_n must coincide, up to a projective transformation, with the image of ν_n . \square

Remark 4.8. Another way to state (23) is saying that the restriction map

$$H^0(A, S^n \Omega_A) \rightarrow H^0(X, S^n \Omega_X)$$

is an isomorphism for all $n \geq 1$, cf. [Deb05, Proposition 13].

Remark 4.9. When M is a principal polarization, namely $(d_1, d_2, d_3) = (1, 1, 1)$, the surface X is a smooth theta divisor of A and we get $p_g(X) = q(X) = 3$, $K_X^2 = 6$. In this case, X is isomorphic to the second symmetric product $\text{Sym}^2 C$, where C is a smooth, non-hyperelliptic curve of genus 3, see [CaCiML98, p. 304].

The next result shows that, if one assumes that the Albanese image is smooth, then the previous counterexamples are the only ones up to finite étale covers. Recall that an abelian cover is a Galois cover with abelian Galois group.

Proposition 4.10. *If Y is a smooth surface of general type with smooth Albanese image, then the following are equivalent.*

- (1) Ω_Y is globally generated and the pluri-cotangent image Y_n has dimension 2 for all $n \geq 1$.

- (2) Ω_Y is globally generated and the 1-cotangent image Y_1 has dimension 2.
- (3) Y is a finite, étale cover of a smooth, ample divisor in an abelian threefold.
- (4) Y is a finite, étale abelian cover of a smooth, ample divisor in an abelian threefold.

Proof. (1) \implies (2) Obvious.

(2) \implies (3) If (2) holds, then by [Rou09, Proposition 2.14] we have $q(Y) = 3$, hence $A := \text{Alb}(Y)$ is an abelian threefold and the Albanese map $a_Y: Y \rightarrow A$ is a local immersion onto its smooth image $X \subset A$. This implies that $Y \rightarrow X$ is a local bihomomorphism between compact complex manifolds, hence an unramified analytic cover ([Lee11, Problem 11-9 p. 303]), which is actually an algebraic cover by GAGA. By adjunction, the surface X satisfies $0 < K_X^2 = X^3$, so the divisor X is ample in A by [BL04, Proposition 4.5.2].

(3) \implies (4) Let $X \subset A$ be a smooth, ample divisor in an abelian threefold and let $f: Y \rightarrow X$ be a finite, étale cover. By Lefschetz hyperplane theorem [Laz04, Theorem 3.1.21] it follows $\pi_1(X) = \pi_1(A) = \mathbb{Z}^6$; thus, since the fundamental group of X is abelian, the cover $f: Y \rightarrow X$ is Galois, with abelian Galois group.

(4) \implies (1) Let $X \subset A$ be a smooth ample divisor in an abelian threefold and let $f: Y \rightarrow X$ be a finite, étale abelian cover. By [Par91, p. 200], there exist non-trivial torsion divisors $L_1, \dots, L_s \in \text{Pic}^0(X)$ such that

$$f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L_1 \oplus \dots \oplus L_s.$$

Since the cover is étale, we have $S^n\Omega_Y = f^*(S^n\Omega_X)$ for all $n \geq 1$. Thus $S^n\Omega_Y$ is globally generated (because $S^n\Omega_X$ is) and, by projection formula, we get

$$f_*S^n\Omega_Y = S^n\Omega_X \oplus (S^n\Omega_X \otimes L_1) \oplus \dots \oplus (S^n\Omega_X \otimes L_s). \quad (26)$$

Since the divisor L_j is not effective and $K_X - L_j$ is ample, we get

$$H^0(X, L_j) = H^1(X, -K_X + L_j) = 0.$$

Thus, tensoring (25) with L_j and passing to cohomology, we deduce $H^0(X, S^n\Omega_X \otimes L_j) = 0$ for all $j \in \{1, \dots, s\}$. Hence (26) yields $H^0(Y, S^n\Omega_Y) = f^*H^0(X, S^n\Omega_X)$, which in turn implies $Y_n = X_n$. By Proposition 4.7 it follows that Y_n has dimension 2 for all $n \geq 1$. \square

