On the geometry of Einstein-type structures

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Abstract

The aim of the paper is to study the geometry of a Riemannian manifold M, with a special structure depending on 3 real parameters, a smooth map φ into a target Riemannian manifold N, and a smooth function f on M itself. We will occasionally let some of the parameters be smooth functions. For a special value of one of them, the structure is obtained by a conformal deformation of a harmonic-Einstein manifold. The setting generalizes various previously studied situations; for instance, Ricci solitons, Ricci harmonic solitons, generalised quasi-Einstein manifolds and so on. One main ingredient of our analysis is the study of certain modified curvature tensors on M, related to the map φ, and to develop a series of results for harmonic-Einstein manifolds that parallel those obtained for Einstein manifolds both some time ago and in the very recent literature. We then turn to locally characterize, via a couple of integrability conditions and mild assumptions on f, the manifold M as a warped product with harmonic-Einstein fibers extending in a very non trivial way a recent result for Ricci solitons. We then consider rigidity and non existence, both in the compact and non-compact cases. This is done via integral formulas and, in the non-compact case, via analytical tools previously introduced by the authors.

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1 Introduction

The aim of this paper is to study the geometry of connected, complete, possibly compact, Riemannian manifolds $(M, \langle\, ,\, \rangle)$ with a (gradient) Einstein-type structure, if any, of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{1.1}$$

where

$$\mathrm{Ric}^{\varphi} := \mathrm{Ric} - \alpha \varphi^{*} \langle\, ,\, \rangle_N$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ a smooth map with tension field $\tau(\varphi)$ and $f, \mu, \lambda \in \mathcal{C}^{\infty}(M)$.

The structure described by (1.1) generalizes some well known particular cases that have been intensively studied by researchers in the last decade. Indeed, for $\mu \equiv 0$, $\lambda \in \mathbb{R}$ and φ constant, (1.1) characterizes gradient Ricci solitons

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda \langle\, ,\, \rangle. \tag{1.2}$$

In case in (1.2) we allow $\lambda \in \mathcal{C}^{\infty}(M)$ we obtain the Ricci almost soliton equation introduced in [38]. Note that when $\lambda(x) = a + bS(x)$ for some constants $a, b \in \mathbb{R}$ and $S(x)$ the scalar curvature of $(M, \langle\, ,\, \rangle)$, the soliton corresponding to (1.2) is called a Ricci-Bourguignon soliton after the recent work of G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri [15]. For a "flow"derivation of the gradient Ricci almost solitons equation in the general case see the work of [21].

In case $\mu \equiv 0$, $\lambda \in \mathbb{R}$ and $\alpha > 0$ the system (1.1) represents Ricci-harmonic solitons introduced by R. Müller, [32]. As expected the concept comes from the study of a combination of the Ricci and harmonic maps flows. We refer to [32] for details and interesting analytic motivations.

For φ and μ constants with $\mu = \frac{1}{k}$, for some $k > 0$ and $\lambda \in \mathbb{R}$, (1.1) describes quasi-Einstein manifolds

$$\mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{k} df \otimes df = \lambda \langle\, ,\, \rangle \tag{1.3}$$

Letting $\mu, \lambda \in \mathcal{C}^{\infty}(M)$ we obtain the generalized quasi-Einstein condition

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\, ,\, \rangle. \tag{1.4}$$

See, for instance, [14] and [4]. Obviously (1.4) extends the quasi-Einstein requirement (1.3) that we shall later consider.

Ricci solitons and quasi-Einstein manifolds are often seen as a perturbation of Einstein manifolds (indeed, the choice of a constant potential in (1.2) and in (1.3) recovers the case of an Einstein metric). Similarly, Einstein-type structures can be seen as a perturbation of harmonic-Einstein manifolds. We recall that a Riemannian manifold $(M, \langle\, ,\, \rangle)$ is said to be harmonic-Einstein if it carries a structure of the type

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = 0, \end{cases} \tag{1.5}$$

We shall see that when $m \geq 3$, λ in (1.5) is necessarily a constant. Clearly (1.5) is obtained from (1.1) in case f is constant.

Note that the first equation of (1.5) can be equivalently rewritten (and we are not using that the metric is Riemannian) as

$$G + \Lambda \langle\, ,\, \rangle = \alpha T, \tag{1.6}$$

where G is the Einstein tensor of $(M, \langle\, ,\, \rangle)$,

$$T = \varphi^{*} \langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{m} \langle\, ,\, \rangle$$

is (up to a sign) the energy stress tensor of the smooth map φ introduced by Baird and Eells in [5] and

$$\Lambda = \frac{m-2}{2}\lambda. \tag{1.7}$$

In case φ is harmonic then T is divergence free and thus four dimensional Lorentzian manifolds that satisfy (1.5) with

$$\alpha = \frac{8\pi\hat{G}}{c^4},$$

where $\hat{G}$ is Newton's gravitational constant and c is the speed of light in vacuum, coincide with solutions of the Einstein fields equations (1.6) with source field the wave map φ and cosmological constant (1.7).

System (1.5) is a starting point in our investigation in the sense that it justifies, in a geometric contest, the interest of studying a structure of the type (1.1). Indeed, as we show in Theorem 2.49 below, if we perform a conformal deformation of the metric $\langle\, ,\, \rangle$ of M, then from (1.5) we obtain a solution of (1.1) for $m \geq 3$ with $\mu = -\frac{1}{m-2}$ and viceversa for an appropriate λ. Thus we can think of the study of (1.1) as of that of (1.5) under conformal deformations of the original metric $\langle\, ,\, \rangle$ of M, up to the freedom of the parameter μ.

This parallels what happens in the study of Einstein and conformally Einstein metrics. This observation suggests to concentrate our study first on the behaviour of Ric^{φ}. Since the latter is defined only in terms of Ric and φ, we push our analysis as far as possible without coupling φ to f, appearing in (1.1), via the condition $\tau(\varphi) = d\varphi(\nabla f)$. The study of Ric^{φ} is realized in Section 2 where we introduce what we have called φ-curvatures and for which we investigate a number of properties similar to those of the usual curvatures derived from the Riemann tensor and its covariant derivatives. As clearly expected the geometry of the map φ comes into the picture but often not so strongly to deviate the behaviour of the φ-curvatures from that of the corresponding Riemannian counterparts.

Almost all the φ-curvatures are formally defined in the way the standard curvatures are introduced using the φ-Ricci tensor instead of the Ricci tensor. More precisely: the φ-scalar curvature, denoted by S^{φ}, is defined as the trace of the φ-Ricci tensor; the φ-Schouten tensor is defined as

$$A^{\varphi} = \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)}\langle\, ,\, \rangle,$$

where $m \geq 2$ is the dimension of M; the φ-Cotton tensor C^{φ} represents the obstruction to the commutation of the covariant derivatives of the φ-Schouten tensor while the φ-Weyl tensor is defined so that the decomposition (2.6) of the Riemannian curvature tensor holds in analogy with the standard one. The only tensor whose definition is different from that probably expected is the φ-Bach tensor B^{φ}. Indeed, its definition is motivated by geometric considerations, notably the integrability conditions (1.9) and Theorem 6.66 below. When φ is a constant map all the φ-curvatures reduce to the standard curvature tensors.

The properties of the φ-curvature tensors parallel those of the Riemannian tensors they generalize. For instance, the φ-Weyl tensor W^{φ} has the same symmetries of the Riemann tensor and its $(1,3)$-version is a conformal invariant, as it can be easily verified by a tedious computation. A relevant difference is that the φ-Cotton, the φ-Weyl and the φ-Bach tensor are not, in general, totally traceless. Their traces are related to the map φ and, clearly, they vanish in case φ is a constant map. We can say more: the φ-Weyl, the φ-Cotton and the φ-Bach tensors are totally traceless if and only if, respectively, φ is constant, φ is conservative (that is, the energy stress tensor related to the map φ is divergence free) and φ is harmonic (with the exceptional case $m = 4$ where φ-Bach is always traceless).

The fact that the above φ-curvatures are not, in general, totally traceless has heavy computational consequences but basic facts are still true. For instance if φ is conservative we are able to recover a generalization of Schur's identity, that relates the divergence of φ-Ricci to the gradient of the φ-scalar curvature. On the contrary the divergence of φ-Weyl is not related with the φ-Cotton tensor as in the case of their standard Riemannian counterparts, see equation (2.65).

We observe that the special system obtained in Theorem 2.49 we mentioned above, that is,

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \dfrac{1}{m-2} df \otimes df = \lambda \langle \, , \, \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{1.8}$$

has some peculiar features. For instance it satisfies the two integrability conditions

$$\begin{cases} C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} = 0 \\ (m-2) B^{\varphi}_{ij} + \dfrac{m-4}{m-2} W^{\varphi}_{tijk} f_t f_k = 0 \end{cases} \tag{1.9}$$

where $m \geq 3$ is the dimension of M and C^{φ}, W^{φ} and B^{φ} are respectively the φ-Cotton, the φ-Weyl and the φ-Bach tensors. When φ is constant the above integrability conditions become the integrability condition for a conformally Einstein metric, that have been proved to be sufficient, under a further mild assumption, to guarantee the existence of a conformally Einstein metric on M by R. Gover and P. Nurowski, [22]. We extend this result to the case of (1.9) showing, under a corresponding mild additional assumption, that they are sufficient conditions to generate a conformally harmonic-Einstein structure on M, see Proposition 2.63. Observe that in (1.8) the coefficient $\mu = -\frac{1}{m-2}$.

In case φ is a constant map and $\mu = 0$ a special form of the integrability conditions in (1.9) (see equations (6.12) and (6.17) for the general case where φ is not constant and $\mu \neq 0$) has been used to study the local geometry of Bach flat gradient Ricci solitons by H.-D. Cao, Q. Chen in [11]. Their results has been extended by G. Catino, P. Mastrolia, D. Monticelli and M. Rigoli to gradient Einstein-type manifolds in Theorem 1.2 of [16]. The latter are structure of the type (1.1) with φ a constant map, $\mu \in \mathbb{R}$ and $\lambda(x) = \rho S(x) + \lambda$ for some real constants ρ and λ.

These results suggest to study (1.1) from the same point of view and in Section 6 we are able to characterize, when $\mu \neq -\frac{1}{m-2}$ (the equality case pertaining to Theorem 2.49), from the adequate integrability conditions and the properness of the function f, the local geometry of a complete Riemannian manifold with a gradient Einstein-type structure and φ-Bach tensor that vanishes along the direction of ∇f. Note that for conformally harmonic-Einstein manifolds the latter requirement is always satisfied, as one can immediately deduce contracting the second equation of (1.9) against ∇f. Our main result, Theorem 6.66 below, is that, in a neighborhood of every regular level set of f, the manifold $(M, \langle \, , \, \rangle)$ is a warped product with $(m-1)$-dimensional harmonic-Einstein fibers, given by the level sets of f. Precisely, we have:

Theorem 1.10. *Let $(M, \langle \, , \, \rangle)$ be a complete, non-compact Riemannian manifold m with an Einstein-type structure as in (6.1). Suppose that $m \geq 3$, that $\alpha > 0$, that $B^{\varphi}(\nabla f, \cdot\,) = 0$ and $\mu \neq 1/(2-m)$ and that f is proper. Then, in a neighborhood of every regular level set of f, the Riemannian manifold $(M, \langle \, , \, \rangle)$ is locally a warped product with $(m-1)$-dimensional harmonic-Einstein fibers.*

We underline that, computationally speaking, this section is a real "tour de force".

In Section 3 we consider the traceless φ-Ricci tensor T^{φ}. In Theorem 3.5 we prove the basic formula (3.6) for $\Delta|T^{\varphi}|$ that we use in the main result of the section; the "gap"property given in Corollary 3.18 that shows that whenever $|T^{\varphi}|$ is sufficiently small, then $(M, \langle \, , \, \rangle)$ carries a harmonic-Einstein type structure, if some necessary conditions are satisfied. One of them involves the largest eigenvalue η^* of the operator $\mathcal{W}^{\varphi} : S^2_0(M) \to S^2_0(M)$ that we define in (2.61). We estimate η^* form above in Proposition 3.22 following an idea of G. Huisken [24] The above Corollary 3.18 also compares with some previous result of ours, [29].

It is well known, from the work of D. S. Kim and Y. H. Kim, [25], that the validity of (1.3) on M yields, via a non-trivial consequence of the second Bianchi identities, the validity of the equation

$$\Delta_f f - k\lambda = -\beta e^{\frac{2}{k} f} \tag{1.11}$$

for some constant $\beta \in \mathbb{R}$. Here Δ_f is the symmetric diffusion operator

$$\Delta_f = \Delta - \langle \nabla f, \nabla \, \rangle.$$

A consequence of (1.11), indeed equivalent to (1.11), is the validity of Hamilton's type identity

$$S + \left(1 - \frac{1}{k}\right)|\nabla f|^2 + (k-m)\lambda = \beta e^{\frac{2}{k}f},$$

where S is the scalar curvature of $(M, \langle\,,\,\rangle)$. Note that we can think of (1.2) as a "limiting"case of (1.3) as $k \to +\infty$. However, as we shall see, the equation companion to (1.11) corresponding to (1.2) is

$$\Delta_f f - m\lambda = \beta - 2\lambda f, \tag{1.12}$$

for some constant β; however, (1.12) is difficult to be interpreted as a "limiting"case of (1.11) as $k \to +\infty$. Observe that (1.12), coupled with (1.2), yields Hamilton's identity for gradient Ricci solitons

$$S + |\nabla f|^2 + 2\lambda f = \beta.$$

In Proposition 7.40 we show that equations (7.43) and (7.44), that correspond to (1.11) and (1.12), hold also for (1.1), where λ is constant (with the exceptional case $\mu = -\frac{1}{m-2}$, where λ may also be a smooth function). The interesting fact is that the smooth map φ and the constant α do not appear in the equations.

From the literature we know various examples of the special structures we just mentioned above. As for their non-existence, for instance in case of quasi-Einstein manifolds, we can refer to the non-existence problem for solutions of equation (1.11). In doing so one might wonder about the constant β.

As a matter of fact, the pairing (1.3), (1.11) has a precise geometric meaning that enables us the shed light on the problem. Towards this aim we go back to an old interesting question considered in A. Besse's book, [8], on the possibility of constructing examples of Einstein manifolds realized as warped product metrics. It is well known that, if $(M^m, \langle\,,\,\rangle)$ and $(\mathbb{P}, \langle\,,\,\rangle_{\mathbb{P}})$ are Riemannian manifolds and we consider on $\bar{M} := M \times \mathbb{P}$ the warped product metric

$$\overline{\langle\,,\,\rangle} := \langle\,,\,\rangle + e^{-\frac{2}{m}f}\langle\,,\,\rangle_{\mathbb{P}}$$

for some function $f \in \mathcal{C}^\infty(M)$, a computation shows that $\overline{\langle\,,\,\rangle}$ is Einstein satisfying

$$\overline{\mathrm{Ric}} = \lambda\overline{\langle\,,\,\rangle}$$

for some $\lambda \in \mathbb{R}$, if and only if $(\mathbb{P}, \langle\,,\,\rangle_{\mathbb{P}})$ is Einstein with

$$\mathrm{Ric}_{\mathbb{P}} = \beta\langle\,,\,\rangle_{\mathbb{P}}$$

for some $\beta \in \mathbb{R}$, and furthermore the following relations hold between λ, β, f and the Ricci tensor Ric of M:

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{m} df \otimes df = \lambda\langle\,,\,\rangle \\ \Delta_f f - m\lambda = -m\beta e^{\frac{2}{m}f}. \end{cases}$$

This setting has been analyzed in detail in [13]. In the second part of Section 7 we shall investigate equations (7.43) and (7.44) on complete, non-compact manifolds mainly with the aid of the weak maximum principle, see for instance, Chapter 4 in the book [1], obtaining a non-existence result.

In Section 4 we develop another technical approach to non existence of Einstein-type structures starting from the following observation: if $\mu \neq 0$, setting $u = e^{-\mu f}$ and tracing the first equation in (1.1) we obtain

$$Lu := \Delta u + \mu(m\lambda - S^\varphi) = 0, \tag{1.13}$$

where $S^\varphi = S - \alpha|d\varphi|^2$ is the φ-scalar curvature. Since $u > 0$, by a well known result of [19] and [31], the operator L is stable or, in other words, its spectral radius $\lambda_L^1(M)$ is non-negative. Thus, instability of L yields a non-existence result for (1.1) at least in case μ is a non-zero constant. Toward this aim we detect appropriate conditions on the coefficient of the linear term in (1.13). This is investigated in Proposition 4.44 below.

A further important problem for equations (7.43) and (7.44) is that of uniqueness of the solution. In Section 8 we produce integral formulas that provide uniqueness in case M is compact. Basically in case of (7.43) the only assumptions are an appropriate lower bound on the Ricci tensor of $(M, \langle\, ,\, \rangle)$ and a range of validity for the parameters, see Theorem 8.3, for equation (7.44) we refer to Theorem 8.19. In the complete, non-compact case we use an unpublished refinement due to G. Albanese, [2], of a previous result of our, [42] Theorem 3.1 and Corollary 3.2, to deal with a very weak superlinearity of the type $t \log t$ for $t >> 1$, that pops up from equation (7.44) after an appropriate "change of variables", see (7.44) and the prototype equation (8.33). As a geometric consequence we obtain, for instance, Corollary 8.42 below that basically compares two Einstein-type structures with $\mu = 0$.

Section 5 is devoted to some results in the compact case where, together with (1.1), we also consider the more general Einstein-type structure

$$\begin{cases} \mathrm{Ric}^{\varphi} + \frac{1}{2}\mathcal{L}_X \langle\, ,\, \rangle = \mu X^{\flat} \otimes X^{\flat} + \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{1.14}$$

for some $X \in \mathfrak{X}(M)$ and with $X^{\flat}$ denoting the 1-form dual to X via the musical isomorphism $^{\flat}$. The compact case is quite rigid once we require constancy of the φ-scalar curvature. Indeed, when $\mu \neq 0$, $\alpha > 0$ and $\lambda, f \in \mathcal{C}^{\infty}(M)$ with f non-constant, a Riemannian manifold with constant φ-scalar curvature that supports an Einstein-type structure as in (1.1) is always isometric to a Euclidean sphere and φ is a constant map, see Theorem 5.22. When $\mu = 0$ the same happens under the same hypothesis for the general structure (1.14), when X is not a Killing vector field, see Theorem 5.14. In proving the mentioned results we extend the well known fact, due to M. Obata, [33], that a compact Einstein manifold with a non-Killing conformal vector field is isometric to a Euclidean sphere, see Lemma 5.2. Our Theorem 5.14 and Theorem 5.22 extend respectively the results of [6] and [7] to the case when, a priori, φ is not constant. Section 5 ends with Theorem 5.47 and Theorem 5.56 where we guarantee the same conclusion in case the φ-Schouten tensor is a Codazzi tensor field (a necessary condition) and one of its normalized k^{th} order symmetric functions in its eigenvalues is a positive constant. We shall see that the φ-scalar curvature is constant if and only the first symmetric function of the eigenvalues of the φ-Schouten tensor is constant, hence we can see these two Theorems as a generalization of Theorem 5.14 and Theorem 5.22. In doing so we use a general formula valid for every 2-times symmetric, covariant Codazzi tensor field T on an m-dimensional Riemannian manifold. Indicating with P_k the k^{th} Newton operator associated to T and with S_k the (non-normalized) k^{th} symmetric function in the eigenvalues of T we obtain equation (5.44) below that reads

$$\mathrm{div}(P_k(\nabla u)) = \sum_{i=0}^{k} (-1)^i S_{k-i} \mathrm{tr}(t^i \circ \mathrm{hess}(u)), \tag{1.15}$$

for $0 \leq k \leq m$, where t and $\mathrm{hess}(u)$ are the endomorphisms of $\mathfrak{X}(M)$ associated to T and $\mathrm{Hess}(u)$ while t^i denotes the i times composition of t with itself. In case the function u satisfies a system of the type

$$\mathrm{Hess}(u) = a\langle\, ,\, \rangle + b du \otimes du + c\varphi^* \langle\, ,\, \rangle_N - dT$$

for some functions $a, b, c, d \in \mathcal{C}^{\infty}(M)$, a computation shows that (1.15) yields

$$\mathrm{div}(P_k(\nabla u)) = c_k(a\sigma_k - d\sigma_{k+1}) + b\langle P_k(\nabla u), \nabla u\rangle + c\langle P_k, \varphi^*\langle\, ,\, \rangle_N\rangle. \tag{1.16}$$

Here $c_k := (m-k)\binom{m}{k}$ and $\sigma_k = \binom{m}{k}^{-1} S_k$. We also observe that for the validity of (1.15), and therefore that of (1.16), we can relax the assumption that T is Codazzi to the property

$$C(X, Y, Z) + C(Z, Y, X) = 0$$

for every $X, Y, Z \in \mathfrak{X}(M)$, where, in a local orthonormal coframe, C is the tensor of components

$$C_{ijk} = T_{ij,k} - T_{ik,j}.$$

Formula (1.16) can be applied in various circumstances. The following example shows its genesis. Let $\overline{M} = I \times_h \mathbb{P}^m$ be a warped product with base space the open interval $I \subseteq \mathbb{R}$ and fiber the m-dimensional Riemannian manifold $(\mathbb{P}, \langle\, , \,\rangle_{\mathbb{P}})$, where $h : I \to \mathbb{R}^+$ is a smooth function. Then the metric on $\overline{M}$ is given by

$$\overline{\langle\, , \,\rangle} = dt^2 + h^2 \langle\, , \,\rangle_{\mathbb{P}}$$

where t is the natural coordinate on I. Given the immersion $\psi : M^m \to \overline{M}$ we define the function $\eta : M \to \mathbb{R}$ by setting

$$\eta(x) := \int_{t_0}^{\pi\circ\psi(x)} h,$$

where $t_0 \in I$ is (arbitrarily) fixed and $\pi : \overline{M} \to I$ is the natural projection on the first factor of the product. A computation shows that

$$\mathrm{Hess}(\eta) = h'(\pi \circ \psi)\langle\, , \,\rangle + h(\pi \circ \psi)\overline{\langle \partial_t, \nu\rangle}\,\overline{\langle \Pi, \nu\rangle}$$

where ν is a local unit normal to ψ and Π is the second fundamental tensor of the immersion. Assume that the immersed hypersurface is one-sided so that the unit normal ν can be chosen globally and let $T := \overline{\langle \Pi, \nu\rangle}$. Then the 2-times covariant, symmetric tensor T is Codazzi provided $\overline{M}$ has constant sectional curvature (note that in this case h is explicitly given). Let the H_k's be the higher order mean curvatures of ψ and let $0 \leq k \leq m-1$. Then, in case M is compact, by integration (1.16) yields

$$\int_M [h'(\pi \circ \psi) H_k + h(\pi \circ \psi)\overline{\langle \partial_t, \nu\rangle} H_{k+1}] = 0,$$

a clear generalization of the Euclidean Hsing-Minkowski's formulas.

We observe that (1.15) is given in Section 4 where, motivated by the results in Section 2 and Section 3, we introduce the notion of Einstein-type structure. In the same section we collect some other formulas instrumental to our study in the subsequent paragraphs. It is worth to mention formulas (4.17) and (4.20), the first one is used in Section 5 to prove Theorem 5.14 while the latter is used in the first part of Section 7 to obtain another important result: an upper and a lower bound for the φ-scalar curvature of a complete, non-compact Riemannian manifold supporting a gradient Einstein-type structure as in (1.1) with α, μ and λ appropriate constants, see Theorem 7.29 (these estimates generalize some previous results proved for generic quasi-Einstein manifolds, [13]).

In what follows we shall freely use the "moving frame" formalism and manifolds will always be tacitly assumed to be connected.

2 φ-curvatures, harmonic-Einstein manifolds and first results

The aim of this section is to introduce the φ-curvatures of a pair $(M, \langle\, , \,\rangle)$ a Riemannian manifold and $\varphi : M \to (N, \langle\, , \,\rangle_N)$ a smooth map. We study some properties. We introduce the concept of harmonic-Einstein manifold and we prove some results. Probably, in a coherent way with our notations, harmonic-Einstein manifolds should be called φ-Einstein manifolds. However, the first terminology has already appeared in the literature so that we have decided to keep it.

Let $(M, \langle\, , \,\rangle)$ and $(N, \langle\, , \,\rangle_N)$ be Riemannian manifolds, $\varphi : M \to (N, \langle\, , \,\rangle_N)$ a smooth map and $\alpha \in \mathbb{R}\backslash\{0\}$. Indicating with Ric the usual Ricci tensor of $(M, \langle\, , \,\rangle)$ we define the φ-Ricci tensor by setting

$$\mathrm{Ric}^{\varphi} := \mathrm{Ric} - \alpha\varphi^*\langle\, , \,\rangle_N. \tag{2.1}$$

The φ-Ricci tensor appears in the work of R. Müller [32] but it was defined and denoted by Ric^{φ} firstly by L. F. Wang in [43]. The φ-scalar curvature S^{φ} is obtained by tracing (2.1), that is,

$$S^{\varphi} := S - \alpha|d\varphi|^2, \tag{2.2}$$

where S is the usual scalar curvature of $(M, \langle\, ,\, \rangle)$ and $|d\varphi|^2$ is the square of the Hilbert-Schmidt norm of the section $d\varphi$ of the vector bundle φ^*TN. We formally introduce the φ-Schouten tensor A^φ in analogy with the standard case

$$A^\varphi := \mathrm{Ric}^\varphi - \frac{S^\varphi}{2(m-1)}\langle\, ,\, \rangle, \tag{2.3}$$

where $m \geq 2$ is the dimension of M. An immediate computation gives the relation of A^φ with the usual Schouten tensor A, that is,

$$A^\varphi = A - \alpha\left(\varphi^*\langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\, ,\, \rangle\right). \tag{2.4}$$

We recall the Kulkarni-Nomizu product of two symmetric 2-covariant tensors, that we shall indicate with the "parrot"operator $\owedge$. It gives rise to a 4-covariant tensor with the same symmetries of Riem, the Riemann curvature tensor. In components, with respect to a local orthonormal coframe, given the 2-covariant symmetric tensors T and V we have

$$(V \owedge T)_{ijkt} := V_{ik}T_{jt} - V_{it}T_{jk} + V_{jt}T_{ik} - V_{jk}T_{it}. \tag{2.5}$$

Then, for $m \geq 3$, the φ-Weyl tensor is defined by

$$W^\varphi := \mathrm{Riem} - \frac{1}{m-2}A^\varphi \owedge \langle\, ,\, \rangle. \tag{2.6}$$

From the standard decomposition of the Riemann curvature tensor we know that, for $m \geq 3$,

$$\mathrm{Riem} = W + \frac{1}{m-2}A \owedge \langle\, ,\, \rangle,$$

and from the distributivity of $\owedge$ with respect to sums, together with (2.4), we deduce the expression of W^φ in terms of W:

$$W^\varphi = W + \frac{\alpha}{m-2}\left(\varphi^*\langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\, ,\, \rangle\right) \owedge \langle\, ,\, \rangle. \tag{2.7}$$

W^φ has been defined as in (2.6) in order to keep the validity of the usual decomposition of the Riemann tensor also in this "φ-case". We note that W^φ has the same symmetries of Riem. However in general W^φ is not totally trace free. Indeed, from (2.7) and the fact that, on the contrary, W is totally trace free we obtain

$$W^\varphi_{kikj} = \alpha\varphi^a_i\varphi^a_j = \alpha(\varphi^*\langle\, ,\, \rangle_N)_{ij}. \tag{2.8}$$

The next result, analogous to Schur's identity, typically shows how the geometry of φ enters into the picture.

Proposition 2.9. *In the above setting we have:*

$$R^\varphi_{ij,i} = \frac{1}{2}S^\varphi_j - \alpha\varphi^a_{ii}\varphi^a_j, \tag{2.10}$$

where φ^a_{ii} are the components of the tension field $\tau(\varphi)$ of the map φ and R^φ_{ij} are the components of the φ-Ricci tensor.

Proof. From (2.2) we have

$$S = S^\varphi + \alpha|d\varphi|^2.$$

Taking its covariant derivative

$$\frac{1}{2}S_j = \frac{1}{2}S^\varphi_j + \alpha\varphi^a_{ij}\varphi^a_i$$

and by the usual Schur's identity we obtain

$$R_{ij,i} = \frac{1}{2}S^\varphi_j + \alpha\varphi^a_{ij}\varphi^a_i. \tag{2.11}$$

Using (2.1) and the above we infer

$$R^{\varphi}_{ij,i} = R_{ij,i} - \alpha\varphi^a_{ii}\varphi^a_j - \alpha\varphi^a_i\varphi^a_{ji}.$$

Therefore, from the symmetries of $\nabla d\varphi$

$$R^{\varphi}_{ij,i} = R_{ij,i} - \alpha\varphi^a_{ij}\varphi^a_i - \alpha\varphi^a_{ii}\varphi^a_j$$

and from (2.11) we deduce (2.10). □

Remark 2.12. If $\tau(\varphi) = 0$ then we have an analogous of the usual Schur's identity

$$R^{\varphi}_{ij,i} = \frac{1}{2} S^{\varphi}_j.$$

The converse holds, that is, the latter implies $\tau(\varphi) = 0$, in case φ is a submersion almost everywhere on M, see page 6 of [5].

Next definition is analogous to that of an Einstein manifold.

Definition 2.13. A Riemannian manifold $(M, \langle\, ,\, \rangle)$ is said to be a *harmonic-Einstein manifold* if there exist $\alpha \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ such that

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{2.14}$$

To have a strict parallelism with the notion of Einstein manifold, in case $m = 2$ we require λ to be constant. Note that for $m \geq 3$ this is automatic because of the following version of Schur's lemma.

Proposition 2.15. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathcal{C}^\infty(M)$ and suppose that for some $\varphi : M \to (N, \langle\, ,\, \rangle_N)$*

$$Ric^{\varphi} = \lambda\langle\, ,\, \rangle. \tag{2.16}$$

Then

$$(m-2)\nabla\lambda = 2\alpha\langle\tau(\varphi), d\varphi\rangle_N. \tag{2.17}$$

In particular, if $\tau(\varphi) = 0$ then λ is constant.

Proof. We trace (2.16) to obtain $S^{\varphi} = m\lambda$ and then

$$S^{\varphi}_j = m\lambda_j. \tag{2.18}$$

On the other hand, taking covariant derivative of (2.16) we have

$$R^{\varphi}_{ij,k} = \lambda_k\delta_{ij}.$$

Tracing with respect to i and k

$$R^{\varphi}_{ij,i} = \lambda_j.$$

We then use (2.10) to obtain

$$(m-2)\lambda_j = 2\alpha\varphi^a_{ii}\varphi^a_j$$

and (2.17) follows at once. □

We next recall the definition of the curvature operator $\mathfrak{R}$ acting on $S^2(\varphi^*TN)$, the space of symmetric 2-covariant tensor fields on φ^*TN, for some $\varphi : M \to (N, \langle\,,\,\rangle_N)$. Let ${}^N R_{acbd}$ denote the components of the curvature tensor of N in a local orthonormal coframe $\{\omega^a\}$, for $1 \le a, b, \ldots \le n$, where n is the dimension of N. Let $\beta = \beta_{ab}\omega^a \otimes \omega^b$ be an element of $S^2(\varphi^*TN)$ and define

$$\mathfrak{R}(\beta) := {}^N R_{acbd}\beta_{cd}\omega^a \otimes \omega^b.$$

It is not difficult to see that, introduced in $S^2(\varphi^*TN)$ the natural inner product $(\,,\,)$, induced by $\langle\,,\,\rangle_N$, the operator $\mathfrak{R} : S^2(\varphi^*TN) \to S^2(\varphi^*TN)$ is self-adjoint and thus diagonalizable. We let $\Lambda(x)$ to denote its largest eigenvalue at $x \in M$. We have

Theorem 2.19. *Let $(M, \langle\,,\,\rangle)$ be a complete, possibly compact, m-dimensional manifold with $m \ge 2$ which is hamonic-Einstein, that is, such that*

$$\begin{cases} Ric^\varphi = \dfrac{S^\varphi}{m}\langle\,,\,\rangle \\ \tau(\varphi) = 0 \end{cases} \tag{2.20}$$

for some $\varphi : M \to (N, \langle\,,\,\rangle_N)$ and some constants $\alpha \in \mathbb{R}$, $\alpha > 0$, and $S^\varphi \in \mathbb{R}$. Assume that

$$\Lambda^* := \sup_{x\in M} \Lambda(x) < \alpha. \tag{2.21}$$

Depending on the sign of the constant S^φ, we have

i) if $S^\varphi \ge 0$, then φ is constant and $(M, \langle\,,\,\rangle)$ is Einstein with scalar curvature $S = S^\varphi$;

ii) if $S^\varphi < 0$, then the energy density $|d\varphi|^2$ satisfies

$$0 \le \sup_M |d\varphi|^2 \le -\frac{S^\varphi}{m(\alpha - \Lambda^*)}.$$

Corollary 2.22. *In the assumption of the Theorem suppose that the manifold is flat harmonic-Einstein, that is, $S^\varphi = 0$. Then φ is constant and $(M, \langle\,,\,\rangle)$ is Ricci flat.*

Remark 2.23. Since $\alpha > 0$, $\mathrm{Ric}^\varphi = 0$ immediately implies that $\mathrm{Ric} \ge 0$ on the complete manifold $(M, \langle\,,\,\rangle)$. In case the harmonic map φ has bounded image and N is simply connected with non-positive sectional curvature by a Theorem of S. Y. Cheng [17] we know that φ is constant and as a consequence $(M, \langle\,,\,\rangle)$ is Ricci flat. The setting of Corollary 2.22 is more general and, in any case, different.

