

A rigidity result for the 3D capillary liquid drop with constant vorticity

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Abstract

We consider the free boundary problem for the Euler equations of fluid dynamics governing the motion of a 3D liquid drop with capillarity σ_0 and nearly spherical shape, under the assumption of constant vorticity $(0, 0, \alpha_0)$. First we study the compatibility of the constant vorticity condition with the evolution in time of the system, showing that, for $\alpha_0 \neq 0$, any smooth solution with convex domain must satisfy a strong geometrical constraint on the shape of the fluid domain, and that the constant vorticity condition (unlike in the irrotational case $\alpha_0 = 0$) does not define an invariant set for the time evolution of the system. Then we focus on the time-independent solutions of the problem and we prove a new rigidity result: starting without assuming any symmetry condition, we show that, if the ratio α_0^2/σ_0 is not too large, then any nearly spherical solution has necessarily cylindrical symmetry, and therefore it is the unique axisymmetric solution already known in literature, the fluid domain is close, but not equal, to a ball, more precisely it is an oblate spheroid, flattened at the poles and bulged at the equator, and each fluid particle moves along a horizontal, circular trajectory with constant angular velocity. To the best of our knowledge, this is the first result for the capillary liquid drop with constant vorticity obtained without assuming cylindrical symmetry.

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1 Introduction

We consider the free boundary problem for the Euler equations of the fluid dynamics

$$\begin{cases} \partial_t u + \langle u, \nabla \rangle u + \nabla p = 0 & \text{in } \Omega_t, \\ \operatorname{div} u = 0 & \text{in } \Omega_t, \\ p = \sigma_0 H_{\partial\Omega_t} & \text{on } \partial\Omega_t, \\ V_t = \langle u, \nu_{\partial\Omega} \rangle & \text{on } \partial\Omega_t, \end{cases} \quad (1.1)$$

with the additional constraint of constant vorticity

$$\operatorname{curl} u = \alpha_0 e_3 \quad \text{in } \Omega_t. \quad (1.2)$$

Here $\Omega_t \subset \mathbb{R}^3$ is a bounded open set, and it is the domain occupied by the fluid, u is the fluid velocity vector field, p is the fluid pressure, $\sigma_0 > 0$ is the capillarity constant, $H_{\partial\Omega_t}$ is the

mean curvature of the boundary $\partial\Omega_t$, V_t is the normal velocity of the boundary $\partial\Omega_t$, $\nu_{\partial\Omega_t}$ is the outer unit normal to the boundary, $\alpha_0 \in \mathbb{R}$ is a vorticity parameter, and $e_3 = (0, 0, 1)$. The unknowns of the problem are Ω_t, u, p .

In this paper we first study the compatibility of the constant vorticity condition with the evolution in time of the system, showing that, for $\alpha_0 \neq 0$, any smooth solution with convex domain must satisfy a strong geometrical constraint on the shape of the fluid domain Ω_t , and that condition (1.2) (unlike in the irrotational case $\alpha_0 = 0$) does not define an invariant set for the time evolution of (1.1). Then we focus on the time-independent solutions of the problem and we prove a new rigidity result: starting without assuming any symmetry condition, we show that, if the ratio α_0^2/σ_0 is not too large, then the solution has necessarily cylindrical symmetry, and therefore it is the unique axisymmetric solution already known in literature, the fluid domain is close, but not equal, to a ball, more precisely, it is an oblate spheroid, flattened at the poles and bulged at the equator, and each fluid particle moves along a horizontal, circular trajectory with constant angular velocity. To the best of our knowledge, this is the first result for the capillary liquid drop with constant vorticity obtained without assuming cylindrical symmetry.

In the next two subsections we introduce and state these results.

1.1 Compatibility of the constant vorticity condition with the time evolution

Before stating our results, we recall the well-known fact that the pressure can be expressed in terms of the other unknowns. Under suitable regularity assumptions, taking the divergence of the first equation in (1.1) and using the second and third equations, one finds that the Laplacian Δp of the pressure in Ω_t and its trace $p|_{\partial\Omega_t}$ at the boundary $\partial\Omega_t$ are determined by Ω_t and u , and therefore p is determined, in terms of Ω_t, u , as the unique solution of that Poisson problem with Dirichlet boundary conditions. For this reason, one can say that the unknowns of the problem are just the domain Ω_t and the velocity vector field u . Existence of a classical solution to the problem (1.1) has been proved in [10] and a continuation principle has been established in [16].

The first result we prove is Theorem 1.1, which says that a smooth solution of (1.1), (1.2) starting from a convex smooth set has a horizontal plane of symmetry, the boundary of the fluid domain is given by the union of two graphs, the velocity vector field does not depend on x_3 and its third component is zero. To simplify the statement, we recall the well-known fact that the barycenter velocity vector $c = \int_{\Omega_t} u \, dx \in \mathbb{R}^3$ is a conserved quantity of system (1.1), that is, c is independent of time along any solution, and therefore the fluid barycenter moves with uniform linear motion $b(t) = b_0 + ct$. Hence the motion of the liquid drop can be decomposed into a trivial, constant velocity drift and a remaining component, which corresponds to using a coordinate frame where the origin of the axes and the barycenter of the fluid coincide. In other words, there is no loss of generality in assuming that the barycenter velocity vector is $\int_{\Omega_t} u \, dx = 0$.

Theorem 1.1. *Let Ω_0 be a strictly convex set with C^5 boundary. Let $u_0 \in H^5(\Omega_0, \mathbb{R}^3)$ be a vector field with $\text{curl } u_0 = \alpha_0 e_3$, with vorticity parameter $\alpha_0 \neq 0$, and barycenter velocity vector $\int_{\Omega_0} u_0 \, dx = 0$. Let (u, Ω_t) be a classical solution of the Cauchy problem for system (1.1) in a time interval $(-\delta, \delta)$, $\delta > 0$, with initial datum (u_0, Ω_0) at time $t = 0$. If u satisfies (1.2) for $t \in (-\delta, \delta)$, then there exists $\lambda_0 \in \mathbb{R}$ such that the domain Ω_0 is symmetric with respect to the plane $\{x_3 = \lambda_0\}$, the surface $\partial\Omega_0 \cap \{x_3 > \lambda_0\}$ is a graph over the plane domain $\Omega_0 \cap \{x_3 = \lambda_0\}$, the vector field u_0 is independent of the variable x_3 , and it has third component $u_{0,3} = 0$.*

The proof of Theorem 1.1 is in Section 3, and it is based on the Alexandrov reflection principle. From the equation of the time evolution of the vorticity and the constant vorticity condition (1.2) one immediately deduces that

$$0 = \langle \text{curl } u, \nabla \rangle u = \alpha_0 \partial_{x_3} u. \quad (1.3)$$

Since $\alpha_0 \neq 0$, identity (1.3) implies that the velocity vector field u is independent of the variable x_3 , and this property of u is used to obtain a symmetry property for the mean curvature $H_{\partial\Omega_t}$ of the boundary $\partial\Omega_t$. Then the symmetry of the curvature, by the Alexandrov reflection principle, leads to the symmetry of the domain Ω_t itself.

Remark 1.2 (*Irrotational solutions have more freedom concerning the geometry of the domain Ω_t*). Irrotational solutions of (1.1) are solutions satisfying the constant vorticity condition (1.2) with parameter $\alpha_0 = 0$. However, for $\alpha_0 = 0$, identity (1.3) becomes $0 = 0$, and the argument of Theorem 1.1 does not apply. Thus, the geometric constraint on the shape of Ω_t does not hold for irrotational solutions. $\square$

Remark 1.3 (*Strict convexity is not strictly necessary*). The assumption of strict convexity of Ω_0 is not really needed in the proof of Theorem 1.1, and it can be replaced by weaker geometric conditions, see Remark 3.1. $\square$

Remark 1.4 (*Local existence for the Cauchy problem*). The existence of a classical solution (Ω_t, u) of the Cauchy problem for system (1.1) is proved by Coutand and Shkoller, see [10, Theorem 1.1] for a statement in terms of Ω_t, u , and [10, Theorem 1.3] for its Lagrangian formulation in spaces of functions defined on the fixed domain $[0, T] \times \Omega_0$. The regularity assumed in Theorem 1.1 is sufficient to apply the existence results in [10]. We observe that, in the proof of Theorem 1.1, the regularity in time of the solution is only used to obtain the vorticity equation $\partial_t w + \langle u, \nabla \rangle w - \langle w, \nabla \rangle u = 0$ for $w = \text{curl } u$ at time $t = 0$ and the identity $\partial_t(\text{curl } u) = \text{curl}(\partial_t u)$ at $t = 0$. $\square$

A consequence of Theorem 1.1 is that the only admissible initial data of the Cauchy problem for (1.1) with the condition (1.2) are pairs (Ω_0, u_0) where the domain Ω_0 is symmetric with respect to the plane $\{x_3 = \lambda_0\}$, for some $\lambda_0 \in \mathbb{R}$, and the velocity vector field u_0 is independent of x_3 . This is a geometric constraint that, in many cases, is not compatible with the time evolution equation, namely the set of pairs (Ω_0, u_0) of this kind is not an invariant set for the dynamics of the capillary drop, as we show, with an explicit example, in the next theorem.

Theorem 1.5. *Let $\Omega_0 = B_1 \subset \mathbb{R}^3$ be the open unit ball, and let $u_0(x) := \frac{\alpha_0}{2}(-x_2, x_1, 0)$. Let (Ω_t, u) be the unique smooth solution of the Cauchy problem for (1.1) in some time interval $(-\delta, \delta)$, $\delta > 0$, with initial data (Ω_0, u_0) at time $t = 0$. Then $\text{curl } u$ is not constant in $(-\delta, \delta)$.*

Theorem 1.5 gives an example of an initial datum (Ω_0, u_0) where (1.2) is satisfied, Ω_0 is symmetric with respect to the plane $\{x_3 = 0\}$, and u_0 depends only on x_1, x_2 , and, nonetheless, the solution (Ω_t, u) starting from (Ω_0, u_0) has nonconstant vorticity in $(-\delta, \delta)$. In other words, we have proved that the Cauchy problem for system (1.1) joint with the additional equation (1.2) is, in general, illposed. The proof of Theorem 1.5 is in Section 3.