Remark 4.11. The argument in the last part of the proof of Proposition 4.10 can be also used in order to construct examples of surfaces of general type having Gauss maps of arbitrarily large degree. For the sake of simplicity, let us just consider finite, étale cyclic covers. Let $X \subset A$ be a smooth ample divisor in an abelian threefold and let L be a non-trivial element of order k in $\text{Pic}^0(X)$. These data define an étale \mathbb{Z}_k -cover $f: Y \rightarrow X$, where

$$f_*\mathcal{O}_Y = \mathcal{O}_X \oplus L \oplus \dots \oplus L^{k-1}.$$

As before, we get $H^0(Y, S^n\Omega_Y) = f^*H^0(X, S^n\Omega_X)$, which in turn implies

$$\mathbb{G}(n, \mathbb{P}(H^0(Y, S^n\Omega_Y))) = \mathbb{G}(n, \mathbb{P}(H^0(X, S^n\Omega_X))).$$

This shows that, for all $n \geq 1$, the n th Gauss map of Y factors through the degree k cover $f: Y \rightarrow X$, and so its degree is a multiple of k . Since k is arbitrary, this construction provides smooth, minimal surfaces of general type Y , all of whose Gauss maps have arbitrarily large degree.

5 Open problems

We end the paper with a couple of open problems.

Open Problem 1. *Are there any examples of minimal surfaces of general type such that $S^n\Omega_X$ is globally generated for some n and $h^0(X, S^n\Omega_X) < \frac{1}{2}(n+1)(n+2)$ holds? If such examples exist, what is the behaviour of the pluri-cotangent map ψ_n ?*

This question was asked by the first Author in the MathOverflow thread [MO430570](#), without any answer so far. It is motivated by the fact that, in all the examples that we are able to compute, if $S^n\Omega_X$ is globally generated then $h^0(X, S^n\Omega_X) \geq \frac{1}{2}(n+1)(n+2)$. The equality is attained, for instance, by finite étale covers of smooth ample divisors in abelian threefolds, see Subsection 4.3.

Open Problem 2. *How should one modify Proposition 4.10 if one removes the smoothness assumption for the Albanese image of Y ? Are there new examples with non-generically finite pluricotangent maps that appear?*

References

- [ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris: *Geometry of algebraic curves*, Springer-Verlag 1985. (Cited on p. 7)
- [Bog78] F. A. Bogomolov: Holomorphic tensors and vectors bundles on projective varieties (in Russian), *Izv. Akad. Nauk. SSSR Ser. Mat.* **42** (1978), 1227-1287; English translation in *Math. USSR-Izv.* **13** (1979), 499-555. (Cited on p. 4)
- [Bom21] E. Bompiani: Proprietà differenziali caratteristiche di enti algebrici, *Rom. Acc. L. Mem.* **26** (1921), 452–474. (Cited on p. 2, 12)
- [BL04] C. Birkenhake, H. Lange: *Complex abelian varieties*, Grundlehren der mathematischen Wissenschaften **302**, Second Edition, Springer 2004. (Cited on p. 20, 22)
- [CaCiML98] F. Catanese, C. Ciliberto and M. M. Lopes: Of the classification of irregular surfaces of general type with non birational bicanonical map, *Trans. of the Amer. Math. Soc.* **350** (1998), 275–308. (Cited on p. 21)