Proof. Since φ is harmonic the Weitzenböck-Bochner formula reads

$$\frac{1}{2}\Delta|d\varphi|^2 = |\nabla d\varphi|^2 + {}^N R_{abcd}\varphi^a_i\varphi^b_j\varphi^c_j\varphi^d_i + R_{ij}\varphi^a_i\varphi^a_j, \tag{2.24}$$

where the indices $a, b, \ldots$ and $i, j, \ldots$ refer, respectively, to local orthonormal coframes on N and M. Having set

$$\beta := \varphi^a_i\varphi^b_i\omega^a \otimes \omega^b,$$

we have

$${}^N R_{abcd}\varphi^a_i\varphi^b_j\varphi^c_j\varphi^d_i = -(\mathfrak{R}(\beta), \beta) \ge -\Lambda|\beta|^2.$$

Observe that

$$|\beta|^2 = |\varphi^*\langle\,,\,\rangle_N|^2$$

and, since from (2.20)

$$\varphi^*\langle\,,\,\rangle_N = \frac{1}{\alpha}\left(\mathrm{Ric} - \frac{S^\varphi}{m}\langle\,,\,\rangle\right), \tag{2.25}$$

using $\Lambda^* < +\infty$ we deduce

$$ {}^N R_{abcd}\varphi^a_i\varphi^b_j\varphi^c_j\varphi^d_i \geq -\frac{\Lambda^*}{\alpha^2}\left|\mathrm{Ric} - \frac{S^\varphi}{m}\langle\, ,\,\rangle\right|^2. \tag{2.26} $$

From (2.24), (2.25), the first equation of (2.20) and (2.26) we then have

$$ \frac{\alpha}{2}\Delta|d\varphi|^2 \geq \left(1 - \frac{\Lambda^*}{\alpha}\right)\left|\mathrm{Ric} - \frac{S^\varphi}{m}\langle\, ,\,\rangle\right|^2 + \frac{S^\varphi}{m}\mathrm{tr}\left(\mathrm{Ric} - \frac{S^\varphi}{m}\langle\, ,\,\rangle\right). \tag{2.27} $$

From Newton's inequality and the first equation of (2.20)

$$ \left|\mathrm{Ric} - \frac{S^\varphi}{m}\langle\, ,\,\rangle\right|^2 \geq \frac{\alpha^2}{m}|d\varphi|^4. $$

Hence (2.27) yields

$$ \frac{1}{2}\Delta\alpha|d\varphi|^2 \geq \left(1 - \frac{\Lambda^*}{\alpha}\right)\frac{\alpha^2}{m}|d\varphi|^4 + \frac{S^\varphi}{m}\alpha|d\varphi|^2 $$

and setting

$$ u := \alpha|d\varphi|^2 $$

we obtain

$$ \frac{1}{2}\Delta u \geq \left(1 - \frac{\Lambda^*}{\alpha}\right)\frac{u^2}{m} + \frac{S^\varphi}{m}u \tag{2.28} $$

where the constant $1 - \frac{\Lambda^*}{\alpha}$ is strictly positive because of (2.21). We now deal with the non-compact case being the compact case simpler. We observe that the first equation of (2.20) and $\alpha > 0$ imply

$$ \mathrm{Ric} \geq \frac{S^\varphi}{m}\langle\, ,\,\rangle $$

where S^φ is constant and therefore completeness of $(M, \langle\, ,\,\rangle)$ yields the validity of the Omori-Yau maximum principle for the Laplace-Beltrami operator Δ. We then apply Theorem 3.6 of [1] to deduce from (2.28) and positivity of $1 - \frac{\Lambda^*}{\alpha}$,

$$ u^* := \sup_M u < +\infty. $$

Then, we apply the Omori-Yau maximum principle again to (2.28) to infer

$$ u^*\left[\left(1 - \frac{\Lambda^*}{\alpha}\right)u^* + \frac{S^\varphi}{m}\right] \leq 0. \tag{2.29} $$

From (2.29) and the definition of u we immediately deduce conclusions $i)$ and $ii)$. □

Suppose $0 < \alpha < \Lambda^*$. In the assumptions of Theorem 2.19 with the further request $\sup|d\varphi|^2 < +\infty$, proceeding in a way analogous to that above we reach the conclusion

$$ \sup_M |d\varphi|^2 \geq \frac{S^\varphi}{m(\Lambda^* - \alpha)} $$

that bears information only in the case $S^\varphi > 0$. In particular we deduce the following gap result:

Theorem 2.30. *Let $(M, \langle\, ,\,\rangle)$ be a complete, possibly compact, m-dimensional manifold with $m \geq 2$ and let $\alpha \in \mathbb{R}$, $\alpha > 0$. Given a constant $\Sigma > 0$, there is no harmonic-Einstein structure as in* (2.20) *on M with $S^\varphi = \Sigma$ and for which*

$$ \sup_M |d\varphi|^2 < \frac{\Sigma}{m(\Lambda^* - \alpha)}. $$

Note that the case $\alpha = \Lambda^*$ can be treated similarly, as we will see below.

Analogously to the standard case we define the φ-Cotton tensor C^φ as the obstruction to the commutativity of the covariant derivative of A^φ, that is, in a local orthonormal coframe,

$$C^\varphi_{ijk} := A^\varphi_{ij,k} - A^\varphi_{ik,j}. \tag{2.31}$$

Using definition (2.3) of A^φ we compute the indicated covariant derivatives in (2.31) to obtain C^φ expressed in terms of the usual Cotton tensor C of $(M, \langle\, ,\, \rangle)$. We have

$$C^\varphi_{ijk} = C_{ijk} - \alpha\left[\varphi^a_{ik}\varphi^a_j - \varphi^a_{ij}\varphi^a_k - \frac{\varphi^a_t}{m-1}(\varphi^a_{tk}\delta_{ij} - \varphi^a_{tj}\delta_{ik})\right]. \tag{2.32}$$

Next relations are obtained by computation

$$C^\varphi_{ikj} = -C^\varphi_{ijk} \quad \text{ and therefore } C^\varphi_{ijj} = 0, \tag{2.33}$$

$$C^\varphi_{jji} = \alpha\varphi^a_{jj}\varphi^a_i = -C^\varphi_{jij}, \tag{2.34}$$

$$C^\varphi_{ijk} + C^\varphi_{jki} + C^\varphi_{kij} = 0. \tag{2.35}$$

Explicitating (2.31) in terms of $R^\varphi_{ij,k}$ we obtain the commutation relations

$$R^\varphi_{ij,k} = R^\varphi_{ik,j} + C^\varphi_{ijk} + \frac{1}{2(m-1)}(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik}), \tag{2.36}$$

that we shall use later (for instance in Theorem 3.5). Next we introduce the φ-Bach tensor B^φ by setting, in a local orthonormal coframe and for $m \geq 3$,

$$(m-2)B^\varphi_{ij} = C^\varphi_{ijk,k} + R^\varphi_{tk}(W^\varphi_{tikj} - \alpha\varphi^a_t\varphi^a_i\delta_{jk}) + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right). \tag{2.37}$$

As remarked in the introduction the above definition of B^φ is motivated by the geometric results we shall obtain with its use. When needed, we shall indicate the term $C^\varphi_{ijk,k}$ as $\operatorname{div}(C^\varphi)$. For the moment we prove

Proposition 2.38. *Let $m \geq 3$; the φ-Bach tensor is symmetric and*

$$tr(B^\varphi) = \alpha\frac{m-4}{(m-2)^2}|\tau(\varphi)|^2. \tag{2.39}$$

In order to prove the Proposition we shall need the commutation formulas

$$A^\varphi_{ik,jk} = A^\varphi_{ki,kj} + R_{kj}A^\varphi_{ki} + R^t_{ijk}A^\varphi_{tk}. \tag{2.40}$$

To show the validity of (2.40) we proceed in a general contest as follows.

Proposition 2.41. *Let T be a 2-times covariant tensor of components T_{ij} with respect to an orthonormal coframe $\{\theta^i\}$, $1 \leq i, j, \ldots \leq m$. Then*

$$T_{ij,kt} = T_{ij,tk} + R^l_{ikt}T_{lj} + R^l_{jkt}T_{il}. \tag{2.42}$$

Proof. We shall use the first and the second structure equations

$$d\theta^i = -\theta^i_k \wedge \theta^k, \quad d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j,$$

where $\{\theta^i_j\}$ are the Levi Civita connection forms associated to the orthonormal coframe $\{\theta^i\}$ and

$$\Theta^i_j = \frac{1}{2}R^i_{jkt}\theta^k \wedge \theta^t$$

are the curvature forms of the metric $\langle\, ,\, \rangle$ of M. By definition, the covariant derivative of T has components $T_{ij,k}$ given by

$$T_{ij,k}\theta^k = dT_{ij} - T_{sj}\theta^s_i - T_{is}\theta^s_j. \tag{2.43}$$

Differentiating the above we have

$$dT_{ij,k} \wedge \theta^k + T_{ij,k}d\theta^k = -dT_{sj} \wedge \theta^s_i - T_{sj}d\theta^s_i - dT_{is} \wedge \theta^s_j - T_{is}d\theta^s_j. \tag{2.44}$$

Next recall that the components of the second covariant derivative of T are

$$T_{ij,kt}\theta^t = dT_{ij,k} - T_{sj,k}\theta^s_i - T_{is,k}\theta^s_j - T_{ij,s}\theta^s_k.$$

We use this information and (2.43) together with the structure equations into (2.44) to infer

$$\begin{aligned}(T_{ij,kt}\theta^t + T_{sj,k}\theta^s_i - T_{is,k}\theta^s_j - T_{ij,s}\theta^s_k) \wedge \theta^k - T_{ij,k}\theta^k_s \wedge \theta^s \\ = -(T_{sj,k}\theta^k + T_{lj}\theta^l_s + T_{sl}\theta^l_j) \wedge \theta^s_i + T_{sj}\theta^s_k \wedge \theta^k_i - T_{sj}\Theta^s_i \\ -(T_{is,k}\theta^k + T_{ks}\theta^k_i + T_{ik}\theta^k_s) \wedge \theta^s_j + T_{is}\theta^s_k \wedge \theta^k_j - T_{is}\Theta^s_j.\end{aligned}$$

Hence,

$$T_{ij,kt}\theta^t \wedge \theta^k = -\frac{1}{2}(T_{sj}R^s_{itk} + T_{is}R^s_{jtk})\theta^t \wedge \theta^k.$$

Skew symmetrizing we obtain

$$T_{ij,kt} - T_{ij,tk} = -T_{sj}R^s_{itk} - T_{is}R^s_{jtk},$$

that is, (2.42). □

Proof (of Proposition 2.38). We rewrite B^φ in the form

$$(m-2)B^\varphi = V + Z$$

where:

$$V_{ij} := C^\varphi_{ijk,k} - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_{kkj}\varphi^a_i, \quad Z_{ij} := R^\varphi_{tk}W^\varphi_{tikj} + \alpha\varphi^a_{ij}\varphi^a_{kk} - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ij}.$$

Since Z is clearly symmetric it remains to show that V shares the same property. To verify this fact, in other words that $V_{ij} = V_{ji}$, we see that, explicitating both sides of the equality, this is equivalent to show that

$$\alpha[\varphi^a_k(R^\varphi_{ik}\varphi^a_i - R^\varphi_{kj}\varphi^a_i) + \varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i] = C^\varphi_{jik,k} - C^\varphi_{ijk,k} = -(C^\varphi_{ijk} - C^\varphi_{jik})_k.$$

By using (2.33) and (2.35) we have

$$-(C^\varphi_{ijk} - C^\varphi_{jik})_k = -(C^\varphi_{ijk} + C^\varphi_{jki})_k = C^\varphi_{kij,k},$$

hence the above equality is equivalent to

$$C^\varphi_{kij,k} = \alpha[\varphi^a_k(R^\varphi_{ik}\varphi^a_j - R^\varphi_{kj}\varphi^a_i) + \varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i]. \tag{2.45}$$

It remains to compute $C^\varphi_{kij,k}$ to verify (2.45). Using (2.31) and (2.40) we have

$$C^\varphi_{kij,k} = A^\varphi_{ki,jk} - A^\varphi_{kj,ik} = (A^\varphi_{ki,kj} + R_{kj}A^\varphi_{ki} + R^t_{ijk}A^\varphi_{kt}) - (A^\varphi_{kj,ki} + R_{ki}A^\varphi_{kj} + R^t_{jik}A^\varphi_{kt}).$$

Hence, with the aid of (2.3), we deduce

$$\begin{aligned}C^\varphi_{kij,k} = \left(R^\varphi_{ki,k} - \frac{S^\varphi_k}{2(m-1)}\delta_{ki}\right)_j + R_{kj}A^\varphi_{ki} + R^t_{ijk}A^\varphi_{kt} \\ - \left(R^\varphi_{kj,k} - \frac{S^\varphi_k}{2(m-1)}\delta_{kj}\right)_i - R_{ki}A^\varphi_{kj} - R^t_{jik}A^\varphi_{kt}.\end{aligned}$$

From (2.10) and the symmetries of Riem we obtain

$$\begin{aligned}C^{\varphi}_{kij,k} =& \left(\frac{1}{2}S^{\varphi}_i - \alpha\varphi^a_{kk}\varphi^a_i - \frac{S^{\varphi}_i}{2(m-1)}\right)_j + R_{kj}A^{\varphi}_{ki} + R^t_{ijk}A^{\varphi}_{kt}\\ &- \left(\frac{1}{2}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j - \frac{S^{\varphi}_j}{2(m-1)}\right)_i - R_{ki}A^{\varphi}_{kj} - R^t_{ijk}A^{\varphi}_{kt}\\ =& \left(\frac{m-2}{2(m-1)}S^{\varphi}_i - \alpha\varphi^a_{kk}\varphi^a_i\right)_j + R_{kj}A^{\varphi}_{ki} - \left(\frac{m-2}{2(m-1)}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j\right)_i - R_{ki}A^{\varphi}_{kj}.\end{aligned}$$

Since $\mathrm{Hess}(S^{\varphi})$ is symmetric we deduce

$$C^{\varphi}_{kij,k} = \alpha(\varphi^a_{kk}\varphi^a_j)_i - \alpha(\varphi^a_{kk}\varphi^a_i)_j + R_{kj}A^{\varphi}_{ki} - R_{ki}A^{\varphi}_{kj}.$$

Using once again (2.3) and the symmetry of $\nabla d\varphi$

$$\begin{aligned}C^{\varphi}_{kij,k} =&\alpha(\varphi^a_{kki}\varphi^a_j + \varphi^a_{kk}\varphi^a_{ji} - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{kk}\varphi^a_{ij})\\ &+ R_{kj}\left(R^{\varphi}_{ki} - \frac{S^{\varphi}}{2(m-1)}\delta_{ki}\right) - R_{ki}\left(R^{\varphi}_{kj} - \frac{S^{\varphi}}{2(m-1)}\delta_{kj}\right)\\ =&\alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + R_{kj}R^{\varphi}_{ki} - \frac{S^{\varphi}}{2(m-1)}R_{ij} - R_{ki}R^{\varphi}_{kj} + \frac{S^{\varphi}}{2(m-1)}R_{ji}.\end{aligned}$$

From (2.1) we finally conclude

$$\begin{aligned}C^{\varphi}_{kij,k} =&\alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + (R^{\varphi}_{kj} + \alpha\varphi^a_k\varphi^a_j)R^{\varphi}_{ki} - (R^{\varphi}_{ki} + \alpha\varphi^a_k\varphi^a_i)R^{\varphi}_{kj}\\ =&\alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + \alpha\varphi^a_k\varphi^a_jR^{\varphi}_{ki} - \alpha\varphi^a_k\varphi^a_iR^{\varphi}_{kj}\\ =&\alpha[\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i + \varphi^a_k(R^{\varphi}_{ki}\varphi^a_j - R^{\varphi}_{kj}\varphi^a_i)],\end{aligned}$$

and this proves the validity of (2.45).

We now compute $\mathrm{tr}(B^{\varphi})$. From (2.37) we have

$$(m-2)B^{\varphi}_{ii} = C^{\varphi}_{iik,k} + R^{\varphi}_{tk}W^{\varphi}_{tiki} - \alpha R^{\varphi}_{ik}\varphi^a_k\varphi^a_i + \alpha\left(|\tau(\varphi)|^2 - \varphi^a_{kki}\varphi^a_i - \frac{m}{m-2}|\tau(\varphi)|^2\right).$$

Then with the aid of (2.34) and (2.8)

$$\begin{aligned}(m-2)B^{\varphi}_{ii} =&\alpha(\varphi^a_{ii}\varphi^a_k)_k + \alpha R^{\varphi}_{tk}\varphi^a_t\varphi^a_k - \alpha R^{\varphi}_{ik}\varphi^a_k\varphi^a_i - \alpha\frac{2}{m-2}|\tau(\varphi)|^2 - \alpha\varphi^a_{kki}\varphi^a_i\\ =&\varphi^a_{iik}\varphi^a_k + |\tau(\varphi)|^2 - \alpha\frac{2}{m-2}|\tau(\varphi)|^2 - \alpha\varphi^a_{kki}\varphi^a_i\\ =&\frac{m-4}{m-2}|\tau(\varphi)|^2,\end{aligned}$$

which is equivalent to (2.39). □

It is well known that the usual Bach tensor B, defined by

$$(m-2)B_{ij} = C_{ijk,k} + R_{tk}W_{tikj},$$

identically vanishes on an Einstein manifold. In the present setting the analogous result is given by the following

Proposition 2.46. *Let $(M, \langle\, ,\,\rangle)$ be a harmonic-Einstein manifold of dimension $m \geq 3$ for some $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Then $B^{\varphi} = 0$, that is, $(M, \langle\, ,\,\rangle)$ is φ-Bach flat.*

Proof. Using definition (2.14) of a harmonic-Einstein manifold we deduce

$$A^{\varphi} = \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)}\langle\, ,\, \rangle = \frac{m-2}{2(m-1)} S^{\varphi}\langle\, ,\, \rangle.$$

Since $m \geq 3$ and φ is harmonic, by Proposition 2.15 S^{φ} is constant. It follows that A^{φ} is parallel, hence is a Codazzi tensor field, and then $C^{\varphi} = 0$. Using (2.8) and once again (2.14) we have

$$R^{\varphi}_{tk}(W^{\varphi}_{tikj} - \alpha\varphi^a_i\varphi^a_t\delta_{jk}) = \frac{S^{\varphi}}{m}(W^{\varphi}_{kikj} - \alpha\varphi^a_i\varphi^a_j) = 0.$$

From the above equation, $C^{\varphi} = 0$ and the fact that φ is harmonic, we deduce

$$(m-2)B^{\varphi}_{ij} = C^{\varphi}_{ijk,k} + R^{\varphi}_{tk}(W^{\varphi}_{tikj} - \alpha\varphi^a_t\varphi^a_i\delta_{jk}) + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right) = 0$$

thus $(M, \langle\, ,\, \rangle)$ is φ-Bach flat. □

Remark 2.47. It possible to prove that, for every Riemannian manifold $(M, \langle\, ,\, \rangle)$ and every $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth map, the tensor B^{φ} is a conformally invariant tensor field in case $m = 4$, where m is the dimension of M. For a proof of this fact see the doctoral thesis of A. Anselli.

Next result is one of the important motivations for the general structure we shall introduce in Section 4. We begin with the following

Definition 2.48. A Riemannian manifold $(M, \langle\, ,\, \rangle)$ of dimension $m \geq 3$ is said to be *conformally harmonic-Einstein* if there exists $\psi \in \mathcal{C}^{\infty}(M)$, $\psi > 0$ on M such that, having defined

$$\widetilde{\langle\, ,\, \rangle} := \psi^2\langle\, ,\, \rangle,$$

the Riemannian manifold $(M, \widetilde{\langle\, ,\, \rangle})$ is harmonic-Einstein.

We then have

Theorem 2.49. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, let $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ be a smooth map and let $\alpha \in \mathbb{R} \setminus \{0\}$. Then there exist $\psi \in \mathcal{C}^{\infty}(M)$, $\psi > 0$ on M and $\Lambda \in \mathcal{C}^{\infty}(M)$ such that, having defined $\widetilde{\langle\, ,\, \rangle} := \psi^2\langle\, ,\, \rangle$, we have*

$$\begin{cases} \widetilde{Ric} - \alpha\widetilde{\varphi}^*\langle\, ,\, \rangle_N = \Lambda\widetilde{\langle\, ,\, \rangle} \\ \tau(\widetilde{\varphi}) = 0, \end{cases} \tag{2.50}$$

where $\widetilde{\varphi}$ denotes the map φ from $(M, \widetilde{\langle\, ,\, \rangle})$ to $(N, \langle\, ,\, \rangle_N)$, if and only if for some $f, \lambda \in \mathcal{C}^{\infty}(M)$

$$\begin{cases} Ric - \alpha\varphi^*\langle\, ,\, \rangle_N + Hess(f) + \frac{1}{m-2}df \otimes df = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases} \tag{2.51}$$

In this case f and ψ are related by

$$\psi = e^{-\frac{f}{m-2}} \tag{2.52}$$

while Λ and λ satisfy

$$\Delta_f f + (m-2)\lambda = (m-2)\Lambda e^{\frac{-2}{m-2}f}. \tag{2.53}$$

Here Δ_f is the symmetric diffusion operator $\Delta - \langle\nabla f, \nabla\,\rangle$.

Remark 2.54. Note that, since $m \geq 3$, Λ is constant by Proposition 2.15.

Remark 2.55. We shall see later, see Remark 6.18, that the system (2.51) satisfies the integrability conditions

$$C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} = 0 \tag{2.56}$$

and

$$(m-2)B^{\varphi}_{ij} + \frac{m-4}{m-2} W^{\varphi}_{tijk} f_t f_k = 0. \tag{2.57}$$

It is worth to observe that (2.57) implies that if $(M, \langle \, , \, \rangle)$ is a four dimensional conformally harmonic-Einstein manifold then it is φ-Bach flat. This partly motivates the definition of B^{φ} given in (2.37). Indeed, in this way the situation parallels that of four dimensional conformally Einstein manifolds that are always Bach flat.

In order to prove the Theorem we shall need the following formula that relates the Ricci tensors $\widetilde{\mathrm{Ric}}$ of $(M, \widetilde{\langle \, , \, \rangle})$ and Ric of $(M, \langle \, , \, \rangle)$ where, for $m \geq 3$,

$$\widetilde{\langle \, , \, \rangle} = e^{-\frac{2}{m-2}f} \langle \, , \, \rangle,$$

From Theorem 1.159 of [8] we have

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{\Delta_f f}{m-2} \langle \, , \, \rangle. \tag{2.58}$$

A second ingredient in the proof is the relation between $\tau(\varphi)$ and $\tau(\widetilde{\varphi})$; from [18], page 161, we have

$$\tau(\widetilde{\varphi}) = e^{\frac{2}{m-2}f} (\tau(\varphi) - d\varphi(\nabla f)). \tag{2.59}$$

Proof (of Thereom 2.49). By (2.59) we deduce that $\tau(\varphi) = d\varphi(\nabla f)$ if and only if $\tau(\widetilde{\varphi}) = 0$. Suppose that (2.50) holds, for some $\Lambda \in \mathbb{R}$, where $\widetilde{\langle \, , \, \rangle} = \psi^2 \langle \, , \, \rangle$ with ψ given by (2.52). Using (2.58) we obtain

$$\mathrm{Ric} - \alpha \varphi^* \langle \, , \, \rangle_N + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{\Delta_f f}{m-2} \langle \, , \, \rangle = \Lambda \widetilde{\langle \, , \, \rangle},$$

that is,

$$\mathrm{Ric} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df - \alpha \varphi^* \langle \, , \, \rangle_N = \left(e^{-\frac{2}{m-2}f} \Lambda - \frac{\Delta_f f}{m-2} \right) \langle \, , \, \rangle,$$

that gives (2.51) once we define λ as in (2.53). Conversely suppose that (2.51) holds for some $f, \lambda \in \mathcal{C}^{\infty}(M)$. Define ψ as in (2.52) and $\widetilde{\langle \, , \, \rangle} = \psi^2 \langle \, , \, \rangle$. From (2.58) and (2.51) we obtain

$$\widetilde{\mathrm{Ric}} - \alpha \varphi^* \langle \, , \, \rangle_N = \lambda \langle \, , \, \rangle + \frac{\Delta_f f}{m-2} \langle \, , \, \rangle = e^{\frac{2}{m-2}f} \left(\lambda + \frac{\Delta_f f}{m-2} \right) \widetilde{\langle \, , \, \rangle},$$

that is, (2.50) with Λ given by (2.53). □

We have just seen that for a conformally harmonic-Einstein manifold $(M, \langle \, , \, \rangle)$ we deduce the validity of the system (2.51) on M and of the two integrability conditions (2.56) and (2.57). Suppose now we are given $f \in \mathcal{C}^{\infty}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and a smooth map $\varphi : M \to N$ for some Riemannian manifold $(N, \langle \, , \, \rangle_N)$ such that (2.56) and (2.57) are satisfied. Does it follow that $(M, \langle \, , \, \rangle)$ is conformally harmonic-Einstein? To answer the question we need to introduce the next genericity condition.

Definition 2.60. Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$ and denote by $S^2_0(M)$ the bundle of the 2-times covariant, symmetric, traceless tensor fields on M. For a smooth map $\varphi : M \to (N, \langle \, , \, \rangle_N)$, we define

$$\mathcal{W}^{\varphi} : S^2_0(M) \to S^2_0(M)$$

by setting for $\beta \in S^2_0(M)$

$$\mathcal{W}^{\varphi}(\beta) = \left[W^{\varphi}_{tikj} - \frac{\alpha}{2} \varphi^a_t (\varphi^a_i \delta_{kj} + \varphi^a_j \delta_{ki}) \right] \beta_{tk} \theta^i \otimes \theta^j, \tag{2.61}$$

where $\beta = \beta_{ij} \theta^i \otimes \theta^j$.

Note that for φ constant $\mathcal{W}^\varphi$ coincides with the endomorphism $\mathcal{W} : S^2_0(M) \to S^2_0(M)$ defined by $\mathcal{W}(\beta) = W_{tikj}\beta_{tk}\theta^i \otimes \theta^j$; a well known endomorphism in the literature.

It is easy to verify that $\mathcal{W}^\varphi$ is well defined, that is, $\mathcal{W}^\varphi(\beta)$ is 2-times covariant, symmetric and traceless for every $\beta \in S^2_0(M)$, and that it is self-adjoint with respect to the standard extension of $\langle\, , \rangle$ to $S^2_0(M)$, that we denote with the same symbol. Thus $\mathcal{W}^\varphi$ is diagonalizable.

Definition 2.62. We say that the pair $(\langle\, , \rangle, \varphi)$ is generic if $d\varphi$ is possibly singular only at isolated points and if $\mathcal{W}^\varphi$ is injective, in other words if all its eigenvalues are non null everywhere on M.

We are now ready to state the following Proposition, that extends a result of A. R. Gover and P. Nurowski [22], Section 2.4, that deals with the conformally Einstein case and that we can consider as the degenerate case where φ is constant.

Proposition 2.63. *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, , \rangle_N)$. Suppose that $(\langle\, , \rangle, \varphi)$ is generic and that the integrability conditions* (2.56) *and* (2.57) *are satisfied for some $f \in \mathcal{C}^\infty(M)$. Then, defining*

$$\widetilde{\langle\, , \rangle} := e^{-\frac{2}{m-2}f}\langle\, , \rangle,$$

the Riemannian manifold $(M, \widetilde{\langle\, , \rangle})$ is harmonic-Einstein.

In the proof of the above statement we will use equation (2.65) proved in the Proposition below. This formula will also be useful later on (for instance in Proposition 6.16).

Proposition 2.64. *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\alpha \in \mathbb{R} \setminus \{0\}$, and $\varphi : M \to (N, \langle\, , \rangle_N)$ a smooth map. Then*

$$W^\varphi_{tijk,t} = \frac{m-3}{m-2}C^\varphi_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}). \tag{2.65}$$

Proof. Observe that from (2.7) we can express W^φ_{tijk} componentwise in the form

$$\begin{aligned} W^\varphi_{tijk} = W_{tijk} &+ \frac{\alpha}{m-2}(\varphi^a_t\varphi^a_j\delta_{ik} - \varphi^a_t\varphi^a_k\delta_{ij} + \varphi^a_i\varphi^a_k\delta_{tj} - \varphi^a_i\varphi^a_j\delta_{tk}) \\ &- \alpha\frac{|d\varphi|^2}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij}). \end{aligned}$$

Taking covariant derivatives, tracing, using the well known formula (see for instance equation (1.87) of [1])

$$W_{tijk,t} = -\frac{m-3}{m-2}C_{ijk}, \tag{2.66}$$

and (2.32) we obtain

$$\begin{aligned} W^\varphi_{tijk,t} =& W_{tijk,t} + \frac{\alpha}{m-2}(\varphi^a_{tt}\varphi^a_j\delta_{ik} + \varphi^a_t\varphi^a_{jt}\delta_{ik} - \varphi^a_{tt}\varphi^a_k\delta_{ij} - \varphi^a_t\varphi^a_{kt}\delta_{ij}) \\ &+ \frac{\alpha}{m-2}(\varphi^a_{ij}\varphi^a_k + \varphi^a_i\varphi^a_{kj} - \varphi^a_{ik}\varphi^a_j - \varphi^a_i\varphi^a_{jk}) + \frac{\alpha}{m-2}\left[-\frac{2\varphi^a_s\varphi^a_{st}}{m-1}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij})\right] \\ =& \frac{m-3}{m-2}C_{ikj} + \frac{\alpha}{m-2}\left[\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}) + \varphi^a_t(\varphi^a_{jt}\delta_{ij} - \varphi^a_{kt}\delta_{ij}) + \varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right] \\ &+ \frac{\alpha}{m-2}\left[-\frac{2}{m-1}\varphi^a_s(\varphi^a_{sj}\delta_{ik} - \varphi^a_{sk}\delta_{ij})\right] \\ =& \frac{m-3}{m-2}C^\varphi_{ikj} + \alpha\frac{m-3}{m-2}\left[\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j - \frac{\varphi^a_t}{m-1}(\varphi^a_{tj}\delta_{ik} - \varphi^a_{tk}\delta_{ij})\right] \\ &+ \frac{\alpha}{m-2}\left[\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}) + \frac{m-3}{m-1}\varphi^a_t(\varphi^a_{jt}\delta_{ij} - \varphi^a_{kt}\delta_{ij}) + \varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right] \\ =& \frac{m-3}{m-2}C^\varphi_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}), \end{aligned}$$

that is, (2.65). ☐

We are now ready to prove Proposition 2.63.

Proof (of Proposition 2.63). We trace (2.56) with respect to i and j and we use (2.8) and (2.34) to obtain, for each $k = 1, \dots, m$,

$$\alpha\varphi^a_k(\varphi^a_{ii} - \varphi^a_i f_i) = 0.$$

Fix $x \in M$. If there exists k such that $\varphi^a_k(x) \neq 0$ then we have the validity of the following equality at x:

$$\tau(\varphi) = d\varphi(\nabla f). \tag{2.67}$$

Otherwise the same holds by continuity. In conclusion (2.67) holds on M. Next, taking the covariant derivative of (2.56), using (2.65) and (2.67), we obtain

$$\begin{aligned}
C^\varphi_{ijk,k} =& -(f_t W^\varphi_{tijk})_k \\
=& - f_{tk}W^\varphi_{tijk} - f_t W^\varphi_{tijk,k} \\
=& - f_{tk}W^\varphi_{tijk} - f_k W^\varphi_{tjik,t} \\
=& - f_{tk}W^\varphi_{tijk} - f_k\left(\frac{m-3}{m-2}C^\varphi_{jki} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{jk}\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_i\delta_{jk} - \varphi^a_k\delta_{ij})\right) \\
=& - f_{tk}W^\varphi_{tijk} - f_k\frac{m-3}{m-2}C^\varphi_{jki} + \alpha(-\varphi^a_{ij}f_k\varphi^a_k + \varphi^a_{jk}f_k\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(-\varphi^a_i f_k\delta_{jk} + \varphi^a_k f_k\delta_{ij}) \\
=& f_{tk}W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha(\varphi^a_{jk}f_k\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) + \frac{\alpha}{m-2}(|\tau(\varphi)|^2\delta_{ij} - \varphi^a_{tt}\varphi^a_i f_j).
\end{aligned}$$

The last formula enables us to express $(m-2)B^\varphi_{ij}$, defined in (2.37), in the form

$$\begin{aligned}
(m-2)B^\varphi_{ij} =& f_{tk}W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha(\varphi^a_{jk}f_k\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) + \frac{\alpha}{m-2}(|\tau(\varphi)|^2\delta_{ij} - \varphi^a_{tt}\varphi^a_i f_j) \\
&+ R^\varphi_{tk}W^\varphi_{tikj} - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right) \\
=& (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha\varphi^a_{jk}f_k\varphi^a_i - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_{kkj}\varphi^a_i,
\end{aligned}$$

and using once again (2.67)

$$\begin{aligned}
(m-2)B^\varphi_{ij} =& (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_k f_{kj}\varphi^a_i \\
=& (R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j.
\end{aligned}$$

Thus the second integrability condition (2.57) can be expressed as

$$(R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j + \frac{m-4}{m-2}W^\varphi_{tijk}f_t f_k = 0,$$

and using once more (2.67) and (2.56)

$$\begin{aligned}
0 =& (R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j + \frac{m-4}{m-2}W^\varphi_{tijk}f_t f_k \\
=& (R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k f_t W^\varphi_{tjki} - \frac{\alpha}{m-2}\varphi^a_t f_t\varphi^a_i f_j + \frac{m-4}{m-2}W^\varphi_{tijk}f_t f_k \\
=& \left(R^\varphi_{tk} + f_{tk} + \frac{1}{m-2}f_t f_k\right)(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}).
\end{aligned}$$

Next we define
$$\lambda := \frac{1}{m}\left(S^{\varphi} + \Delta f + \frac{|\nabla f|^2}{m-2}\right),$$
so that the 2-times covariant symmetric tensor field
$$\beta := \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df - \lambda \langle \, , \, \rangle$$
is traceless. From the above identity and from (2.8) we then have
$$\begin{aligned}(W^{\varphi}_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\beta_{tk} =&(W^{\varphi}_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\left(R^{\varphi}_{tk} + f_{tk} + \frac{1}{m-2}f_tf_k - \lambda\delta_{tk}\right)\\ =&(W^{\varphi}_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\left(R^{\varphi}_{tk} + f_{tk} + \frac{1}{m-2}f_tf_k\right) - \lambda(W^{\varphi}_{kikj} - \alpha\varphi^a_i\varphi^a_j) = 0.\end{aligned}$$
Interchanging the role of i and j in the above equation we get
$$0 = (W^{\varphi}_{tjki} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk} = (W^{\varphi}_{tikj} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk}.$$
Summing up the last two formulas
$$\begin{aligned}0 =&(W^{\varphi}_{tikj} - \alpha\varphi^a_i\varphi^a_t\delta_{jk})\beta_{tk} + (W^{\varphi}_{tikj} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk}\\ =&[2W^{\varphi}_{tikj} - \alpha\varphi^a_t(\varphi^a_i\delta_{tj} + \varphi^a_j\delta_{ti})]\beta_{tk}\\ =&2\left(W^{\varphi}_{tikj} - \frac{1}{2}\alpha\varphi^a_t(\varphi^a_i\delta_{tj} + \varphi^a_j\delta_{ti})\right)\beta_{tk}.\end{aligned}$$
Hence,
$$\mathcal{W}^{\varphi}(\beta) = \left(W^{\varphi}_{tikj} - \frac{1}{2}\alpha\varphi^a_t(\varphi^a_i\delta_{tj} + \varphi^a_j\delta_{ti})\right)\beta_{tk}\theta^i \otimes \theta^j = 0.$$
Thus, since $\mathcal{W}^{\varphi}$ is injective, $\beta = 0$, that is,
$$\mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df = \lambda \langle \, , \, \rangle.$$
The latter together with (2.67) and Theorem 2.49 show that $(M, \widetilde{\langle \, , \, \rangle})$ is harmonic-Einstein. □

3 A gap result for harmonic-Einstein manifolds

The aim of this section is to prove Theorem 3.12 below when we give a gap result for $||T||_{L^\infty}$.

Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold of dimension m, $\varphi : M \to (N, \langle \, , \, \rangle_N)$ a smooth map, $\alpha \in \mathbb{R} \setminus \{0\}$ and set T^{φ} to denote the traceless part of the φ-Ricci tensor, that is,
$$T^{\varphi} := \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{m}\langle \, , \, \rangle. \tag{3.1}$$
Of course when φ is constant $T^{\varphi} \equiv T$ the usual traceless Ricci tensor. Let the operator $\mathcal{W}^{\varphi}$ be defined as in (2.61) and observe that for every $\beta \in S^2_0(M)$
$$\langle \mathcal{W}^{\varphi}(\beta), \beta \rangle = W^{\varphi}_{tikj}\beta_{tk}\beta_{ij} - \alpha\varphi^a_i\varphi^a_j\beta_{ik}\beta_{kj}, \tag{3.2}$$
where $\beta = \beta_{ij}\theta^i \otimes \theta^j$. We also set
$$\mathrm{div}(C^{\varphi}) := C^{\varphi}_{ijk,k}\theta^i \otimes \theta^j. \tag{3.3}$$

and

$$\mathrm{tr}(C^\varphi) = C^\varphi_{kki}\theta^i. \tag{3.4}$$

Note that, from (2.34),

$$\mathrm{tr}(C^\varphi)_{i,j} = \alpha(\varphi^a_{kk}\varphi^a_i)_j = \alpha(\varphi^a_{kki}\varphi^a_j + \varphi^a_{kk}\varphi^a_{ij}).$$

Next result is computational but far from trivial.