Remark 1.6 (*Illposedness is a 3-dimensional phenomenon*). The illposedness of system (1.1), (1.2) shown in Theorem 1.5 is a genuinely 3-dimensional phenomenon: this is due to the fact that the vorticity equation in two dimensions is $\partial_t w + \langle u, \nabla \rangle w = 0$ and, therefore, the constant vorticity property is preserved during the evolution. In fact, in dimension 2, the solution of the Cauchy problem with initial data (Ω_0, u_0) , where Ω_0 is the unit ball of $\mathbb{R}^2$ (the unit disc) and $u_0 = \frac{\alpha_0}{2}(-x_2, x_1)$, is the time-independent solution $(\Omega_t, u(t)) = (\Omega_0, u_0)$. $\square$

1.2 Rigidity result for time-independent solutions

The aim of the second, and main, part of the paper is to investigate solutions $\Omega_t = \Omega$, u of problem (1.1)-(1.2) with

$$\Omega, u = \text{independent of time.} \quad (1.4)$$

In the irrotational case, that is, for $\alpha_0 = 0$, the only solutions (Ω, u) with Ω of class C^2 , time-independent and simply connected (with u , in principle, possibly depending on time) are solutions where $u = 0$ and Ω is a ball. The argument is rather simple, and we recall it here. Assume that (Ω, u) solve (1.1), with $\text{curl } u = 0$ and Ω time-independent, C^2 and simply connected. Since the boundary $\partial\Omega$ does not change in time, its normal velocity V_t is zero, and, from the fourth equation in (1.1), the vector u has zero normal component at the boundary $\partial\Omega$. Hence, for any scalar function f , one has $\int_{\partial\Omega} f \langle u, \nu_{\partial\Omega} \rangle d\sigma = 0$, and, by the second equation in (1.1) and the divergence theorem, $\int_{\Omega} \langle \nabla f, u \rangle dx = \int_{\Omega} \text{div}(fu) dx = 0$. In other words, the vector field u is orthogonal in $L^2(\Omega, \mathbb{R}^3)$ to any gradient vector field. On the other hand, since $\text{curl } u$ is zero and Ω is simply connected, there exists a scalar function Φ , called the velocity potential, such that $u = \nabla \Phi$. Taking $f = \Phi$ in the orthogonality property above, we get $0 = \int_{\Omega} \langle \nabla f, u \rangle dx = \int_{\Omega} |\nabla \Phi|^2 dx$, that is, $u = \nabla \Phi = 0$. Since $u = 0$, the first equation in (1.1) gives $\nabla p = 0$. Hence the pressure has the same value (possibly depending on t) at all points of Ω . On the other hand, $\sigma_0 H_{\partial\Omega}$ is independent of time, and, by the third equation in (1.1), $p|_{\partial\Omega}$ is independent of time too. Hence p is a constant. The third equation in (1.1) also says

that the mean curvature of $\partial\Omega$ is constant, and this, by Alexandrov's Theorem, implies that Ω is a ball.

When $\alpha_0 \neq 0$ the problem becomes more challenging, and the simple argument above fails. As we have seen above, an important difference is that being irrotational is a condition that is preserved along the motion and that gives no special geometric constraint on the shape of the fluid domain, while, on the contrary, the condition of nonzero constant vorticity implies strong geometric constraints on the shape of the domain Ω_t and on the velocity vector field u (Theorem 1.1) and, in general, it is not preserved along the motion and can lead to illposed Cauchy problems (Theorem 1.5). These constraints play a role also in the study of time-independent solutions.

For time-independent (Ω, u) , problem (1.1)-(1.2) becomes

$$\begin{cases} \langle u, \nabla \rangle u = -\nabla p & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ \operatorname{curl} u = \alpha_0 e_3 & \text{in } \Omega, \\ p = \sigma_0 H_{\partial\Omega} & \text{on } \partial\Omega, \\ \langle u, \nu_{\partial\Omega} \rangle = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.5)$$

The main result of the paper is the following rigidity theorem.

Theorem 1.7. *Let Ω be a strictly convex open set with C^2 boundary and $u \in C^2(\Omega, \mathbb{R}^3) \cap C(\bar{\Omega}, \mathbb{R}^3)$. Suppose that (Ω, u) is a solution of (1.4), (1.5). Then*

- (i) *There exists a real constant λ_0 , a strictly convex open set $D \subset \mathbb{R}^2$ and a real function $f \in C^2(D) \cap C(\bar{D})$, with $f > 0$ in D and $f = 0$ on ∂D , such that*

$$\Omega = \{(x', x_3) \in \mathbb{R}^3 : x' \in D, |x_3 - \lambda_0| < f(x')\}.$$

Moreover the vector field u is independent of x_3 , and it has third component $u_3 = 0$.

- (ii) *If*

$$\frac{\alpha_0^2}{\sigma_0} |D|^{\frac{3}{2}} \leq \sqrt{2\pi^3}, \quad (1.6)$$

then D is a disk, $\partial\Omega$ is a surface of revolution, and the velocity vector field is $u = \frac{\alpha_0}{2}(-x_2, x_1, 0)$.

Remark 1.8 (*Strict convexity is not really necessary*). As we highlight in Remark 4.16 below, the assumption of strict convexity is actually not necessary to prove Theorem 1.7, and the convexity of Ω would be sufficient. Nonetheless, we give the proof in details assuming Ω strictly convex to simplify the proofs and the notations, and we briefly outline the steps that lead to prove Theorem 1.7 when Ω is just convex. $\square$

Remark 1.9 (*Regularity assumption*). The regularity for (Ω, u) required in Theorem 1.7 is only due to geometric arguments, and it is weaker than that in Theorem 1.1, where (Ω, u) is the initial datum of a Cauchy problem. $\square$

Remark 1.10 (*Formula of the surface $\partial\Omega$*). In the second item of Theorem 1.7, the function f giving the profile of the surface $\partial\Omega$ can be determined from an explicit identity, see (4.16), leading to the axisymmetric shape of the capillary drop already obtained in [24]. Concerning the connection between Theorem 1.7 and [24], see also the discussion below. $\square$

Theorem 1.7 is related to the results of [24, 39], where the calculus of variations is used to study the shape of rotating fluids. To discuss this connection, a clarification about the use of the expression “stationary solution” is in order:

- in [24, 39], an open set $\Omega \subset \mathbb{R}^3$ of class C^2 is said to be a ‘stationary rotating solution’ if it is a critical point of the shape functional

$$\mathcal{L}(\Omega) = \sigma_0 P(\Omega) - \gamma_0 \int_{\Omega} |x'|^2 dx, \quad (1.7)$$

where $P(\Omega)$ is the perimeter of Ω , among all sets of finite perimeter with fixed volume, with $\gamma_0 \in \mathbb{R}$;

- in the present paper, a pair (Ω, u) is said to be a ‘steady solution’, or ‘stationary solution’, if Ω is an open bounded set with C^2 boundary, $u \in C^2(\Omega)$, and (Ω, u) satisfies (1.4), (1.5).

In (1.7) we used the notation $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$. We emphasize that, with respect to the full problems studied in the papers [24, 39], in (1.7) we are making a big simplification: in [39] the author also considers the possibility of self-gravitation, while in [24] rotating stationary surfaces (i.e., not necessarily boundaries of smooth sets) are studied. We note that the definition in [24, 39] only deals with the fluid domain Ω , as the velocity vector field u does not appear in it, and that the adjective “stationary” in that definition reminds the expression “stationary point of a functional”. On the other hand, the meaning of “stationary” used in the present paper is rather standard in fluid dynamics, and includes both the fluid domain Ω and the velocity vector field u . We also recall that, in the fluid dynamics literature, the adjectives ‘steady’ and ‘stationary’ are often used synonymously, e.g., in the book [1] of Arnold and Khesin.

Now let us comment on the differences between these two definitions and the connection between the results. Computing the first variation of $\mathcal{L}(\Omega)$, one finds that a stationary rotating solution Ω in the sense of [24, 39], that is, a critical point of the functional $\mathcal{L}$, satisfies the Euler-Lagrange equation

$$\sigma_0 H_{\partial\Omega} - \gamma_0 |x'|^2 = \lambda \quad \text{on } \partial\Omega, \quad (1.8)$$

where λ is a Lagrange multiplier that appears because of the volume constraint. On the other hand, we start with the free boundary problem for the Euler equations of fluid dynamics with constant vorticity, and we prove (see identities (4.8), (4.9) and Lemma 4.3 below) that, if (Ω, u) solves (1.4)-(1.5), then there exists a function g , independent of x_3 , harmonic in Ω , and a constant c such that

$$\frac{1}{2} \left| \nabla g - \frac{\alpha_0}{2} (x', 0) \right|^2 + \alpha_0 \left(g - \frac{\alpha_0}{4} |x'|^2 \right) + \sigma_0 H_{\partial\Omega} = c \quad \text{on } \partial\Omega. \quad (1.9)$$

The main difference between (1.8) and (1.9) is the presence of the function g . In fact, our problem contains both geometry (the shape of the fluid domain Ω) and dynamics (the velocity vector field u , which in (1.9) is expressed in terms of g), while problem (1.8) considers only the geometry part. If g is a constant, then (1.9) becomes (1.8) with $\gamma_0 = \alpha_0^2/8$ and $\lambda = c - \alpha_0 g$. In this paper we *prove* that, when (1.6) holds, D is a disk, and from this we *deduce* that g is a constant. Thus, Theorem 1.7 states that if the equatorial planar region D is convex and the ratio α_0^2/σ_0 satisfies (1.6), then the domain Ω of the solution (Ω, u) of (1.4), (1.5) and the critical point of the shape functional (1.7) coincide.