- [Deb05] O. Debarre: Varieties with ample cotangent bundle, *Compositio Math.* **141** (2005), 1445-1459. (Cited on p. [17](#), [21](#))
- [EGAIV] A. Grothendieck, with J. Dieudonné: *Eléments de Géométrie Algébrique IV. Étude locale des schémas et des morphismes de schémas*, Troisième partie, *Inst. Hautes Études Sci. Publ. Math.* **28** (1966), 1–255 (Cited on p. [13](#))
- [Eis94] D. Eisenbud: *Commutative algebra with a view toward algebraic geometry*, Graduate Texts in Mathematics **150**, Springer 1994. (Cited on p. [21](#))
- [Fried98] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Universitext, Springer 1998. (Cited on p. [13](#), [14](#))
- [Ha77] R. Hartshorne: *Algebraic Geometry*, Graduate Texts in Mathematics **52**, Springer 1977. (Cited on p. [11](#), [18](#))
- [Ion05] P. Ionescu: Birational geometry of rationally connected manifolds via quasi-lines, *Projective varieties with unexpected properties*, de Gruyter 2005, 317–335. (Cited on p. [12](#))
- [Kl69] S. L. Kleiman: Ample vector bundles on algebraic surfaces, *Proc. American Math. Soc.* **21** (1969), 673–676. (Cited on p. [7](#))
- [Kob80] S. Kobayashi: First Chern class and holomorphic tensor fields, *Nagoya Math. J.* **77** (1980), 5–11. (Cited on p. [4](#), [5](#))
- [Laz04] R. Lazarsfeld: *Positivity in algebraic geometry I-II*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **48**, Springer 2004. (Cited on p. [3](#), [6](#), [7](#), [8](#), [16](#), [22](#))
- [Lee11] J. M. Lee: *Introduction to Topological Manifolds*, Graduate Texts in Mathematics **202**, Springer 2011. (Cited on p. [22](#))
- [MU19] E. Mistretta, S. Urbinati: Iitaka fibrations for vector bundles, *Int. Math. Res. Not.* vol. **2019**, no. 7, 2223–2240. (Cited on p. [2](#), [4](#), [9](#), [10](#))
- [Mum99] D. Mumford: *The red book of varieties and schemes*. Second, expanded edition. Includes the Michigan lectures (1974) on curves and their Jacobians. With contributions by Enrico Arbarello. *Lecture Notes in Mathematics* **1358**, Springer 1999. (Cited on p. [11](#))
- [Par91] R. Pardini: Abelian covers of algebraic varieties, *J. reine angew. Math.* **417** (1991), 191-213. (Cited on p. [22](#))
- [PR13] L. Pirio, F. Russo: Varieties n -covered by curves of degree δ , *Comment. Math. Helvet.* **88** (2013), 715–757. (Cited on p. [2](#), [9](#), [12](#))
- [PT13] L. Pirio, J. M. Trépreau: Sur les variétés $X \subset \mathbb{P}^N$ telles que par n points passe une courbe de X de degré donné, *Bull. Soc. math. France* **141** (1) (2013), p. 131–195. (Cited on p. [12](#))

- [PY07] G. Prasad, S.K. Yeung: Fake projective planes, *Invent. Math.* **168** (2007), 321–370. (Cited on p. 7)
- [Reid77] M. Reid: Bogomolov’s theorem $c_1^2 \leq 4c_2$, *Proceedings of the International Symposium on Algebraic Geometry - Kyoto 1977*, 623–642, Kinokuniya Book Store, Tokyo 1978. (Cited on p. 6)
- [Rou09] X. Roulleau: L’application cotangente des surfaces de type générale, *Geom. Dedicata* **142** (2009), 151–171. (Cited on p. 1, 2, 7, 8, 9, 12, 19, 22)
- [RouRous13] X. Roulleau, E. Rousseau: Canonical surfaces with big cotangent bundle, *Duke Math. J.* **163**, no. 7 (2014), 1337–1351. (Cited on p. 4, 8)

Francesco Polizzi

Dipartimento di Matematica e Informatica, Università della Calabria
Cubo 30B, 87036 Arcavacata di Rende (Cosenza), Italy.
E-mail address: francesco.polizzi@unical.it

Xavier Roulleau

Laboratoire angevin de recherche en mathématiques, LAREMA, UMR 6093 du
CNRS, UNIV. Angers, SFR MathStic, 2 Bd Lavoisier, 49045 Angers Cedex 01,
France
E-mail address: Xavier.Roulleau@univ-angers.fr