Theorem 3.5. *In the above setting and for $m \geq 3$ we have*

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 =& |\nabla T^\varphi|^2 + \frac{m-2}{2(m-1)} tr(T^\varphi \circ Hess(S^\varphi)) + \frac{m}{m-2} tr[(T^\varphi)^3] + \frac{S^\varphi}{m-1}|T^\varphi|^2 \\ &+ tr(div(C^\varphi) \circ T^\varphi) - \langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle - tr(T^\varphi \circ \nabla tr(C^\varphi))\end{aligned} \tag{3.6}$$

Proof. A simple calculation shows the validity of

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + T^\varphi_{ij,kk}T^\varphi_{ij}.$$

From (3.1),

$$T^\varphi_{ij,kk} = R^\varphi_{ij,kk} - \frac{\Delta S^\varphi}{m}\delta_{ij},$$

and since T^φ is traceless the formula above can be rewritten as

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + R^\varphi_{ij,kk}T^\varphi_{ij}. \tag{3.7}$$

Now we want to evaluate $R^\varphi_{ij,kk}$. First we derive the following commutation relation, alternative to (2.36),

$$R^\varphi_{ij,k} = R^\varphi_{ik,j} + R^t_{ikj,t} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j). \tag{3.8}$$

To prove it we use the second Bianchi identity and the definition (2.1) of the φ-Ricci tensor

$$\begin{aligned}R^t_{ijk,t} =& -R^t_{ikt,j} - R^t_{itj,k} = R_{ik,j} - R_{ij,k} \\ =& R^\varphi_{ik,j} + \alpha(\varphi^a_i\varphi^a_k)_j - R^\varphi_{ij,k} - \alpha(\varphi^a_i\varphi^a_j)_k \\ =& R^\varphi_{ik,j} - R^\varphi_{ij,k} + \alpha(\varphi^a_{ij}\varphi^a_k + \varphi^a_i\varphi^a_{kj}) - \alpha(\varphi^a_{ik}\varphi^a_j + \varphi^a_i\varphi^a_{jk}) \\ =& R^\varphi_{ik,j} - R^\varphi_{ij,k} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j).\end{aligned}$$

To compute the coefficients of $\Delta\mathrm{Ric}^\varphi$ we then use (3.8), together with (2.42), (2.10) and (2.1) to get:

$$\begin{aligned}R^\varphi_{ij,kk} \overset{(3.8)}{=}& [R^\varphi_{ik,j} + R^t_{ikj,t} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j)]_k \\ =& R^\varphi_{ik,jk} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\ \overset{(2.42)}{=}& R^\varphi_{ik,kj} + R^t_{ijk}R^\varphi_{tk} + R^t_{kjk}R^\varphi_{it} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\ \overset{(2.10)}{=}& \left(\frac{1}{2}S^\varphi_i - \alpha\varphi^a_{kk}\varphi^a_i\right)_j + R^t_{ijk}R^\varphi_{tk} + R_{tj}R^\varphi_{it} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\ \overset{(2.1)}{=}& \frac{1}{2}S^\varphi_{ij} - \alpha(\varphi^a_{kkj}\varphi^a_i + \varphi^a_{kk}\varphi^a_{ij}) + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + R^t_{ikj,tk} \\ &+ \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\ =& \frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + R^t_{ikj,tk} + \alpha(-\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}).\end{aligned}$$

Exploiting the commutation relation (2.36)

$$
\begin{aligned}
R^t_{ikj,tk} =&[R^\varphi_{ij,k} - R^\varphi_{ik,j} + \alpha(\varphi^a_{ik}\varphi^a_j - \varphi^a_{ij}\varphi^a_k)]_k \\
=&\left[C^\varphi_{ijk} + 12(m-1)(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik})\right]_k + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}) \\
=&C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}),
\end{aligned}
$$

and inserting into the above we obtain

$$
\begin{aligned}
R^\varphi_{ij,kk} =&\frac{m-2}{2(m-1)}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + \frac{\Delta S^\varphi}{2(m-1)}\delta_{ij} \\
&+ \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}).
\end{aligned}
\tag{3.9}
$$

Indeed,

$$
\begin{aligned}
R^\varphi_{ij,kk} =&\frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + \alpha(-\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\
&+ C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}) \\
=&\frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j - \alpha\varphi^a_{kkj}\varphi^a_i \\
&+ C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) - \alpha\varphi^a_{ij}\varphi^a_{kk} \\
=&\frac{m-2}{2(m-1)}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + \frac{\Delta S^\varphi}{2(m-1)}\delta_{ij} + \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}).
\end{aligned}
$$

Using the decomposition (2.6), that in components reads

$$
R_{tijk} = W^\varphi_{tijk} + \frac{1}{m-2}(R^\varphi_{tj}\delta_{ik} - R^\varphi_{tk}\delta_{ij} + R^\varphi_{ik}\delta_{tj} - R^\varphi_{ij}\delta_{tk}) - \frac{S^\varphi}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij}),
$$

we obtain

$$
\begin{aligned}
R^t_{ijk}R^\varphi_{tk} =&W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(R^\varphi_{tj}\delta_{ik} - R^\varphi_{tk}\delta_{ij} + R^\varphi_{ik}\delta_{tj} - R^\varphi_{ij}\delta_{tk})R^\varphi_{tk} \\
&- \frac{S^\varphi}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij})R^\varphi_{tk} \\
=&W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(R^\varphi_{tj}R^\varphi_{ti} - |\mathrm{Ric}^\varphi|^2\delta_{ij} + R^\varphi_{ik}R^\varphi_{jk} - R^\varphi_{ij}S^\varphi) \\
&- \frac{S^\varphi}{(m-1)(m-2)}(R^\varphi_{ij} - S^\varphi\delta_{ij}) \\
=&W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(2R^\varphi_{kj}R^\varphi_{ki} - |\mathrm{Ric}^\varphi|^2\delta_{ij} - R^\varphi_{ij}S^\varphi) - \frac{S^\varphi}{(m-1)(m-2)}(R^\varphi_{ij} - S^\varphi\delta_{ij}) \\
=&W^\varphi_{tijk}R^\varphi_{tk} + \frac{2}{m-2}R^\varphi_{ik}R^\varphi_{kj} - \frac{1}{m-2}|\mathrm{Ric}^\varphi|^2\delta_{ij} \\
&+ \frac{(S^\varphi)^2}{(m-1)(m-2)}\delta_{ij} - \frac{m}{(m-1)(m-2)}S^\varphi R^\varphi_{ij}.
\end{aligned}
$$

Inserting the last formula in (3.9) we obtain

$$
\begin{aligned}
R^\varphi_{ij,kk} =&\frac{m-2}{2(m-1)}S^\varphi_{ij} + \frac{m}{m-2}R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + W^\varphi_{tijk}R^\varphi_{tk} - \frac{m}{(m-1)(m-2)}S^\varphi R^\varphi_{ij} \\
&+ \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) \\
&+ \left[\frac{(S^\varphi)^2}{(m-1)(m-2)} + \frac{1}{2(m-1)}\Delta S^\varphi - \frac{1}{m-2}|\mathrm{Ric}^\varphi|^2\right]\delta_{ij}.
\end{aligned}
\tag{3.10}
$$

Using the fact that T^φ is traceless, from (3.10), we infer

$$\begin{aligned}R^\varphi_{ij,kk}T^\varphi_{ij} =& \frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij} + \frac{m}{m-2}T^\varphi_{ij}R^\varphi_{kj}R^\varphi_{ik} + T^\varphi_{ij}C^\varphi_{ijk,k} + W^\varphi_{tijk}T^\varphi_{ij}R^\varphi_{tk} \\ &- \frac{m}{(m-1)(m-2)}S^\varphi T^\varphi_{ij}R^\varphi_{ij} + \alpha T^\varphi_{ij}(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}).\end{aligned} \tag{3.11}$$

The following relations can be easily deduced from (3.1)

$$R^\varphi_{kj}R^\varphi_{ik}T^\varphi_{ij} = T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij} + \frac{2S^\varphi}{m}|T^\varphi|^2,$$

$$R^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij} = T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij} + \frac{S^\varphi}{m}T^\varphi_{ij}\varphi^a_i\varphi^a_j,$$

$$T^\varphi_{ij}R^\varphi_{tk}W^\varphi_{tijk} = T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk} - \alpha\frac{S^\varphi}{m}T^\varphi_{ij}\varphi^a_i\varphi^a_j.$$

Using them all in (3.11) we conclude that

$$\begin{aligned}R^\varphi_{ij,kk}T^\varphi_{ij} =& \frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij} + \frac{m}{m-2}T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij} + \frac{1}{m-1}S^\varphi|T^\varphi|^2 + T^\varphi_{ij}C^\varphi_{ijk,k} \\ &+ T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk} + \alpha T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij} - \alpha T^\varphi_{ij}(\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ij}\varphi^a_{kk}).\end{aligned}$$

Inserting the last formula in (3.7) we finally obtain

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 =& |\nabla T^\varphi|^2 + \frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij} + \frac{m}{m-2}T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij} + \frac{1}{m-1}S^\varphi|T^\varphi|^2 + T^\varphi_{ij}C^\varphi_{ijk,k} \\ &+ T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk} + \alpha T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij} - \alpha T^\varphi_{ij}(\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ij}\varphi^a_{kk}),\end{aligned}$$

that is, (3.6). ☐

We let $\eta(x)$ denote the largest eigenvalue of $\mathcal{W}^\varphi : S^2_0(M) \to S^2_0(M)$ at $x \in M$ and we set

$$\eta^* := \sup_M \eta.$$

We are now ready to prove the following

Theorem 3.12. *Let $(M, \langle\, ,\, \rangle)$ be a stochastically complete Riemannian manifold of dimension $m \geq 3$ and let $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ be a smooth map, $\alpha \in \mathbb{R}$, $\alpha > 0$. Assume*

i) S^φ is constant.

ii) φ is harmonic.

iii) $div(C^\varphi) = 0$.

Then, either $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein or

$$\sup_M |T^\varphi| \geq \sqrt{\frac{m-1}{m}}\left(\frac{S^\varphi}{m-1} - \eta^*\right). \tag{3.13}$$

Remark 3.14. Note that by Proposition 2.15, Definition 2.13 and (3.3), conditions *i*), *ii*) and *iii*) are necessary for $(M, \langle\, ,\, \rangle)$ to be harmonic-Einstein. Furthermore (3.13) is not empty only if

$$S^\varphi > (m-1)\eta^*. \tag{3.15}$$

Proof. First of all note that if $\eta^* = +\infty$ then (3.13) holds true. Thus we can suppose $\eta^* < +\infty$. In the assumptions of the Theorem $\mathrm{div}(C^\varphi) = 0$ and, since φ is harmonic, by (3.4) and (2.34), $\mathrm{tr}(C^\varphi) = 0$. Thus equation (3.6) becomes

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + \frac{m}{m-2}\mathrm{tr}[(T^\varphi)^3] + \frac{S^\varphi}{m-1}|T^\varphi|^2 - \langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle. \tag{3.16}$$

Since T^φ is traceless, Okumura's inequality, [34], gives the validity of

$$\mathrm{tr}[(T^\varphi)^3] \geq -\frac{m-2}{\sqrt{m(m-1)}}|T^\varphi|^3.$$

Furthermore, from the estimates on the largest eigenvalue of $\mathcal{W}^\varphi$

$$\langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle \leq \eta^*|T^\varphi|^2.$$

Inserting these informations in (3.16) and setting $u := |T^\varphi|^2$ we deduce the validity of the differential inequality

$$\frac{1}{2}\Delta u \geq \left(\frac{S^\varphi}{m-1} - \eta^* - \frac{m}{\sqrt{m(m-1)}}\sqrt{u}\right)u. \tag{3.17}$$

If $u^* := \sup_M u = +\infty$ then (3.13) is obviously satisfied. Thus let $u^* < +\infty$. Since stochastically completeness is equivalent to the validity of the weak maximum principle for the Laplace-Beltrami operator, see [35], applying the latter to (3.17) we obtain

$$0 \geq \left(\frac{S^\varphi}{m-1} - \eta^* - \frac{m}{\sqrt{m(m-1)}}\sqrt{u^*}\right)u^*.$$

Thus either $u^* = 0$, that is, $T^\varphi = 0$ on M and $(M, \langle\, ,\,\rangle)$ is harmonic-Einstein or

$$\frac{S^\varphi}{m-1} - \eta^* - \sqrt{\frac{mu^*}{m-1}} \leq 0.$$

The latter inequality implies (3.13). $\square$

As a consequence we obtain the following "gap"result for $|T^\varphi|^2$.

Corollary 3.18. *Under the assumptions of Theorem 3.12 together with* (3.15) *suppose that*

$$\sup_M |T^\varphi|^2 < \frac{m-1}{m}\left(\frac{S^\varphi}{m-1} - \eta^*\right)^2, \tag{3.19}$$

then $(M, \langle\, ,\,\rangle)$ *is harmonic-Einstein.*

In particular when φ is constant we deduce

Corollary 3.20. *Let* $(M, \langle\, ,\,\rangle)$ *be a stochastically complete manifold of dimension* $m \geq 3$*. Assume that the scalar curvature is constant and* $\mathrm{div}C = 0$*. If*

$$\sup_M |T|^2 < \frac{m-1}{m}\left(\frac{S}{m-1} - \eta^*\right)^2$$

then $(M, \langle\, ,\,\rangle)$ *is Einstein.*

Note that in this case $\eta^* = \sup_M \eta$ is the largest eigenvalue of $\mathcal{W} = \mathcal{W}^\varphi : S^2_0(M) \to S^2_0(M)$ (see Definition 2.60).

Remark 3.21. A result in the spirit of Theorem 3.12 but with different assumptions is given in Theorem A of [27].

To conclude this Section we provide an estimate for η^* in the following

Proposition 3.22. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ a smooth map and $\alpha > 0$. If*

$$|W^\varphi|^* := \sup_M |W^\varphi| < +\infty \quad \text{and} \quad (|d\varphi|^2)^* := \sup_M |d\varphi|^2 < +\infty,$$

then

$$\eta^* \leq \sqrt{\frac{m-2}{2(m-1)}|W^\varphi|^* + \frac{\alpha}{m-2}(|d\varphi|^2)^*} \tag{3.23}$$

Proof. We set: for every $\beta \in S_0^2(M)$, $\beta = \beta_{ij}\theta^i \otimes \theta^j$,

$$\mathcal{W}(\beta) := W_{tikj}\beta_{tk}\theta^i \otimes \theta^j.$$

Then $\mathcal{W} : S_0^2(M) \to S_0^2(M)$ is well defined and self-adjoint with respect to the standard extension of $\langle\, ,\, \rangle$ to $S_0^2(M)$. Moreover from Huisken's inequality (see Lemma 2.9 in [24] or also Proposition 8.8 in [1], whose proof can be extended, with the notation there, to the case where $T \in S_0^2(M)$)

$$|\langle \mathcal{W}(\beta), \beta\rangle| \leq \sqrt{\frac{m-2}{2(m-1)}|W|^2}|\beta|^2. \tag{3.24}$$

From (3.2) and the decomposition (2.7) we get

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle = \langle \mathcal{W}(\beta), \beta\rangle - \alpha\frac{m}{m-2}|d\varphi(\beta)|^2 + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2|\beta|^2, \tag{3.25}$$

where, in local coordinates,

$$d\varphi(\beta) = \varphi^a_j\beta_{ij}\theta^i \otimes E_a.$$

From (3.25) we deduce

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle \leq \langle \mathcal{W}(\beta), \beta\rangle + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2|\beta|^2$$

and using (3.24) we have

$$|\langle \mathcal{W}^\varphi(\beta), \beta\rangle| \leq \left(\sqrt{\frac{m-2}{2(m-1)}|W|^2} + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2\right)|\beta|^2. \tag{3.26}$$

To obtain (3.23) we need the following relation between $|W|^2$ and $|W^\varphi|^2$:

$$|W^\varphi|^2 = |W|^2 + \frac{4\alpha^2}{m-2}|\varphi^*\langle\, ,\, \rangle_N|^2 - \frac{2\alpha^2}{(m-1)(m-2)}|d\varphi|^4. \tag{3.27}$$

To prove (3.27) we use (2.7) and the symmetries of W^φ to get

$$\begin{aligned}
|W^\varphi|^2 =& W^\varphi_{tikj}W^\varphi_{tikj} \\
=& W^\varphi_{tikj}\left[W_{tikj} + \frac{\alpha}{m-2}(\varphi^a_t\varphi^a_k\delta_{ij} - \varphi^a_t\varphi^a_j\delta_{ik} + \varphi^a_i\varphi^a_j\delta_{tk} - \varphi^a_i\varphi^a_k\delta_{tj}) - \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2(\delta_{tk}\delta_{ij} - \delta_{tj}\delta_{ik})\right] \\
=& W^\varphi_{tikj}W_{tikj} + \frac{4\alpha}{m-2}W^\varphi_{tiki}\varphi^a_t\varphi^a_k - \frac{2\alpha}{(m-1)(m-2)}|d\varphi|^2W^\varphi_{kiki},
\end{aligned}$$

and we conclude using (2.7), the fact that W is totally trace free and (2.8). From (3.27) we obtain

$$|W|^2 \leq |W^\varphi|^2 + \frac{2\alpha^2}{(m-1)(m-2)}|d\varphi|^4,$$

so that

$$\sqrt{\frac{m-2}{2(m-1)}|W|^2} \leq \sqrt{\frac{m-2}{2(m-1)}|W^\varphi|^2 + \left(\frac{\alpha}{m-1}|d\varphi|^2\right)^2} \leq \sqrt{\frac{m-2}{2(m-1)}}|W^\varphi| + \frac{\alpha}{m-1}|d\varphi|^2. \tag{3.28}$$

Plugging the above into (3.26) we get

$$|\langle \mathcal{W}^\varphi(\beta), \beta\rangle| \leq \left(\sqrt{\frac{m-2}{2(m-1)}}|W^\varphi| + \frac{\alpha}{m-2}|d\varphi|^2\right)|\beta|^2,$$

and then (3.23) holds. □

4 The general structure, formulas and a "spectral" non-existence result

The geometric considerations discussed in Sections 2 and 3 justify the introduction of the following general structure on a Riemannian manifold. In what follow $\mathfrak{X}(M)$ will denote the $\mathcal{C}^\infty(M)$-module of the vector fields on M.

Definition 4.1. We say that the Riemannian manifold $(M, \langle\,,\,\rangle)$ carries an *Einstein-type structure* if there exist $X \in \mathfrak{X}(M)$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$ for some Riemannian manifold $(N, \langle\,,\,\rangle_N)$, and functions $\alpha, \lambda, \mu \in \mathcal{C}^\infty(M)$ such that

$$\begin{cases} \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle - \mu X^\flat \otimes X^\flat - \alpha\varphi^*\langle\,,\,\rangle_N = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{4.2}$$

where ${}^\flat : \mathfrak{X}(M) \to \bigwedge^1(M)$ is the musical isomorphism and $\mathcal{L}_X\langle\,,\,\rangle$ denotes the Lie derivative of the metric along the vector field X.

In case $X = \nabla f$ for some $f \in \mathcal{C}^\infty(M)$ we say that $(M, \langle\,,\,\rangle)$ carries a gradient Einstein-type structure. In case the Einstein-type structure is gradient (4.2) takes the form

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df - \alpha\varphi^*\langle\,,\,\rangle_N = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases} \tag{4.3}$$

When α is a constant the term $\mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N$ will be simply written as Ric^φ, following the notation introduced in Section 2.

The following commutation relations, valid for every $Y \in \mathfrak{X}(M)$, are proved in [1], equations (8.9) and (8.25) respectively

$$Y^i_{jk} - Y^i_{kj} = Y^t R_{tijk}, \tag{4.4}$$

$$Y^i_{jkl} - Y^i_{jlk} = Y^t_j R_{tikl} + Y^i_t R^t_{jkl}. \tag{4.5}$$

We shall need them in the proof of the following

Proposition 4.6. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension m with an Einstein-type structure as in (4.2), with $\alpha \in \mathbb{R}\setminus\{0\}$, $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^\infty(M)$, $\mu \in \mathbb{R}$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$. Then in a local orthonormal coframe the following hold,*

$$R^\varphi_{ij,k} - R^\varphi_{ik,j} + R_{tijk}X^t + \frac{1}{2}(X^j_k - X^k_j)_i = \mu[X^i_k X^j - X^i_j X^k + X^i(X^j_k - X^k_j)] + \lambda_k\delta_{ij} - \lambda_j\delta_{ik}, \tag{4.7}$$

$$\frac{1}{2}S^\varphi_k - R^\varphi_{ik}X^i + \frac{1}{2}(X^i_k - X^k_i)_i = \mu\left[\frac{1}{2}(X^i_k + X^k_i)X^i + \frac{3}{2}(X^i_k - X^k_i)X^i - X^i_iX^k\right] + (m-1)\lambda_k, \tag{4.8}$$

$$\begin{aligned}\frac{1}{2}\Delta_{(1+2\mu)X}S^\varphi + (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) + \left[\frac{(m-1)\mu+1}{m}S^\varphi - \mu(m-1)\lambda\right](S^\varphi - m\lambda)\\ = (m-1)\Delta_{2\mu X}\lambda + \frac{\mu}{2}D,\end{aligned} \tag{4.9}$$

where

$$D := 2[(X^i_k - X^k_i)X^i]_k + (X^i_k - X^k_i)X^i_k. \tag{4.10}$$

Here Δ_Y, for $Y \in \mathfrak{X}(M)$, stands for the operator $\Delta - \langle Y, \nabla\ \rangle$.

Proof. In a local orthonormal coframe (4.2) is given by

$$\begin{cases} R^\varphi_{ij} + \frac{1}{2}(X^i_j + X^j_i) = \mu X^iX^j + \lambda\delta_{ij} \\ \varphi^a_{ii} = \varphi^a_i X^i. \end{cases} \tag{4.11}$$

The covariant derivative of the first equation in (4.11) is

$$R^\varphi_{ij,k} + \frac{1}{2}(X^i_{jk} + X^j_{ik}) = \mu(X^i_kX^j + X^iX^j_k) + \lambda_k\delta_{ij}.$$

Inverting the role of j and k and subtracting we obtain

$$R^\varphi_{ij,k} - R^\varphi_{ik,j} + \frac{1}{2}(X^i_{jk} - X^i_{kj} + X^j_{ik} - X^k_{ij}) = \mu(X^i_kX^j - X^i_jX^k + X^iX^j_k - X^iX^k_j) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik}.$$

Using three times (4.4) and the first Bianchi identity we deduce

$$\frac{1}{2}(X^i_{jk} - X^i_{kj} + X^j_{ik} - X^k_{ij}) = R_{tijk}X^t + \frac{1}{2}(X^j_k - X^k_j)_i.$$

Plugging into the above we have

$$R^\varphi_{ij,k} - R^\varphi_{ik,j} + R_{tijk}X^t + \frac{1}{2}(X^j_k - X^k_j)_i = \mu[X^i_kX^j - X^i_jX^k + X^i(X^j_k - X^k_j)] + \lambda_k\delta_{ij} - \lambda_j\delta_{ik},$$

that is (4.7). Tracing on i and j in (4.7) we get

$$S^\varphi_k - R^\varphi_{ik,i} - R_{ik}X^i + \frac{1}{2}(X^i_k - X^k_i)_i = \mu(2X^i_kX^i - X^i_iX^k - X^iX^k_i) + (m-1)\lambda_k.$$

Using (2.10), the second equation of (4.11) and the definition (2.1) of Ric^φ we infer

$$R^\varphi_{ik,i} + R_{ik}X^i = \frac{1}{2}S^\varphi_k - \alpha\varphi^a_{ii}\varphi^a_k + R_{ik}X^i = \frac{1}{2}S^\varphi_k + R^\varphi_{ik}X^i$$

and inserting into the above we obtain (4.8).

Tracing the first equation of (4.11) we deduce

$$S^\varphi + X^i_i = \mu|X|^2 + m\lambda, \tag{4.12}$$

using it together with the first equation of (4.11) in (4.8) we get

$$\begin{aligned}\frac{1}{2}S^\varphi_k - R^\varphi_{ik}X^i + \frac{1}{2}(X^i_k - X^k_i)_i =& \mu\left[(-R^\varphi_{ik} + \mu X^iX^k + \lambda\delta_{ik})X^i + (S^\varphi - \mu|X|^2 - m\lambda)X^k\right]\\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)X^i + (m-1)\lambda_k,\end{aligned}$$

or, equivalently,

$$\frac{1}{2}S^{\varphi}_k + \frac{1}{2}(X^i_k - X^k_i)_i = (1-\mu)R^{\varphi}_{ik}X^i + \mu(S^{\varphi} - (m-1)\lambda)X^k + \mu\frac{3}{2}(X^i_k - X^k_i)X^i + (m-1)\lambda_k, \tag{4.13}$$

Contracting (4.13) with X^k we obtain

$$\frac{1}{2}S^{\varphi}_k X^k + \frac{1}{2}(X^i_k - X^k_i)_i X^k = (1-\mu)R^{\varphi}_{ik}X^i X^k + \mu(S^{\varphi} - (m-1)\lambda)|X|^2 + (m-1)\lambda_k X^k,$$

thus

$$(1-\mu)\mu R^{\varphi}_{ik}X^i X^k = \frac{\mu}{2}S^{\varphi}_k X^k + \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^{\varphi} - (m-1)\lambda)\mu|X|^2 - (m-1)\mu\lambda_k X^k. \tag{4.14}$$

From (4.5) easily follows

$$X^i_{kik} = X^k_{iik},$$

then taking the divergence of (4.13) and exploiting the above commutation relation we have

$$\begin{aligned}\frac{1}{2}S^{\varphi}_{kk} =& (1-\mu)(R^{\varphi}_{ik,k}X^i + R^{\varphi}_{ik}X^i_k) + \mu(S^{\varphi}_k - (m-1)\lambda_k)X^k + \mu(S^{\varphi} - (m-1)\lambda)X^k_k \\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)\lambda_{kk}.\end{aligned} \tag{4.15}$$

From the first equation of (4.11) we infer

$$|\mathrm{Ric}^{\varphi}|^2 + R^{\varphi}_{ik}X^i_k = \mu R^{\varphi}_{ik}X^i X^k + \lambda S^{\varphi},$$

using the definition (3.1) the above is equivalent to

$$|T^{\varphi}|^2 + \frac{(S^{\varphi})^2}{m} + R^{\varphi}_{ik}X^i_k = \mu R^{\varphi}_{ik}X^i X^k + \lambda S^{\varphi},$$

that is,

$$R^{\varphi}_{ik}X^i_k = -|T^{\varphi}|^2 - \frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) + \mu R^{\varphi}_{ik}X^i X^k.$$

Using the above formula and (2.10) we deduce

$$\begin{aligned}(1-\mu)(R^{\varphi}_{ik,k}X^i + R^{\varphi}_{ik}X^i_k) =& \left(\frac{1}{2} - \frac{\mu}{2}\right)S^{\varphi}_i X^i - (1-\mu)\alpha\varphi^a_{kk}\varphi^a_i X^i - (1-\mu)|T^{\varphi}|^2 \\ &- (1-\mu)\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) + \mu(1-\mu)R^{\varphi}_{ik}X^i X^k\end{aligned}$$

and from the second equation of (4.11) and (4.14) it follows

$$\begin{aligned}(1-\mu)(R^{\varphi}_{ik,k}X^i + R^{\varphi}_{ik}X^i_k) =& \frac{1}{2}S^{\varphi}_i X^i - (1-\mu)(|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2) - (1-\mu)\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^{\varphi} - (m-1)\lambda)\mu|X|^2 - (m-1)\mu\lambda_k X^k.\end{aligned}$$

Inserting the latter into (4.15) we obtain

$$\begin{aligned}\frac{1}{2}S^{\varphi}_{kk} =& \frac{1+2\mu}{2}S^{\varphi}_i X^i - (1-\mu)(|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2) - (1-\mu)\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^{\varphi} - (m-1)\lambda)(-X^k_k + \mu|X|^2) - 2(m-1)\mu\lambda_k X^k \\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)\lambda_{kk},\end{aligned}$$

that, using (4.12) can be written as

$$\begin{aligned}\frac{1}{2}S^{\varphi}_{kk} =& \frac{1+2\mu}{2}S^{\varphi}_{i}X^{i} - (1-\mu)(|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2) - \left[(1-\mu)\frac{S^{\varphi}}{m} + \mu S^{\varphi} - \mu(m-1)\lambda\right](S^{\varphi} - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k + \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)(\lambda_{kk} - 2\mu\lambda_k X^k),\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\Delta_{(1+2\mu)X}S^{\varphi} + (1-\mu)(|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2) + \left[\frac{(m-1)\mu+1}{m}S^{\varphi} - \mu(m-1)\lambda\right](S^{\varphi} - m\lambda) \\ &\qquad = (m-1)\Delta_{2\mu X}\lambda + \frac{\mu}{2}[(X^i_k - X^k_i)_i X^k + 3(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k].\end{aligned}$$

We then conclude the validity of (4.9), since

$$\begin{aligned}(X^i_k - X^k_i)_i X^k + 3(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k &= 2(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k \\ &= 2[(X^i_k - X^k_i)X^i]_k + (X^i_k - X^k_i)X^i_k \\ &= D.\end{aligned}$$

□

Remark 4.16. In case $\mu = 0$ equation (4.9) can be rewritten as

$$\frac{1}{2}\Delta_X S^{\varphi} + \alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2 + \frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) = (m-1)\Delta\lambda. \tag{4.17}$$

Observe that when $X = \nabla f$, or more generally in case ∇X is symmetric, equation (4.7) becomes

$$R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} + R_{tijk}f_t = \mu(f_{ik}f_j - f_{ij}f_k) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik} \tag{4.18}$$

and (4.8) becomes

$$\frac{1}{2}S^{\varphi}_k - R^{\varphi}_{ik}f_i = \mu(f_{ik}f_i - \Delta f f_k) + (m-1)\lambda_k, \tag{4.19}$$

moreover D defined in (4.10) vanishes identically and thus (4.9) takes the form

$$\frac{1}{2}\Delta_{(1+2\mu)f}S^{\varphi} + (1-\mu)(|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2) + \left[\frac{(m-1)\mu+1}{m}S^{\varphi} - \mu(m-1)\lambda\right](S^{\varphi} - m\lambda) = (m-1)\Delta_{2\mu f}\lambda, \tag{4.20}$$

that shall be used in Theorem 7.29 of Section 7.

Bochner's type formula (4.22), contained in the Proposition below, will be used later on in the proof of Proposition 7.62.

Proposition 4.21. *Let $(M, \langle\, ,\,\rangle)$ be an m-dimensional Riemannian manifold with an Einstein-type structure as in* (4.3) *with $\lambda, f \in \mathcal{C}^{\infty}(M)$, $\alpha, \mu \in \mathbb{R}$ and $\varphi : M \to (N, \langle\, ,\,\rangle_N)$. Then*

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |Hess(f)|^2 + \alpha|\tau(\varphi)|^2 + (2\mu\lambda m - \lambda - 2\mu S^{\varphi})|\nabla f|^2 + \mu(2\mu-1)|\nabla f|^4 - (m-2)\langle\nabla\lambda, \nabla f\rangle. \tag{4.22}$$

Proof. From Bochner's formula

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \langle\nabla\Delta f, \nabla f\rangle + \mathrm{Ric}(\nabla f, \nabla f). \tag{4.23}$$

Using the definition (2.1) of Ric^{φ} and the second equation of (4.3) we obtain

$$\mathrm{Ric}(\nabla f, \nabla f) = \mathrm{Ric}^{\varphi}(\nabla f, \nabla f) + \alpha|\tau(\varphi)|^2. \tag{4.24}$$

Tracing the first equation of (4.3) we get

$$\Delta f = -S^{\varphi} + \mu|\nabla f|^2 + m\lambda, \tag{4.25}$$

so that,

$$\langle \nabla\Delta f, \nabla f\rangle = -\langle \nabla S^{\varphi}, \nabla f\rangle + \mu\langle \nabla|\nabla f|^2, \nabla f\rangle + m\langle \nabla\lambda, \nabla f\rangle.$$

Contracting (4.19) with ∇f we deduce

$$\langle \nabla S^{\varphi}, \nabla f\rangle = 2\mathrm{Ric}^{\varphi}(\nabla f, \nabla f) + 2\mu[\mathrm{Hess}(f)(\nabla f, \nabla f) - \Delta f|\nabla f|^2] + 2(m-1)\langle \nabla\lambda, \nabla f\rangle$$

inserting into the above and observing that

$$\langle \nabla|\nabla f|^2, \nabla f\rangle = 2\mathrm{Hess}(f)(\nabla f, \nabla f), \tag{4.26}$$

we have

$$\langle \nabla\Delta f, \nabla f\rangle = -2\mathrm{Ric}^{\varphi}(\nabla f, \nabla f) + 2\mu\Delta f|\nabla f|^2 - (m-2)\langle \nabla\lambda, \nabla, f\rangle. \tag{4.27}$$

We plug (4.27) and (4.24) into (4.23) to infer

$$\frac{1}{2}\Delta|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + 2\mu\Delta f|\nabla f|^2 - \mathrm{Ric}^{\varphi}(\nabla f, \nabla f) - (m-2)\langle \nabla\lambda, \nabla, f\rangle.$$

Using the first equation of (4.3),(4.25) and (4.26) we obtain

$$\begin{aligned}2\mu\Delta f|\nabla f|^2 - \mathrm{Ric}^{\varphi}(\nabla f, \nabla f) =& 2\mu(-S^{\varphi} + \mu|\nabla f|^2 + m\lambda)|\nabla f|^2 + (\mathrm{Hess}(f) - \mu df \otimes df - \lambda\langle\, ,\, \rangle)(\nabla f, \nabla f)\\ =& (2\mu m\lambda - 2\mu S^{\varphi} - \lambda)|\nabla f|^2 + \mu(2\mu - 1)|\nabla f|^4 + \frac{1}{2}\langle \nabla|\nabla f|^2, \nabla f\rangle,\end{aligned}$$

replacing into the above we get (4.22). ☐

Next Proposition shall be used in the proof of Lemma 5.2.