Sketch of the proof. While it is clear that the problem of finding critical points Ω of the shape functional (1.7) is a problem of calculus of variations, it is less evident that studying solutions (Ω, u) of (1.4), (1.5) is also a problem of calculus of variations. The first step towards the proof of Theorem 1.7 is to show that the set Ω must have a horizontal plane of symmetry, while u must be independent of x_3 (see Theorem 1.1, now applied to time-independent solutions).

At this point, since the domain Ω is assumed to be bounded and convex, it may seem natural to write the equations on a spherical reference manifold, that is, in terms of functions defined on $\mathbb{S}^2$, like in [2, 3]. However, this turns out to be not a convenient choice, because the symmetry properties proved in Theorem 1.1 are not easy to handle when using nearly spherical sets. On the contrary, we find it more convenient to parametrize the boundary of Ω as the graph of a function f (and its opposite $-f$) over the equatorial planar region D , see Lemma 4.3.

Observe that the symmetry of Ω , joint with the assumption that Ω is smooth, immediately implies that the gradient of the profile function f must blow up when approaching ∂D . This suggests that it is hard to use a strategy based on the implicit function theorem in Banach spaces. In fact, one might be tempted to prove Theorem 1.7 by showing that the trajectory $\alpha_0 \mapsto (\Omega(\alpha_0), u(\alpha_0))$, for values of the vorticity parameter α_0 in a neighborhood of zero, contains the only steady solutions close to $(\Omega(0), u(0))$. In principle, this makes sense because we already observed that, for $\alpha_0 = 0$, the only solution $\Omega(0)$ is a ball, and $u(0) = 0$. However, since the gradient of the profile function f blows up without an evident quantitative control, it is hard to guess the correct function space where the implicit function theorem could be set.

Parametrizing $\partial\Omega$ as the graph of $\pm f$ over D , we obtain a system of equations where all the unknown functions are defined on the unknown equatorial planar region D . We underline that this dimensional reduction is not the standard Craig-Sulem formulation of fluid dynamics as a problem on the boundary $\partial\Omega$.

The next step is to analyze the properties of the velocity field u , also using the time-independence assumption. An accurate investigation of the problem shows that the torsion function v_D of the equatorial planar region D , i.e., the solution of the problem

$$\Delta v_D = -1 \quad \text{in } D, \quad v_D \in H_0^1(D),$$

plays an essential role. Indeed, we show that D must support a solution to an overdetermined problem, namely a solution to a second order elliptic problem with both Dirichlet and Neumann-type conditions on ∂D , which is

$$\frac{\alpha_0^2}{2} |\nabla v_D|^2 - \frac{\sigma_0 \beta}{|\nabla v_D|} + \sigma_0 H_{\partial D} = c \quad \text{on } \partial D, \quad (1.10)$$

where β, c are real constants, and $H_{\partial D}$ is the curvature of the boundary ∂D . It is not obvious that (1.10) is strong enough to infer that D is a disk, mainly because the constants β and c seem to be related to the nature of the original problem (which is three dimensional) and therefore to the profile function f . Nevertheless, an investigation of the smoothness assumption provides explicit formulas for β and c as functions of the planar set D , see Theorem 4.2 and Lemmas 4.9, 4.10. To conclude, we use special properties of the torsion function of a planar convex set to find an efficient bound for the constant β (see (4.40)). This allows us to conclude that, if (1.6) holds, then (1.10) is sufficiently strong to guarantee that D must be a disk. Once we have obtained this rigidity property, i.e., that D is a disk, it is easy to check that the profile function is radial and it can be computed by quadrature, see Remark 4.4.

Related literature. The axisymmetric equilibria configurations for capillary drops with constant vorticity were discovered experimentally by Plateau in [30] and computed by Poincaré [31], Rayleigh [32] and Chandrasekhar [7], where their stability properties were also studied, and more recently by Wente [39] and Lopez [24].

Concerning the free-boundary problems for fluids with constant vorticity, we mention for instance [37, 36, 28] for the Craig-Sulem formulation of the 2D ocean problem with or without general bottom topography, and [19] for the 2D capillary drop problem with constant vorticity. The existence of global solutions for fluids with constant vorticity has been studied in [9, 8, 38, 11], and in [4, 6] for periodic and quasi-periodic travelling waves. Travelling drops and bubbles in a two-fluid interaction have been constructed in [27], where an overdetermined problem also appears.

Concerning rigidity results for the 3D fluid dynamics, we mention [37] for the 3D ocean problem with constant vorticity, and [29] where localizable 3D Euler flows in bounded domains are proved to be axisymmetric, with rotationally symmetric domain whose transverse section is a disk or an annulus with convex boundary curves.

Structure of the paper. In Section 2 we report some definitions and preliminary results. In Section 3 we prove Theorems 1.1, 1.5, which also give a first reduction. In Section 4 we show that if D is convex, then the problem of finding a solution of (1.4), (1.5) is connected to the overdetermined problem (1.10) for the torsion function. In Section 4.3 we conclude the proof of Theorem 1.7.

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2 Notations and preliminary results

We start by briefly recalling the notion of mean curvature, torsion function and some useful formulas. For an exhaustive treatment of the subject, we refer, e.g., to [26].

Let $d = 2$ or $d = 3$. For a bounded open set $E \subset \mathbb{R}^d$, we denote by $\nu_{\partial E}(x)$ the exterior unit normal. If E is of class C^2 , then there exists $\delta > 0$ such that, defining $I_\delta(\partial E) := \{y \in$

$\mathbb{R}^d : \text{dist}(x, \partial E) < \delta\}$, there exists $\tilde{\nu} \in C^1(I_\delta(\partial E))$ that satisfies $\tilde{\nu} = \nu_{\partial E}$ on ∂E . The mean curvature at $x \in \partial E$ is defined as the tangential divergence of the normal, namely

$$H_{\partial E} = \text{div}_{\partial E} \nu_{\partial E} = \text{div} \tilde{\nu} - \langle (D\tilde{\nu}) \nu_{\partial E}, \nu_{\partial E} \rangle,$$

where $D\tilde{\nu}$ is the Jacobian matrix of the vector field $\tilde{\nu}$. In dimension $d = 2$, for a bounded convex open set $E \subset \mathbb{R}^2$, we recall the Gauss-Bonnet formula

$$\int_{\partial E} H_{\partial E} d\sigma = 2\pi. \quad (2.1)$$

Observe that, with this definition, the mean curvature is positive if E is convex, and $H_{\partial B} = d - 1$, where B is the unit ball of $\mathbb{R}^d$, and its boundary $\partial B = \mathbb{S}^{d-1}$ is the unit sphere. If E is locally the subgraph of a $C^2(\mathbb{R}^{d-1})$ function f , at a point $(x', f(x'))$ it holds

$$H_{\partial E}(x', f(x')) = -\text{div} \left(\frac{\nabla f(x')}{\sqrt{1 + |\nabla f(x')|^2}} \right) =: H(f)(x'). \quad (2.2)$$

Moreover, if there exists a smooth function $g \in C_c^2(\mathbb{R}^d)$ such that $\Omega = \{x \in \mathbb{R}^d : g(x) > s\}$ and s is a regular value for g (i.e., $\nabla g(x) \neq 0$ for all x such that $g(x) = s$), then for any $x \in \partial E$, it holds

$$H_{\partial E}(x) = -\text{div} \left(\frac{\nabla g(x)}{|\nabla g(x)|} \right). \quad (2.3)$$

We recall the tangential divergence theorem

$$\int_{\partial E} \text{div}_{\partial E} X d\sigma = \int_{\partial E} H_{\partial E} \langle X, \nu_{\partial E} \rangle d\sigma \quad (2.4)$$

for any vector field $X \in C^1(\mathbb{R}^d)$. Another useful integral identity involving the curvature is

$$\int_{\partial E} H_{\partial E} \langle \nabla u, \nu_{\partial E} \rangle^2 d\sigma = \int_E \left((\Delta u)^2 - |\nabla^2 u|^2 \right) dx, \quad (2.5)$$

where $u = 0$ on ∂E , $\nabla^2 u = D^2 u$ is the Hessian matrix, and $|\nabla^2 u|^2$ is the sum of the square of its entries. This is a special case of Reilly formula [33]; the special form (2.5) is, e.g., in [34].

We recall an important symmetry result regarding the mean curvature of a set. A first result in this direction is due to Li [20] and it has been later generalized in the subsequent papers [21, 22, 23]).

Theorem 2.1 (Li, Yan, Yao [23]). *Let $E \subset \mathbb{R}^d$ be a bounded open set of class C^2 such that, for any $x \in \partial E$ with $(\nu_{\partial E}(x))_d = 0$, there exists a vertical cylinder C_r tangent to ∂E at x and such that $\overline{C_r} \cap \overline{E} \subset \partial E$. Assume further that for any two points $(x', a), (x', b) \in \partial E$ such that $\{(x', \lambda a + (1 - \lambda)b) : \lambda \in (0, 1)\} \subset E$ it holds*

$$H_{\partial E}(x', a) = H_{\partial E}(x', b).$$

Then there exists $\lambda \in \mathbb{R}$ such that E is symmetric with respect to the plane $\{x_d = \lambda\}$.

Actually, in [23] Theorem 2.1 is stated in a more general context, assuming that the mean curvature is nonincreasing in the x_d direction; here, instead, we state the simplified version that is used in the present paper.

We also recall the notion of torsion function, which is an essential tool for our analysis. Let $E \subset \mathbb{R}^d$ be an open bounded set. We denote by v_E the torsion function of the set E , which is the unique solution of

$$\begin{cases} \Delta v_E = -1 & \text{in } E, \\ v_E \in H_0^1(E). \end{cases}$$

Some properties of this function are immediate to obtain. In fact, by maximum principle, it is immediate to show that $v_E(x) \geq 0$ and equality holds if and only if $x \in \partial E$. Moreover, since v_E is constant on ∂E , for any $x \in \partial E$ it holds

$$\nabla v_E(x) = -|\nabla v_E(x)| \nu_{\partial E}(x). \quad (2.6)$$

Other useful identities about the torsion function are

$$\int_E v_E dx = \int_E v_E (-\Delta v_E) dx = \int_E |\nabla v_E|^2 dx, \quad (2.7)$$

$$\int_{\partial E} |\nabla v_E| d\sigma = \int_{\partial E} -\langle \nabla v_E, \nu_{\partial E} \rangle d\sigma = - \int_E \Delta v_E dx = |E|. \quad (2.8)$$

A rigidity result concerning the torsion function is the following theorem by Serrin.

Theorem 2.2 (Serrin [35], Theorem 3). *Let E be a bounded open set in $\mathbb{R}^d$ with C^3 boundary. Let v_E be its torsion function. If there exists a nonincreasing function f such that*

$$\frac{\partial v_E}{\partial \nu_{\partial E}} = f(H_{\partial E}) \quad \text{on } \partial E, \quad (2.9)$$

then E is a ball. If $|\nabla v_E|$ is a nondecreasing function of $H_{\partial E}$ on ∂E , then E is a ball.

When E is convex, the torsion function has some very special properties. To this regard, we state the following theorems.

Theorem 2.3 (Makar-Limanov [25], Kawohl [17], Kennington [18]). *Let E be a bounded convex open set with C^2 boundary. Then v_E is $1/2$ -concave, meaning that $\sqrt{v_E}$ is concave.*

Theorem 2.3 has been first proved in [25] for $d = 2$, then in [17] and [18] for $d > 2$. Finally, we also recall the gradient estimate for the torsion function of planar convex open sets.

Theorem 2.4 (Classical; see, e.g., [5, 15]). *Let $E \subset \mathbb{R}^2$ be an open convex set. Then the torsion function v_E satisfies*

$$\|\nabla v_E\|_{L^\infty(E)} \leq c|E|^{\frac{1}{2}}$$

for some universal constant $c < (2\pi)^{-\frac{1}{2}}$.

The proof of Theorem 2.4 is quite classical; many works in the recent years have been devoted to find the optimal value of the constant c , since it is related to the Hermite-Hadamard inequality. We mention, for instance, the papers [5] and [15].

3 Time evolution and constant vorticity

In this section we prove our results about the compatibility of the constant vorticity condition (1.2) with the equations (1.1) for the time evolution of the capillary drop, namely we prove Theorems 1.1 and 1.5.

Proof of Theorem 1.1. The existence of a $\delta > 0$ such that a solution to (1.1) exists in $(-\delta, \delta)$ is guaranteed by [10, Theorem 1.1]. Assume that the velocity field u has constant vorticity $w = \text{curl } u = \alpha_0 e_3$ for $t \in (-\delta, \delta)$. This implies that

$$\begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix} = \alpha_0 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (3.1)$$

Moreover, the vector field $w = \text{curl } u$ satisfies the vorticity equation

$$\partial_t w + \langle u, \nabla \rangle w = \langle w, \nabla \rangle u.$$

Since $w = \alpha_0 e_3$ is constant (independent of t and x), we have

$$0 = \langle w, \nabla \rangle u = \alpha_0 \begin{pmatrix} \partial_3 u_1 \\ \partial_3 u_2 \\ \partial_3 u_3 \end{pmatrix}. \quad (3.2)$$

Since $\alpha_0 \neq 0$, the velocity field u does not depend on the third variable, i.e., $u(t, x_1, x_2, x_3) = u(t, x_1, x_2)$. Thus, (3.1) implies that $\partial_2 u_3$ and $\partial_1 u_3$ vanish, so that $\nabla u_3 = 0$, that is, $u_3(t, x) = u_3(t)$ is independent of x . This implies that $u_3 = 0$, because $\int_{\Omega_t} u = 0$. Since $u_3 = 0$, from the first equation in (1.1) we deduce that

$$\partial_3 p = 0, \quad (3.3)$$

whence $p(t, x_1, x_2, x_3) = p(t, x_1, x_2)$. Moreover, the third equation in (1.1) implies the following symmetry property:

$$\text{if } (x_1, x_2, x_3) \text{ and } (x_1, x_2, y_3) \in \partial\Omega_t \implies H_{\partial\Omega_t}(x_1, x_2, x_3) = H_{\partial\Omega_t}(x_1, x_2, y_3). \quad (3.4)$$

Since Ω_0 is convex, the assumption of Theorem 2.1 are satisfied and hence we find that there exists $\lambda_0 \in \mathbb{R}$ such that the set Ω_0 is symmetric with respect to the plane $\{x_3 = \lambda_0\}$. Since Ω_0 is symmetric and strictly convex, there exists a concave function

$$f \in C^5(D, \mathbb{R}) \cap C(\overline{D}, \mathbb{R}), \quad f > 0 \text{ in } D, \quad f = 0 \text{ on } \partial D,$$

where D is the planar set $\Omega_0 \cap \{x_3 = \lambda_0\}$, or, more precisely,

$$D := \{x' \in \mathbb{R}^2 : (x', \lambda_0) \in \Omega_0\},$$

such that

$$\partial\Omega_0 = \{(x', \lambda_0 + f(x')) : x' \in D\} \cup \{(x', \lambda_0 - f(x')) : x' \in D\} \cup \Sigma_0,$$

where

$$\Sigma_0 = \partial D \times \{\lambda_0\}.$$

In fact, from the convexity of Ω_0 it is immediate to show that $\partial\Omega_0 \cap \{x_3 > \lambda_0\}$ is a graph over D . The regularity of the function f comes from the regularity of the boundary $\partial\Omega$. $\square$

Proof of Theorem 1.5. Let $\Omega_0 = B_1$, $u_0 = \frac{\alpha_0}{2}(-x_2, x_1, 0)$, and let (Ω_t, u) be the unique smooth solution of the Cauchy problem for (1.1) in some time interval $(-\delta, \delta)$, $\delta > 0$, with initial data (Ω_0, u_0) at time $t = 0$. We calculate

$$\operatorname{div}(u_0 \cdot \nabla u_0) = -\frac{\alpha_0^2}{4} \operatorname{div} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} = -\frac{\alpha_0^2}{2}.$$

Hence, taking the divergence of the first equation in (1.1), and recalling that the curvature of the unit sphere $\mathbb{S}^2$ is 2, we get that the pressure at time $t = 0$ verifies

$$\begin{cases} \Delta p = \alpha_0^2/2 & \text{in } \Omega_0 = B_1, \\ p = 2\sigma_0 & \text{on } \partial\Omega_0 = \mathbb{S}^2. \end{cases}$$

The solution to this problem is

$$p(0, x) = \frac{\alpha_0^2}{12}(x_1^2 + x_2^2 + x_3^2 - 1) + 2\sigma_0,$$

which depends nontrivially on x_3 , with $\partial_3 p(0, x) = (\alpha_0^2/6)x_3$. If $\operatorname{curl} u$ is constant in the time interval $(-\delta, \delta)$, then, as proved above, (3.3) holds, and this is a contradiction. $\square$

Remark 3.1 (Regularity and convexity assumptions). As already explained in the introduction, the study of time-independent solutions requires much less regularity. In fact, since there is no need of using the short time existence result of Coutand and Shkoller, the assumption Ω of class C^2 and strictly convex is sufficient to guarantee that Ω_0 admits a plane of symmetry, which is enough to make the argument of the next sections work.

We also observe that the strict convexity of Ω_0 is actually not needed, but it makes the presentation less heavy. This is discussed in Remark 4.16 below. $\square$

4 An overdetermined problem and rigidity

In this section we begin the study of the time-independent problem (1.5), starting with the following basic observation.

Lemma 4.1. *Assume the hypotheses of Theorem 1.7. Then item (i) of Theorem 1.7 holds.*

Proof. Repeat the proof of Theorem 1.1. The only difference is that now Ω and u are of class C^2 and independent of time. $\square$

By item (i) of Theorem 1.7, the domain Ω is symmetric with respect to the plane $\{x_3 = \lambda_0\}$. Without loss of generality, we assume that the barycenter of Ω is at the origin, and therefore $\lambda_0 = 0$, that is, Ω is symmetric with respect to the plane $\{x_3 = 0\}$.

The starting point of our analysis is following theorem about the torsion function.

Theorem 4.2 (An overdetermined problem for the torsion function). *Let (Ω, u) satisfy the assumptions of Theorem 1.7, and suppose that the barycenter of Ω is at the origin. Let λ_0, D, f be given by item (i) of Theorem 1.7, so that $\lambda_0 = 0$,*

$$D = \{x' \in \mathbb{R}^2 : (x', 0) \in \Omega\}, \quad \partial\Omega \cap \{x_3 > 0\} = \{(x', f(x')) : x' \in D\}. \quad (4.1)$$

Then the torsion function $v_D \in H_0^1(D)$ of the planar set D , that is, the solution of the problem

$$\begin{cases} \Delta v_D = -1 & \text{in } D, \\ v_D = 0 & \text{on } \partial D, \end{cases} \quad (4.2)$$

also satisfies the equation

$$\frac{\alpha_0^2}{2} |\nabla v_D|^2 - \frac{\sigma_0 \beta}{|\nabla v_D|} + \sigma_0 H_{\partial D} = c \quad \text{on } \partial D, \quad (4.3)$$

where $H_{\partial D}$ is the curvature of the boundary ∂D (which is a planar curve), and β and c are real constants depending only on D .