Proposition 4.28. *Let $(M, \langle\, ,\, \rangle)$ and $(N, \langle\, ,\, \rangle_N)$ be Riemannian manifolds, $\varphi : M \to N$, $\alpha \in \mathbb{R} \setminus \{0\}$. Let X be a conformal vector field on M, satisfying*

$$\tau(\varphi) = d\varphi(X). \tag{4.29}$$

Setting

$$\eta := \frac{1}{m} div(X), \tag{4.30}$$

we have

$$\Delta\eta + \frac{S^{\varphi}}{m-1}\eta + \frac{1}{2(m-1)}\langle \nabla S^{\varphi}, X\rangle + \frac{\alpha}{m-1}\langle \nabla\tau(\varphi), d\varphi\rangle = 0. \tag{4.31}$$

Remark 4.32. In case φ is a constant map, so that (4.29) is automatically satisfied, we get the well known formula

$$\Delta\eta + \frac{S}{m-1}\eta + \frac{1}{2(m-1)}\langle \nabla S, X\rangle = 0. \tag{4.33}$$

Proof. To prove (4.31) we shall use (4.33). However, for the sake of completeness we first give a proof of the latter. Since X is conformal

$$\frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = \eta\langle\, ,\, \rangle. \tag{4.34}$$

We rewrite (4.34) in local form with respect to an orthonormal coframe as

$$X^i_j + X^j_i = 2\eta\delta_{ij}. \tag{4.35}$$

Observe that, from (4.35)

$$R_{ij}X^i_j + R_{ij}X^j_i = 2S\eta,$$

and since Ric is symmetric

$$R_{ij}X^i_j = \eta S. \tag{4.36}$$

Moreover from Schur identity

$$R_{ji,j} = \frac{1}{2}S_i. \tag{4.37}$$

Clearly, from (4.30)

$$\Delta\eta = \frac{1}{m}\Delta(\mathrm{div}(X)), \tag{4.38}$$

using (4.4) we compute

$$\begin{aligned}\Delta(\mathrm{div}(X)) =& (X^i_i)_{jj} = (X^i_{ij})_j = (X^i_{ji} + R_{kiij}X^k)_j \\ =& X^i_{jij} - (R_{ij}X^i)_j \\ =& (X^i_j)_{ij} - R_{ij,j}X^i - R_{ij}X^i_j.\end{aligned}$$

With the aid of (4.35), (4.37) and (4.36) the latter can be rewritten in the form

$$\begin{aligned}\Delta(\mathrm{div}(X)) =& (-X^j_i + 2\eta\delta_{ij})_{ij} - \frac{1}{2}S_iX^i - S\eta \\ =& -X^j_{iij} + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta.\end{aligned}$$

Using (4.5) we obtain

$$X^j_{iij} = X^j_{iji} + R_{kjij}X^k_i + R^k_{iij}X^j_k = X^j_{iji} + R_{ki}X^k_i - R_{kj}X^j_k = X^j_{iji},$$

and inserting into the last equality we infer

$$\Delta(\mathrm{div}(X)) = -(X^j_{ij})_i + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta.$$

Using once again (4.4), (4.35), (4.37) and (4.36):

$$\begin{aligned}\Delta(\mathrm{div}(X)) =& -(X^j_{ji} + R_{kjij}X^k)_i + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta \\ =& -X^j_{jii} - (R_{ki}X^k)_i + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta \\ =& -\Delta\mathrm{div}(X) - R_{ki,i}X^k - R_{ki}X^k_i + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta \\ =& -\Delta\mathrm{div}(X) - \frac{1}{2}S_iX^i - S\eta + 2\Delta\eta - \frac{1}{2}S_iX^i - S\eta \\ =& -\Delta\mathrm{div}(X) - S_iX^i - 2S\eta + 2\Delta\eta,\end{aligned}$$

that is,

$$\Delta(\mathrm{div}(X)) = \Delta\eta - \frac{1}{2}S_iX^i - S\eta.$$

Then replacing in (4.38) we get

$$\Delta\eta = \frac{1}{m}\left(\Delta\eta - \frac{1}{2}\langle\nabla S, X\rangle - S\eta\right),$$

that is, (4.33).

Next we obtain (4.31) from (4.33). Towards this aim we observe that, from (2.2)

$$S = S^{\varphi} + \alpha|d\varphi|^2, \tag{4.39}$$

and then

$$S_i = S^{\varphi}_i + 2\alpha\varphi^a_{ki}\varphi^a_k.$$

Thus,

$$S_iX^i = S^{\varphi}_i X^i + 2\alpha\varphi^a_{ki}\varphi^a_k X^i. \tag{4.40}$$

Using the symmetry of $\nabla d\varphi$ and (4.29)

$$\varphi^a_{ki}X^i = \varphi^a_{ik}X^i = (\varphi^a_i X^i)_k - \varphi^a_i X^i_k = \varphi^a_{iik} - \varphi^a_i X^i_k. \tag{4.41}$$

Inserting (4.41) in (4.40) we obtain

$$S_iX^i = S^{\varphi}_i X^i + 2\alpha\varphi^a_{iik}\varphi^a_k - 2\alpha\varphi^a_i\varphi^a_k X^i_k. \tag{4.42}$$

Moreover, from (4.35) and the symmetry of $\varphi^*\langle\,,\,\rangle_N$

$$\varphi^a_i\varphi^a_j X^i_j = \eta|d\varphi|^2,$$

hence putting it in (4.42)

$$S_iX^i = S^{\varphi}_i X^i + 2\alpha\varphi^a_{iik}\varphi^a_k - 2\alpha\eta|d\varphi|^2,$$

that is,

$$\frac{1}{2}\langle\nabla S, X\rangle = \frac{1}{2}\langle\nabla S^{\varphi}, X\rangle + \alpha\langle\nabla\tau(\varphi), d\varphi\rangle - \alpha\eta|d\varphi|^2. \tag{4.43}$$

Using (4.43) and (4.39) in (4.33) we finally obtain (4.31). □

We now present a general non-existence result, based on spectral considerations, for gradient Einstein-type structures with $\mu \neq 0$.

Proposition 4.44. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension m. For $r \in \mathbb{R}^+$, let*

$$v(r) := vol(\partial B_r), \quad A(r) := \frac{\mu}{v(r)}\int_{\partial B_r}(m\lambda - S^{\varphi}),$$

where B_r is the geodesic ball of radius r centered at $o \in M$. Let $z \in Lip_{loc}(\mathbb{R}^+_0)$ be a solution of the Cauchy problem

$$\begin{cases} (vz')' + Av = 0 & \text{on } \mathbb{R}^+ \\ z(0^+) = z_0 > 0, & (vz')(0^+) = 0. \end{cases} \tag{4.45}$$

Suppose that z admits a first zero $R_0 \in \mathbb{R}^+$. Then there exist no $f, \lambda \in \mathcal{C}^\infty(M)$ and $\alpha, \mu \in \mathbb{R}\setminus\{0\}$, such that

$$Ric^{\varphi} + Hess(f) - \mu df \otimes df = \lambda\langle\,,\,\rangle. \tag{4.46}$$

Proof. By contradiction assume the existence of $f, \lambda \in \mathcal{C}^\infty(M)$ and $\alpha, \mu \in \mathbb{R}\setminus\{0\}$, such that (4.46) holds. Since $\mu \neq 0$, the positive function $u := e^{-\mu f}$ satisfies

$$\mathrm{Hess}(f) - \mu df \otimes df = -\frac{\mathrm{Hess}(u)}{\mu u},$$

and (4.46) can be rewritten as

$$\mathrm{Ric}^{\varphi} - \frac{\mathrm{Hess}(u)}{\mu u} = \lambda\langle\,,\,\rangle.$$

Taking the trace of the above we obtain $Lu = 0$, where

$$Lu := \Delta u + q(x)u, \quad q := \mu(m\lambda - S^\varphi).$$

Since $u > 0$, by a well known result of [19] and [31], the operator L is stable or, in other words, its spectral radius $\lambda_1^L(M)$ is non-negative.

Now we prove that under our assumptions $\lambda_L^1(M) < 0$, obtaining the desired contradiction. Observe that $v \in L^\infty_{loc}(\mathbb{R}_0^+)$, $v > 0$ on $\mathbb{R}^+$ and $v^{-1} \in L^\infty_{loc})(\mathbb{R}^+)$ by Proposition 1.6 of [10]. By Proposition 3.2 of [10] the solution of (4.45) is in $Lip_{loc}(\mathbb{R}_0^+)$ and its possible zeroes are isolated. Suppose that z admits a first zero $R_0 \in \mathbb{R}^+$. We define

$$\psi := z \circ r,$$

where r is the distance function from the fixed origin $o \in M$. We consider the Rayleigh quotient

$$Q(\psi) := \left(\int_{B_{R_0}} \psi^2\right)^{-1} \int_{B_{R_0}} (|\nabla\psi|^2 - q\psi^2).$$

From the co-area formula and Gauss lemma we get

$$Q(\psi) = \left(\int_0^{R_0} z^2 v\right)^{-1} \int_0^{R_0} [(z')^2 v - Avz^2].$$

Integrating by parts and using (4.45) we obtain

$$\int_0^{R_0} (z')^2 v = zz'v|_0^{R_0} - \int_0^{R_0} z(vz') = \int_0^{R_0} Avz^2,$$

so that $Q(\psi) = 0$. Then $\lambda_1^L(B_{R_0}) \le 0$ and by monotonicity of the eigenvalues of L we infer $\lambda_1^L(M) < 0$. □

It remains to determine some sufficient conditions under which a solution z of (4.45), always existing by Proposition 3.2 of [10], admits a first zero. From Corollary 5.2 of [10], if $A \ge 0$ on $\mathbb{R}^+$, $A \not\equiv 0$ and either $g^{-1} \notin L^1(+\infty)$ or otherwise there exist $r > R > 0$ such that $A \not\equiv 0$ on $[0, R]$ and

$$\int_R^r (\sqrt{A} - \sqrt{\chi_g}) > -\frac{1}{2}\left(\log\int_0^R Av + \log\int_R^{+\infty}\frac{1}{g}\right), \tag{4.47}$$

z has a first zero. Here $g \in L^\infty_{loc}(\mathbb{R}_0^+)$ is such that $g^{-1} \in L^\infty_{loc}(\mathbb{R}^+)$ and $0 \le v \le g$ on $\mathbb{R}_0^+$, while χ_g is the critical curve relative to g defined by

$$\chi_g(r) = \left\{2g(r)\int_r^{+\infty}\frac{1}{g}\right\}^{-2}.$$

Note that (4.47) can be rewritten as

$$\int_R^r (\sqrt{A} - \sqrt{\chi_g}) > -\frac{1}{2}\left(\log\int_{B_R} \mu(m\lambda - S^\varphi) + \log\int_R^{+\infty}\frac{1}{g}\right).$$

Observe that the existence of a first zero is “a fortiori” guaranteed by an oscillatory condition. For instance, from Corollary 2.9 of [26], if for some $r_0 \in \mathbb{R}^+$

$$\mu \lim_{r\to+\infty} \int_{B_r \setminus B_{r_0}} (m\lambda - S^\varphi) = +\infty, \tag{4.48}$$

then every solution of (4.45) is oscillatory. By way of example, we have

Proposition 4.49. *Suppose $S^{\varphi} \leq m\lambda$ on M,*

$$v(r) \leq Cr^{\theta}, \tag{4.50}$$

for some constants $C > 0$ and $\theta \in \mathbb{R}$ and, in case $\theta > 1$, that for some $R \in \mathbb{R}^{+}$ and for some constant D

$$\int_{\partial B_r} \mu(m\lambda - S^{\varphi}) \geq \frac{D^2}{r^{\gamma}} v(r) \quad \text{for } r \geq R, \tag{4.51}$$

with either $\gamma < 2$ or $\gamma = 2$ and $D > \frac{\theta-1}{2}$. Then a solution z of (4.45) admits a first zero.

Proof. From (4.50) we can choose

$$g(r) = Cr^{\theta}.$$

Clearly $g^{-1} \notin L^1(+\infty)$ if and only if $\theta \leq 1$. In case $\theta > 1$

$$\chi_g(r) = \left(\frac{\theta - 1}{2r}\right)^2.$$

Hence (4.47) can be rewritten as

$$\int_R^r \sqrt{A} - \frac{\theta-1}{2}(\log r - \log R) > -\frac{1}{2}\log\int_{B_R} \mu(m\lambda - S^{\varphi}) - \frac{1}{2}\log\frac{R^{1-\theta}}{C(\theta-1)},$$

that is,

$$\int_R^r \sqrt{A} - \frac{\theta-1}{2}\log r > \frac{1}{2}\log C + \frac{1}{2}\log(\theta-1) - \frac{1}{2}\log\int_{B_R} \mu(m\lambda - S^{\varphi}).$$

From (4.51) and the definition of A we immediately see that

$$\sqrt{A(r)} \geq \frac{D}{r^{\frac{\gamma}{2}}} \quad \text{for } r^{\geq} R. \tag{4.52}$$

Using (4.52), to obtain the validity of (4.47) for some $r \geq R$ it is sufficient that

$$D\int_R^r \frac{ds}{s} - \frac{\theta-1}{2}\log r > \frac{1}{2}\log C + \frac{1}{2}\log(\theta-1) - \frac{1}{2}\log\int_{B_R} \mu(m\lambda - S^{\varphi}),$$

that is,

$$D\,(\log r - \log R) - \frac{\theta-1}{2}\log r > \log(R^D\sqrt{C(\theta-1)}) - \frac{1}{2}\log\int_{B_R} \mu(m\lambda - S^{\varphi}),$$

or equivalently,

$$D\log r - \frac{\theta-1}{2}\log r > D\log R + \log(R^D\sqrt{C(\theta-1)}) - \frac{1}{2}\log\int_{B_R} \mu(m\lambda - S^{\varphi}). \tag{4.53}$$

Since $D > \frac{\theta-1}{2}$ there exists r large enough such that (4.53) holds. Then, from the discussion before the Proposition, we conclude the proof. Observe that $A \not\equiv 0$ on $[0, R]$ is guaranteed by the fact that $S^{\varphi} \not\equiv m\lambda$ on B_R that, in turns, is guaranteed by (4.51). □

Remark 4.54. We consider, in case $\theta > 1$, the limiting case $v(r) = Cr^{\theta}$. Inserting this information into (4.51) we obtain

$$\int_{\partial B_r} \mu(m\lambda - S^{\varphi}) \geq CD^2 r^{\theta-\gamma} \quad \text{for } r \geq R.$$

An immediate computation and the fact that $\theta - \gamma + 1 > 0$, since $\gamma \leq 2$ and $\theta > 1$, shows that

$$\int_R^r \mu(m\lambda - S^{\varphi}) \geq \frac{CD^2}{\theta - \gamma + 1}(r^{\theta-\gamma+1} - R^{\theta-\gamma+1})$$

and therefore the integral diverges as $r \to +\infty$. This means that condition (4.48) is satisfied and the solution is even oscillatory.

As another example we give

Proposition 4.55. *Suppose $S^\varphi \geq m\lambda$ on M,*

$$v(r) \leq \Lambda \exp\{ar^\alpha \log^\beta r\}, \tag{4.56}$$

for some constants $\Lambda, a, \alpha > 0$ and $\beta \geq 0$ and

$$\int_{\partial B_r} \mu(m\lambda - S^\varphi) \geq \frac{9a^2}{4} r^{2(\alpha-1)} \log^{2(\beta-1)} r(\alpha \log r + \beta)^2 v(r) \tag{4.57}$$

Then a solution z of (4.45) admits a first zero.

Proof. The proof is similar to that of Proposition 4.49. From (4.56) we can choose

$$g(r) = \Lambda \exp\{ar^\alpha \log^\beta r\}.$$

Clearly $g^{-1} \notin L^1(+\infty)$. We claim that the validity, for some r and R large enough, of

$$\int_R^r \sqrt{A} - ar^\alpha \log^\beta r > -\frac{1}{2} \log \int_{B_R} \mu(m\lambda - S^\varphi) + \frac{1}{2} - \frac{3a}{2} R^\alpha \log^\beta R \tag{4.58}$$

implies the validity of (4.47). Indeed, if we define

$$\widetilde{\chi}_g(t) := \left(\frac{g'(t)}{2g(t)} \right)^2,$$

then

$$\sqrt{\widetilde{\chi}_g(t)} \sim \sqrt{\chi_g(t)} \quad \text{for } t \to +\infty,$$

see (4.4) of [9]. In particular, if R is large enough, then for every $t \geq R$,

$$\sqrt{\chi_g(t)} < 2\sqrt{\widetilde{\chi}_g(t)}.$$

Then we deduce

$$\int_R^r \sqrt{\chi_g} < 2 \int_R^r \sqrt{\widetilde{\chi}_g} = \log g(r) - \log g(R) = ar^\alpha \log^\beta r - aR^\alpha \log^\beta R,$$

so that

$$\int_R^r \sqrt{A} - \int_R^r \sqrt{\chi_g} > \int_R^r \sqrt{A} - ar^\alpha \log^\beta r + aR^\alpha \log^\beta R. \tag{4.59}$$

Moreover

$$-\frac{1}{2} \log \int_R^{+\infty} \frac{1}{v} \sim \frac{a}{2} t^\alpha \log^\beta t \quad \text{for } t \to +\infty,$$

hence for R large enough we have

$$-\frac{1}{2} \log \int_R^{+\infty} \frac{1}{v} < \frac{1}{2}(1 - aR^\alpha \log^\beta R). \tag{4.60}$$

Using (4.59) and (4.60) we deduce the validity of the claim.

Clearly (4.56) implies

$$\sqrt{A(t)} \geq \frac{3a}{2} t^{\alpha-1} \log^{\beta-1} t(\alpha \log t + \beta) = \frac{3a}{2}(t^\alpha \log^\beta t)'.$$

Using the above, the validity of (4.58) is implied by the validity of

$$\frac{a}{2} r^\alpha \log^\beta r > -\frac{1}{2} \log \int_{B_R} \mu(m\lambda - S^\varphi) + \frac{1}{2}. \tag{4.61}$$

The right hand side of (4.61) above is monotone decreasing in R, then it is sufficient that (4.61) holds for some $R = R_0$ to obtain that it holds also for all $R \geq R_0$. Then we may fix R such that $A \not\equiv 0$ on $[0, R]$, clearly for r large enough we obtain the validity of (4.61). Then we can conclude the proof, as in Proposition 4.49. □

5 Some results in the compact case

Our first aim is to extend the well known fact that a compact Einstein manifold that admits a non-Killing conformal vector field is isometric to a Euclidean sphere. To do so we first recall

Theorem 5.1 (**Licherowicz-Obata**). *Let $(M, \langle\,,\,\rangle)$ be a compact Riemannian manifold of dimension m satisfying for some $\kappa \in \mathbb{R}$*

$$Ric \geq (m-1)\kappa\langle\,,\,\rangle.$$

Let $u \in \mathcal{C}^\infty(M)$ be a non-constant eigenfunction of $-\Delta$ relative to the eigenvalue $\lambda \in \mathbb{R}$, that is,

$$\Delta u + \lambda u = 0.$$

Then

$$\lambda \geq m\kappa,$$

equality holding if and only if $(M, \langle\,,\,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ of constant sectional curvature $\kappa > 0$.

With the aid of formula (4.31) we are able to prove

Lemma 5.2. *Let $(M, \langle\,,\,\rangle)$ be a compact, harmonic-Einstein manifold of dimension $m \geq 2$ with $\alpha > 0$, that is, for some $\alpha \in \mathbb{R}$, $\alpha > 0$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$ we have*

$$\begin{cases} Ric^\varphi = \dfrac{S^\varphi}{m}\langle\,,\,\rangle \\ \tau(\varphi) = 0. \end{cases} \tag{5.3}$$

If there exists a non-Killing conformal vector field $X \in \mathfrak{X}(M)$ such that

$$d\varphi(X) = 0, \tag{5.4}$$

then φ is constant and $(M, \langle\,,\,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature

$$\kappa := \frac{S^\varphi}{m(m-1)} > 0. \tag{5.5}$$

Furthermore there exists $h \in \mathcal{C}^\infty(M)$ satisfying

$$\mathrm{Hess}(h) + \kappa h\langle\,,\,\rangle = 0, \tag{5.6}$$

and

$$\frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle = Hess(h). \tag{5.7}$$

Remark 5.8. As expected, for φ constant, we obtain the classical result on Einstein manifolds mentioned at the beginning of the section.

Proof. Let $X \in \mathfrak{X}(M)$ be a non-Killing conformal vector field, that is,

$$\mathcal{L}_X\langle\,,\,\rangle = 2\eta\langle\,,\,\rangle, \tag{5.9}$$

for some $\eta \in \mathcal{C}^\infty(M)$, $\eta \not\equiv 0$. Observe that (5.4) is essential for the coupling condition (4.29) $\tau(\varphi) = d\varphi(X)$ of Proposition 4.28 since by the second equation of (5.3) φ is harmonic. Since S^φ is constant, by Proposition 2.15 and (5.3) for $m \geq 3$ and by assumption for $m = 2$ and since φ is harmonic, formula (4.31) becomes

$$\Delta\eta + \frac{S^\varphi}{m-1}\eta = 0. \tag{5.10}$$

By integration we obtain

$$\int_M |\nabla\eta|^2 = \frac{S^\varphi}{m-1}\int_M \eta^2,$$

and thus $S^\varphi \geq 0$. Suppose by contradiction that $S^\varphi = 0$ then η is harmonic on the compact Riemannian manifold $(M, \langle\,,\,\rangle)$, hence it is constant. Taking the trace of (5.9) we get

$$\mathrm{div}(X) = m\eta$$

and since η is constant, integrating over M we deduce also that $\eta = 0$. But since $\eta \not\equiv 0$ we obtain a contradiction. We have therefore proved that $S^\varphi > 0$. From the first equation in (5.3), $\alpha > 0$ and the fact that $\langle\,,\,\rangle_N$ is a Riemannian metric on N we obtain

$$\mathrm{Ric} \geq \frac{S^\varphi}{m}\langle\,,\,\rangle. \tag{5.11}$$

Since X is not Killing, η does not vanish identically on M and from (5.10) and $S^\varphi > 0$ we deduce that η cannot be a constant. The validity of (5.10) and (5.11) allows us to apply Lichnerowicz-Obata Theorem (see Theorem 5.1) to deduce that $(M, \langle\,,\,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by (5.5). We now observe that, from the first equation in (5.3) and the fact that we have now equality in (5.11), because of (5.5) and the isometry, we have

$$\frac{S^\varphi}{m}\langle\,,\,\rangle = \mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N = \frac{S^\varphi}{m}\langle\,,\,\rangle - \alpha\varphi^*\langle\,,\,\rangle_N,$$

and since $\alpha \neq 0$

$$\varphi^*\langle\,,\,\rangle_N = 0.$$

Thus φ is constant. To obtain (5.7), if $X = \nabla h + Y$ is the Hodge-de Rham decomposition of X, where $Y \in \mathfrak{X}(M)$ is a divergence free vector field and $h \in \mathcal{C}^\infty(M)$, we only need to show that Y is Killing. Observe that (5.10) can be rewritten as

$$\Delta\eta + m\kappa\eta = 0,$$

and, as proved by Obata in [33], this implies the validity of

$$\mathrm{Hess}(\eta) + \kappa\eta\langle\,,\,\rangle = 0. \tag{5.12}$$

Then, from (5.9)

$$\frac{1}{2}\mathcal{L}_X g = -\frac{1}{\kappa}\mathrm{Hess}(\eta). \tag{5.13}$$

Using the Hodge-de Rham decomposition of X in (5.13) we obtain

$$\frac{1}{2}\mathcal{L}_Y g + \mathrm{Hess}(h) = -\frac{1}{\kappa}\mathrm{Hess}(\eta)$$

and taking the trace, since Y is divergence free, we deduce

$$\Delta h = -\frac{1}{\kappa}\Delta\eta.$$

This implies that the function

$$\zeta := h + \frac{1}{\kappa}\eta$$

is harmonic on M and, since M is compact, is constant. From (5.13)

$$\frac{1}{2}\mathcal{L}_X g = \mathrm{Hess}(h),$$

in particular Y is a Killing vector field. Observe also that, since (5.12) holds, up to a translation h solves (5.6). Then the claim, since in the Hodge-de Rham decomposition h is determined up to an additive constant. □

As an application of the above Lemma we prove two rigidity results, that distinguish between the cases $\mu = 0$ and $\mu \neq 0$. In case $\mu = 0$ we are able to study a general Einstein-type structure while for $\mu \neq 0$ we restrict ourselves to the gradient case.

Theorem 5.14. *Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure of the form*

$$\begin{cases} Ric^{\varphi} + \frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{5.15}$$

for some $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^{\infty}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$. Assume that $\alpha > 0$ and that S^{φ} is constant. Then $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein, more precisely, (5.15) *reduces to*

$$\begin{cases} Ric^{\varphi} = \frac{S^{\varphi}}{m}\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{5.16}$$

Suppose also that X is not a Killing vector field. Then φ is constant and $(M, \langle\, ,\, \rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by (5.5). *Furthermore there exists $h \in \mathcal{C}^{\infty}(M)$ such that $X = \nabla h$ and*

$$Hess(h) + \kappa h\langle\, ,\, \rangle = 0. \tag{5.17}$$

Proof. We recall that for $\mu = 0$ we have the validity of (4.17), that is,

$$\frac{1}{2}\Delta_X S^{\varphi} = -\alpha|\tau(\varphi)|^2 - |T^{\varphi}|^2 - (S^{\varphi} - m\lambda)\frac{S^{\varphi}}{m} + (m-1)\Delta\lambda.$$

Tracing the first equation of (5.15) we obtain

$$S^{\varphi} - m\lambda = -\mathrm{div}(\mathrm{X}), \tag{5.18}$$

thus inserting into the above we get

$$\frac{1}{2}\Delta S^{\varphi} = \frac{1}{2}\langle X, \nabla S^{\varphi}\rangle - \alpha|\tau(\varphi)|^2 - |T^{\varphi}|^2 + \frac{S^{\varphi}}{m}\mathrm{div}(X) + (m-1)\Delta\lambda.$$

Integrating over M, using the divergence theorem and integrating by parts we infer

$$\frac{m-2}{2m}\int_M \langle X, \nabla S^{\varphi}\rangle = \int_M (|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2).$$

Since $\alpha > 0$ and, in case $m \geq 3$, S^{φ} is constant we deduce the validity of

$$\begin{cases} \mathrm{Ric}^{\varphi} = \frac{S^{\varphi}}{m}\langle\, ,\, \rangle \\ \tau(\varphi) = 0, \end{cases} \tag{5.19}$$

so that $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. Now recall that S^{φ} is constant (this is needed for the case $m = 2$) and X is not a Killing vector field. Comparing the first equation of (5.19) with the first equation of (5.15) we deduce

$$\mathcal{L}_X\langle\, ,\, \rangle = \frac{2}{m}\eta\langle\, ,\, \rangle, \tag{5.20}$$

where

$$\eta := m\lambda - S^{\varphi}. \tag{5.21}$$

Thus X is in particular a conformal vector field on M. Comparing the second equation of (5.19) with the second equation of (5.15) we deduce

$$d\varphi(X) = 0.$$

Note that since S^φ is constant we can apply Lemma 5.2 to conclude first that φ is constant, that $(M, \langle\, , \rangle)$ is isometric to the Euclidean sphere of constant sectional curvature κ given by (5.5) and finally that

$$\frac{1}{2}\mathcal{L}_X\langle\, , \rangle = \mathrm{Hess}(h)$$

for some $h \in \mathcal{C}^\infty(M)$ such that (5.17) holds. □

The next result is the analogous of Theorem 5.14 in case $\mu \neq 0$. However its proof is not based on equation (4.20), as probably expected, but on the powerful identity (5.29) below.

Theorem 5.22. *Let $(M, \langle\, , \rangle)$ be a compact manifold of dimension $m \geq 2$ with a gradient Einstein-type structure of the form*

$$\begin{cases} Ric^\varphi + Hess(f) - \mu df \otimes df = \lambda\langle\, , \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{5.23}$$

for some $f, \lambda \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R}\setminus\{0\}$, $\mu \in \mathbb{R}$ and $\varphi : M \to (N, \langle\, , \rangle_N)$. Assume that S^φ is constant, $\mu \neq 0$ and $\alpha > 0$. Then the structure (5.23) *is harmonic-Einstein, that is,* (5.23) *reduces to* (5.16).

Assume also that f is non-constant. Then φ is constant and $(M, \langle\, , \rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by (5.5).

Proof. Let

$$u := e^{-\mu f} \tag{5.24}$$

and let T^φ be the traceless φ-Ricci tensor, defined in (3.1). We compute $\mathrm{div}(T^\varphi(\nabla u, \cdot)^\sharp)$. Exploiting the definition of T^φ, in a local orthonormal coframe, we have

$$(T^\varphi_{ij}u_i)_j = T^\varphi_{ij,j}u_i + T^\varphi_{ij}u_{ij} = R^\varphi_{ij,j}u_i - \frac{S^\varphi_i}{m}u_i + T^\varphi_{ij}u_{ij}. \tag{5.25}$$

Using (5.24) a computation yields

$$u_i = -\mu u f_i, \quad u_{ij} = -\mu u(f_{ij} - \mu f_i f_j), \tag{5.26}$$

so that, using the first equation of (5.23),

$$u_{ij} = \mu u(R^\varphi_{ij} - \lambda\delta_{ij}). \tag{5.27}$$

Moreover from (2.10), the first equation of (5.26) and the second equation of (5.23)

$$R^\varphi_{ij,j}u_i = \frac{1}{2}S^\varphi_i u_i - \alpha\varphi^a_{jj}\varphi^a_i u_i = \frac{1}{2}S^\varphi_i u_i + \mu u\alpha\varphi^a_{jj}\varphi^a_i f_i = \frac{1}{2}S^\varphi_i u_i + \mu u\alpha\varphi^a_{ii}\varphi^a_{jj}. \tag{5.28}$$

Inserting (5.27) and (5.28) into (5.25), since T^φ is traceless, we obtain

$$\begin{aligned}(T^\varphi_{ij}u_i)_j =& \frac{1}{2}S^\varphi_i u_i + \mu u\alpha\varphi^a_{ii}\varphi^a_{jj} - \frac{S^\varphi_i}{m} + \mu T^\varphi_{ij}(R^\varphi_{ij} - \lambda\delta_{ij})u \\ =& \frac{m-2}{2m}S^\varphi_i u_i + \mu(\alpha\varphi^a_{ii}\varphi^a_{jj} + T^\varphi_{ij}T^\varphi_{ij}),\end{aligned}$$

that is, in global notation

$$\mathrm{div}(T^\varphi(\nabla u, \cdot)^\sharp) = \frac{m-2}{2m}\langle\nabla S^\varphi, \nabla u\rangle + \mu(\alpha|\tau(\varphi)|^2 + |T^\varphi|^2)u. \tag{5.29}$$

Since S^φ is constant for $m \geq 3$, integrating over M and using the divergence theorem we deduce

$$\mu\int_M (\alpha|\tau(\varphi)|^2 + |T^\varphi|^2)u = 0.$$

From $\mu \neq 0$, $\alpha > 0$ and $u > 0$ on M we obtain $T^\varphi = 0$ and $\tau(\varphi) = 0$, that is, the equations in (5.16). Suppose that S^φ is constant and that f is non-constant. Now from (5.27), using the first equation of (5.16), we infer

$$\mathrm{Hess}(u) = \mu \left(\frac{S^\varphi}{m} - \lambda \right) u \langle \, , \, \rangle,$$

so that ∇u is a conformal vector field. Since f is non-constant then also u is non-constant. From the first equation of (5.26) and the second equation of (5.16)

$$d\varphi(\nabla u) = -\mu u d\varphi(\nabla f) = 0.$$

We now conclude as in the proof of Theorem 5.14. □

Next we present two more rigidity results, again distinguishing between the cases $\mu = 0$ and $\mu \neq 0$. In both results we assume that the manifold is φ-Cotton flat. In Section 6 we shall produce examples where this happens. Towards this aim we need to introduce a general formula for a 2-times covariant, symmetric tensor field T on a Riemannian manifold $(M, \langle \, , \, \rangle)$ of dimension m. For $x \in M$ fixed, we set

$$\lambda_1 \leq \ldots \leq \lambda_m,$$

to denote the (possibly coinciding) eigenvalues of T at x and we consider the elementary symmetric functions

$$S_0 := 1, \quad S_k := \sum_{1 \leq i_1 < \ldots < i_k \leq m} \lambda_{i_1} \ldots \lambda_{i_k} \quad \text{for } 1 \leq k \leq m. \tag{5.30}$$

In other words the S_k's are the coefficients of the polynomial expansion

$$\det(I + \lambda T) = \sum_{k=0}^{m} S_k \lambda^k,$$

where I is the identity. As usual we normalize the S_k's by setting

$$S_k = \binom{m}{k} \sigma_k.$$

In this way we obtain the validity of Newton's inequalities in the form

$$\sigma_{k-1}\sigma_{k+1} \leq \sigma_k^2 \quad \text{for } 1 \leq k \leq m-1. \tag{5.31}$$

Furthermore, if $\sigma_{k-1} \neq 0$ at x, equality holds in (5.31) if and only if all the eigenvalues of T at x are equal. Considering the σ_k's as functions on M, from [23], we deduce that if for some k, $1 \leq k \leq m$, we have $\sigma_k > 0$ everywhere on M then, for $1 \leq i \leq k$, $\sigma_i > 0$ on M and furthermore, Gårding's inequalities hold, that is,

$$\sigma_1 \geq \sigma_2^{\frac{1}{2}} \geq \ldots \geq \sigma_{k-1}^{\frac{1}{k-1}} \geq \sigma_k^{\frac{1}{k}}, \tag{5.32}$$

with equality at a point $x \in M$ at some stage of the chain if and only if T has equal eigenvalues at x. The next Lemma follows directly by (5.32) and will be used in Theorem 5.47.

Lemma 5.33. *In the notations above suppose that $\sigma_k > 0$ on M for some $2 \leq k \leq m-1$, where $m \geq 3$ is the dimension of M. Then*

$$\sigma_1\sigma_k - \sigma_{k+1} \geq 0 \tag{5.34}$$

with equality holding at a point $x \in M$ if and only if T is proportional to the metric at x.

Proof. Since $\sigma_k > 0$ on M, we have the validity of (5.32). From $\sigma_{k-1} > 0$ on M and Newton's inequalities (5.31)

$$\sigma_{k+1} = \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_{k-1}} \leq \frac{\sigma_k^2}{\sigma_{k-1}} = \sigma_k \frac{\sigma_k}{\sigma_{k-1}}.$$

We claim

$$\frac{\sigma_k}{\sigma_{k-1}} \leq \sigma_1,$$

and since $\sigma_k > 0$, from the above we obtain

$$\sigma_{k+1} \leq \sigma_k \sigma_1,$$

that is (5.34). It remains to prove the claim. We use Gårding's inequalities twice and $\sigma_1, \sigma_k > 0$ to deduce

$$\sigma_k = \sigma_k^{\frac{1}{k}} \sigma_k^{\frac{k-1}{k}} \leq \sigma_1 \sigma_k^{\frac{k-1}{k}} \leq \sigma_1 \sigma_{k-1}.$$

Since $\sigma_{k-1} > 0$ this implies the claim. Observe that the equality holds at a point if and only if T is proportional to the metric at that point since the equality forces Newton's inequality and Gårding's inequalities to be equalities at that point. □

Associated with T one considers the Newton endomorphisms

$$P_k = P_k(T) : \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \text{for } 0 \leq k \leq m,$$

inductively defined by

$$P_0 := I, \quad P_k := S_k I - t \circ P_{k-1} \quad \text{for } 1 \leq k \leq m, \tag{5.35}$$

where $t : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the endomorphism induced by T. Note that $P_m = 0$ on M and, having set

$$c_k := (m-k)\binom{m}{k}, \tag{5.36}$$

we have

$$\operatorname{tr}(P_k) = (m-k)S_k = c_k \sigma_k, \quad \operatorname{tr}(t \circ P_{k-1}) = kS_k = c_{k-1}\sigma_k. \tag{5.37}$$

The Newton's endomorphisms give rise to a family of second order differential operators L_k defined as follows. Setting $\operatorname{hess}(u)$ for the endomorphism induced by $\operatorname{Hess}(u)$, where $u \in \mathcal{C}^2(M)$,

$$L_k u := \operatorname{tr}(P_k \circ \operatorname{hess}(u)). \tag{5.38}$$

A computation shows that L_k can be written in the form:

$$L_k u = \operatorname{div}(P_k(\nabla u)) - \langle \operatorname{div}(P_k), \nabla u \rangle. \tag{5.39}$$

Obviously,

$$\operatorname{div}(P_0) = 0 = \operatorname{div}(P_m). \tag{5.40}$$

To compute $\operatorname{div}(P_k)$ for the remaining values of k we introduce the 3-times covariant tensor field C of components

$$C_{ijk} := T_{ij,k} - T_{ik,j}. \tag{5.41}$$

A recursive computation shows that, for $1 \leq k \leq m-1$

$$\operatorname{div}(P_k)_j = -\operatorname{div}(P_{k-1})_i T_{ij} - C_{ijs}(P_k)_{is}. \tag{5.42}$$

Explicitating (5.42) one sees that $\operatorname{div}(P_k) = 0$ for all $1 \leq k \leq m-1$ if and only $C_{ijs} + C_{sji} = 0$, that is, the tensor C is skew symmetric in the first and the third entries. In particular when T is a Codazzi tensor field all the Newton's endomorphisms are divergence free. Hence in the assumption

$$C(X,Y,Z) = -C(Z,Y,X) \quad \text{for all } X,Y,Z \in \mathfrak{X}(M), \tag{5.43}$$

equation (5.39) becomes

$$\mathrm{tr}(P_k \circ \mathrm{hess}(u)) = L_k u = \mathrm{div}(P_k(\nabla u)),$$

that we can also rewrite in the useful form

$$\mathrm{div}(P_k(\nabla u)) = \sum_{i=0}^{k} (-1)^i S_{k-i} \mathrm{tr}(t^i \circ \mathrm{hess}(u)). \tag{5.44}$$

We remark that, having fixed the 2-times covariant tensor field T, we can also define an operator

$$\widetilde{L}_k : \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \text{for } 0 \leq k \leq m,$$

by setting, for every $Z \in \mathfrak{X}(M)$

$$\widetilde{L}_k(Z) := \frac{1}{2}\mathrm{tr}(P_k \circ l_Z), \tag{5.45}$$

where $l_Z : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the endomorphism associated to $\mathcal{L}_Z\langle\, , \,\rangle$, the Lie derivative of the metric in the direction of Z. A computation yields a formula analogous to (5.39), that is,

$$\widetilde{L}_k(Z) = \mathrm{div}(P_k(Z)) - \langle \mathrm{div}(P_k), Z \rangle,$$

hence under assumption (5.43)

$$\widetilde{L}_k(Z) = \mathrm{div}(P_k(Z)).$$

We then obtain the following generalization of (5.44)

$$\mathrm{div}(P_k(Z)) = \frac{1}{2} \sum_{i=0}^{k} (-1)^i S_{k-i} \mathrm{tr}(t^i \circ l_Z). \tag{5.46}$$

In the following we will denote by σ_k^φ the normalized k^{th} symmetric function of the eigenvalues of the φ-Schouten tensor. We begin with the case $\mu = 0$.