The proof of Theorem 4.2 is divided in two parts: in Subsection 4.1 we prove that v_D satisfies (4.3), while in Subsection 4.2 we compute the values of the constants β and c and show that they depend only on D .

4.1 A geometric-analytic condition

The main goal of this subsection is to prove the first part of Theorem 4.2. Let Ω, u, D, f be like in Theorem 4.2. We introduce the matrices

$$\mathcal{J}_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{J}_2 := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (4.4)$$

We decompose the vector field u into its irrotational and rotation part,

$$u = v + \frac{\alpha_0}{2} \mathcal{J}_3 x, \quad \mathcal{J}_3 x = \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}, \quad (4.5)$$

where v is defined as the difference $v := u - (\alpha_0/2) \mathcal{J}_3 x$. Since u is independent of x_3 and it has third component $u_3 = 0$, also v is independent of x_3 and $v_3 = 0$. Moreover, $\operatorname{div}(\mathcal{J}_3 x) = 0$ and $\operatorname{curl}(\mathcal{J}_3 x) = 2e_3$, so that both $\operatorname{div} v$ and $\operatorname{curl} v$ vanish in Ω . We define the planar vector fields

$$\begin{aligned} \tilde{u} : D &\rightarrow \mathbb{R}^2, & \tilde{v} : D &\rightarrow \mathbb{R}^2, \\ \tilde{u}(x') &= \begin{pmatrix} \tilde{u}_1(x') \\ \tilde{u}_2(x') \end{pmatrix} = \begin{pmatrix} u_1(x', 0) \\ u_2(x', 0) \end{pmatrix}, & \tilde{v}(x') &= \begin{pmatrix} \tilde{v}_1(x') \\ \tilde{v}_2(x') \end{pmatrix} = \begin{pmatrix} v_1(x', 0) \\ v_2(x', 0) \end{pmatrix}, \end{aligned} \quad (4.6)$$

and we note that

$$\operatorname{div}_2 \tilde{u} = \operatorname{div}_2 \tilde{v} = 0, \quad \operatorname{curl}_2 \tilde{u} = \alpha_0, \quad \operatorname{curl}_2 \tilde{v} = 0 \quad \text{in } D,$$

where div_2 and curl_2 are the $\mathbb{R}^2$ -divergence and $\mathbb{R}^2$ -curl, i.e.,

$$\operatorname{div}_2 F := \partial_{x_1} F_1 + \partial_{x_2} F_2, \quad \operatorname{curl}_2 F := \partial_{x_1} F_2 - \partial_{x_2} F_1$$

for any planar vector field $F \in C^1(D, \mathbb{R}^2)$. Since $D \subset \mathbb{R}^2$ is convex, and therefore simply connected, and $\operatorname{curl}_2 \tilde{v} = 0$ in D , there exists a scalar function $\phi : D \rightarrow \mathbb{R}$ such that

$$\tilde{v} = \nabla \phi \quad \text{in } D. \quad (4.7)$$

The function ϕ is harmonic in D , because $\Delta\phi = \operatorname{div} \nabla\phi = \operatorname{div} \tilde{v} = 0$ in D , and therefore there exists a function $g : D \rightarrow \mathbb{R}$ such that

$$\nabla g = \mathcal{J}_2 \nabla \phi \quad \text{in } D. \quad (4.8)$$

Both ϕ and g are functions of two variables $(x_1, x_2) = x' \in D$, and they can be trivially extended to functions defined in $D \times \mathbb{R}$, and hence in Ω . We also denote ϕ, g the extended functions,

$$\phi : \Omega \rightarrow \mathbb{R}, \quad \phi(x) = \phi(x', x_3) := \phi(x'), \quad g : \Omega \rightarrow \mathbb{R}, \quad g(x) = g(x', x_3) := g(x').$$

Of course $\partial_{x_3}\phi = \partial_{x_3}g = 0$. Thus

$$v = \begin{pmatrix} v_1 \\ v_2 \\ 0 \end{pmatrix} = \nabla \phi = \begin{pmatrix} \partial_{x_1}\phi \\ \partial_{x_2}\phi \\ 0 \end{pmatrix} \quad \text{in } \Omega,$$

and the decomposition (4.5) becomes

$$u = \nabla \phi + \frac{\alpha_0}{2} \mathcal{J}_3 x \quad \text{in } \Omega, \quad \tilde{u} = \nabla \phi + \frac{\alpha_0}{2} \mathcal{J}_2 x' \quad \text{in } D. \quad (4.9)$$

The first step towards the proof of Theorem 4.2 is to deduce a system of equations for f and ϕ on the planar set D . This is the content of the next lemma.

Lemma 4.3. *Let Ω, u, D, f be like in Theorem 4.2, and let $\tilde{u}, \tilde{v}, \phi, g$ be the functions defined in (4.6), (4.7), (4.8). Then*

$$\frac{1}{2} |\tilde{u}|^2 + \alpha_0 g - \frac{\alpha_0^2}{4} |x'|^2 + \sigma_0 H(f) = c_1 \quad \text{in } D, \quad (4.10)$$

$$\langle \tilde{u}, \nabla f \rangle = \langle \nabla \phi + \frac{\alpha_0}{2} \mathcal{J}_2 x', \nabla f \rangle = 0 \quad \text{in } D, \quad (4.11)$$

$$\langle \tilde{u}, \nu_{\partial D} \rangle = \langle \nabla \phi + \frac{\alpha_0}{2} \mathcal{J}_2 x', \nu_{\partial D} \rangle = 0 \quad \text{on } \partial D, \quad (4.12)$$

for some constant $c_1 \in \mathbb{R}$, where $H(f)$ is the mean curvature operator defined in (2.2).

Proof. For any planar vector field $F \in C^1(D, \mathbb{R}^2)$, one has

$$\langle F, \nabla \rangle F - \frac{\nabla(|F|^2)}{2} = \begin{pmatrix} F_2 \partial_2 F_1 - F_2 \partial_1 F_2 \\ F_1 \partial_1 F_2 - F_1 \partial_2 F_1 \end{pmatrix} = (\partial_1 F_2 - \partial_2 F_1) \begin{pmatrix} -F_2 \\ F_1 \end{pmatrix} = (\operatorname{curl}_2 F) \mathcal{J}_2 F.$$

Applying this identity to $\tilde{u}$, recalling that $\operatorname{curl}_2 \tilde{u} = \alpha_0$, and using (4.9), (4.8), we get

$$\langle \tilde{u}, \nabla \rangle \tilde{u} - \frac{\nabla(|\tilde{u}|^2)}{2} = \alpha_0 \mathcal{J}_2 \tilde{u} = \alpha_0 \mathcal{J}_2 \left(\nabla \phi + \frac{\alpha_0}{2} \mathcal{J}_2 x' \right) = \alpha_0 \nabla g - \frac{\alpha_0^2}{2} x' = \nabla \left(\alpha_0 g - \frac{\alpha_0^2}{4} |x'|^2 \right).$$

Hence the first equation in (1.5) on $\Omega \cap \{x_3 = 0\} = D \times \{0\}$ gives

$$\nabla \left(\frac{|\tilde{u}|^2}{2} + \alpha_0 g - \frac{\alpha_0^2}{4} |x'|^2 + \tilde{p} \right) = 0 \quad \text{in } D,$$

where $\tilde{p} : D \rightarrow \mathbb{R}$ is the function $\tilde{p}(x') := p(x', 0)$. As a consequence, there exists a constant $c_1 \in \mathbb{R}$ such that

$$\frac{|\tilde{u}|^2}{2} + \alpha_0 g - \frac{\alpha_0^2}{4} |x'|^2 + \tilde{p} = c_1 \quad \text{in } D.$$

Since $u_3 = 0$, from the first equation in (1.5) it follows that the pressure p is independent of x_3 . This means that the pressure is constant along any vertical line, and, in particular, for any $x' \in D$, the value of the pressure at the point $(x', 0) \in \Omega \cap \{x_3 = 0\}$ is equal to its value at the point $(x', f(x')) \in \partial\Omega \cap \{x_3 > 0\}$. Thus, by the fourth equation in (1.5),

$$\tilde{p}(x') = p(x', 0) = p(x', f(x')) = \sigma_0 H_{\partial\Omega}(x', f(x')) = \sigma_0 H(f)(x'),$$

where $H(f)$ is defined in (2.2). This proves (4.10).

The fifth equation in (1.5) is the orthogonality property $\langle u, \nu_{\partial\Omega} \rangle = 0$ on $\partial\Omega$. The exterior normal vector to the boundary $\partial\Omega$ at the point $(x', f(x')) \in \partial\Omega$, with $x' \in D$, is

$$\nu_{\partial\Omega}(x', f(x')) = \frac{1}{\sqrt{1 + |\nabla f(x')|^2}} \begin{pmatrix} -\nabla f(x') \\ 1 \end{pmatrix}. \quad (4.13)$$

Since u is independent of x_3 and its third component is $u_3 = 0$, recalling the definition (4.6) of $\tilde{u}$, one has

$$u(x', f(x')) = u(x', 0) = \begin{pmatrix} \tilde{u}(x') \\ 0 \end{pmatrix}.$$

Hence the identity $\langle u, \nu_{\partial\Omega} \rangle = 0$ on $\partial\Omega$ implies that $\langle \tilde{u}, \nabla f \rangle = 0$ in D . Moreover $\tilde{u}$ is given by the second identity in (4.9), and this gives (4.11).

Identity (4.12) is obtained from the fifth identity of (1.5) on $\partial\Omega \cap \{x_3 = 0\} = \partial D \times \{0\}$, observing that, since Ω is symmetric and smooth, one has

$$\nu_{\partial\Omega}(x', 0) = \begin{pmatrix} \nu_{\partial D}(x') \\ 0 \end{pmatrix} \text{ for } (x', 0) \in \partial\Omega \cap \{x_3 = 0\} = \partial D \times \{0\}, \quad (4.14)$$

and using (4.9), as above, to replace $\tilde{u}$. $\square$

From now on, if there is no risk of ambiguity, we write x instead of x' when referring to points in $D \subset \mathbb{R}^2$.

Remark 4.4 (*The axisymmetric case*). If D is the disk B_ρ of center 0 and radius $\rho > 0$, system (4.10), (4.11), (4.12) becomes easy to solve. Indeed, (4.12) becomes

$$\langle \nabla \phi, \nu_{\partial B} \rangle = 0 \quad \text{on } \partial B_\rho,$$

because $\nu_{\partial B_\rho}(x) = x/\rho$ and $\langle \mathcal{J}_2 x, x \rangle = 0$. Hence ϕ is harmonic in the disk B_ρ with zero Neumann boundary data, and therefore ϕ is constant. Thus $\nabla \phi = 0$ in B_ρ , and $\nabla g = \mathcal{J}_2 \nabla \phi = 0$, whence g is also a constant, $g = g_0 \in \mathbb{R}$. Since $\nabla \phi = 0$, from (4.9) we get $\tilde{u} = (\alpha_0/2)\mathcal{J}_2 x$ in B_ρ , whence $|\tilde{u}|^2 = (\alpha_0^2/4)|x|^2$, and (4.10) becomes

$$-\frac{\alpha_0^2}{8}|x|^2 + \sigma_0 H(f) = c_1 - \alpha_0 g_0 \quad \text{in } B_\rho. \quad (4.15)$$

Since $\nabla \phi = 0$, (4.11) gives $\langle \nabla f, \mathcal{J}_2 x \rangle = 0$ in B_ρ , which implies that f is a radial function.

Since f is radial and $D = B_\rho$ is a disk, equation (4.15) can be easily transformed into an ODE, which is the one studied by Lopez in [24], Section 4. For sake of completeness, we now compute the exact solution in this case. We denote $r = |x|$. Since f is radial, we have $f(x) = f_0(r)$, where $f_0(s) := f(s, 0)$, and

$$\nabla f(x) = f_0'(r) \frac{x}{r}, \quad \frac{\nabla f(x)}{\sqrt{1 + |\nabla f(x)|^2}} = \psi(r)x, \quad \psi(r) := \frac{f_0'(r)}{r\sqrt{1 + f_0'(r)^2}}.$$

Since $\operatorname{div}(\psi(r)x) = r\psi'(r) + 2\psi(r)$, recalling the definition (2.2) of $H(f)$, (4.15) becomes

$$r\psi'(r) + 2\psi(r) + \frac{\alpha_0^2}{8\sigma_0}r^2 + \frac{c_1 - \alpha_0 g_0}{\sigma_0} = 0.$$

This is a linear ODE. Its general solution in the interval $r \in (0, \infty)$ is

$$\psi(r) = \frac{\gamma_1}{r^2} - \frac{\alpha_0^2}{32\sigma_0}r^2 - \frac{c_1 - \alpha_0 g_0}{2\sigma_0}, \quad \gamma_1 \in \mathbb{R}.$$

Recalling the definition of $\psi(r)$, and multiplying by r , we have

$$\frac{f_0'(r)}{\sqrt{1 + f_0'(r)^2}} = r\psi(r) = \frac{\gamma_1}{r} - \frac{\alpha_0^2}{32\sigma_0}r^3 - \frac{c_1 - \alpha_0 g_0}{2\sigma_0}r. \quad (4.16)$$

Since the quantity on the left is bounded, we find that $\gamma_1 = 0$. This implies that $f_0'(0) = 0$. Since the tangent plane to $\partial\Omega$ is vertical at points on $\partial\Omega \cap \{x_3 = 0\} = \partial B_\rho \times \{0\}$, we have to

impose that $\lim_{r \rightarrow \rho^-} f'_0(r) = -\infty$. This holds if the quantity on the right in (4.16) is -1 at $r = \rho$, i.e.,

$$\frac{c_1 - \alpha_0 g_0}{2\sigma_0} = \frac{1}{\rho} - \frac{\alpha_0^2 \rho^2}{32\sigma_0}.$$

Thus,

$$\frac{f'_0(r)}{\sqrt{1 + f'_0(r)^2}} = -\frac{\alpha_0^2}{32\sigma_0} r^3 - \left(\frac{1}{\rho} - \frac{\alpha_0^2 \rho^2}{32\sigma_0}\right) r. \quad (4.17)$$

One has to invert the formula for $f'_0(r)$, and the solution $f_0(r)$ can be found by quadrature after imposing the boundary condition $f_0(\rho) = 0$. $\square$

Remark 4.5 (*A threshold for the shape of monotone axisymmetric profiles*). From formula (4.17) we note that $f'_0(r)$ is negative for all $r \in (0, \rho)$ if the coefficient of r is ≤ 0 , that is, if

$$\frac{\alpha_0^2 \rho^3}{32\sigma_0} \leq 1.$$

This is a threshold for the shape of an axisymmetric rotating capillary drop: when the radius ρ becomes too large, or the vorticity α_0 too strong, or the capillarity σ_0 too weak, then the profile function $f_0(r)$ is no longer a strictly monotone decreasing function of the distance r from the axis of symmetry. This means that the fluid domain Ω ceases to be a convex set. This is a well-known phenomenon in physics literature. $\square$

After the last remarks, now we go back to the general case. When D is not a disk, we also have to deal with the irrotational part $\nabla\phi$ of the velocity field. This makes the computations heavier. However, when the set D is convex, the level sets of the torsion function of D play a role analogous to that of polar coordinates in Remark 4.4, as we show in the following Lemma.

Lemma 4.6. *Let Ω, u, D, f be like in Theorem 4.2, and let v_D be the torsion function of D . Then there exist a constant $c \in \mathbb{R}$ and a function*

$$\tilde{f} : [0, M] \rightarrow [0, \infty), \quad M := \max_D v_D = \|v_D\|_{L^\infty(D)},$$

with $\tilde{f} \in C^2((0, M)) \cap C([0, M])$, $\tilde{f}(0) = 0$, such that $f = \tilde{f} \circ v_D$ in $\overline{D}$, and

$$\frac{\alpha_0^2}{2} |\nabla v_D|^2 + \alpha_0^2 v_D - \sigma_0 \operatorname{div} \left(\frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + |\tilde{f}'(v_D) \nabla v_D|^2}} \right) = c \quad \text{in } D. \quad (4.18)$$

Proof. In D , let

$$g_1 := g - \frac{\alpha_0}{4} |x|^2. \quad (4.19)$$

Recalling (4.8) and (4.9),

$$\nabla g_1 = \nabla g - \frac{\alpha_0}{2} x = \mathcal{J}_2(\nabla\phi + \frac{\alpha_0}{2} \mathcal{J}_2 x) = \mathcal{J}_2 \tilde{u} \quad \text{in } D. \quad (4.20)$$

By the orthogonality property (4.12),

$$\langle \nabla g_1, \mathcal{J}_2 \nu_{\partial D} \rangle = \langle \mathcal{J}_2 \tilde{u}, \mathcal{J}_2 \nu_{\partial D} \rangle = \langle \tilde{u}, \nu_{\partial D} \rangle = 0 \quad \text{on } \partial D.$$

Since $\mathcal{J}_2 \nu_{\partial D}$ is tangent to ∂D , the last orthogonality property implies that g_1 is constant along the boundary ∂D , that is, there exists a constant $k \in \mathbb{R}$ such that $g_1 = k$ on ∂D . Thus, since g is harmonic in D , the function g_1 solves the Dirichlet problem

$$\begin{cases} \Delta g_1 = -\alpha_0 & \text{in } D, \\ g_1 = k & \text{on } \partial D. \end{cases}$$

Hence g_1 is a translation of a multiple of the torsion function of D . In fact, the function $g_2 := (g_1 - k)/\alpha_0$ satisfies $\Delta g_2 = -1$ in D and $g_2 = 0$ on ∂D , namely g_2 solves (4.2), and therefore it is the torsion function v_D of the set D . Thus,

$$v_D = \frac{g_1 - k}{\alpha_0}, \quad g_1 = k + \alpha_0 v_D. \quad (4.21)$$

Since D is convex, by Theorem 2.3, $\sqrt{v_D}$ is a concave function. Moreover, the superlevel sets $\{v_D \geq s\}$ are convex for any $s \in [0, M]$, v_D has a unique critical point $x_* \in D$, which is

the maximum point (i.e., $v_D(x_*) = M$), and the level sets of v_D are smooth closed curves. Hence the vectors $\nabla g_1 = \alpha_0 \nabla v_D$ and $\mathcal{J}_2 \nabla g_1 = \alpha_0 \mathcal{J}_2 \nabla v_D$, except at the point x_* , are nonzero orthogonal vectors. For any $s \in [0, M)$, consider the level set

$$\mathcal{E}_s := \{x \in D : v_D(x) = s\} = \{x \in D : g_1(x) = k + \alpha_0 s\}. \quad (4.22)$$

For any point $x \in \mathcal{E}_s$, the vector $\nabla g_1(x)$ is normal to $\mathcal{E}_s$ at x , and therefore $\mathcal{J}_2 \nabla g_1(x)$ is tangent to $\mathcal{E}_s$ at x . By (4.20), $\mathcal{J}_2 \nabla g_1 = \mathcal{J}_2^2 \tilde{u} = -\tilde{u}$, and, by (4.11),

$$\langle \nabla f, \mathcal{J}_2 \nabla g_1 \rangle = -\langle \nabla f, \tilde{u} \rangle = 0 \quad \text{in } D.$$

Since $\mathcal{J}_2 \nabla g_1$ is tangent to $\mathcal{E}_s$, the last orthogonality property implies that f is constant along the curve $\mathcal{E}_s$. Denote $\tilde{f}(s)$ the value of f on $\mathcal{E}_s$, that is, $f(x) = \tilde{f}(s)$ for all $x \in \mathcal{E}_s$. Then $f(x) = \tilde{f}(s) = \tilde{f}(v_D(x))$ for all $x \in \mathcal{E}_s$, for all $s \in [0, M)$. Also, $f(x_*) = \tilde{f}(M)$. This proves that $f = \tilde{f} \circ v_D$ in the closure of D . The regularity of $\tilde{f}$ comes from that of f and from the properties of the torsion function v_D already recalled. One has $\tilde{f}(0) = 0$ because $f = 0$ on the boundary ∂D , which is the level set $\mathcal{E}_0$.