Theorem 5.47. *Let $(M, \langle\, , \,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with an Einstein-type structure of the form (5.15) with $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, , \,\rangle_N)$. Suppose that $(M, \langle\, , \,\rangle)$ is φ-Cotton flat, that is,*

$$C^\varphi = 0, \tag{5.48}$$

that X is non-Killing and that σ_k is a positive constant for some $k = 2, \ldots, m-1$. Then φ is constant and $(M, \langle\, , \,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature

$$\kappa = \frac{2(\sigma_k^\varphi)^{\frac{1}{k}}}{m-2}. \tag{5.49}$$

Furthermore there exists $h \in \mathcal{C}^\infty(M)$ such that $X = \nabla h$ and (5.17) holds.

Remark 5.50. If $m = 2$ then $A^\varphi = T^\varphi$, hence $\sigma_1^\varphi = 0$ and thus, from Newton's inequality $\sigma_2^\varphi \leq 0$. This motivates the hypothesis $m \geq 3$.

Proof. Since (5.48) holds the φ-Schouten tensor A^φ is a Codazzi tensor field. Then (5.46) holds with $Z = X$ and $T = A^\varphi$. Expressing the first equation of (5.15) in terms of A^φ we obtain

$$\frac{1}{2}\mathcal{L}_X\langle\, , \,\rangle = -\frac{S^\varphi}{2(m-1)}\langle\, , \,\rangle - A^\varphi + \lambda\langle\, , \,\rangle,$$

so that

$$\frac{1}{2} l_X = \left(\lambda - \frac{S^\varphi}{2(m-1)}\right) I - a^\varphi, \tag{5.51}$$

where l_X and a^φ denotes the endomorphisms of $\mathfrak{X}(M)$ induced by $\mathcal{L}_X\langle\, ,\,\rangle$ and A^φ respectively. Inserting (5.51) in (5.46) with $Z = X$ and $T = A^\varphi$, a computation using (5.37) yields, via (5.46),

$$\operatorname{div}(P_k^\varphi(X)) = c_k\left[\left(\lambda - \frac{S^\varphi}{2(m-1)}\right)\sigma_k^\varphi - \sigma_{k+1}^\varphi\right], \tag{5.52}$$

where c_k is defined in (5.36) and P_k^φ is the k^{th} Newton's endomorphism associated to A^φ. Since we are assuming that $\sigma_k > 0$, from Lemma 5.33 we deduce the validity of

$$\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi \geq 0, \tag{5.53}$$

equality holding at a point if and only if at that point A^φ, and therefore $\operatorname{Ric}^\varphi$, is proportional to the metric. Since M is compact by the Hodge-de Rham decomposition

$$X = \nabla h + Y,$$

for some $h \in \mathcal{C}^\infty(M)$ and $Y \in \mathfrak{X}(M)$ with $\operatorname{div}(Y) = 0$. Thus, tracing the first equation of (5.15)

$$S^\varphi + \Delta h = m\lambda,$$

that can be rewritten as

$$\sigma_1^\varphi + \frac{\Delta h}{m} = \lambda - \frac{S^\varphi}{2(m-1)}.$$

Indeed, tracing (2.3),

$$\sigma_1^\varphi = \frac{\operatorname{tr}(A^\varphi)}{m} = \frac{S^\varphi}{m} - \frac{S^\varphi}{2(m-1)}. \tag{5.54}$$

Substituting in (5.52) we have

$$\operatorname{div}(P_k^\varphi)(X) = c_k\left(\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi + \frac{\sigma_k^\varphi}{m}\Delta h\right).$$

Integrating on M, since σ_k^φ is constant, we infer

$$\int_M (\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi) = 0.$$

By (5.53) we deduce that the equality holds in (5.53) on all of M, and A^φ is a trivial Codazzi tensor field. In particular S^φ is constant and $\operatorname{Ric}^\varphi$ is proportional to the metric on M. ombining it with Lemma **??** we deduce Since A^φ is Codazzi, that is, $C^\varphi \equiv 0$, we infer that φ is harmonic. Indeed, we recall that, from (2.34),

$$(\operatorname{tr}C^\varphi)_i = \alpha\varphi^a_{kk}\varphi^a_i$$

while from the second of (5.15)

$$\varphi^a_{ii} = \varphi^a_i X^i.$$

Thus, inserting into the above yields

$$(\operatorname{tr}C^\varphi)(X) = (\operatorname{tr}C^\varphi)_i X^i = \alpha\varphi^a_{kk}\varphi^a_{ii} = \alpha|\tau(\varphi)|^2.$$

Hence, $(M, \langle\, ,\,\rangle)$ is harmonic-Einstein with S^φ constant. Now we can conclude as in Theorem 5.14. Observe also that, since A^φ is proportional to the metric, using (5.54) and the constancy of φ

$$\frac{m-2}{2m(m-1)}S = \sigma_1^\varphi = (\sigma_2^\varphi)^{\frac{1}{2}} = \ldots = (\sigma_m^\varphi)^{\frac{1}{m}}.$$

Thus we have

$$\kappa = \frac{S}{m(m-1)} = \frac{2(\sigma_k^\varphi)^{\frac{1}{k}}}{m-2},$$

as in (5.49). $\square$

Remark 5.55. For φ constant and X non-Killing Theorem 5.47 can be considered an extension of the classical result obtained at the beginning of the Section, that is, Lemma 5.2 with φ constant, to higher order symmetric functions of the eigenvalues of the Schouten tensor.

In Theorem 5.47 we dealt with the case $\mu = 0$ and with a general vector field X. Now we consider the case $\mu \neq 0$ but we restrict ourselves to the gradient case, $X = \nabla f$ for some $f \in \mathcal{C}^\infty(M)$. We have

Theorem 5.56. *Let $(M, \langle\,,\,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with an Einstein-type structure of the form*

$$\begin{cases} Ric^\varphi + Hess(f) - \mu df \otimes df = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{5.57}$$

with $f, \lambda \in \mathcal{C}^\infty(M)$, $\mu, \alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$. Suppose that $(M, \langle\,,\,\rangle)$ is φ-Cotton flat, that is (5.48) *holds, that f is non-constant and that σ_k^φ is a positive constant for some $k = 2, \ldots, m-1$. Then φ is constant and $(M, \langle\,,\,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by* (5.49)*.*

Proof. We set

$$u := e^{-\mu f}.$$

Then, from (5.57), we deduce

$$\mathrm{Ric}^\varphi - \frac{1}{\mu u}\mathrm{Hess}(u) = \lambda\langle\,,\,\rangle.$$

The above is equivalent, using the definition of A^φ, to

$$\mathrm{Hess}(u) = \mu u\left[A^\varphi - \left(\lambda - \frac{S^\varphi}{2(m-1)}\right)\langle\,,\,\rangle\right].$$

Then, as in the proof of Theorem 5.47, we obtain

$$\mathrm{div}(P_k^\varphi(\nabla u)) = \mu c_k\left[u(\sigma_{k+1}^\varphi - \sigma_1^\varphi\sigma_k^\varphi) + \frac{\sigma_k^\varphi}{m\mu}\Delta u\right].$$

Using constancy of σ_k and integrating on M we obtain

$$\mu c_k \int_M u(\sigma_{k+1}^\varphi - \sigma_1^\varphi\sigma_k^\varphi) = 0$$

and since $u > 0$ and $\mu \neq 0$,

$$\sigma_{k+1}^\varphi - \sigma_1^\varphi\sigma_k^\varphi = 0, \qquad \text{on } M.$$

We now conclude as in Theorem 5.47, observing that ∇u cannot be a Killing vector field because u is non-constant on M. ☐

Similarly to what expressed in Remark 5.55 for Theorem 5.47 we have

Corollary 5.58. *Let $(M, \langle\,,\,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ which is a quasi-Einstein manifold (see equation* (1.3) *in the Introduction). Suppose that $(M, \langle\,,\,\rangle)$ is Cotton flat and that the normalized k-th symmetric function σ_k of the Schouten tensor A^φ is a positive constant for some $k = 2, \ldots, m$ and that f is non-constant. Then $(M, \langle\,,\,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ of constant sectional curvature $\kappa = \frac{2(\sigma_k)^{1/k}}{m-2}$.*

Remark 5.59. Observe that, since

$$\sigma_1^\varphi = \frac{m-2}{2(m-1)}S^\varphi,$$

Theorem 5.14 and Theorem 5.22 can be interpreted as the case $k = 1$ of Theorem 5.47 and Theorem 5.56, respectively. In this case the assumptions of φ-Cotton flatness and on the sign of the curvature are unnecessary.

6 Gradient Einstein-type structure with vanishing conditions on the φ-Bach tensor

In this section we shall consider a Riemannian manifold $(M, \langle\, ,\, \rangle)$ with an Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{6.1}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu \in \mathbb{R}$, $\lambda, f \in \mathcal{C}^{\infty}(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$.

Our aim is to prove the structure Theorem 6.66 below, generalizing Theorem 1.2 of [16]. In the following we shall use (4.18) and (4.19), which we report here for the reader's convenience

$$R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} = f_t R_{tikj} + \mu(f_{ik} f_j - f_{ij} f_k) + \lambda_k \delta_{ij} - \lambda_j \delta_{ik}, \tag{6.2}$$

$$\frac{1}{2} S^{\varphi}_i = R^{\varphi}_{ki} f_k + \mu(f_{ki} f_k - \Delta f f_i) + (m-1)\lambda_i. \tag{6.3}$$

We now come to the definition of the tensor D^{φ} that shall reveal essential in our study.

Definition 6.4. Let $m \geq 3$. In a local orthonormal coframe we let the components of D^{φ} be given by

$$D^{\varphi}_{ijk} := \frac{1}{m-2}\left[R^{\varphi}_{ij} f_k - R^{\varphi}_{ik} f_j + \frac{1}{m-1} f_t (R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}) - \frac{S^{\varphi}}{m-1}(f_k \delta_{ij} - f_j \delta_{ik})\right]. \tag{6.5}$$

We observe that if φ is a constant map then D^{φ} coincides with the tensor D defined in [11], with a different convention. The following properties are easily verified by computation.

Proposition 6.6. *In the present setting, with $m \geq 3$, the tensor D^{φ} is skew-symmetric in the last two indices and it is totally trace free, that is,*

$$D^{\varphi}_{ikj} = -D^{\varphi}_{ijk}, \tag{6.7}$$

$$D^{\varphi}_{kii} = D^{\varphi}_{iki} = D^{\varphi}_{iik} = 0. \tag{6.8}$$

An essential feature of D^{φ} is that it can be expressed purely in terms of the potential function f. Indeed, we have the following

Proposition 6.9. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$D^{\varphi}_{ijk} = \frac{1}{m-2}\left[f_{ik} f_j - f_{ij} f_k + \frac{1}{m-1} f_t (f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) - \frac{\Delta f}{m-1}(f_j \delta_{ik} - f_k \delta_{ij})\right]. \tag{6.10}$$

Proof. The proof is computational, using (6.1). Indeed tracing (6.1)

$$S^{\varphi} + \Delta f = \mu|\nabla f|^2 + m\lambda,$$

hence using it in the definition (6.5), together with (6.1), we obtain

$$\begin{aligned} D^{\varphi}_{ijk} =& \frac{1}{m-2}\left[(-f_{ij} + \mu f_i f_j + \lambda\delta_{ij}) f_k - (-f_{ik} + \mu f_i f_k + \lambda\delta_{ik}) f_j\right] \\ &+ \frac{1}{(m-1)(m-2)} f_t[(-f_{tk} + \mu f_t f_k + \lambda\delta_{tk})\delta_{ij} - (-f_{tj} + \mu f_t f_j + \lambda\delta_{tj})\delta_{ik}] \\ &- \frac{-\Delta f + \mu|\nabla f|^2 + m\lambda}{(m-1)(m-2)}(f_k\delta_{ij} - f_j\delta_{ik}) \\ =& \frac{1}{m-2}\left[f_{ik} f_j - f_{ij} f_k + \frac{1}{m-1} f_t (f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) - \frac{\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij})\right], \end{aligned}$$

that is (6.10). ∎

Now we prove the first integrability condition of the system (6.1).

Proposition 6.11. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} = [1 + (m-2)\mu] D^{\varphi}_{ijk}. \tag{6.12}$$

Remark 6.13. A long and tedious computation shows that the left hand side tensor of components $C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk}$ of (6.12) in the metric $\langle \, , \rangle$, is exactly the φ-Cotton tensor $\widetilde{C}^{\varphi}$ of the conformally deformed metric

$$\widetilde{\langle \, , \rangle} = e^{-\frac{2f}{m-2}} \langle \, , \rangle.$$

Indeed,

$$e^{-\frac{3f}{m-2}} \widetilde{C}^{\widetilde{\varphi}}_{ijk} = C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk}$$

so that

$$\widetilde{C}^{\widetilde{\varphi}} = \widetilde{C}^{\widetilde{\varphi}}_{ijk} \widetilde{\theta}^i \widetilde{\theta}^j \widetilde{\theta}^k = C^{\varphi}.$$

With $\widetilde{\varphi}$ we mean the map $\varphi : M \to (N, \langle \, , \rangle_N)$ but where now on M we consider the Riemannian metric $\widetilde{\langle \, , \rangle}$.

Proof. Using (6.2) in (2.36) we obtain

$$C^{\varphi}_{ijk} + \frac{1}{2(m-1)} \left(S^{\varphi}_k \delta_{ij} - S^{\varphi}_j \delta_{ik} \right) + f_t R^t_{ijk} - \mu (f_{ik} f_j - f_{ij} f_k) - \lambda_k \delta_{ij} + \lambda_j \delta_{ik} = 0. \tag{6.14}$$

We claim the validity of

$$R^t_{ijk} f_t = W^{\varphi}_{tijk} f_t - D^{\varphi}_{ijk} - \frac{f_t}{m-1} (R^{\varphi}_{tk} \delta_{ij} - R^{\varphi}_{tj} \delta_{ik}). \tag{6.15}$$

We postpone its proof and we complete the proof of (6.12). Inserting (6.15) in (6.14) we obtain

$$\begin{aligned} 0 =& C^{\varphi}_{ijk} + W^{\varphi}_{tijk} f_t - D^{\varphi}_{ijk} + \frac{1}{2(m-1)} (S^{\varphi}_k \delta_{ij} - S^{\varphi}_j \delta_{ik}) \\ & - \mu (f_{ik} f_j - f_{ij} f_k) - \lambda_k \delta_{ij} + \lambda_j \delta_{ik} - \frac{f_t}{m-1} (R^{\varphi}_{tk} \delta_{ij} - R^{\varphi}_{tj} \delta_{ik}). \end{aligned}$$

Using (6.3) we deduce

$$\begin{aligned} \frac{1}{2(m-1)} (S^{\varphi}_k \delta_{ij} - S^{\varphi}_j \delta_{ik}) =& \frac{1}{m-1} (R^{\varphi}_{tk} f_t + \mu (f_{tk} f_t - \Delta f f_k) + (m-1) \lambda_k) \delta_{ij} \\ & - \frac{1}{m-1} (R^{\varphi}_{tj} f_t + \mu (f_{tj} f_t - \Delta f f_j) + (m-1) \lambda_j) \delta_{ik} \\ =& \frac{f_t}{m-1} (R^{\varphi}_{tk} \delta_{ij} - R^{\varphi}_{tj} \delta_{ik}) + \mu \frac{f_t}{m-1} (f_{tk} \delta_{ij} - f_{tj} \delta_{ik}) \\ & + \mu \frac{\Delta f}{m-1} (f_j \delta_{ik} - f_k \delta_{ij}) + \lambda_k \delta_{ij} - \lambda_j \delta_{ik}, \end{aligned}$$

and by plugging it into the above identity we infer

$$\begin{aligned} 0 =& C^{\varphi}_{ijk} + W^{\varphi}_{tijk} f_t - D^{\varphi}_{ijk} \\ & - \mu \left[f_{ik} f_j - f_{ij} f_k + \frac{f_t}{m-1} (f_{tj} \delta_{ik} - f_{tk} \delta_{ij}) + \frac{\Delta f}{m-1} (f_k \delta_{ij} - f_j \delta_{ik}) \right], \end{aligned}$$

that implies (6.12), using (6.10). It remains to prove (6.15). Explicitating (2.3) in (2.6) we obtain

$$R_{tijk} - W^{\varphi}_{tijk} = \frac{1}{m-2} \left[R^{\varphi}_{tj} \delta_{ik} - R^{\varphi}_{tk} \delta_{ij} + R^{\varphi}_{ik} \delta_{tj} - R^{\varphi}_{ij} \delta_{tk} - \frac{S^{\varphi}}{m-1} (\delta_{tj} \delta_{ik} - \delta_{tk} \delta_{ij}) \right],$$

then, using (6.5), we deduce

$$\begin{aligned}R_{tijk}f_t - W^{\varphi}_{tijk}f_t =&\frac{1}{m-2}\left[f_t(R^{\varphi}_{tj}\delta_{ik} - R^{\varphi}_{tk}\delta_{ij}) + R^{\varphi}_{ik}f_j - R^{\varphi}_{ij}f_k - \frac{S^{\varphi}}{m-1}(\delta_{ik}f_j - \delta_{ij}f_k)\right]\\ =&-\frac{1}{m-2}\left[R^{\varphi}_{ij}f_k - R^{\varphi}_{ik}f_j + \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}) - \frac{S^{\varphi}}{m-1}(\delta_{ij}f_k - \delta_{ik}f_j)\right]\\ &-\frac{1}{m-2}\left(1-\frac{1}{m-1}\right)f_t(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik})\\ =&-D^{\varphi}_{ijk} - \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik})\end{aligned}$$

that easily implies (6.15). □

The second integrability condition follows by taking the divergence of (6.12). Indeed we have the following

Proposition 6.16. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$\begin{aligned}&(m-2)B^{\varphi}_{ij} + \frac{m-4}{m-2}W^{\varphi}_{tijk}f_tf_k\\ &= [1+(m-2)\mu]\left(D^{\varphi}_{ijk,k} - \frac{1}{m-2}W^{\varphi}_{tijk}f_tf_k - \frac{m-3}{m-2}D^{\varphi}_{jki}f_k - \frac{\alpha}{m-2}\varphi^a_{kk}\varphi^a_i f_j\right).\end{aligned} \tag{6.17}$$

Proof. We take the divergence of (6.12) and we use (6.1) and (2.65), together with (2.37), to obtain

$$\begin{aligned}[1+(m-2)\mu]D^{\varphi}_{ijk,k} =&(C^{\varphi}_{ijk} + f_tW^{\varphi}_{tijk})_k\\ =&C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} + f_tW^{\varphi}_{tijk,k}\\ =&C^{\varphi}_{ijk,k} + (-R^{\varphi}_{tk} + \mu f_tf_k + \lambda\delta_{tk})W^{\varphi}_{tijk} + f_kW^{\varphi}_{tjik,t}\\ =&C^{\varphi}_{ijk} + R^{\varphi}_{tk}W^{\varphi}_{tikj} + \mu W^{\varphi}_{tijk}f_tf_k + \lambda W^{\varphi}_{kijk}\\ &+\left[\frac{m-3}{m-2}C^{\varphi}_{jki} + \alpha(\varphi_{ji}\varphi^a_k - \varphi^a_{jk}\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_i\delta_{jk} - \varphi^a_k\delta_{ji})\right]f_k\\ =&(m-2)B^{\varphi}_{ij} + \alpha R^{\varphi}_{kj}\varphi^a_k\varphi^a_i - \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right)\\ &+\mu W^{\varphi}_{tijk}f_tf_k - \alpha\lambda\varphi^a_i\varphi^a_j\\ &+\frac{m-3}{m-2}C^{\varphi}_{jki}f_k + \alpha(\varphi_{ji}\varphi^a_{kk} - \varphi^a_{jk}f_k\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_if_j - \varphi^a_{kk}\delta_{ji})\\ =&(m-2)B^{\varphi}_{ij} + \alpha\left(R^{\varphi}_{kj}\varphi^a_k + \varphi^a_{kkj} - \lambda\varphi^a_j - \varphi^a_{jk}f_k + \frac{1}{m-2}\varphi^a_{kk}f_j\right)\varphi^a_i\\ &+\mu W^{\varphi}_{tijk}f_tf_k + \frac{m-3}{m-2}C^{\varphi}_{jki}f_k.\end{aligned}$$

Observe that from (6.1) we deduce the validity of

$$R^{\varphi}_{jk}\varphi^a_k + f_{jk}\varphi^a_k = \mu\varphi^a_{kk}f_j + \lambda\varphi^a_j,$$

and by plugging it into the above, together with (6.12), we obtain

$$\begin{aligned}[1+(m-2)\mu]D^{\varphi}_{ijk,k} =&(m-2)B^{\varphi}_{ij} + \alpha\left(-f_{jk}\varphi^a_k + \mu\varphi^a_{kk}f_j + (\varphi^a_kf_k)_j - \varphi^a_{jk}f_k + \frac{1}{m-2}\varphi^a_{kk}f_j\right)\varphi^a_i\\ &+\mu W^{\varphi}_{tijk}f_tf_k + \frac{m-3}{m-2}C^{\varphi}_{jki}f_k\\ =&(m-2)B^{\varphi}_{ij} + \frac{\alpha}{m-2}[1+(m-2)\mu]\varphi^a_{kk}f_j\varphi^a_i\\ &+\mu W^{\varphi}_{tijk}f_tf_k - \frac{m-3}{m-2}W^{\varphi}_{tjki}f_tf_k + \frac{m-3}{m-2}[1+(m-2)\mu]D^{\varphi}_{jki}f_k,\end{aligned}$$

and thus (6.17) follows. □

Remark 6.18. In case $\mu = -\frac{1}{m-2}$ from (6.12) and (6.17) we respectively obtain (2.56) and (2.57). Furthermore, when φ is constant, (6.12) and (6.17) extend respectively (4-5) and (4-6) of [16], with $\alpha = \beta = 1$. Observe, however, that the normalization $\alpha = \beta = 1$ that we adopt here is inessential.

In what follows we shall assume the following vanishing condition on φ-Bach

$$B^{\varphi}(\nabla f, \cdot) = 0 \tag{6.19}$$

and the non-degeneracy condition

$$\mu \neq -\frac{1}{m-2}, \tag{6.20}$$

we shall comment on this in Remark 6.31. Our aim is now to prove the following

Proposition 6.21. *In the present setting, with $m \geq 3$, assume* (6.19) *and define the vector field $Y \in \mathfrak{X}(M)$ of components*

$$Y^j := -D^{\varphi}_{ijk} f_i f_k. \tag{6.22}$$

Then, if (6.20) *holds, we have*

$$\frac{m-2}{2}|D^{\varphi}|^2 + \frac{\alpha}{m-2}|\tau(\varphi)|^2|\nabla f|^2 = div(Y). \tag{6.23}$$

Proof. Observe that (6.19) componentwise reads

$$B^{\varphi}_{ij} f_i = 0. \tag{6.24}$$

From (6.17) and the symmetries of W^{φ} and D^{φ} (see (6.7)), using also (6.1) we deduce

$$(m-2)B^{\varphi}_{ij} f_i = [1 + (m-2)\mu]\left(D^{\varphi}_{ijk,k} f_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2 f_j\right).$$

Since (6.20) holds, then (6.24) implies

$$D^{\varphi}_{ijk,k} f_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2 f_j = 0.$$

Contracting it with f_j we then deduce

$$D^{\varphi}_{ijk,k} f_i f_j - \frac{\alpha}{m-2}|\tau(\varphi)|^2|\nabla f|^2 = 0. \tag{6.25}$$

To proceed we first prove the identity

$$|D^{\varphi}|^2 = \frac{2}{m-2} D^{\varphi}_{ijk} R^{\varphi}_{ij} f_k. \tag{6.26}$$

It can be proved using the definition (6.5) of D^{φ} and its properties (6.7) and (6.8) as follows:

$$\begin{aligned}
|D^{\varphi}|^2 =& D^{\varphi}_{ijk} D^{\varphi}_{ijk} \\
=& \frac{1}{m-2} D^{\varphi}_{ijk}\left[R^{\varphi}_{ij} f_k - R^{\varphi}_{ik} f_j + \frac{1}{m-1} f_t (R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}) - \frac{S^{\varphi}}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\right] \\
=& \frac{1}{m-2}\left[D^{\varphi}_{ijk}(R^{\varphi}_{ij} f_k - R^{\varphi}_{ik} f_j) + \frac{1}{m-1} f_t (D^{\varphi}_{iik} R^{\varphi}_{tk} - D^{\varphi}_{iji} R^{\varphi}_{tj}) - \frac{S^{\varphi}}{m-1}(f_k D^{\varphi}_{iik} - f_j D^{\varphi}_{iji})\right] \\
=& \frac{1}{m-2} D^{\varphi}_{ijk} R^{\varphi}_{ij} f_k - \frac{1}{m-2} D^{\varphi}_{ikj} R^{\varphi}_{ij} f_k \\
=& \frac{2}{m-2} D^{\varphi}_{ijk} R^{\varphi}_{ij} f_k.
\end{aligned}$$

To obtain (6.23) from (6.25) we observe that, using (6.7), (6.1) and (6.8)

$$
\begin{aligned}
D^{\varphi}_{ijk,k}f_if_j =&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}f_{ik}f_j - D^{\varphi}_{ijk}f_if_{jk}\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}f_{ik}f_j\\
=&(D^{\varphi}_{ijk}f_if_j)_k + D^{\varphi}_{ijk}f_{ij}f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k + D^{\varphi}_{ijk}(-R^{\varphi}_{ij} + \mu f_if_j + \lambda\delta_{ij})f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}R^{\varphi}_{ij}f_k + \mu D^{\varphi}_{ijk}f_if_jf_k + \lambda D^{\varphi}_{iik}f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}R^{\varphi}_{ij}f_k,
\end{aligned}
$$

and thus we conclude using (6.22) and (6.26). □

We are now ready to prove the first important result of this section.

Theorem 6.27. *Let $(M, \langle\, ,\, \rangle)$ be a complete, non-compact Riemannian manifold of dimension m with an Einstein-type structure as in* (6.1). *Suppose that $m \geq 3$, that $\alpha > 0$, that* (6.20) *and* (6.19) *hold and that f is proper. Then $D^{\varphi} = 0$ and φ is harmonic.*

Proof. Let c be a regular value of f and let Σ_c and Ω_c be its corresponding sublevel hypersurface and set, that is

$$
\Omega_c := \{x \in M : f(x) \leq c\}, \quad \Sigma_c := \{x \in M : f(x) = c\} = \partial\Omega_c. \tag{6.28}
$$

Integrating (6.23) on M, that holds since we are assuming the validity of (6.19), and applying the divergence theorem

$$
\frac{m-2}{2}\int_{\Omega_c}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{\Omega_c}|\tau(\varphi)|^2|\nabla f|^2 = \int_{\Sigma_c}\langle Y, \nu\rangle,
$$

where ν is the outward unit normal to Σ_c and Y is the vector field with components defined by (6.22). Since ν is in the direction of ∇f and since, using (6.7)

$$
\langle Y, \nabla f\rangle = Y^k f_k = D^{\varphi}_{ijk}f_if_jf_k = 0,
$$

we obtain

$$
\frac{m-2}{2}\int_{\Omega_c}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{\Omega_c}|\tau(\varphi)|^2|\nabla f|^2 = 0.
$$

Since c is an arbitrary regular point of f we conclude

$$
\frac{m-2}{2}\int_{M}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{M}|\tau(\varphi)|^2|\nabla f|^2 = 0.
$$

and since $\alpha > 0$ and, using the second equation in (6.1), the vanishing of $|\tau(\varphi)|^2|\nabla f|^2$ is equivalent to the harmonicity of φ, the thesis follows at once. □

Remark 6.29. Note that we can give the vector field Y the following remarkable form:

$$
(m-1)Y = \mathrm{Ric}^{\varphi}(\nabla f, \nabla f)\nabla f - |\nabla f|^2\mathrm{Ric}^{\varphi}(\nabla f, \cdot)^{\sharp}. \tag{6.30}
$$

Indeed, from the definition (6.5) of D^{φ}

$$
\begin{aligned}
D^{\varphi}_{ijk}f_i =&\frac{1}{m-2}\left[R^{\varphi}_{ij}f_if_k - R^{\varphi}_{ik}f_if_j + \frac{1}{m-1}f_t(R^{\varphi}_{tk}f_j - R^{\varphi}_{tj}f_k) - \frac{S^{\varphi}}{m-1}(f_kf_j - f_jf_k)\right]\\
=&\frac{1}{m-2}\left[\left(1 - \frac{1}{m-1}\right)R^{\varphi}_{ij}f_if_k - \left(1 - \frac{1}{m-1}\right)R^{\varphi}_{ik}f_if_j\right]\\
=&\frac{1}{m-1}f_i(R^{\varphi}_{ij}f_k - R^{\varphi}_{ik}f_j).
\end{aligned}
$$

Therefore we have

$$Y^k = D^\varphi_{ijk} f_i f_j = \frac{1}{m-1}(R^\varphi_{ij} f_i f_j f_k - R^\varphi_{ik} f_i |\nabla f|^2),$$

that is (6.30).

Remark 6.31. Observe that in the degenerate case where $\mu = -\frac{1}{m-2}$, that is, when $(M, \langle\, ,\, \rangle)$ is a conformally harmonic-Einstein manifold by Theorem 2.49, the condition (6.19) is always satisfied. It follows by contracting the second integrability condition (2.57) with f_i, using the skew symmetry of W^φ in the first two indexes. Observe that a sufficient condition to guarantee (6.19) is that $(M, \langle\, ,\, \rangle)$ is φ-Bach flat, that is, $B^\varphi = 0$. In case $m \neq 4$ this requirement is quite strong, since from Proposition 2.38 it implies φ is a harmonic map. On the contrary in case $m = 4$ it seems a reasonable assumption, since B^φ is traceless.

Our aim is now to analize the consequences of Theorem 6.27, that is, the two simultaneous conditions

$$i)\ D^\varphi = 0, \quad ii)\ \tau(\varphi) = 0,$$

on the geometry of the level hypersurface $\Sigma_c = \partial\Omega_c$, defined as in (6.28), for a regular value of f. We fix the indexes ranges

$$1 \leq i, j, \ldots \leq m, \quad 1 \leq a, b, \ldots \leq m-1, \quad 1 \leq A, B, \ldots \leq n.$$

With respect to a local orthonormal coframe on M we have

$$\begin{cases} R^\varphi_{ij} + f_{ij} = \mu f_i f_j + \lambda \delta_{ij}, \\ \varphi^A_{ii} = 0 = \varphi^A_i f_i, \\ D^\varphi_{ijk} = 0. \end{cases} \tag{6.32}$$

The following Proposition provides the relation between the norm of D^φ and the curvature of the level hypersurfaces of f, it uses only the first and the last equation of (6.32).

Proposition 6.33. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$ that satisfies the first equation of (6.32). Let c be a regular value of f and let Σ_c be the corresponding level hypersurface. For $p \in \Sigma_c$ choose a local first order frame along f, that is a local orthonormal frame $\{e_i\}$ such that $e_1, \ldots, e_{m-1}$ are tangent to Σ_c and*

$$e_m = \frac{\nabla f}{|\nabla f|}.$$

Then, at p,

$$\frac{m-2}{2|\nabla f|^2}|D^\varphi|^2 = |\mathring{h}|^2 |\nabla f|^2 + \frac{m-2}{m-1} R^\varphi_{am} R^\varphi_{am}, \tag{6.34}$$

where $\mathring{h}$ is the traceless part of h, the second fundamental form of Σ_c.