Now we write identity (4.10) in terms of $\tilde{f}$ and v_D . First, by (4.20) and (4.21), one has $\mathcal{J}_2 \tilde{u} = \nabla g_1 = \alpha_0 \nabla v_D$, whence $|\tilde{u}|^2 = \alpha_0^2 |\nabla v_D|^2$. Second, by (4.19) and (4.21), one has $g = g_1 + \frac{\alpha_0}{4} |x|^2 = k + \alpha_0 v_D + \frac{\alpha_0}{4} |x|^2$. Third, to write $H(f)$, we use the chain rule $\nabla f = \tilde{f}'(v_D) \nabla v_D$. Plugging these identities into (4.10) gives (4.18) with $c = c_1 - \alpha_0 k$. $\square$

Now we show that the gradient of the torsion function on the boundary of a C^2 set admits a lower bound. The proof is rather standard and it is based on the maximum principle.

Lemma 4.7. *Let $D \subset \mathbb{R}^2$ be a bounded open set of class C^2 , and let v_D be its torsion function. Then $v_D \in C^1(\overline{D})$ and there exist two constants $C_0, C_1 \in \mathbb{R}$ such that*

$$0 < C_0 \leq \inf_{\partial D} |\nabla v_D| \leq \sup_{\partial D} |\nabla v_D| \leq C_1 < \infty.$$

Proof. The regularity of v_D can be obtained by the classical regularity theory of elliptic operators, and the existence of the upper bound C_1 is a direct consequence of it. We prove the lower bound C_0 .

By the regularity and compactness of the boundary ∂D , there exists $r_0 > 0$ such that for every point $x_0 \in \partial D$ there exists a point $x_1 \in D$ such that the open disc $B_{r_0}(x_1)$ is contained in the open set D , the circle $\partial B_{r_0}(x_1)$ and the boundary ∂D have one point in common, which is x_0 , and no other common points, and $\partial B_{r_0}(x_1)$ is tangent to ∂D at x_0 .

Let $x_0 \in \partial D$, and let $B := B_{r_0}(x_1)$ be the tangent disc described above. The torsion function v_B of the ball B is explicit, and it is

$$v_B(x) = \frac{r_0^2 - |x - x_1|^2}{4}.$$

The difference $u := v_D - v_B$ satisfies

$$\begin{cases} \Delta u = 0 & \text{in } B, \\ u \geq 0 & \text{on } \partial B, \end{cases}$$

because $v_D \geq 0$ in $\overline{D}$, and, in particular, on ∂B , while $v_B = 0$ on ∂B . Also, $u(x_0) = 0$ because $x_0 \in \partial D \cap \partial B$. Since u is harmonic in B , by the maximum principle we have

$$0 = u(x_0) = \min_{\partial B} u = \min_B u.$$

Moreover u is not constant in B because $u > 0$ on $(\partial B) \cap D = (\partial B) \setminus \{x_0\}$. Since x_0 is a minimum point for the harmonic function u , by Hopf Lemma we have

$$\frac{\partial}{\partial \nu} u(x_0) < 0.$$

Since ∂B is tangent to ∂D from inside, the two curves have the same normal $\nu_{\partial D}(x_0) = \nu_{\partial B}(x_0)$ at x_0 . Therefore

$$\langle \nabla v_D(x_0), \nu_{\partial D}(x_0) \rangle - \langle \nabla v_B(x_0), \nu_{\partial B}(x_0) \rangle = \langle \nabla u(x_0), \nu_{\partial B}(x_0) \rangle < 0.$$

By (2.6), we have $\nabla v_D(x_0) = -|\nabla v_D(x_0)|\nu_{\partial D}(x_0)$ and $\nabla v_B(x_0) = -|\nabla v_B(x_0)|\nu_{\partial B}(x_0)$. Hence

$$|\nabla v_D(x_0)| > |\nabla v_B(x_0)| = \frac{r_0}{2}.$$

This proves the lower bound with $C_0 = r_0/2$, uniformly in $x_0 \in \partial D$. $\square$

We are now in position to prove the first part of Theorem 4.2. We compute

$$\begin{aligned} & \operatorname{div} \left(\frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2}} \right) \\ &= \frac{\tilde{f}''(v_D) |\nabla v_D|^2 - \tilde{f}'(v_D)}{\sqrt{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2}} - \frac{(\tilde{f}'(v_D))^2 \tilde{f}''(v_D) |\nabla v_D|^4 + \frac{1}{2} (\tilde{f}'(v_D))^3 \langle \nabla(|\nabla v_D|^2), \nabla v_D \rangle}{\{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2\}^{\frac{3}{2}}} \\ &= \frac{\tilde{f}''(v_D) |\nabla v_D|^2 - \tilde{f}'(v_D) \{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2\} - \frac{1}{2} (\tilde{f}'(v_D))^3 \langle \nabla(|\nabla v_D|^2), \nabla v_D \rangle}{\{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2\}^{\frac{3}{2}}}. \end{aligned}$$

As already observed above, by the maximum principle, the function v_D is positive and by classic PDE results it is smooth. Moreover, the only critical point of v_D is at the maximum of v_D , $v_D(x_*) = M$, since the torsion function of a convex domain is $1/2$ -concave. Therefore, every $s \in (0, M)$ is a regular value, i.e., for every s , on the level set $\mathcal{E}_s = \{v_D = s\}$ the gradient of v_D does not vanish, the curve $\mathcal{E}_s$ is smooth (by implicit function theorem) and connected, since $\sqrt{v_D}$ is concave. The curve $\mathcal{E}_s$ defined in (4.22) is a level set of the function v_D , and therefore, by (2.3), its curvature is

$$H_{\mathcal{E}_s} = -\operatorname{div} \left(\frac{\nabla v_D}{|\nabla v_D|} \right) = - \left(\frac{\Delta v_D}{|\nabla v_D|} - \frac{\langle \nabla(|\nabla v_D|), \nabla v_D \rangle}{|\nabla v_D|^2} \right) = \frac{|\nabla v_D| + \langle \nabla(|\nabla v_D|), \nabla v_D \rangle}{|\nabla v_D|^2}.$$

Hence

$$\frac{1}{2} \langle \nabla(|\nabla v_D|^2), \nabla v_D \rangle = |\nabla v_D| \langle \nabla(|\nabla v_D|), \nabla v_D \rangle = |\nabla v_D|^3 H_{\mathcal{E}_s} - |\nabla v_D|^2,$$

whence

$$\operatorname{div} \left(\frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2}} \right) = \frac{\tilde{f}''(v_D) |\nabla v_D|^2 - \tilde{f}'(v_D) - (\tilde{f}'(v_D))^3 |\nabla v_D|^3 H_{\mathcal{E}_s}}{\{1 + (\tilde{f}'(v_D))^2 |\nabla v_D|^2\}^{\frac{3}{2}}}.$$

Thus identity (4.18) on the level set $\mathcal{E}_s$, $s \in (0, M)$, becomes

$$\frac{\alpha_0^2}{2} |\nabla v_D|^2 + \alpha_0^2 s - \sigma_0 \frac{\tilde{f}''(s) |\nabla v_D|^2 - \tilde{f}'(s) - (\tilde{f}'(s))^3 |\nabla v_D|^3 H_{\mathcal{E}_s}}{\{1 + (\tilde{f}'(s))^2 |\nabla v_D|^2\}^{\frac{3}{2}}} = c \quad \text{on } \mathcal{E}_s. \quad (4.23)$$

Now we prove some limit properties for the function $\tilde{f}$ close to the level $s = 0$, related to properties of the profile function f close to $\partial\Omega \cap \{x_3 = 0\} = \partial D \times \{0\}$.

Lemma 4.8. *The function $\tilde{f}$ obtained in Lemma 4.6 satisfies*

$$\lim_{s \rightarrow 0^+} \tilde{f}'(s) = \infty \quad (4.24)$$

and

$$\beta := \lim_{s \rightarrow 0^+} \frac{\tilde{f}''(s)}{(\tilde{f}'(s))^3} \in \mathbb{R}. \quad (4.25)$$

Proof. By Lemma 4.7, $|\nabla v_D|$ is bounded from below and above on ∂D , and therefore, by continuity, also on a neighborhood of ∂D in D . Consider a point $x_0 \in \partial D$. Let φ be the solution of the Cauchy problem

$$\varphi'(t) = \frac{(\nabla v_D)(\varphi(t))}{|(\nabla v_D)(\varphi(t))|^2}, \quad \varphi(0) = x_0$$

in the interval $t \in [0, \delta]$, where $\delta > 0$ is sufficiently small to have $\varphi(t) \in D$ for all $t \in (0, \delta]$. The composition $\psi(t) := v_D(\varphi(t))$ satisfies $\psi(0) = v_D(x_0) = 0$ and $\psi'(t) = \langle (\nabla v_D)(\varphi(t)), \varphi'(t) \rangle = 1$. Hence $\psi(t) = t$ on $[0, \delta]$, namely

$$v_D(\varphi(t)) = t \quad \forall t \in [0, \delta],$$

so that $\varphi(t)$ belongs to the level set $\mathcal{E}_t$, for all $t \in [0, \delta]$. The point $(\varphi(t), f(\varphi(t)))$ on the boundary $\partial\Omega$ converges, as $t \rightarrow 0$, to the point $(\varphi(0), f(\varphi(0))) = (x_0, 0) \in \partial\Omega$. Hence the normal to the boundary $\partial\Omega$ at the point $(\varphi(t), f(\varphi(t)))$, which is given in (4.13), converges, as $t \rightarrow 0$, to the normal to $\partial\Omega$ at the point $(x_0, 0)$, which is given in (4.14). In particular, the third component of these normal vectors satisfies

$$\lim_{t \rightarrow 0} \frac{1}{\sqrt{1 + |(\nabla f)(\varphi(t))|^2}} = 0.$$

Hence $|(\nabla f)(\varphi(t))| \rightarrow \infty$ as $t \rightarrow 0$. Since $f = \tilde{f} \circ v_D$, one has $\nabla f(x) = \tilde{f}'(v_D(x)) \nabla v_D(x)$, and

$$(\nabla f)(\varphi(t)) = \tilde{f}'(v_D(\varphi(t))) (\nabla v_D)(\varphi(t)) = \tilde{f}'(t) (\nabla v_D)(\varphi(t)).$$