Proof. First we compute $|D^\varphi|^2$ on M. A long and tedius computation yields the validity, where $\nabla f \neq 0$, of the following

$$\begin{aligned} \frac{m-2}{2|\nabla f|^2}|D^\varphi|^2 = & |\mathrm{Ric}^\varphi|^2 - \frac{m}{m-1} R^\varphi_{ma} R^\varphi_{ma} - \frac{m}{m-1}(R^\varphi_{mm})^2 \\ & - \frac{1}{m-1}(S^\varphi)^2 + \frac{2}{m-1} S^\varphi R^\varphi_{mm}. \end{aligned} \tag{6.35}$$

Let c be a regular value of f, $p \in \Sigma_c$ and $\{e_i\}$ a local first order frame along f, then

$$f_a = 0, \quad f_m = |\nabla f|. \tag{6.36}$$

Let h be the second fundamental form of Σ_c, then (see proof of Proposition 6.1 of [16])

$$h_{ab} = -\theta^m_a(e_b) = -\frac{f_{ab}}{|\nabla f|}. \tag{6.37}$$

Using the first equation of (6.32), that holds by hypothesis,

$$h_{ab} = \frac{1}{|\nabla f|}(R^{\varphi}_{ab} - \mu f_a f_b - \lambda\delta_{ab}) = \frac{1}{|\nabla f|}(R^{\varphi}_{ab} - \lambda\delta_{ab}). \tag{6.38}$$

The mean curvature h is defined as

$$\mathrm{h} := \frac{h_{aa}}{m-1}.$$

Tracing (6.38) we deduce the validity of

$$\mathrm{h} = \frac{1}{|\nabla f|}\left(\frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1} - \lambda\right). \tag{6.39}$$

We denote by $\mathring{h}$ the traceless part of h, that is,

$$\mathring{h}_{ab} := h_{ab} - \mathrm{h}\delta_{ab}.$$

Using (6.38) and (6.39) we obtain

$$\begin{aligned} |\mathring{h}|^2 =& |h|^2 - (m-1)\left(\frac{h_{aa}}{m-1}\right)^2 \\ =& \frac{1}{|\nabla f|^2}\left[|\mathrm{Ric}^{\varphi}|^2 - 2R^{\varphi}_{am}R^{\varphi}_{am} - \frac{m}{m-1}(R^{\varphi}_{mm})^2 - \frac{1}{m-1}(S^{\varphi})^2 + \frac{2}{m-1}S^{\varphi}R^{\varphi}_{mm}\right]. \end{aligned}$$

By plugging it in (6.35) we deduce the validity of (6.34). □

Remark 6.40. In the assumptions of Proposition 6.33, if $D^{\varphi} = 0$ then Σ_c is totally umbilical, that is

$$\mathring{h} = 0,$$

or equivalently

$$h_{ab} = \frac{1}{|\nabla f|}\left(\frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1} - \lambda\right)\delta_{ab}, \tag{6.41}$$

and for every $a = 1, \ldots, m-1$

$$R^{\varphi}_{am} = 0. \tag{6.42}$$

Then, by plugging (6.41) in (6.38) we obtain

$$\frac{1}{|\nabla f|}(R^{\varphi}_{ab} - \lambda\delta_{ab}) = \frac{1}{|\nabla f|}\left(\frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1} - \lambda\right)\delta_{ab},$$

that is,

$$R^{\varphi}_{ab} = \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{ab}. \tag{6.43}$$

In the following Proposition also the second equation of (6.32) comes into play.

Proposition 6.44. *In the assumptions and the notations above with $D^{\varphi} \equiv 0$ on Σ_c, that is, all the equations of (6.32) are satisfied and c is a regular value of f, the quantities $|\nabla f|$, h, S^{φ} and λ are constant on each connected component of Σ_c. In particular Σ_c is totally umbilical hypersurface of $(M, \langle\, ,\,\rangle)$ with constant mean curvature. Moreover ${}^{\Sigma_c}S^{\varphi}$ is constant on Σ_c, where ${}^{\Sigma_c}S^{\varphi}$ is the φ-scalar curvature of the Riemannian manifold $(\Sigma_c, \langle\, ,\,\rangle_{\Sigma_c})$, where $\langle\, ,\,\rangle_{\Sigma_c}$ is the metric induced on Σ_c and where we are considering the restriction of φ on Σ_c.*

Proof. We use the notations of Proposition 6.33. Using the first equation of (6.32) and the fact that (6.36) holds in the chosen frame we obtain

$$\frac{|\nabla f|_a^2}{2} = f_{ia} f_i = (-R_{ia}^{\varphi} + \mu f_i f_a + \lambda\delta_{ia}) f_i = -R_{ba}^{\varphi} f_b - R_{ma}^{\varphi} |\nabla f| + \lambda f_a.$$

Using in it once again (6.36), together with (6.42) we deduce $|\nabla f|_a^2 = 0$. Hence $|\nabla f|$ is constant on Σ_c. By Codazzi equation, the definition (2.1), and (6.42) and the fact that $h_{ab} = \mathrm{h}\delta_{ab}$, we have

$$(m-2)\mathrm{h}_b = R_{mb} = R_{mb}^{\varphi} + \alpha\varphi_m^A \varphi_b^A = \alpha\varphi_m^A\varphi_b^A. \tag{6.45}$$

Now observe that, using the second equation of (6.32) and (6.36)

$$\varphi_m^A |\nabla f| = \varphi_m^A f_m = \varphi_i^A f_i - \varphi_a^A f_a = \varphi_{ii}^A = 0. \tag{6.46}$$

Then, since $|\nabla f|$ is constant on Σ_c, using (6.46)

$$(|\nabla f|\mathrm{h})_b = |\nabla f|\mathrm{h}_b = \frac{\alpha}{m-2}\varphi_b^A\varphi_m^A|\nabla f| = 0.$$

Hence $|\nabla f|\mathrm{h}$ is constant on Σ_c and, since $|\nabla f|$ is constant on Σ_c, it implies that Σ_c has constant mean curvature h. Using (6.3) with $i = b$, (6.36), (6.42) and also the first equation of (6.32) we conclude

$$\begin{aligned}
\frac{1}{2}S_b^{\varphi} =& R_{kb}^{\varphi} f_k + \mu(f_{kb} f_k - \Delta f f_b) + (m-1)\lambda_b \\
=& R_{ab}^{\varphi} f_a + R_{mb}^{\varphi}|\nabla f| + \mu(f_{ab} f_a + f_{mb}|\nabla f|) + (m-1)\lambda_b \\
=& R_{mb}^{\varphi}|\nabla f| + \mu f_{mb}|\nabla f| + (m-1)\lambda_b \\
=& \mu(-R_{mb}^{\varphi} + \mu f_m f_b + \lambda\delta_{mb})|\nabla f| + (m-1)\lambda_b \\
=& (m-1)\lambda_b.
\end{aligned}$$

It follows that

$$\frac{1}{2}S^{\varphi} - (m-1)\lambda \tag{6.47}$$

is constant on Σ_c. In particular, if we show that S^{φ} is constant on Σ_c we can conclude also that λ is constant on Σ_c. To show that S^{φ} is constant on Σ_c we first observe that (6.39) can be rewritten as

$$|\nabla f|\mathrm{h} = \frac{S^{\varphi} - R_{mm}^{\varphi}}{m-1} - \lambda. \tag{6.48}$$

Hence we obtain

$$(m-1)|\nabla f|\mathrm{h} = S^{\varphi} - R_{mm}^{\varphi} - (m-1)\lambda = \left(\frac{1}{2}S^{\varphi} - (m-1)\lambda\right) + \frac{1}{2}S^{\varphi} - R_{mm}^{\varphi},$$

and since both $|\nabla f|\mathrm{h}$ and (6.47) are constants on Σ_c we can conclude that also

$$\frac{1}{2}S^{\varphi} - R_{mm}^{\varphi}$$

is constant on Σ_c. Then it is sufficient to show that R_{mm}^{φ} is constant to obtain that S^{φ} is constant and conclude the proof. At this purpose, observe that uisng the first equation of (6.32),(6.43) and (6.48)

$$f_{aa} = -R_{aa}^{\varphi} + \mu f_a f_a + \lambda\delta_{aa} = -\left(\frac{S^{\varphi} - R_{mm}^{\varphi}}{m-1}\right)(m-1) + (m-1)\lambda = (m-1)|\nabla f|\mathrm{h},$$

hence using (6.3) with $i = m$ and (6.36) we can conclude

$$
\begin{aligned}
\frac{1}{2}S^{\varphi}_{m} =& R^{\varphi}_{km}f_k + \mu(f_{km}f_k - \Delta f f_m) + (m-1)\lambda_m\\
=& R^{\varphi}_{am}f_a + R^{\varphi}_{mm}|\nabla f| + \mu(f_{am}f_a + f_{mm}|\nabla f| - \Delta f|\nabla f|) + (m-1)\lambda_m\\
=& R^{\varphi}_{mm}|\nabla f| + \mu(f_{mm} - \Delta f)|\nabla f| + (m-1)\lambda_m\\
=& R^{\varphi}_{mm}|\nabla f| - \mu f_{aa}|\nabla f| + (m-1)\lambda_m\\
=& R^{\varphi}_{mm}|\nabla f| + \mu(m-1)h|\nabla f|^2 + (m-1)\lambda_m.
\end{aligned}
$$

Since h and $|\nabla f|$ are constants on Σ_c we deduce that

$$
\frac{1}{2}S^{\varphi}_{m} - R^{\varphi}_{mm}|\nabla f| - (m-1)\lambda_m
$$

is constant on Σ_c, then, using once again that $|\nabla f|$ and also that (6.47) are constants on Σ_c,

$$
\begin{aligned}
0 =& \left(\frac{1}{2}S^{\varphi}_{m} - R^{\varphi}_{mm}|\nabla f| - (m-1)\lambda_m\right)_a\\
=& \left(\frac{1}{2}S^{\varphi} - (m-1)\lambda\right)_{am} - R^{\varphi}_{mm,a}|\nabla f|\\
=& - R^{\varphi}_{mm,a}|\nabla f|.
\end{aligned}
$$

Thus R^{φ}_{mm} is constant on Σ_c and the proof of the constancy of S^{φ} on Σ_c is concluded. Now it remains to show that ${}^{\Sigma_c}S^{\varphi}$ is constant on Σ_c. By Gauss formula, since the immersion is totally umbilical,

$$
{}^{\Sigma_c}S = S - 2R_{mm} + (m-1)(m-2)\mathrm{h}^2,
$$

hence, from the definitions (2.2) and (2.1)

$$
\begin{aligned}
{}^{\Sigma_c}S^{\varphi} =& {}^{\Sigma_c}S - \alpha\varphi^A_a\varphi^A_a\\
=& S - 2R_{mm} + (m-1)(m-2)\mathrm{h}^2 - \alpha|d\varphi|^2 + \alpha\varphi^A_m\varphi^A_m\\
=& S^{\varphi} - 2R^{\varphi}_{mm} - 2\alpha\varphi^A_m\varphi^A_m + (m-1)(m-2)\mathrm{h}^2 + \alpha\varphi^{\alpha}_m\varphi^{\alpha}_m\\
=& S^{\varphi} - 2R^{\varphi}_{mm} + (m-1)(m-2)\mathrm{h}^2 - \alpha\varphi^A_m\varphi^A_m.
\end{aligned}
$$

Then we have, using (6.46):

$$
{}^{\Sigma_c}S^{\varphi}|\nabla f| = [S^{\varphi} - 2R^{\varphi}_{mm} + (m-1)(m-2)\mathrm{h}^2]|\nabla f|,
$$

and thus we can conclude that ${}^{\Sigma_c}S^{\varphi}$ is constant. □

Our aim now it to show that Σ_c is harmonic-Einstein with respect the the induced metric and the restriction of φ, for a regular value c of f. To show it we need the following result, that has an importance also on its own.

Proposition 6.49. *In the assumptions above, if f is non-constant, then $(M, \langle\, ,\, \rangle)$ is φ-Cotton flat.*

Proof. We want to prove that $C^{\varphi} = 0$. By analiticity it is sufficient to prove the result on $\{x \in M : \nabla f(x) \neq 0\}$. We take a local first order frame $\{e_i\}$ along f. By the first integrability condition (6.12), since we are assuming the validity of the third equation of (6.32) we deduce

$$
C^{\varphi}_{ijk} = -f_t W^{\varphi}_{tijk}. \tag{6.50}
$$

Hence, by the symmetries of W^{φ} and using (6.36)

$$
0 = -f_i f_t W^{\varphi}_{tijk} = f_i C^{\varphi}_{ijk} = f_a C^{\varphi}_{ajk} + |\nabla f| C^{\varphi}_{mjk} = C^{\varphi}_{mjk}|\nabla f|.
$$

Then $C^{\varphi}_{mjk} = 0$. Since Σ_c is totally umbilical with constant mean curvature h is parallel, that is $h_{ab,c} = 0$. Then, from Codazzi's equation

$$R_{mabc} = 0, \tag{6.51}$$

indeed

$$-R_{mabc} = h_{ab,c} - h_{ac,b} = 0.$$

But then, explicitating the decomposition (2.6)

$$0 = R_{mabc} = W^{\varphi}_{mabc} + \frac{1}{m-2}\left[R^{\varphi}_{mb}\delta_{ac} - R^{\varphi}_{mc}\delta_{ab} + R^{\varphi}_{ac}\delta_{mb} - R^{\varphi}_{ab}\delta_{mc} - \frac{S^{\varphi}}{m-1}(\delta_{mb}\delta_{ac} - \delta_{mc}\delta_{ab})\right],$$

and since (6.42) holds we conclude from the above equality:

$$W^{\varphi}_{mabc} = 0. \tag{6.52}$$

Therefore, from (6.50), using (6.36) and (6.52) we obtain

$$C^{\varphi}_{abc} = -f_t W^{\varphi}_{tabc} = -f_d W^{\varphi}_{dabc} - |\nabla f| W^{\varphi}_{mabc} = 0.$$

By the symmetries of C^{φ} it remains only to prove $C^{\varphi}_{amb} = 0$. First of all observe that

$$R^{\varphi}_{am,k}\theta^k = \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a, \tag{6.53}$$

in fact from the definition of covariant derivative, since (6.42) holds,

$$\begin{aligned}
0 =& dR^{\varphi}_{am} \\
=& R^{\varphi}_{km}\theta^k_a + R^{\varphi}_{ak}\theta^k_m + R^{\varphi}_{am,k}\theta^k \\
=& R^{\varphi}_{bm}\theta^b_a + R^{\varphi}_{mm}\theta^m_a + R^{\varphi}_{ab}\theta^b_m + R^{\varphi}_{am}\theta^m_m + R^{\varphi}_{am,k}\theta^k \\
=& R^{\varphi}_{mm}\theta^m_a + R^{\varphi}_{ab}\theta^b_m + R^{\varphi}_{am,k}\theta^k
\end{aligned}$$

and thus, using also (6.43) from the above equality we obtain

$$\begin{aligned}
R^{\varphi}_{am,k}\theta^k =& - R^{\varphi}_{mm}\theta^m_a - R^{\varphi}_{ab}\theta^b_m \\
=& - R^{\varphi}_{mm}\theta^m_a - \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{ab}\theta^b_m \\
=& - R^{\varphi}_{mm}\theta^m_a - \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\theta^a_m \\
=& \left(-R^{\varphi}_{mm} + \frac{S^{\varphi}}{m-1} - \frac{R^{\varphi}_{mm}}{m-1}\right)\theta^m_a \\
=& \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a,
\end{aligned}$$

that is (6.53). Now we are going to prove

$$R^{\varphi}_{am,m} = 0. \tag{6.54}$$

Observe that, by taking $i = a$ and $j = m$ in the first equation of (6.32) we obtain

$$R^{\varphi}_{am} + f_{am} = \mu f_a f_m,$$

and thus, using (6.42) and (6.36), we deduce

$$f_{am} = 0. \tag{6.55}$$

Moreover, taking the covariant derivative of the first equation of (6.32) we infer

$$R^{\varphi}_{ij,k} + f_{ijk} = \mu f_{ik} f_j + \mu f_i f_{jk} + \lambda_k \delta_{ij}$$

that for $i = m = k$ and $j = a$ reads as

$$R^{\varphi}_{am,m} + f_{mam} = \mu f_{mm} f_a + \mu f_m f_{am}.$$

Then, using (6.55) and the commutation relation (1.116) of [1], that is

$$f_{ijk} = f_{ikj} + R^t_{ijk} f_t,$$

together with (6.51),

$$R^{\varphi}_{am,m} = -(\Delta f - f_{bb})_a. \tag{6.56}$$

Indeed

$$R^{\varphi}_{am,m} = -f_{mam} = -(f_{mma} + f_i R^i_{mam}) = -(f_{mma} + f_b R^b_{mam} + f_m R^m_{mam}) = -f_{mma} = -(\Delta f - f_{bb})_a.$$

Since, taking the trace of the first equation of (6.32) we obtain

$$S^{\varphi} + \Delta f = \mu |\nabla f|^2 + m\lambda$$

and since S^{φ}, $|\nabla f|$ and λ are constant on Σ_c we deduce from the above equality that also Δf is constant on Σ_c. Moreover, from the first equation of (6.32), (6.43) and (6.36)

$$\begin{aligned} f_{ab} =& - R^{\varphi}_{ab} + \mu f_a f_b + \lambda \delta_{ab} \\ =& - \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1} \delta_{ab} + \lambda \delta_{ab} \\ =& - \frac{1}{m-1} (S^{\varphi} - R^{\varphi}_{mm} - (m-1)\lambda) \delta_{ab}, \end{aligned}$$

that is,

$$f_{ab} = -\frac{1}{m-1} (S^{\varphi} - R^{\varphi}_{mm} - (m-1)\lambda) \delta_{ab}. \tag{6.57}$$

Tracing (6.57) we have

$$f_{aa} = -(S^{\varphi} - R^{\varphi}_{mm} - (m-1)\lambda), \tag{6.58}$$

and thus also f_{aa} is constant on Σ_c. Then we can conclude from (6.56) the valdity of (6.54), since both Δf and f_{bb} are constants on Σ_c. Using (6.54) and (6.53) we infer

$$R^{\varphi}_{am,b} \theta^b = R^{\varphi}_{am,k} \theta^k = \frac{S^{\varphi} - m R^{\varphi}_{mm}}{m-1} \theta^m_a.$$

Using (6.37) in the above equality we deduce the validity of

$$R^{\varphi}_{am,b} = \frac{S^{\varphi} - m R^{\varphi}_{mm}}{m-1} \theta^m_a(e_b) = \frac{1}{|\nabla f|} \frac{m R^{\varphi}_{mm} - S^{\varphi}}{m-1} f_{ab}. \tag{6.59}$$

Then we finally obtain, using (2.36), (6.43) and (6.59)

$$\begin{aligned} C^{\varphi}_{abm} =& R^{\varphi}_{ab,m} - R^{\varphi}_{am,b} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} \\ =& \left(\frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1} \delta_{ab} \right)_m + \frac{1}{|\nabla f|} \frac{S^{\varphi} - m R^{\varphi}_{mm}}{m-1} f_{ab} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} \\ =& \frac{S^{\varphi}_m - R^{\varphi}_{mm,m}}{m-1} \delta_{ab} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} + \frac{1}{|\nabla f|} \frac{S^{\varphi} - m R^{\varphi}_{mm}}{m-1} f_{ab} \\ =& \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} - \frac{1}{m-1} R^{\varphi}_{mm,m} \delta_{ab} + \frac{1}{|\nabla f|} \frac{S^{\varphi} - m R^{\varphi}_{mm}}{m-1} f_{ab}. \end{aligned}$$

Moreover, since φ is harmonic, from (2.10),

$$S^{\varphi}_{m} = 2R^{\varphi}_{im,i} = 2R^{\varphi}_{am,a} + 2R^{\varphi}_{mm,m}.$$

By inserting it in the above equality we deduce the validity of

$$C^{\varphi}_{abm} = \frac{1}{m-1}R^{\varphi}_{cm,c}\delta_{ab} + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}f_{ab}. \tag{6.60}$$

Taking the trace of (6.59) and using (6.58) we have

$$R^{\varphi}_{am,a} = \frac{1}{|\nabla f|}\frac{mR^{\varphi}_{mm} - S^{\varphi}}{m-1}f_{aa} = \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}(S^{\varphi} - R^{\varphi}_{mm} - (m-1)\lambda). \tag{6.61}$$

On the other hand, using (6.57) we obtain

$$\frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}f_{ab} = -\frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{(m-1)^2}(S^{\varphi} - R^{\varphi}_{mm} - (m-1)\lambda)\delta_{ab}. \tag{6.62}$$

Using (6.61) and (6.62) in (6.60) we conclude

$$C^{\varphi}_{abm} = \frac{1}{m-1}R^{\varphi}_{cm,c}\delta_{ab} + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}f_{ab} = 0,$$

then the proof is completed. □

We are now able to prove the following Proposition, as claimed before.

Proposition 6.63. *In the assumptions above, Σ_c is harmonic Einstein with respect the induced metric for every regular value c of f.*

Proof. First of all observe that $C^{\varphi} = 0$ from Proposition 6.49, hence using also the third equation (6.32) the first integrability condition (6.12) implies

$$0 = C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} = |\nabla f| W^{\varphi}_{mijk} \qquad \text{on } \Sigma_c,$$

thus

$$W^{\varphi}_{mijk} = 0. \tag{6.64}$$

From the decomposition (2.6), using (6.64) we obtain

$$R_{mamb} = \frac{1}{m-2}\left(R^{\varphi}_{ab} + R^{\varphi}_{mm}\delta_{ab} - \frac{S^{\varphi}}{m-1}\delta_{ab}\right), \tag{6.65}$$

indeed

$$\begin{aligned}
R_{mamb} =& W^{\varphi}_{mamb} + \frac{1}{m-2}(A^{\varphi}_{mm}\delta_{ab} - A^{\varphi}_{mb}\delta_{am} + A^{\varphi}_{ab} - A^{\varphi}_{ma}\delta_{bm})\\
=&\frac{1}{m-2}\left(R^{\varphi}_{mm}\delta_{ab} - \frac{S^{\varphi}}{2(m-1)}\delta_{ab} + R^{\varphi}_{ab} - \frac{S^{\varphi}}{2(m-1)}\delta_{ab}\right)\\
=&\frac{1}{m-2}\left(R^{\varphi}_{ab} + R^{\varphi}_{mm}\delta_{ab} - \frac{S^{\varphi}}{m-1}\delta_{ab}\right).
\end{aligned}$$

Using (6.43) we then have

$$R^{\varphi}_{ab} = \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{ab},$$

and by plugging it in the above we obtain

$$R_{mamb} = \frac{1}{m-2}\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1}\delta_{ab} + R^\varphi_{mm}\delta_{ab} - \frac{S^\varphi}{m-1}\delta_{ab}\right) = \frac{R^\varphi_{mm}}{m-1}\delta_{ab}.$$

By Gauss formula

$$^{\Sigma_c}R_{ac} = R_{ac} - R_{amcm} + (m-2)\mathrm{h}^2\delta_{ac},$$

then using (2.1)

$$^{\Sigma_c}R^\varphi_{ac} = {}^{\Sigma_c}R_{ac} - \alpha\varphi^A_a\varphi^A_c = R^\varphi_{ac} - R_{amcm} + (m-2)\mathrm{h}^2\delta_{ac}.$$

By inserting (6.65) and using (6.43) in the above we obtain

$$\begin{aligned} ^{\Sigma_c}R^\varphi_{ac} =& R^\varphi_{ac} - R_{amcm} + (m-2)\mathrm{h}^2\delta_{ac} \\ =& \frac{S^\varphi - R^\varphi_{mm}}{m-1}\delta_{ac} - \frac{R^\varphi_{mm}}{m-1}\delta_{ac} + (m-2)\mathrm{h}^2\delta_{ac} \\ =& \left[\frac{S^\varphi - 2R^\varphi_{mm}}{m-1} + (m-2)\mathrm{h}^2\right]\delta_{ac}. \end{aligned}$$

Thus, denoting by $^{\Sigma_c}\mathrm{Ric}^\varphi$, $\langle\, ,\,\rangle_{\Sigma_c}$ and $^{\Sigma_c}S^\varphi$ the φ-Ricci curvature, the induced metric and the φ-scalar curvature of Σ_c (where we are considering the restriction of φ on Σ_c)

$$^{\Sigma_c}\mathrm{Ric}^\varphi = \frac{^{\Sigma_c}S^\varphi}{m-1}\langle\, ,\,\rangle_{\Sigma_c}.$$

We conclude that $(\Sigma_c, \langle\, ,\,\rangle_{\Sigma_c})$ is harmonic-Einstein since $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ is harmonic and thus from $d\varphi(\nabla f) = 0$ we see that $\varphi : (\Sigma_c, \langle\, ,\,\rangle_{\Sigma_c}) \to (N, \langle\, ,\,\rangle_N)$ is harmonic too. □

We are now ready to prove the most important result of this section.

Theorem 6.66. *Let $(M, \langle\, ,\,\rangle)$ be a complete, non-compact Riemannian manifold of dimension m with an Einstein-type structure as in* (6.1)*. Suppose that $m \geq 3$, that $\alpha > 0$, that $\mu \neq 1/(2-m)$ and $B^\varphi(\nabla f,\ )$ hold and that f is proper. Then, in a neighborhood of every regular level set of f, the Riemannian manifold $(M, \langle\, ,\,\rangle)$ is locally a warped product with $(m-1)$-dimensional harmonic-Einstein fibers.*

Proof. Our assumptions permits to apply Theorem 6.27 to deduce that φ must be harmonic and D^φ must vanish on M. Let Σ be a regular level set of f, that is $|\nabla f| \neq 0$ on Σ (it exists by Sard's theorem, since f is non-constant). In a neighborhood $\mathcal{U}$ of Σ which does not contain any critical point of f the potential function f only depends on the signed distance r to the hypersurface Σ. Hence, by a suitable change of variable, we can express the metric tensor g as

$$dr \otimes dr + g_{ab}d\theta^a \otimes d\theta^b,$$

where $g_{ab} = g_{ab}(r, \theta)$ and $r \in (r_*, r^*)$ for some maximal $r_* \in [-\infty, 0)$ and $r^* \in (0, +\infty]$, where $\theta^2, \ldots, \theta^m$ is any local coordinates system on the level surface Σ. Since, as proved in Proposition 6.44, Σ is totally umbilical and has constant mean curvature

$$\frac{\partial g_{ab}}{\partial r} = -2h_{ab} = \phi g_{ab}, \quad \phi(r) = -2\mathrm{h}(r).$$

Thus we deduce the validity of

$$g_{ab}(r, \theta) = e^{\Phi(r)}g_{ab}(0, \theta), \quad \Phi(r) = \int_0^r \phi.$$

This proves that on $\mathcal{U}$ the metric g takes the form of a warped product metric

$$dr \otimes dr + w^2\langle\, ,\,\rangle_\Sigma,$$

where w is a positive function on (r_*, r^*) and $\langle\, ,\,\rangle_\Sigma$ is the metric induced on $\langle\, ,\,\rangle$ by g, which is harmonic-Einstein by Proposition 6.63. □

7 The complete case

In this section we consider a complete Riemannian manifold $(M, \langle\,,\,\rangle)$ with a gradient Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{7.1}$$

where $\varphi : M \to (N, \langle\,,\,\rangle_N)$ is a smooth map, $\alpha, \mu, \lambda \in \mathbb{R}$ with $\alpha \neq 0$, $f \in \mathcal{C}^{\infty}(M)$.

Recall that the validity of a system of the type

$$\mathrm{Ric} + \mathrm{Hess}(v) - \frac{1}{\gamma} dv \otimes dv \geq -(\gamma + m - 1) G(r) \langle\,,\,\rangle, \tag{7.2}$$

for some $\gamma \in \mathbb{R}^+$ and some continuous function $G : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, implies some restriction on the volume growth of geodesic balls. The same applies to the simpler system

$$\mathrm{Ric} + \mathrm{Hess}(v) \geq -(\gamma + m - 1) G(r) \langle\,,\,\rangle.$$

Here $r(x) := \mathrm{dist}_M(x, o)$ is the geodesic distance of $x \in M$ to a fixed origin $o \in M$.

Indeed, in case $\gamma > 0$, the left hand side of (7.2) is the generalized Bakry-Émery Ricci tensor Ric^{γ}_v of $(M, \langle\,,\,\rangle)$ introduced by Z. Qian in [40], so that we can write (7.2) in the form

$$\mathrm{Ric}^{\gamma}_v \geq -(\gamma + m - 1) G(r) \langle\,,\,\rangle. \tag{7.3}$$

Inequality (7.3) enables us to estimate from above the weighted volume of geodesic balls

$$\mathrm{vol}_v(B_r) := \int_{B_r} e^{-v},$$

via Theorem 2.4 of [28] whenever G has an appropriate behaviour at infinity (see (7.6)). Of course in the estimate a role is played by the parameter γ. Indeed, let g be a positive solution (if any) of

$$\begin{cases} g'' - Gg \geq 0 \text{ on } \mathbb{R}^+_0 \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \tag{7.4}$$

Then (7.3), together with completeness of $(M, \langle\,,\,\rangle)$, implies, via Theorem 2.4 of [28], that for r large enough,

$$\mathrm{vol}_v(\partial B_r) \leq C g^{\gamma+m-1}(r) \quad \text{and} \quad \mathrm{vol}_v(B_r) \leq D + C \int_0^r g^{\gamma+m-1}, \tag{7.5}$$

for some constants $C, D > 0$. Note that, and this is important, the upper bound in (7.5) only depends on G via g but not on v.

Assuming that $G \in \mathcal{C}^1(\mathbb{R}^+_0)$ is positive and satisfies

$$\inf_{\mathbb{R}^+_0} \frac{G'}{G^{\frac{3}{2}}} > -\infty, \tag{7.6}$$

by choosing

$$g(t) = \frac{1}{D\sqrt{G(0)}} \left(e^{D \int_0^t \sqrt{G}} - 1 \right) \tag{7.7}$$

for a positive constant D large enough as a solution of (7.4), from Proposition 2.3 of [28] we have

$$\Delta_v r(x) \leq C \sqrt{G(r(x))} \quad \text{for } r(x) >> 1 \tag{7.8}$$

for some large enough positive constant C. As a consequence, for $r_0 \in \mathbb{R}^+$ there exists $A, B, C \in \mathbb{R}^+$ such that, for every $r \geq r_0$

$$\mathrm{vol}_v(\partial B_r) \leq e^{C\int_{r_0}^{r}\sqrt{G}}$$

and

$$\mathrm{vol}_v(B_r) \leq A + B\int_{r_0}^{r} e^{C\int_{r_0}^{t}\sqrt{G}}dt.$$

Note that, in case

$$\mathrm{Ric}_v \geq -(m-1)G(r)\langle\, ,\, \rangle,$$

that, for the sake of brevity we shall indicate as the case $\gamma = +\infty$, the estimates corresponding to (7.5) are given in Proposition 8.11 of [1], that is,

$$\mathrm{vol}_v(\partial B_r) \leq e^{C(r-\varepsilon)+\int_{\varepsilon}^{r}\left(\int_{\varepsilon}^{t}(m-1)G\right)dt} \tag{7.9}$$

for some constants $\varepsilon, C > 0$ and $r \geq \varepsilon$ and, as a consequence,

$$\mathrm{vol}_v(B_r) \leq D + \int_0^r e^{Cs+\int_{\varepsilon}^{s}\left(\int_{\varepsilon}^{t}(m-1)G\right)dt}ds \tag{7.10}$$

with C, ε as above, $D > 0$ a constant and $r \in \mathbb{R}_0^+$.

In particular, when $G \equiv \Sigma$ for some $\Sigma \in \mathbb{R}$, that is,

$$\mathrm{Ric}_v^{\gamma} \geq -(\gamma + m - 1)\Sigma\langle\, ,\, \rangle \tag{7.11}$$

we have: if $\gamma > 0$ and $\Sigma \geq 0$, (7.4) admits a positive solution h such that $h(r) = e^{\sqrt{\Sigma}r}$ for $r >> 1$, so that the second estimate in (7.5) yields

$$\mathrm{vol}_v(B_r) \leq D + Ce^{(\gamma+m-1)\sqrt{\Sigma}r} \quad \text{for } r >> 1 \tag{7.12}$$

and some constants $C, D > 0$ while, if $\gamma = +\infty$ and $G \equiv \Sigma$ for some $\Sigma \in \mathbb{R}$, that is,

$$\mathrm{Ric}_v \geq -(m-1)\Sigma\langle\, ,\, \rangle \tag{7.13}$$

from (7.9) and (7.10) we respectively obtain the estimates

$$\mathrm{vol}_v(\partial B_r) \leq e^{\frac{m-1}{2}\Sigma r^2 + Cr} \quad \text{and} \quad \mathrm{vol}_v(B_r) \leq D + \int_0^r e^{\frac{(m-1)\Sigma}{2}t^2+Ct}dt \quad \text{for } r >> 1 \tag{7.14}$$

and some constants $C, D > 0$.

We point out that for if $\gamma > 0$ and $\Sigma < 0$, Qian, Theorem 5 in [40], shows that the complete manifold $(M, \langle\, ,\, \rangle)$ satisfying (7.11) has to be compact. For $\gamma = +\infty$ and $\Sigma < 0$ a complete Riemannian manifold $(M, \langle\, ,\, \rangle)$ satisfying (7.14) is not necessarily compact (to see this it is sufficient to consider the Gaussian shrinker gradient Ricci soliton structure on Euclidean space). Nevertheless, the following Proposition holds.

Proposition 7.15. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold such that* (7.13) *holds for some $v \in \mathcal{C}^{\infty}(M)$ and for some constant $\Sigma < 0$. Then $(M, \langle\, ,\, \rangle)$ is Δ_v-parabolic.*

Recall that $(M, \langle\, ,\, \rangle)$ is said to be Δ_v-parabolic if every bounded above Δ_v-subharmonic function on M is constant.

To prove the above Proposition we observe that Theorem A of [41] can be easily adapted in the weighted setting, obtaining

Theorem 7.16. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold, let $v \in \mathcal{C}^{\infty}(M)$ and assume that*

$$vol_v(\partial B_r)^{-1} \notin L^1(+\infty). \tag{7.17}$$

Then M is Δ_v-parabolic.

Proof (of Proposition 7.15). Our assumptions imply the validity of the first of (7.14). Thus, since $\Sigma < 0$ we deduce the validity of (7.17). $\square$

For $\Sigma \geq 0$ we have

Proposition 7.18. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold, $v \in \mathcal{C}^\infty(M)$ and $\Sigma \geq 0$. Assume either (7.12) holds or that the second inequality of (7.14) holds. Then we have the validity of the weak maximum principle at infinity for Δ_v. As a consequence, the L^1-Liouville property for Δ_v-subharmonic functions holds.*

Recall that the L^1-Liouville property for Δ_v-subharmonic functions holds if every $u \in Lip_{loc}(M)$ solution of $\Delta_v u \leq 0$ on M and satisfying $0 \leq u \in L^1(M, e^{-v})$ is constant.

Proof. From Theorem 9 of [39], the validity of the weak maximum principle at infinity for Δ_v is guaranteed in case

$$\frac{r}{\log \mathrm{vol}_v(B_r)} \notin L^1(+\infty). \tag{7.19}$$

Since $\gamma > 0$, (7.11) implies

$$\mathrm{Ric}_v \geq -(\gamma + m - 1)\Sigma\langle\, ,\, \rangle.$$

Then we can assume the validity of (7.13) in both cases. As remarked above we get (7.14) for some constants $C, D > 0$, so that, by a computation we obtain that (7.19) holds. Now the validity of the L^1-Liouville property for Δ_v-subharmonic functions can be deduced from Theorem 24 of [39]. $\square$

In the presence of a gradient-Einstein type structure on a complete Riemannian manifold we naturally have the validity of a system of the type (7.2), as we now show.

Proposition 7.20. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold with a gradient Einstein-type structure as in (7.1) for some $f \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle\, ,\, \rangle_N)$, $\alpha \in \mathbb{R}\setminus\{0\}$ and $\mu, \lambda \in \mathbb{R}$. Let $o \in M$ be a fixed origin and $r(x) := dist_M(x, o)$ the geodesic distance of $x \in M$ from o. Let $K : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ such that*

$$|d\varphi|^2 \leq K(r) \qquad \text{if } \alpha < 0 \tag{7.21}$$

and $F : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be such that

$$|\nabla f|^2 \leq F(r) \qquad \text{if } \alpha < 0 \text{ and } \mu < 0. \tag{7.22}$$

Then, denoting with t_+ and t_- the positive and negative part of $t \in \mathbb{R}$, using the conventions $\frac{1}{0} = +\infty$, $(+\infty)_+ = 0$ and

$$Ric_f^{+\infty} = Ric_f,$$

we have

$$Ric_f^{\frac{1}{\mu_+}} \geq -\left(\left(\frac{1}{\mu}\right)_+ + m - 1\right) G(r)\langle\, ,\, \rangle, \tag{7.23}$$

where

$$G = -\frac{\lambda - \mu_- F - \alpha_- K}{\left(\frac{1}{\mu}\right)_+ + m - 1}. \tag{7.24}$$

Proof. The following inequalities hold, in the sense of quadratic forms,

$$0 \leq \varphi^*\langle\, ,\, \rangle_N \leq |d\varphi|^2\langle\, ,\, \rangle.$$

Hence using the first equation of (7.1) we obtain, in case $\alpha > 0$

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq \lambda\langle\, ,\, \rangle$$

while in case $\alpha < 0$, using (7.21),

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq (\lambda + \alpha K(r))\langle\,,\,\rangle.$$

From the above we conclude

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq (\lambda - \alpha_- K(r))\langle\,,\,\rangle. \tag{7.25}$$

In case $\mu = 0$, (7.25) gives

$$\mathrm{Ric}_f^{+\infty} = \mathrm{Ric}_f \geq (\lambda - \alpha_- K(r))\langle\,,\,\rangle,$$

and, in case $\mu > 0$, (7.25) gives

$$\mathrm{Ric}_f^{\frac{1}{\mu}} \geq (\lambda - \alpha_- K(r))\langle\,,\,\rangle.$$

Moreover

$$df \otimes df \leq |\nabla f|^2 \langle\,,\,\rangle$$

and thus, in case $\mu < 0$, from (7.25), using (7.22), we get

$$\mathrm{Ric}_f \geq (\lambda - \alpha_- K(r) + \mu F(r))\langle\,,\,\rangle.$$

We then conclude the validity of (7.23). ☐

As an application of Proposition 7.20 we have

Proposition 7.26. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold with a gradient Einstein-type structure as in* (7.1) *for some $f \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\mu, \lambda \in \mathbb{R}$. In case $\alpha < 0$ assume*

$$(|d\varphi|^2)^* := \sup_M |d\varphi|^2 < +\infty$$

and in case $\alpha, \mu < 0$ assume

$$(|\nabla f|^2)^* := \sup_M |\nabla f|^2 < +\infty.$$

Then

$$Ric_f^{\frac{1}{\mu^+}} \geq -\left(\left(\frac{1}{\mu}\right)_+ + m - 1\right)\Sigma\langle\,,\,\rangle \tag{7.27}$$

with

$$\Sigma := -\frac{\lambda - \alpha_-(|d\varphi|^2)^* - \mu_-(|\nabla f|^2)^*}{\left(\frac{1}{\mu}\right)_+ + m - 1} \in \mathbb{R}, \tag{7.28}$$

where we are using the convention $(+\infty)_- = 0$. In particular

- *i) The weak maximum principle at infinity for Δ_f and the L^1-Liouville property for Δ_f-subharmonic functions hold;*
- *ii) In case $\mu > 0$, if $\alpha, \lambda > 0$ or if $\alpha < 0$ and $\lambda > |\alpha|(|d\varphi|^2)^*$ then M is compact.*
- *iii) In case $\mu = 0$, if $\alpha, \lambda > 0$ or if $\alpha < 0$ and $\lambda > |\alpha|(|d\varphi|^2)^*$ then $(M, \langle\,,\,\rangle)$ is parabolic with respect to Δ_f.*
- *iv) In case $\mu < 0$, if $\alpha > 0$ and $\lambda > |\mu|(|\nabla f|^2)^*$ or if $\alpha < 0$ and $\lambda > |\mu|(|\nabla f|^2)^* + |\alpha|(|d\varphi|^2)^*$ then $(M, \langle\,,\,\rangle)$ is Δ_f-parabolic.*

Proof. The validity of (7.27) with Σ given by (7.28) follows immediately from Proposition 7.20 by choosing, in case $\alpha < 0$, $K \equiv (|d\varphi|^2)^*$ and, in case $\mu < 0$, $F \equiv (|\nabla f|^2)^*$. Then *i)* follows from Proposition 7.18, *ii)* from Theorem 5 of [40] and finally *iii)* and *iv)* follows from Proposition 7.15. ☐

Now we give, as a consequence of (4.20), a two-sided estimate on $S^{\varphi}_* := \inf_M S^{\varphi}$. Precisely we prove

Theorem 7.29. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold of dimension m with a gradient Einstein-type structure as in (7.1) with $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu, \lambda \in \mathbb{R}$, $f \in \mathcal{C}^{\infty}(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ a smooth map. Assume $\alpha > 0$ and $0 < \mu \leq 1$. If $\lambda \leq 0$ assume that $f_* > -\infty$ or that the smallest eigenvalue of Hess(f) is bounded from below.*

i) If $\lambda > 0$ then M is compact and

$$\frac{(m-1)\mu}{1+(m-1)\mu} m\lambda \leq S^{\varphi}_* \leq m\lambda.$$

If $\mu \neq 1$, then

$$\frac{(m-1)\mu}{1+(m-1)\mu} m\lambda < S^{\varphi}_* \leq m\lambda.$$

Furthermore, $S^{\varphi}_* = m\lambda$, that is, $S^{\varphi}(x_0) = m\lambda$ for some $x_0 \in M$, if and only $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein and f is constant.

ii) If $\lambda = 0$ then

$$S^{\varphi}_* = 0.$$

Moreover, if $\mu \neq 1$, either $S^{\varphi} > 0$ on M or otherwise $(M, \langle\, ,\, \rangle)$ is harmonic Einstein with $S^{\varphi} \equiv 0$ and f is constant.

iii) If $\lambda < 0$ then

$$m\lambda \leq S^{\varphi}_* \leq \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda.$$

If $\mu \neq 1$, then $S^{\varphi}(x_0) = m\lambda$ for some $x_0 \in M$ if and only $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein and f is constant.

Proof. Since λ is constant, equation (4.20) can be written in the form

$$\frac{1}{2}\Delta_{(1+2\mu)f} S^{\varphi} = (\mu-1)(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2) - \frac{(m-1)\mu+1}{m}(S^{\varphi} - m\lambda)\left(S^{\varphi} - \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda\right). \tag{7.30}$$

We set $u := -S^{\varphi}$ so that (7.30) takes the form

$$\frac{1}{2}\Delta_{(1+2\mu)f} u = (1-\mu)(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2) + \frac{(m-1)\mu+1}{m}(u + m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda\right). \tag{7.31}$$

Since $\mu \leq 1$ we deduce

$$\frac{1}{2}\Delta_{(1+2\mu)f} u \geq \frac{(m-1)\mu+1}{m}(u + m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda\right)$$

on M. We now set

$$g := (1+2\mu)f$$

so that

$$\frac{1}{2}\Delta_g u \geq \frac{(m-1)\mu+1}{m}(u + m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda\right), \tag{7.32}$$

or equivalently, in terms of S^{φ},

$$\frac{1}{2}\Delta_g S^{\varphi} \leq -\frac{(m-1)\mu+1}{m}(S^{\varphi} - m\lambda)\left(S^{\varphi} - \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda\right). \tag{7.33}$$

i) If $\lambda > 0$ then, from Proposition 7.26 ii), M is compact and since $S^\varphi_* = S^\varphi(x_0)$ for some $x_0 \in M$, from (7.33) we deduce

$$\frac{(m-1)\mu}{1+(m-1)} m\lambda \le S^\varphi_* \le m\lambda.$$

We now show that the first inequality is strict if $\mu \neq 1$. Indeed, by contradiction suppose $S^\varphi_* = \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda$. Because of (7.33) the non-negative function

$$v := S^\varphi - \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda$$

satisfies

$$\frac{1}{2}\Delta_g v \le -\frac{(m-1)\mu+1}{m}\left(v - \frac{1}{1+(m-1)\mu} m\lambda\right) v = -\frac{(m-1)\mu+1}{m} v^2 + \lambda v \le \lambda v.$$

Since M is compact v attains its minimum and from the minimum principle, see page 35 of [20], we deduce that v vanishes identically. Hence

$$S^\varphi \equiv \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda. \tag{7.34}$$

From (7.30) by integration we then deduce

$$(1-\mu)(\alpha|\tau(\varphi)|^2 + |T^\varphi|^2) = 0,$$

so that, since $\mu < 1$ and $\alpha > 0$, $(M, \langle\, ,\, \rangle)$ is a harmonic-Einstein manifold. From the first equation of (7.1) we infer

$$\frac{S^\varphi}{m}\langle\, ,\, \rangle + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\, ,\, \rangle,$$

that implies, using (7.34),

$$\mathrm{Hess}(f) - \mu df \otimes df = \frac{\lambda}{1+(m-1)\mu}\langle\, ,\, \rangle > 0.$$

Since M is compact this gives a contradiction in the point of absolute maximum of f.

Suppose now that $S^\varphi_* = m\lambda$. Then

$$S^\varphi \ge S^\varphi_* = m\lambda \ge \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda,$$

hence from (7.33) we deduce

$$\frac{1}{2}\Delta_g S^\varphi \le 0.$$

Since M is compact we infer that $S^\varphi \equiv S^\varphi_*$. Once again from (7.30) we obtain that $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein and from the first equation of (7.1) we have

$$\mathrm{Hess}(f) - \mu df \otimes df = 0. \tag{7.35}$$

Tracing (7.35) gives $\Delta f = \mu|\nabla f|^2 \ge 0$ and since M is compact and $\mu > 0$, f must be constant.

If $\lambda \le 0$ we show that the weak maximum principle hold for Δ_g if $f_* > -\infty$ or if the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below. Suppose $f_* > -\infty$, then $\mathrm{vol}_g(B_r) \le e^{-2\mu f_*}\mathrm{vol}_f(B_r)$ and thus

$$\frac{r}{\log \mathrm{vol}_f B_r} \notin L^1(+\infty),$$

because of equation (7.1) and the estimate (7.14) valid with $v = f$ and $\Sigma = 0$. From Proposition 3.17 of [36] we obtain the validity of the weak maximum principle for Δ_g. Now suppose that the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below. The first equation of (7.1) can be written in terms of g as

$$\mathrm{Ric} + \mathrm{Hess}(g) - \frac{\mu}{(1+2\mu)^2} dg \otimes dg = \lambda\langle\, ,\, \rangle + 2\mu\mathrm{Hess}(f) + \alpha\varphi^*\langle\, ,\, \rangle_N,$$

so that, using $\alpha, \mu > 0$ and since the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below,

$$\mathrm{Ric}_g^\gamma \geq \lambda\langle\, ,\, \rangle, \quad \text{with} \quad \gamma := \frac{(1+2\mu)^2}{\mu} > 0.$$

Then, from Proposition 7.26, the weak maximum principle for Δ_g also holds in this case. Using Theorem 4.2 of [1], since $\lambda \leq 0$, from (7.32) we deduce for $u^* := \sup_M u$,

$$-\frac{(m-1)\mu}{1+(m-1)\mu} m\lambda \leq u^* \leq -m\lambda$$

of course when $u^* < +\infty$. But this is the case because of Theorem 4.1 of [1], (7.32) and the conditions on the parameters. From the above we immediately infer

$$m\lambda \leq S_*^\varphi \leq \frac{(m-1)\mu}{1+(m-1)\mu} m\lambda.$$

ii) Let $\lambda = 0$, the bounds on S_*^φ gives $S_*^\varphi = 0$. In this case (7.33) gives $\Delta_g S^\varphi \leq 0$ so that either $S^\varphi > 0$ on M or $S^\varphi \equiv 0$. In the latter case, if $\mu \neq 1$, from (7.30), we obtain that $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein and thus, from the first equation of (7.1), once again we deduce (7.35). Then or f is constant or otherwise the positive function $u := e^{-\mu f}$ satisfies $\mathrm{Hess}(u) = 0$. This is not possible since by a Cheeger-Gromoll type argument, see for instance case $i1$) in the proof of Proposition 8.13 of [1], there are no positive non-constant affine function.

iii) Let $\lambda < 0$. The estimates on S_*^φ have been obtained above. If $S^\varphi(x_0) = m\lambda$ for some $x_0 \in M$ then, from (7.33), the non-negative function $v := S^\varphi - m\lambda$ satifies,

$$\frac{1}{2}\Delta_g v \leq -\frac{(m-1)\mu+1}{m} v \left(v + \frac{1}{1+(m-1)\mu} m\lambda\right) = -\frac{(m-1)\mu+1}{m} v^2 - \lambda v \leq -\lambda v,$$

that is,

$$\Delta_g v + 2\lambda v \leq 0,$$

so that, since v attains its minimum, from the minimum principle $v \equiv 0$. Then $S^\varphi \equiv m\lambda$ and then, as before, in case $\mu \neq 1$ from (7.30) we infer that $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. From (7.1) we deduce that satisfies (7.35). As in ii) above, we deduce that f is constant. □

Note that in case $\mu = 0$ and φ is constant (7.1) yields the Ricci soliton system

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda\langle\, ,\, \rangle. \tag{7.36}$$

In this situation we have the well known identity due to Hamilton,

$$\nabla S = 2\mathrm{Ric}(\nabla f, \cdot)^\sharp. \tag{7.37}$$

The latter, in turns, gives rise to the celebrated Hamilton identity

$$S + |\nabla f|^2 - 2\lambda f = \Lambda, \tag{7.38}$$

for some constant $\Lambda \in \mathbb{R}$. Note that in case $\lambda \neq 0$ one can add a constant to f to obtain $\Lambda = 0$. We shall generalize (7.37) and (7.38) to the Einstein-type structure (7.1). The equation corresponding to (7.37), with λ non-constant, is given in a local orthonormal coframe by (4.19), which we report here for the sake of convenience

$$\frac{1}{2}S^{\varphi}_j = R^{\varphi}_{kj}f_k - \mu\Delta f f_j + \mu f_k f_{kj} + (m-1)\lambda_j. \tag{7.39}$$

Observe that for $\mu = 0$ and for λ and φ constants (7.39) reduces to (7.37). Next we extend (7.38) in the following

Proposition 7.40. *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold with an Einstein-type structure as in (7.1) with λ constant. Then there exists $\Lambda \in \mathbb{R}$ such that, if $\mu \neq 0$:*

$$S^{\varphi} - (\mu - 1)|\nabla f|^2 + \left(\frac{1}{\mu} - m\right)\lambda = \frac{\Lambda}{\mu}e^{2\mu f}, \tag{7.41}$$

and if $\mu = 0$:

$$S^{\varphi} + |\nabla f|^2 - 2\lambda f = m\lambda - \Lambda. \tag{7.42}$$

As a consequence we have the validity of the following equations, if $\mu \neq 0$:

$$\Delta_f f = \frac{\lambda}{\mu} - \frac{\Lambda}{\mu}e^{2\mu f}, \tag{7.43}$$

and if $\mu = 0$:

$$\Delta_f f = \Lambda - 2\lambda f. \tag{7.44}$$

Remark 7.45. Observe that in (7.43) and (7.44) the map $\varphi : M \to (N, \langle\, , \rangle_N)$ and the constant α of Ric^{φ} do not appear. This observation enables us to extends many results on quasi-Einstein manifolds to our more general structure.

Proof. We claim the validity of the following equation

$$(\Delta_f f + (m-2)\lambda)_j - 2f_j(\mu\Delta_f f - \lambda) = 0. \tag{7.46}$$

Towards this aim we trace the first equation of (7.1) to obtain

$$m\lambda = S^{\varphi} + \Delta f - \mu|\nabla f|^2. \tag{7.47}$$

Taking the covariant derivative and inserting into (7.39) we deduce

$$\frac{1}{2}S^{\varphi}_j = R^{\varphi}_{kj}f_k - \mu\Delta f f_j + \mu f_k f_{kj} - \lambda_j + (S^{\varphi} + \Delta f - \mu|\nabla f|^2)_j,$$

that is,

$$\frac{1}{2}S^{\varphi}_j + (\Delta f)_j + R^{\varphi}_{ij}f_i = \mu\Delta f f_j + \mu f_{ij}f_i + \lambda_j.$$

From the first equation of (7.1) we infer

$$R^{\varphi}_{ij}f_i + f_{ij}f_i = \mu|\nabla f|^2 f_j + \lambda f_j,$$

and replacing into the above yields

$$\frac{1}{2}S^{\varphi}_j + (\Delta f)_j - f_{ij}f_i + \mu|\nabla f|^2 f_j + \lambda f_j = \mu\Delta f f_j + \mu f_{ij}f_i + \lambda_j,$$

that is,

$$S^{\varphi}_j = -2(\Delta f)_j + 2(1+\mu)f_{ij}f_i - 2\mu|\nabla f|^2 f_j - 2\lambda f_j + 2\mu\Delta f f_j + 2\lambda_j. \tag{7.48}$$

The covariant derivative of (7.47) yields

$$S^{\varphi}_j + (\Delta f)_j = 2\mu f_i f_{ij} + m\lambda_j, \tag{7.49}$$

and by inserting (7.48) into (7.49) we obtain

$$-2(\Delta f)_j + 2(1+\mu) f_{ij} f_i - 2\mu|\nabla f|^2 f_j - 2\lambda f_j + 2\mu \Delta f f_j + 2\lambda_j + (\Delta f)_j = 2\mu f_i f_{ij} + m\lambda_j,$$

that implies (7.46). Now, assuming λ constant (7.46) can be rewritten as

$$(\Delta_f f)_j - 2f_j(\mu\Delta_f f - \lambda) = 0. \tag{7.50}$$

If $\mu \neq 0$ from (7.50) we deduce

$$\left(\Delta_f f - \frac{\lambda}{\mu}\right)_j - 2\mu f_j \left(\Delta_f f - \frac{\lambda}{\mu}\right) = 0.$$

It follows that the function

$$v := \left(\Delta_f f - \frac{\lambda}{\mu}\right) e^{-2\mu f}$$

is a constant, say $-\frac{\Lambda}{\mu}$, on M. Indeed,

$$v_j = \left[\left(\Delta_f f - \frac{\lambda}{\mu}\right)_j - 2\mu f_j \left(\Delta_f f - \frac{\lambda}{\mu}\right)\right] e^{-2\mu f} = 0.$$

Observe that since $v = -\frac{\Lambda}{\mu}$ we have the validity of (7.43). To deduce (7.41) it is sufficient to use (7.47) in the form

$$\Delta_f f = -S^{\varphi} + (\mu - 1)|\nabla f|^2 + m\lambda \tag{7.51}$$

so that inserting it in the definition of v we have

$$-\frac{\Lambda}{\mu} = v = \left(-S^{\varphi} + (\mu-1)|\nabla f|^2 + m\lambda - \frac{\lambda}{\mu}\right) e^{-2\mu f}.$$

If $\mu = 0$, (7.50) becomes

$$(\Delta_f f)_j + 2\lambda f_j = 0,$$

and thus, since λ is constant,

$$(\Delta_f f + 2\lambda f)_j = 0.$$

Then the function

$$v := \Delta_f f + 2\lambda f$$

is constant on M. Choosing Λ such that $v = \Lambda + m\lambda$ we obtain (7.44). Using (7.51) in the above gives

$$\Lambda + m\lambda = -S^{\varphi} - |\nabla f|^2 + m\lambda + 2\lambda f,$$

that is, (7.42). □

Remark 7.52. In the above proof we use the first equation of (7.1) except in only one point, precisely when we use (7.39) at the very beginning of the argument.

Remark 7.53. It is worth to observe that when $m \geq 3$ and

$$\mu = -\frac{1}{m-2}, \tag{7.54}$$

or equivalently when $(M, \langle\,,\,\rangle)$ is conformally harmonic-Einstein, equations (7.41) and (7.43) holds for $\lambda \in \mathcal{C}^\infty(M)$, see Theorem 2.49. This can also be seen directly, in fact from the proof of the Proposition above, in case (7.54) holds, equation (7.46) becomes

$$(\Delta_f f + (m-2)\lambda)_j - 2f_j\left(-\frac{1}{m-2}\Delta_f f - \lambda\right) = 0,$$

that is,

$$(\Delta_f f + (m-2)\lambda)_j + \frac{2}{m-2}f_j(\Delta_f f + (m-2)\lambda) = 0.$$

Then, setting

$$v := (\Delta_f f + (m-2)\lambda)e^{\frac{2}{m-2}f},$$

it is easy to see that v is constant on M immediately obtaining the validity of (7.41) and (7.43), without assuming constancy of λ .

In the proof of the Proposition above, in case $\mu \neq 0$ we face the problem of the choice of the constant Λ. Considering the case $\mu = \frac{1}{d}$ for some positive integer d, it is possible to prove (for a proof see the doctoral thesis of A. Anselli)

Theorem 7.55. *Let $(M, \langle\,,\,\rangle)$ and $(F, \langle\,,\,\rangle_F)$ be Riemannian manifolds of dimension m and d respectively. Let $f \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$ be a smooth map. Denote by $\bar{M}$ the warped product $M \times_u F$, where $u = e^{-\frac{f}{d}}$, and let $\Phi := \varphi \circ \pi_M : \bar{M} \to (N, \langle\,,\,\rangle_N)$. Then $(\bar{M}, \overline{\langle\,,\,\rangle})$ satisfies,*

$$\begin{cases} \overline{Ric} - \alpha\Phi^*\langle\,,\,\rangle_N = \lambda\bar{g} \\ \tau(\Phi) = 0, \end{cases} \tag{7.56}$$

for some constant λ, if and only if $(M, \langle\,,\,\rangle)$ satisfies

$$\begin{cases} Ric - \alpha\varphi^*\langle\,,\,\rangle_N + Hess(f) - \frac{1}{d}df \otimes df = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases}$$

and $(F, \langle\,,\,\rangle_F)$ satisfies

$${}^F Ric = \Lambda\langle\,,\,\rangle_F, \tag{7.57}$$

where Λ is the constant given by

$$\Delta_f f = d\lambda - d\Lambda e^{2\mu f}, \tag{7.58}$$

Note that (7.58) is exactly (7.43) with $\mu = \frac{1}{d}$. Observe moreover that 4-dimensional Lorentzian manifold satisfying (7.56) are natural examples of static spacetime satisfying the Einstein equation with energy momentum tensor given by the energy-stress tensor of the wave map (harmonic map) Φ and vanishing cosmological constant. See the doctoral thesis of A. Anselli for more details.

We now provide some triviality results for gradient Einstein-type structure with potential function f satisfying $|\nabla(e^{-\frac{f}{p}})| \in L^p(M)$ for some $1 < p < +\infty$. To prove the next Proposition we shall use

Theorem 7.59 (Theorem 1.1 of [37])**.** *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold and let $f \in \mathcal{C}^\infty(M)$. Assume that $u \in Lip_{loc}(M)$ satisfy*

$$u\Delta_f u \geq 0 \quad \text{weakly on } M. \tag{7.60}$$

If, for some $p \in (1, +\infty)$,

$$\left(\int_{\partial B_r} |u|^p e^{-f}\right)^{-1} \notin L^1(+\infty), \tag{7.61}$$

then u is constant.

Proposition 7.62. *Let $(M, \langle\, ,\, \rangle)$ be a complete, non compact Riemannian manifold of dimension m with a gradient Einstein-type structure as in* (7.1) *with $\alpha \in \mathbb{R}\setminus\{0\}$, $\mu, \lambda \in \mathbb{R}$, $f \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ a smooth map. Suppose $|\nabla(e^{-\frac{f}{p}})| \in L^p(M)$, or equivalently $|\nabla f| \in L^p(M, e^{-f})$, for some $p \in (1, +\infty)$, $\alpha > 0$ and that one the following conditions is satisfied*

i) $\mu > \frac{1}{2}$, $S^\varphi \le \left(m - \frac{1}{2\mu}\right)\lambda$ *on* M*;*

ii) $\mu = \frac{1}{2}$, $S^\varphi < (m-1)\lambda$ *on* M*;*

iii) $\mu = 0$, $\lambda < 0$*;*

iv) $\mu < 0$, $S^\varphi \ge \left(m - \frac{1}{2\mu}\right)\lambda$ *on* M*.*

Then f is constant and $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein.

Proof. Since $\lambda \in \mathbb{R}$ equation (4.22) becomes

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + (2\mu\lambda m - \lambda - 2\mu S^\varphi)|\nabla f|^2 + \mu(2\mu - 1)|\nabla f|^4. \tag{7.63}$$

Recall that, from Kato's inequality,

$$|\nabla|\nabla f||^2 \le |\mathrm{Hess}(f)|^2 \quad \text{weakly on } M.$$

Then we infer

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\nabla f|\Delta_f|\nabla f| + |\nabla|\nabla f||^2 \le |\nabla f|\Delta_f|\nabla f| + |\mathrm{Hess}(f)|^2 \quad \text{weakly on } M.$$

Combining the above with (7.63) and using $\alpha > 0$, we obtain

$$|\nabla f|\Delta_f|\nabla f| \ge (2\mu m\lambda - 2\mu S^\varphi - \lambda)|\nabla f|^2 + \mu(2\mu - 1)|\nabla f|^4 \quad \text{weakly on } M.$$

From the above, if anyone of $i)$, $ii)$, $iii)$ or $iv)$ holds then it is easy to show the validity of one of the following inequalities for some positive constant c,

$$|\nabla f|\Delta_f|\nabla f| \ge c|\nabla f|^2 \quad \text{or} \quad |\nabla f|\Delta_f|\nabla f| \ge c|\nabla f|^4 \quad \text{weakly on } M, \tag{7.64}$$

Then we are in position to apply Theorem 7.59 with the choice of $u = |\nabla f|$, observing that $|\nabla f| \in L^p(M, e^{-f})$ guarantee the validity of (7.61) and that (7.60) holds. We then conclude that $|\nabla f|$ is constant and therefore from (7.64) f is constant. As a consequence of (7.63) we deduce

$$\alpha|\tau(\varphi)|^2 = 0$$

and $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. □

Remark 7.65. Consider the assumptions of Proposition 7.62 but instead of one of $i), ii), iii)$ or $iv)$, assume now

v) $\mu > 0$, $\lambda < 0$, $\Lambda < 0$ and

$$f_* \ge \frac{1}{2\mu}\log\left(\frac{\lambda}{2\Lambda}\right),$$

where $f_* := \inf_M f$ and Λ is the constant appearing in (7.43).

Rewrite (7.63) using the trace of the first equation in (7.1), as

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + |\nabla f|^2(2\mu\Delta f - \lambda - \mu|\nabla f|^2),$$

or equivalently,

$$\frac{1}{2}\Delta_f|\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + |\nabla f|^2(2\mu\Delta_f f + \mu|\nabla f|^2 - \lambda).$$

Therefore

$$|\nabla f|\Delta_f|\nabla f| \geq |\nabla f|^2(2\mu\Delta_f f + \mu|\nabla f|^2 - \lambda) \quad \text{weakly on } M. \tag{7.66}$$

We use (7.43) to obtain, from (7.66),

$$|\nabla f|\Delta_f|\nabla f| \geq (\lambda - 2\Lambda e^{2\mu f} + \mu|\nabla f|^2)|\nabla f|^2 \quad \text{weakly on } M.$$

The hypothesis on λ, Λ and f_* guarantee the validity of

$$\lambda - 2\Lambda e^{2\mu f} \geq 0,$$

hence from the above we get

$$|\nabla f|\Delta_f|\nabla f| \geq \mu|\nabla f|^4 \quad \text{weakly on } M,$$

and thus we can conclude, as in the proof of Proposition 7.62 that f is constant and $\tau(\varphi) \equiv 0$.

Proposition 7.40 motivates the study of non-existence results or triviality results on a Riemannian manifold $(M, \langle\, ,\, \rangle)$ for solutions of differential inequalities of the form

$$\Delta_v v \geq \rho + \beta e^{2\delta v}, \tag{7.67}$$

for some constants $\rho, \beta, \delta \in \mathbb{R}$, possibly coupled with a system of the type

$$\mathrm{Ric}_v^\gamma \geq -(\gamma + m - 1)G(r)\langle\, ,\, \rangle \tag{7.68}$$

for some $\gamma \in \mathbb{R}^+$ and some function $G : \mathbb{R}_0^+ \to \mathbb{R}_0^+$. Here a first result in this direction.

Theorem 7.69. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold of dimension m and let $G \in \mathcal{C}^1(\mathbb{R}_0^+)$ be a non-decreasing function with $G(0) > 0$ and $\frac{1}{\sqrt{G}} \notin L^1(+\infty)$. Assume $\gamma, \beta, \delta > 0$. If $\rho \geq 0$ there are no solutions v of* (7.67) *satisfying* (7.68)*, while if $\rho < 0$ there are no solutions of* (7.67) *satisfying both* (7.68) *and*

$$v^* := \sup_M v > \frac{1}{2\delta}\log\left(-\frac{\rho}{\beta}\right).$$

Proof. We first deal with the non-compact case. We claim the hypothesis on G imply the validity of the Omori-Yau maximum principle for the operator Δ_v. To prove this we first observe that, as reported at the beginning of this section, under the milder hypotheses that $G \in \mathcal{C}^1(\mathbb{R}_0^+)$, $G > 0$ on $\mathbb{R}_0^+$ and

$$\inf_{\mathbb{R}_0^+} \frac{G'}{G^{\frac{3}{2}}} > -\infty,$$

choosing g as in (7.7), g satisfies (7.4) and then, using Proposition 2.3 of [28], we deduce

$$\Delta_v r(x) \leq C\sqrt{G(r(x))} \quad \text{ for } r(x) >> 1$$

for some positive constant C large enough. Moreover, since G is non-decreasing,

$$|\nabla r| = 1 \leq C\sqrt{G(r)},$$

again for some constant $C > 0$. Then an application of Theorem 3.2 and Remark 3.3 of [1] with the choices (in the notation of [1]) $\gamma(x) = r(x)$ and $G(\gamma)$ given by $\sqrt{G(r)}$, the present G, gives the validity of the claim.

Let now $\beta, \delta > 0$. Choosing $f(t) = \rho + \beta e^{2\delta t}$ and $F = f$ on $[a, +\infty)$, with $a = 0$ in case $\rho \geq 0$ and $a = \frac{1}{2\delta}\log\left(-\frac{\rho}{\beta}\right)$ otherwise, we can apply Theorem 3.6 of [1] to deduce $v^* < +\infty$ and

$$\rho + \beta e^{2\delta v^*} \leq 0.$$

Thus, for $\rho < 0$ we infer

$$v^* \leq \frac{1}{2\delta}\log\left(-\frac{\rho}{\beta}\right)$$

while for $\rho \geq 0$

$$0 < e^{2\delta v^*} \leq -\frac{\rho}{\beta} \leq 0$$

yielding a contradiction so that, in this case, v cannot exists.

The compact case, since v has to attain a maximum v^* on M, follows immediately by the above reasoning. □

Corollary 7.70. *Let $(M, \langle\, , \rangle)$ be a complete manifold supporting a gradient Einstein-type structure as in (7.1) with λ a non-negative constant, $\mu > 0$, $\alpha \geq 0$. Then f satisfies (7.43) with $\Lambda \geq 0$.*

Proof. Since $\alpha \geq 0$ from the first in (7.1) we have $\mathrm{Ric}_f^{1/\mu} \geq 0$. Proceeding as in the proof of Theorem 7.69, $f^* < +\infty$ and

$$\frac{\lambda}{\mu} - \frac{\Lambda}{\mu} e^{2\mu f^*} \leq 0.$$

This yields a contradiction in case $\Lambda < 0$. □

We observe that J. Case, see [12], proves non-existence of non-constant solutions of the equation

$$\Delta_v v = \beta e^{2\delta v}, \tag{7.71}$$

on a complete Riemannian manifold $(M, \langle\, , \rangle)$ for $\beta, \delta \geq 0$ and under the assumption

$$\mathrm{Ric}_v^\gamma \geq 0,$$

for some $\gamma > 0$ (as a matter of fact he also considers the case $\gamma = +\infty$, but this case can be dealt similarly and we skip it for the sake of brevity). His proof is based on a conformal change of metric together with a gradient estimate (note that for the latter one needs to consider an equation as in (7.71) instead of a differential inequality). Our Theorem 7.69 recover Case's result when $\beta, \delta > 0$ and $\rho = 0$ even in case of the differential inequality (7.67). However, we can obtain and in fact extend his full result for (7.67) with the equality sign with the aid of the following trick.

Consider on M the equation

$$\Delta v = |\nabla v|^2 + \rho + \beta e^{2\delta v}, \tag{7.72}$$

and suppose that

$$\mathrm{Ric}_v^\gamma \geq -(\gamma + m - 1)\lambda\langle\, , \rangle, \tag{7.73}$$

for some constant $\lambda > 0$. Referring to (7.3) with $G = \lambda$ we find a solution g of (7.4) that for $t >> 1$ is given by

$$g(t) = e^{\sqrt{\lambda} t}.$$

In this case, as $r \to +\infty$

$$\mathrm{vol}_v(B_r) \leq D + E e^{(\gamma + m - 1)\sqrt{\lambda} r} \tag{7.74}$$

for some constants $D, E > 0$. Next, we recall Bochner's formula

$$\frac{1}{2}\Delta|\nabla v|^2 = |\mathrm{Hess}(v)|^2 + \mathrm{Ric}(\nabla v, \nabla v) + \langle \nabla \Delta v, \nabla v\rangle. \tag{7.75}$$

Using equation (7.72) we obtain

$$\langle \nabla \Delta v, \nabla v\rangle = \langle \nabla v, \nabla |\nabla v|^2\rangle + 2\delta\beta e^{2\delta v}|\nabla v|^2 = 2\mathrm{Hess}(v)(\nabla v, \nabla v) + 2\delta\beta e^{2\delta v}|\nabla v|^2.$$

We insert the above into (7.75) and we use (7.73) to finally get

$$\frac{1}{2}\Delta_v|\nabla v|^2 \geq |\mathrm{Hess}(v)|^2 + [2\delta\beta e^{2\delta v} - (\gamma + m - 1)\lambda]|\nabla v|^2 + \frac{1}{\gamma}|\nabla v|^4. \tag{7.76}$$

We are now ready to prove the following

Theorem 7.77. *Let $(M, \langle\, ,\, \rangle)$ be a complete, possibly compact Riemannian manifold. Let $v \in \mathcal{C}^2(M)$ and suppose that for some $\gamma > 0$ the modified Bakry-Emery Ricci tensor satisfy*

$$Ric_v^\gamma \geq 0. \tag{7.78}$$

Let the product $\delta\beta \geq 0$ and $\rho \in \mathbb{R}$. If v is a solution of (7.72) then v is constant.

Proof. First we analyse the non-compact case. Fix any $\lambda > 0$, then (7.78) implies the validity of (7.73). Next set $u := |\nabla v|^2$. Then, since by hypothesis $\delta\beta \geq 0$, from (7.76) we deduce the validity of the following differential inequality

$$\frac{1}{2}\Delta_v u \geq -(\gamma + m - 1)\lambda u + \frac{1}{\gamma}u^2.$$

From (7.74) we infer

$$\lim_{r\to+\infty} \frac{\log \mathrm{vol}_v(B_r)}{r^2} = 0.$$

Applying Theorem 4.2 of [1] we deduce

$$u^* := \sup_M u < +\infty.$$

Then Theorem 4.1 of [1] yields the validity of the inequality

$$u^*\left[\frac{1}{\gamma}u^* - (\gamma + m - 1)\lambda\right] \leq 0,$$

so that

$$0 \leq u^* \leq \gamma(\gamma + m - 1)\lambda.$$

Since $\lambda > 0$ was arbitrary we infer $u^* = 0$ completing the proof.

From the above reasoning we see that the compact case is immediate. □

Now we give a consequence of Theorem 7.77.

Corollary 7.79. *Let $(M, \langle\, ,\, \rangle)$ and $(\mathbb{P}, \langle\, ,\, \rangle_{\mathbb{P}})$ be complete manifolds such that $M \times \mathbb{P}$ has an Einstein warped product structure of the type*

$$\overline{\langle\, ,\, \rangle} = \langle\, ,\, \rangle + e^{-\frac{2}{d}u}\langle\, ,\, \rangle_{\mathbb{P}},$$

where $u \in \mathcal{C}^\infty(M)$ and m, d are respectively the dimensions of M and $\mathbb{P}$. If $(M \times \mathbb{P}, \overline{\langle\, ,\, \rangle})$ has non-negative scalar curvature, then $(\mathbb{P}, \langle\, ,\, \rangle_{\mathbb{P}})$ is Einstein with non-negative scalar curvature.

Proof. Since $(M \times \mathbb{P}, \overline{\langle\,,\,\rangle})$ is Einstein with non-negative scalar curvature we have $\overline{\mathrm{Ric}} = \lambda\overline{\langle\,,\,\rangle}$ for some constant $\lambda \geq 0$. It is well known that $(\mathbb{P}, \langle\,,\,\rangle_{\mathbb{P}})$ must be Einstein, that is

$$\mathrm{Ric}_{\mathbb{P}} = \zeta\langle\,,\,\rangle_{\mathbb{P}},$$

for some $\zeta \in \mathbb{R}$ and also that the following system holds

$$\begin{cases} \mathrm{Ric}_u^d = \mathrm{Ric} + \mathrm{Hess}(u) - \dfrac{1}{d}du \otimes du = \lambda\langle\,,\,\rangle \\ \Delta_u u - \dfrac{d}{m+d} = -\zeta e^{\frac{2}{d}u}. \end{cases}$$

Thus

$$\mathrm{Ric}_u^d \geq 0 \quad \text{and} \quad \Delta_u u = \frac{d}{m+d} - \zeta e^{\frac{2}{d}u}.$$

It follows that if $\mathbb{P}$ has negative scalar curvature, that is, if $\zeta < 0$ Theorem 7.77 gives a contradiction. Indeed if u is constant then

$$\zeta = \frac{d}{m+d}e^{-\frac{2}{d}u} \geq 0. \qquad \square$$

Remark 7.80. The above Corollary gives a partial answer to a question posed by A. Besse [8].

Next we consider the differential inequality (7.67) not paired with (7.2). Note that (7.67) (that is justified by the geometric setting of equation (7.43)) immediately yields the validity of the differential inequality

$$\Delta v \geq \rho + \beta e^{2\delta v} \quad \text{on } M. \tag{7.81}$$

The advantage of (7.81) over (7.67) is that the former can be treated with the aid of the weak maximum principle for the Laplace Beltrami operator. More precisely with Theorem 4.2 of [1] we prove that a solution v of (7.81) satisfies $v^* < +\infty$ and then, with Theorem 4.1 of [1], we arrive to prove non-existence for $\beta, \delta > 0$ and $\rho \geq 0$.

Explicitly we have

Theorem 7.82. *Let $(M, \langle\,,\,\rangle)$ be a complete, possibly compact Riemannian manifold satisfying*

$$\liminf_{r\to+\infty} \frac{\log vol(B_r)}{r^2} < +\infty, \tag{7.83}$$

and let $\rho, \beta, \delta \in \mathbb{R}$ with $\beta, \delta > 0$. If $\rho \geq 0$ then (7.81) *has no solutions, while if $\rho < 0$ there are no solutions of* (7.81) *satisfying*

$$v^* := \sup_M v > \frac{1}{2\delta}\log\left(-\frac{\rho}{\beta}\right).$$

Remark 7.84. By using Theorem 3.7 and Theorem 3.8 of [3] we see that the conclusions of Theorem 7.82 remain valid when M has a non-empty boundary ∂M by adding to (7.81) the boundary condition:

$$\partial_\nu v \leq 0 \text{ on } \partial M,$$

where ν is the outward unit normal to ∂M. Observe that completeness of $(M, \langle\,,\,\rangle)$ in this case has to be intended in the Cauchy sense. The same observation applies to Theorem 7.88 below. Observe that operating the substitution

$$u := e^{-f},$$

in case $\mu = 0$ equation (7.44) becomes

$$\Delta u + \Lambda u + 2\lambda u \log u = 0, \tag{7.85}$$

while in case $\mu \neq 0$ equation (7.43) becomes

$$\Delta u + \frac{\lambda}{\mu}u - \Lambda u^{1-2\mu} = 0. \tag{7.86}$$

In case $m \geq 3$ and (7.54) holds, (7.85) and (7.86) are valid also in case λ is non-constant.

Now we analyse the differential inequality

$$\Delta u + \rho u + \beta u^{1-2\delta} \geq 0, \tag{7.87}$$

Recall that in the geometric case we have equalities in (7.86). Non-existence for (7.87) can be obtained with the parameters satisfying $\beta > 0, \delta < 0$ and $\rho \leq 0$. Indeed, with the techniques used above we prove the validity of the following Theorem.

Theorem 7.88. *Let $(M, \langle\, ,\, \rangle)$ be a complete, possibly compact Riemannian manifold satisfying* (7.83) *and let $\rho, \beta, \delta \in \mathbb{R}$ with $\beta > 0$ and $\delta < 0$. If $\rho \geq 0$ the differential inequality* (7.87) *has no solutions u satisfying*

$$u^* := \sup_M u > 0,$$

while if $\rho < 0$ there are no solutions u satisfying

$$u^* < \left(-\frac{\rho}{\beta}\right)^{\frac{1}{2\delta}}.$$

As a consequence of Theorems 7.82 and 7.88 we deduce the following

Corollary 7.89. *Let $(M, \langle\, ,\, \rangle)$ be a complete, possibly compact Riemannian manifold satisfying* (7.83). *Then $(M, \langle\, ,\, \rangle)$ has no Einstein-type structure as in* (7.1) *for some constants α, μ and λ with $\mu > 0$, $\lambda \geq 0$ and $\Lambda < 0$. Here Λ is the constant of Proposition 7.40.*

We conclude this section by considering the non-existence of solutions of (7.72) by means of the spectral properties of the operator $L := \Delta - 2\delta\rho$. We set

$$u := e^{2\delta v}$$

and we switch to the equation

$$\Delta u - 2\delta\rho u - 2\delta\beta u^2 = \left(1 + \frac{1}{2\delta}\right)\frac{|\nabla u|^2}{u},$$

equivalent to (7.72). Since $u > 0$ the above is, in turn, equivalent to

$$u\Delta u - 2\delta\rho u^2 - 2\delta\beta u^3 = \left(1 + \frac{1}{2\delta}\right)|\nabla u|^2. \tag{7.90}$$

We let $\lambda_1^L(M)$ to denote the spectral radius of L and we observe that, by Rayleigh variational characterization,

$$\lambda_1^L(M) = \inf_{\substack{\psi \in \mathcal{C}^\infty(M) \\ \psi \not\equiv 0}} \frac{\int_M (|\nabla\psi|^2 + 2\delta\rho\psi^2)}{\int_M \psi^2} = 2\delta\rho + \lambda_1^\Delta(M), \tag{7.91}$$

with the obvious meaning of the notation. We next recall Theorem 3.3 of [30] taking the opportunity to correct some typos there.

Theorem 7.92. *Let $(M, \langle\, ,\, \rangle)$ be a complete manifold, $a(x), b(x) \in \mathcal{C}^0(M)$ and suppose $b(x) \geq 0$. Let $H > 0$, $K > -1$ and $A \in \mathbb{R}$ be constants satisfying:*

$$A \leq H(K+1) - 1. \tag{7.93}$$

Assume that there exists $\psi \in \mathcal{C}^2(M)$, $\psi > 0$, solution of the differential inequality

$$\Delta\psi + Ha(x)\psi \leq -K\frac{|\nabla\psi|^2}{\psi} \quad \text{on } M.$$

Then for any constant $\sigma > -1$ *the differential inequality*

$$u\Delta u + a(x)u^2 - b(x)u^{\sigma+1} \geq -A|\nabla u|^2$$

has no non-negative solutions $u \in \mathcal{C}^2(M)$ *satisfying*

$$\operatorname{supp} u \cap \{x \in M : b(x) > 0\} \neq \varnothing,$$

and

$$\left(\int_{\partial B_r} \psi^{\frac{\tau+1}{H}(2-p)} u^{p(\tau+1)} \right)^{-1} \notin L^1(+\infty) \tag{7.94}$$

for some constants $p > 1$ *and* $\tau > -1$ *such that*

$$A \leq \tau \leq H(K+1) - 1.$$

We are now ready to prove

Theorem 7.95. *Let* $(M, \langle\, ,\, \rangle)$ *be a complete, non-compact manifold and* $\rho, \delta, \beta \in \mathbb{R}$ *constants such that*

$$\delta\beta > 0 \text{ and either } 2\delta \leq -1 \text{ or } 2\delta > 0.$$

Suppose that

$$\lambda_1^{\Delta}(M) \geq -2\delta\rho. \tag{7.96}$$

Then there exists no solution v *of equation* (7.72) *on* M *satisfying*

$$e^{\gamma v} \in L^1(M), \tag{7.97}$$

for some constant $\gamma > 0$ *with*

$$-\frac{1}{\delta} \leq \gamma \leq 2.$$

Proof. From (7.96) and (7.91) it follows that $\lambda_1^L(M) \geq 0$, where $L = \Delta - 2\delta\rho$ as above. By a result of Fischer-Colbrie and Schoen in [19] there exists a smooth solution $\psi > 0$ on M of

$$L\psi = 0.$$

Setting $u := e^{2\delta v}$ we have the validity of (7.90) and we apply Theorem 7.92 with the choices

$$a(x) = -2\delta\rho, \quad b(x) = 2\delta\beta > 0, \quad A = -\left(1 + \frac{1}{2\delta}\right), \quad K = 0, \quad H = 1, \quad p = 2, \quad \tau = \frac{\gamma}{2} - 1, \quad \sigma = 1.$$

The requests on δ and β in the statement show that the chosen parameter satisfy the required inequalities of Theorem 7.92 and that (7.97) implies the validity of the corresponding (7.94). Since $2\beta\delta > 0$ then $\{x \in M : b(x) > 0\} = M$, but $u > 0$ and thus the conclusion follows at once. □

Remark 7.98. We note that in case $2\delta \leq -1$ equation (7.90) yields

$$\Delta u \leq 2\delta\rho u. \tag{7.99}$$

By Barta's theorem

$$\lambda_1^{\Delta}(M) \geq \inf_M \left(-\frac{\Delta u}{u} \right) = -2\delta\rho,$$

so that, in this case, assumption (7.96) is automatically satisfied.

As a geometric application of Theorem 7.95, using Remark 7.98, we get

Corollary 7.100. *Let $(M, \langle\,,\,\rangle)$ be a complete, non-compact manifold. Then $(M, \langle\,,\,\rangle)$ has no Einstein-type structure as in* (7.1) *for some constants α, μ, λ in case $\Lambda < 0$, Λ the constant appearing in* (7.41)*, and either one of the following conditions hold*

i) μ satisfies

$$\mu \leq -\frac{1}{2}$$

and (7.97) *holds for some constant γ such that*

$$-\frac{1}{\mu} \leq \gamma \leq 2;$$

ii) $\mu > 0$, $\lambda_1^{\Delta}(M) \geq -2\lambda$ and (7.97) *holds for for some constant γ such that*

$$0 < \gamma \leq 2;$$

We end the section with the following

Proposition 7.101. *Let $(M, \langle\,,\,\rangle)$ be a complete, non-compact manifold satisfying*

$$vol(\partial B_r) \leq Ce^{ar}, \tag{7.102}$$

for some constants $C > 0$ and $a \geq 0$. Let $2\delta \leq -1$, $\beta \geq 0$ and suppose that

$$a^2 + 8\delta\rho < 0. \tag{7.103}$$

Then equation (7.72) *has no solutions on M.*

Proof. Let v be a solution of (7.72) so that $u := e^{2\delta v}$ is a solution of (7.90). The choice of the parameter δ yields the validity of (7.99) on M and therefore of (7.96). On the other hand by Theorem 6.8 of [10] and (7.102) we have

$$\lambda_1^{\Delta}(M) \leq \frac{a^2}{4}.$$

Putting together the latter and (7.96) we obtain

$$a^2 \geq -8\delta\rho,$$

contradicting (7.103). □

8 Some uniqueness results

We first prove a uniqueness result in the compact case for the equation

$$\Delta_v v = \rho + \beta e^{2\delta v}, \tag{8.1}$$

where $\rho, \beta, \delta \in \mathbb{R}$. It is clear that when $-\frac{\rho}{\beta} > 0$ the constant function

$$v := \log\left(-\frac{\rho}{\beta}\right)^{\frac{1}{2\delta}} \tag{8.2}$$

is a solution of (8.1).

Theorem 8.3. *Let $(M, \langle\, , \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ and let $v \in \mathcal{C}^2(M)$ be a solution of (8.1) on M for some constants ρ, δ and $\beta \neq 0$. Assume*

$$Ric \geq 2\frac{m-1}{m}\rho\delta\langle\, , \rangle \tag{8.4}$$

and if $m = 2$:

$$\delta < \frac{1}{2} \tag{8.5}$$

while if $m \geq 3$ either

$$i)\ \delta > \frac{1}{2} \quad or \quad ii)\ \delta < -\frac{2}{m-2}. \tag{8.6}$$

Then $-\frac{\rho}{\beta} > 0$ and v is given by (8.2).

Proof. We fix $\tau \in \mathbb{R} \setminus \{0\}$ and we perform the change of variable

$$u = e^{\frac{v}{\tau}}$$

to obtain from (8.1) the validity on M of

$$\Delta u = \zeta \frac{|\nabla u|^2}{u} + g(u), \tag{8.7}$$

with

$$\zeta := 1 + \tau, \quad g(u) := \frac{1}{\tau}(\rho u + \beta u^{1+2\tau\delta}). \tag{8.8}$$

For $\sigma \in \mathbb{R}$ we define the vector field

$$V = u^\sigma \left(\frac{1}{2}\nabla|\nabla u|^2 - \frac{\Delta u}{m}\nabla u\right) - \left(\frac{\sigma}{2} + \frac{m-1}{m}\zeta\right) u^{\sigma-1}|\nabla u|^2\nabla u.$$

After a long computation, using Bochner's formula and (8.7) we obtain

$$\begin{aligned}
\mathrm{div}V = {} & u^\sigma \left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m}\right] + u^\sigma \mathrm{Ric}(\nabla u, \nabla u) \\
& - \frac{u^{\sigma-2}}{2m}[2(m-1)\zeta^2 + 3m\sigma\zeta + m\sigma(\sigma-1)]|\nabla u|^4 \\
& - \frac{u^{\sigma-1}}{2m}\{[(m+2)\sigma + 2(m-1)\zeta]g(u) - 2(m-1)ug'(u)\}|\nabla u|^2.
\end{aligned}$$

Next we insert the expression of $g(u)$, $g'(u)$ and the value of ζ given by (8.8) into the above to obtain

$$\begin{aligned}
\mathrm{div}V = {} & u^\sigma \left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m}\right] - \frac{u^{\sigma-2}}{2m}A|\nabla u|^4 - \frac{u^{\sigma+2\tau\delta}}{2m}B|\nabla u|^2 \\
& + \frac{u^\sigma}{2m}[2m\mathrm{Ric}(\nabla u, \nabla u) + D|\nabla u|^2],
\end{aligned} \tag{8.9}$$

where the coefficients A, B, D are given by

$$\begin{cases}
A := m\sigma^2 + (3\tau + 2)m\sigma + 2(m-1)(1+\tau)^2 \\
B := [(m+2)\sigma + 2(m-1)\tau(1-2\delta)]\dfrac{\beta}{\tau} \\
D := -[2(m-1)\tau + (m+2)\sigma]\dfrac{\rho}{\tau}.
\end{cases}$$

We integrate (8.9) on M and using the divergence theorem we infer

$$\begin{aligned} 0 = &\int_M 2mu^\sigma \left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m}\right] - A\int_M u^{\sigma-2}|\nabla u|^4 - B\int_M u^{\sigma+2\tau\delta}|\nabla u|^2 \\ &+ \int_M u^\sigma[2m\mathrm{Ric}(\nabla u, \nabla u) + D|\nabla u|^2]. \end{aligned} \tag{8.10}$$

We need to find values of the parameters σ, τ such that $A \leq 0$, $B \leq 0$ and minimize D on these values to impose the condition on the Ricci curvature tensor to obtain

$$2m\mathrm{Ric}(\nabla u, \nabla u) + D|\nabla u|^2 \geq 0. \tag{8.11}$$

Towards this aim we let

$$y := 1 + \frac{1}{\tau}, \quad \gamma := -\frac{\sigma}{\tau}.$$

Note that $y \neq 1$ and it is well defined since $\tau \neq 0$. Rewriting A, B and D in terms of y and γ we see that

$$\begin{cases} A \leq 0 & \text{if and only if } \; 2\dfrac{m-1}{m}y^2 - 2\gamma y + \gamma^2 - \gamma \leq 0 \\ B \leq 0 & \text{if and only if } \; 2\beta\dfrac{m-1}{m+2}(1-2\delta) \leq \beta\gamma \end{cases} \tag{8.12}$$

and (8.11) is implied by

$$\frac{2m}{m+2}\mathrm{Ric} \geq \rho\left(2\frac{m-1}{m+2} - \gamma\right)\langle\,,\,\rangle. \tag{8.13}$$

Since

$$|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \geq 0$$

by Newton's inequality, to deduce from (8.10) that u (and therefore v) is constant it is enough to have one strict inequality in one of the two inequalities of (8.12). Next we choose

$$\gamma := 2\frac{m-1}{m+2}(1-2\delta). \tag{8.14}$$

With this choice the second inequality of (8.12) is always satisfied, independently of β, with the equality sign. Furthermore (8.13) becomes exactly assumption (8.4). With γ as in (8.14), $A < 0$ if and only if we can choose $y \neq 1$ such that

$$(m+2)y^2 - 2m(1+2\delta)y + \frac{m}{m+2}(1-2\delta)[2(m-1)(1-2\delta) - m - 2] < 0.$$

This is the case if the discriminant of polynomial expression in y is positive. Setting $x := 1-2\delta$ this amounts to show that

$$x[m+2-(m-2)x] > 0,$$

a fact guaranteed by the requirements in (8.5) if $m = 2$ and by (8.6) if $m \geq 3$. Since v is constant from (8.1) we obtain the conclusion of the Theorem. □

Going back to the geometric origins of equation (8.1), see equation (7.41) of Proposition 7.40, we deduce the next

Corollary 8.15. *Let $(M, \langle\,,\,\rangle)$ be a compact manifold of dimension $m \geq 2$ with a gradient Einstein type structure of the form*

$$\begin{cases} Ric^\varphi + Hess(f) - \mu df \otimes df = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases}$$

for some $\alpha, \mu, \lambda \in \mathbb{R}$, $\alpha \neq 0$. *Assume that* $\Lambda \neq 0$, *where* Λ *is the constant appearing on* (7.41), *that*

$$Ric \geq 2\frac{m-1}{m}\lambda\langle\, , \rangle \tag{8.16}$$

and that, for $m = 2$

$$\mu < \frac{1}{2}$$

while for $m \geq 3$ *either*

$$i)\ \mu < -\frac{2}{m-2} \qquad or \qquad ii)\ \mu > \frac{1}{2}.$$

Then f *is constant so that* $S^{\varphi} = m\lambda \neq 0$ *and*

$$\begin{cases} Ric^{\varphi} = \lambda\langle\, , \rangle \\ \tau(\varphi) = 0. \end{cases}$$

In particular $(M, \langle\, , \rangle)$ *is harmonic-Einstein.*

Next we prove a uniqueness result in the compact case for the equation

$$\Delta_v v = \beta - 2\lambda v, \tag{8.17}$$

where $\lambda, \beta \in \mathbb{R}$. It is clear that when $\lambda \neq 0$ the constant function

$$v := \frac{\beta}{2\lambda} \tag{8.18}$$

is a solution of (8.17).

Theorem 8.19. *Let* $(M, \langle\, , \rangle)$ *be a compact Riemannian manifold of dimension* $m \geq 2$ *and let* $v \in \mathcal{C}^2(M)$ *be a solution of* (8.17) *on* M *for some constants* λ *and* $\beta \neq 0$. *Assume that* (8.16) *holds. Then* $\lambda \neq 0$ *and* v *is given by* (8.18).

The proof of the Theorem is postponed. Before, we add a couple of observations.

Remark 8.20. The case $\Lambda = 0$ is simpler, with the only restriction $\mu \neq 0$. Indeed for $\Lambda = 0$, $mu \neq 0$, equation (7.43) becomes $\Delta_f f = \lambda/\mu$. Hence the function $u = e^{-f} > 0$ solves

$$\Delta u = -\frac{\lambda}{\mu}u.$$

Compactness of M implies that u and therefore f is constant. Since $u > 0$, we must have $\lambda = 0$ and

$$\begin{cases} \mathrm{Ric}^{\varphi} = 0 \\ \tau(\varphi) = 0. \end{cases}$$

In particular $(M, \langle\, , \rangle)$ is harmonic-Einstein.

Putting together Corollary 8.15 with Remark 8.20 we obtain a result independent of the constant Λ. Precisely we have

Corollary 8.21. *Let* $(M, \langle\, , \rangle)$ *be a compact manifold of dimension* $m \geq 3$ *with a gradient Einstein-type structure of the form*

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda\langle\, , \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases}$$

for some $\alpha, \mu, \lambda \in \mathbb{R}$, $\alpha \neq 0$. *Assume that*

$$\mathrm{Ric} \geq 2\frac{m-1}{m}\lambda\langle\,,\,\rangle$$

and

$$\mu > \frac{1}{2}.$$

Then f *is constant and* φ *is harmonic, so that*

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = 0. \end{cases}$$

In particular $(M, \langle\,,\,\rangle)$ *is harmonic-Einstein.*

of Theorem 8.19. We fix $\tau \in \mathbb{R} \setminus \{0\}$ and we perform the change of variable

$$u = e^{\frac{v}{\tau}}$$

to obtain from (8.17) the validity on M of

$$\Delta u = \zeta\frac{|\nabla u|^2}{u} + g(u), \tag{8.22}$$

with

$$\zeta := 1 + \tau, \quad g(u) := \frac{1}{\tau}(\beta u - 2\lambda\tau u \log u). \tag{8.23}$$

For $\sigma \in \mathbb{R}$ we define the vector field

$$V = u^{\sigma}\left(\frac{1}{2}\nabla|\nabla u|^2 - \frac{\Delta u}{m}\nabla u\right) - \left(\frac{\sigma}{2} + \frac{m-1}{m}\zeta\right)u^{\sigma-1}|\nabla u|^2\nabla u.$$

As in the proof of Theorem 8.3, using Bochner's formula and (8.22), we obtain

$$\begin{aligned} \mathrm{div}V = {} & u^{\sigma}\left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m}\right] + u^{\sigma}\mathrm{Ric}(\nabla u, \nabla u) \\ & - \frac{u^{\sigma-2}}{2m}[2(m-1)\zeta^2 + 3m\sigma\zeta + m\sigma(\sigma-1)]|\nabla u|^4 \\ & - \frac{u^{\sigma-1}}{2m}\{[(m+2)\sigma + 2(m-1)\zeta]g(u) - 2(m-1)ug'(u)\}|\nabla u|^2. \end{aligned}$$

Next we insert the expression of $g(u)$, $g'(u)$ and the value of ζ given by (8.23) into the above to obtain

$$\begin{aligned} \mathrm{div}V = {} & u^{\sigma}\left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m}\right] - \frac{u^{\sigma-2}}{2m}A|\nabla u|^4 + \frac{u^{\sigma}}{2m}\left(2\lambda\log u - \frac{\beta}{\tau}\right)B|\nabla u|^2 \\ & + u^{\sigma}[\mathrm{Ric}(\nabla u, \nabla u) - D|\nabla u|^2], \end{aligned}$$

where

$$\begin{cases} A := m\sigma^2 + (3\tau+2)m\sigma + 2(m-1)(1+\tau)^2 \\ B := (m+2)\sigma + 2(m-1)\tau \\ D := 2\frac{m-1}{m}\lambda. \end{cases}$$

By choosing

$$\sigma = -\frac{2(m-1)\tau}{m+2} \tag{8.24}$$

we obtain $B = 0$ and thus the above can be rewritten as

$$\mathrm{div} V = u^\sigma \left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right] - \frac{u^{\sigma-2}}{2m} A |\nabla u|^4 + u^\sigma [\mathrm{Ric}(\nabla u, \nabla u) - D|\nabla u|^2], \tag{8.25}$$

We integrate (8.25) on M and using the divergence theorem we infer

$$0 = \int_M u^\sigma \left[|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \right] - \frac{A}{2m} \int_M u^{\sigma-2} |\nabla u|^4 + \int_M u^\sigma [\mathrm{Ric}(\nabla u, \nabla u) - D|\nabla u|^2]. \tag{8.26}$$

We need to find values of the parameter τ such that $A \le 0$. Towards this aim we let

$$y := 1 + \frac{1}{\tau}, \quad \gamma := -\frac{\sigma}{\tau}.$$

Note that $y \neq 1$ and it is well defined since $\tau \neq 0$. Rewriting A in terms of y, as in the proof of Theorem 8.3 above, we have

$$A \le 0 \quad \text{if and only if} \quad 2\frac{m-1}{m} y^2 - 2\gamma y + \gamma^2 - \gamma \le 0. \tag{8.27}$$

Since

$$|\mathrm{Hess}(u)|^2 - \frac{(\Delta u)^2}{m} \ge 0$$

by Newton's inequality and

$$\mathrm{Ric}(\nabla u, \nabla u) - D|\nabla u|^2 \ge 0$$

by (8.16), to deduce from (8.26) that u (and therefore v) is constant, it is enough to have a strict inequality in (8.27). Observe that, with the choice of σ given by (8.24),

$$\gamma = 2\frac{m-1}{m+2}.$$

With this choice $A < 0$ if and only if we can choose $y \neq 1$ such that

$$(m+2)y^2 - 2my + \frac{m(m-4)}{m+2} < 0.$$

This is the case since the discriminant of the polynomial expression in y is given by $4m$ and it is positive. Observe that since $\beta \neq 0$ and v is constant then $\lambda \neq 0$ and thus v is given by (8.1). □

Going back to the geometric origins of equation (8.17), see equation (7.42) of Proposition 7.40, we deduce the next

Corollary 8.28. *Let $(M, \langle\, , \rangle)$ be a compact manifold of dimension $m \ge 2$ with a gradient Einstein type structure of the form*

$$\begin{cases} Ric^\varphi + Hess(f) = \lambda \langle\, , \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{8.29}$$

for some $\alpha, \lambda \in \mathbb{R}$, $\alpha \neq 0$. Assume that $\Lambda \neq 0$, where Λ is the constant appearing on (7.42) *and that* (8.16) *holds. Then f is constant so that $S^\varphi = m\lambda \neq 0$ and*

$$\begin{cases} Ric^\varphi = \lambda \langle\, , \rangle \\ \tau(\varphi) = 0. \end{cases}$$

In particular $(M, \langle\, , \rangle)$ is harmonic-Einstein.

Remark 8.30. Observe that if $\lambda \neq 0$ then the request $\Lambda \neq 0$ is not a restriction because one can add a constant to f to obtain $\Lambda \neq 0$. However, observe that if $\lambda = 0$ and $\Lambda = 0$ then (7.42) becomes $\Delta_f f = 0$. Thus the function $u = e^{-f} > 0$ is harmonic on M. Compactness of M implies that u is constant. So is f, and we reach the same conclusion of Corollary 8.28 with $\lambda = 0$.

Putting together Corollary 8.28 and Remark 8.30 we have the validity of

Corollary 8.31. *Let $(M, \langle\, ,\, \rangle)$ be a compact manifold of dimension $m \geq 2$ with a gradient Einstein-type structure of the form* (8.29) *for some $\alpha, \lambda \in \mathbb{R}$, $\alpha \neq 0$, and such that* (8.16) *holds. Then f is constant and φ is harmonic, so that*

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases}$$

In particular, $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein.

Remark 8.32. In case $\lambda > 0$ then (8.16) implies via Myers' Theorem the compactness of M. Observe also that (8.16) is equivalent to

$$\alpha\varphi^*\langle\, ,\, \rangle_N \geq \mathrm{Hess}(f) + \frac{m-2}{m}\lambda\langle\, ,\, \rangle.$$

We now come to analyze the uniqueness of the second geometric equation in Proposition 7.40, that is, (7.44). It is worth to consider it in the form (7.85). We begin with the prototype equation, for $u > 0$

$$\Delta u + \rho u - \beta u \log u = 0 \quad \text{ on } M. \tag{8.33}$$

Note that the positive constant $e^{\frac{\rho}{\beta}}$ is a solution of (8.33), when $\beta \neq 0$. To show uniqueness we shall use an unpublished comparison result due to G. Albanese in [2]. It extends some previous work in [42] to the case of a very weak superlinearity $u\hat{f}(u)$ including the case of $\hat{f}(u) = \log u$ as in (8.33). For $\hat{f} \in \mathcal{C}^1(\mathbb{R}^+)$ we require

$$\begin{cases} i) & \lim\limits_{t\to+\infty} \hat{f}(t) = +\infty \\ ii) & \liminf\limits_{t\to+\infty} t^{1+\varepsilon}\hat{f}'(t) > 0 \;\text{ for every } \varepsilon > 0 \\ iii) & \hat{f}' \text{ is positive and } \zeta\text{-decreasing on } \mathbb{R}^+ \\ iv) & \lim\limits_{t\to+\infty} \dfrac{\hat{f}(t)}{t^2\hat{f}'(t)} = 0, \end{cases} \tag{8.34}$$

where ζ-decreasing on $\mathbb{R}^+$ means that the constant ζ satisfies $0 < \zeta \leq 1$ and for every $t \in \mathbb{R}^+$

$$\inf_{s\in(0,t]} \hat{f}'(s) \geq \zeta\hat{f}'(t).$$

Theorem 8.35 (**[2]**)**.** *Let $(M, \langle\, ,\, \rangle, e^{-h})$ be a complete, non-compact weighted manifold, $\zeta \in \mathbb{R}$, $\tau \geq 0$ and $\mu \in [0, 1]$ satisfying the condition*

$$2 + \zeta + \tau\mu > 0.$$

Let $a(x), b(x) \in \mathcal{C}^0(M)$, suppose that $b(x) > 0$ on M, that there exists a constant $C > 0$ such that for $r(x) >> 1$

$$b(x) \geq Cr(x)^{\zeta}$$

and finally that

$$\sup_{x\in M} \frac{a_-(x)}{b(x)} r(x)^{\tau(1-\mu)} < +\infty,$$

where a_- is the negative part of a. Let $\hat{f} \in \mathcal{C}^1(\mathbb{R}^+)$ satisfy (8.34) *and let $u, v \in \mathcal{C}^2(M)$ be positive solutions on M of*

$$\Delta_h u + a(x)u - b(x)u\hat{f}(u) \geq 0 \geq \Delta_h v + a(x)v - b(x)v\hat{f}(v)$$

such that for some constant $B \geq 1$ and for $r(x) >> 1$

$$v(x) \geq \frac{1}{B} r(x)^{\tau}, \quad u(x) \leq B r(x)^{\tau}.$$

Assume

$$\liminf_{r \to +\infty} \frac{\log vol_h(B_r)}{r^{2+\zeta+\tau\mu}} < +\infty.$$

Then $u \leq v$ on M.

Note that the function

$$\hat{f}(t) := \log t, t \in \mathbb{R}^+$$

satisfies (8.34). From Theorem 8.35 and the latter observation we deduce the following

Proposition 8.36. *Let $(M, \langle\, , \rangle)$ be a complete, non-compact, manifold and let $\beta > 0$ and $\rho \geq 0$. Assume that for some constant $\tau \geq 0$*

$$\liminf_{r \to +\infty} \frac{\log volB_r}{r^{2+\tau}} < +\infty. \tag{8.37}$$

Then equation (8.33) has at most one positive solution u satisfying, for some constant $B \geq 1$ and for $r >> 1$

$$\frac{1}{B} r(x)^{\tau} \leq u(x) \leq B r(x)^{\tau}. \tag{8.38}$$

Note that, in case $\tau = 0$ the assumption $\rho \geq 0$ can be relaxed to $\rho \in \mathbb{R}$. In particular in this setting the only bounded and bounded away from zero positive solution u of (8.33) is $u = e^{\frac{\rho}{\beta}}$.

Suppose now that on the complete, non-compact Riemannian manifold $(M, \langle\, , \rangle)$ we have an Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric} - \alpha\varphi^*\langle\, , \rangle_N + \mathrm{Hess}(f) = \lambda\langle\, , \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{8.39}$$

for some $\alpha, \lambda \in \mathbb{R}$ and $\varphi : M \to (N, \langle\, , \rangle_N)$. Assume $\lambda \neq 0$. Then, by adding a constant to f we can always suppose that equation (7.44) has the form

$$\Delta_f f + 2\lambda f = 0.$$

Thus (7.85) for $u = e^{-f}$ becomes

$$\Delta u + 2\lambda u \log u = 0. \tag{8.40}$$

Observe also that the constant α and the smooth map φ do not appear into (8.40). Thus if we have a second Einstein-type structure

$$\begin{cases} \mathrm{Ric} - \bar{\alpha}\bar{\varphi}^*\langle\, , \rangle_{\bar{N}} + \mathrm{Hess}(g) = \lambda\langle\, , \rangle \\ \tau(\bar{\varphi}) = d\bar{\varphi}(\nabla g) \end{cases} \tag{8.41}$$

for some $\bar{\alpha} \in \mathbb{R}$ and $\bar{\varphi} : M \to (\bar{N}, \langle\, , \rangle_{\bar{N}})$, up to adding a constant to g the function $v := e^{-g}$ satisfies (8.40) again. From Proposition 8.36 we then deduce the next

Corollary 8.42. *Let $(M, \langle\, , \rangle)$ be a complete, non-compact, manifold, $\alpha, \bar{\alpha} \in \mathbb{R}^+$, $\varphi : M \to (N, \langle\, , \rangle_N)$ and $\bar{\varphi} : M \to (\bar{N}, \langle\, , \rangle_{\bar{N}})$ smooth maps and $f, g \in \mathcal{C}^{\infty}(M)$ potential functions on M realizing the two Einstein-type structures (8.39) and (8.41) with $\lambda < 0$. Suppose that, for some constants $B \geq 1$ and $\tau \geq 0$, for $r(x) >> 1$*

$$-\log B - \tau \log r(x) \leq f(x), g(x) \leq \log B - \tau \log r(x).$$

If

$$\liminf_{r \to +\infty} \frac{\log volB_r}{r^{2+\tau}} < +\infty$$

then

$$f = g + C \quad \text{on } M,$$

for some constant $C \in \mathbb{R}$.

It remains to analyze the geometric equation (7.43) that we consider in the form (7.86). Thus, the prototype equation is

$$\Delta u + \rho u - \beta u^{1-2\mu}, \qquad u > 0 \tag{8.43}$$

with $\mu, \rho, \beta \in \mathbb{R}$. Note that the non-linearity can be written in the form $u\hat{f}(u)$ with $\hat{f}(t) = t^{-2\mu}$. Thus the requests appearing in (8.34) are satisfied if and only if $\mu < 0$. Applying Theorem 8.35 we have

Theorem 8.44. *Let $(M, \langle\, , \rangle)$ be a complete, non-compact manifold. Let $\mu < 0$, $\beta > 0$ and $\tau > 0$. Assume that* (8.37) *holds. Then equation* (8.43) *has at most one positive solution u satisfying* (8.38) *for some constant $B \geq 1$ and $r(x) >> 1$. In particular, if $\rho > 0$ the only bounded and bounded away from zero positive solution u of* (8.43) *is*

$$u = \left(\frac{\rho}{\beta}\right)^{\frac{1}{2\mu}}.$$

Geometric conclusions similar to those contained in Corollary 8.42 are left to the interested reader.

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