Since $|\nabla v_D|$ is bounded, $|\tilde{f}'(t)| \rightarrow \infty$ as $t \rightarrow 0$. Hence $|\tilde{f}'| > 1$ in some interval $(0, \delta_1)$, and, since $\tilde{f}'$ is continuous, it cannot change sign in $(0, \delta_1)$. Moreover $\tilde{f}(0) = 0$, $\tilde{f} > 0$ on $(0, \delta_1)$, whence $\tilde{f}' > 0$ at a point, and therefore at all points of $(0, \delta_1)$. Hence $|\tilde{f}'| = \tilde{f}'$, which gives (4.24). Identity (4.23) at the point $\varphi(s)$, with $s \in (0, \delta]$, gives

$$\frac{\tilde{f}''(s)}{(\tilde{f}'(s))^3} = \frac{V(s)\mu(s)}{\sigma_0} \left\{ \frac{\alpha_0^2}{2} V^2(s) + \alpha_0^2 s + \frac{\sigma_0}{\mu(s)(\tilde{f}'(s))^2 V^3(s)} + \frac{\sigma_0 H_{\mathcal{E}_s}(\varphi(s))}{\mu(s)} - c \right\},$$

where

$$V(s) := |(\nabla v_D)(\varphi(s))|, \quad \mu(s) := \left(1 + \frac{1}{(\tilde{f}'(s))^2 V^2(s)} \right)^{\frac{3}{2}}.$$

For $s \rightarrow 0$, one has

$$\tilde{f}'(s) \rightarrow \infty, \quad V(s) \rightarrow |\nabla v_D(x_0)|, \quad \mu(s) \rightarrow 1, \quad H_{\mathcal{E}_s}(\varphi(s)) \rightarrow H_{\partial D}(x_0),$$

whence

$$\lim_{s \rightarrow 0^+} \frac{\tilde{f}''(s)}{(\tilde{f}'(s))^3} = \frac{|\nabla v_D(x_0)|}{\sigma_0} \left\{ \frac{\alpha_0^2}{2} |\nabla v_D(x_0)|^2 + \sigma_0 H_{\partial D}(x_0) - c \right\}, \quad (4.26)$$

which is a finite limit. $\square$

Taking the limit as $s \rightarrow 0^+$ in (4.23), or using directly formula (4.26) of the constant β , we obtain (4.3). To complete the proof of Theorem 4.2, it remains to prove that the constants β and c depend only on the set D .

4.2 The constants β and c

The problem in (4.3) is related to the famous Serrin overdetermined problem. We now investigate further this link, by computing the values of the constants β and c . The definition of the constant β in (4.25) in terms of the profile function f suggests that β depends on the three-dimensional nature of the problem. We show that this is not (completely) true, because, assuming that the drop Ω is smooth, the tangent plane to the boundary $\partial\Omega$ at all points of $\partial\Omega \cap \{x_3 = 0\}$ is vertical, the normal at those points is horizontal, and, thanks to this geometrical observation, we show that the knowledge of the planar set D is sufficient to compute the value of β . In the following lemma we compute the value of the constant c . Recall that we are assuming that the barycenter of Ω is at the origin of $\mathbb{R}^3$, hence the origin of $\mathbb{R}^2$ is in D . Also recall that $P(D)$ denotes the perimeter of the set D .

Lemma 4.9. *The constant c in (4.18) is uniquely determined by the formula*

$$c = \frac{1}{|D|} \left(\frac{3\alpha_0^2}{2} \int_D |\nabla v_D|^2 dx + \sigma_0 P(D) \right). \quad (4.27)$$

Proof. For $s \in (0, M]$, we consider the superlevel set $D_s := \{x \in D : v_D(x) > s\}$, and we integrate (4.18) over D_s ,

$$\frac{\alpha_0^2}{2} \int_{D_s} |\nabla v_D|^2 dx + \alpha_0^2 \int_{D_s} v_D dx - \sigma_0 \int_{D_s} \operatorname{div} \left(\frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + |\tilde{f}'(v_D) \nabla v_D|^2}} \right) dx = c |D_s|.$$

The boundary ∂D_s is the level set $\mathcal{E}_s$ in (4.22) of the torsion function v_D , the gradient ∇v_D is orthogonal to the curve $\mathcal{E}_s$, and the outward unit normal to $\mathcal{E}_s$ is

$$\nu_{\partial D_s} = -\frac{\nabla v_D}{|\nabla v_D|} \quad \text{on } \partial D_s.$$

From the divergence theorem,

$$\begin{aligned} \int_{D_s} \operatorname{div} \left(\frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + |\tilde{f}'(v_D) \nabla v_D|^2}} \right) dx &= \int_{\partial D_s} \left\langle \frac{\tilde{f}'(v_D) \nabla v_D}{\sqrt{1 + |\tilde{f}'(v_D) \nabla v_D|^2}}, \nu_{\partial D_s} \right\rangle d\sigma \\ &= - \int_{\partial D_s} \frac{\tilde{f}'(s) |\nabla v_D|}{\sqrt{1 + |\tilde{f}'(s) \nabla v_D|^2}} d\sigma. \end{aligned}$$

By (4.24) and Lemma 4.7, the function in the last integral converges to 1 uniformly as $s \rightarrow 0^+$, and therefore the integral converges to $\int_{\partial D} 1 d\sigma = P(D)$. Taking the limit as $s \rightarrow 0^+$, we obtain

$$\frac{\alpha_0^2}{2} \int_D |\nabla v_D|^2 dx + \alpha_0^2 \int_D v_D dx + \sigma_0 P(D) = c|D|.$$

By (2.7), we obtain (4.27). $\square$

As observed in Lemma 4.9, the constant c can be computed just from the knowledge of D . At this point, we are also able to compute the value of the constant β , showing that also β is given only in terms of quantities related to the planar set D . This completes the proof of Theorem 4.2.

Lemma 4.10. *Assume the hypotheses of Theorem 4.2. The constant β defined in (4.25) is given by*

$$\beta = - \left(\sigma_0 \int_{\partial D} \frac{\langle x, \nu_{\partial D} \rangle}{|\nabla v_D|} d\sigma \right)^{-1} \left\{ \sigma_0 P(D) + \alpha_0^2 \int_D |\nabla v_D|^2 dx \right\}. \quad (4.28)$$

Proof. Multiplying (4.3) by $\langle x, \nu_{\partial D} \rangle$ and integrating along the curve ∂D , we get

$$\frac{\alpha_0^2}{2} \int_{\partial D} |\nabla v_D|^2 \langle x, \nu \rangle d\sigma - \sigma_0 \beta \int_{\partial D} \frac{\langle x, \nu \rangle}{|\nabla v_D|} d\sigma + \sigma_0 \int_{\partial D} H_{\partial D} \langle x, \nu \rangle d\sigma = c \int_{\partial D} \langle x, \nu \rangle d\sigma. \quad (4.29)$$

We calculate each term. By the divergence theorem,

$$\begin{aligned} \int_{\partial D} |\nabla v_D|^2 \langle x, \nu_{\partial D} \rangle d\sigma &= \int_D \operatorname{div} \{ x |\nabla v_D|^2 \} dx \\ &= \int_D (\operatorname{div} x) |\nabla v_D|^2 dx + \int_D \langle x, \nabla (|\nabla v_D|^2) \rangle dx \\ &= 2 \int_D |\nabla v_D|^2 dx + 2 \int_D \langle (\nabla^2 v_D)(\nabla v_D), x \rangle dx, \end{aligned}$$

where $\nabla^2 v_D$ is the Hessian matrix of v_D . From the general identity

$$\operatorname{div} \{ \langle \nabla w, x \rangle \nabla w \} = \langle \nabla w, x \rangle \Delta w + |\nabla w|^2 + \langle (\nabla^2 w)(\nabla w), x \rangle$$

applied to the torsion function v_D , using the identity $\Delta w_D = -1$, we find

$$\int_D \langle (\nabla^2 v_D)(\nabla v_D), x \rangle dx = \int_D \operatorname{div} \{ \langle \nabla v_D, x \rangle \nabla v_D \} dx + \int_D \langle \nabla v_D, x \rangle dx - \int_D |\nabla v_D|^2 dx.$$

By the divergence theorem and formula (2.6) for the gradient of v_D ,

$$\int_D \operatorname{div} \{ \langle \nabla v_D, x \rangle \nabla v_D \} dx = \int_{\partial D} \langle \nabla v_D, x \rangle \langle \nabla v_D, \nu_{\partial D} \rangle d\sigma = \int_{\partial D} |\nabla v_D|^2 \langle x, \nu_{\partial D} \rangle d\sigma.$$

By the divergence theorem applied to the vector field xv_D , which has divergence $\operatorname{div}(xv_D) = (\operatorname{div} x)v_D + \langle x, \nabla v_D \rangle = 2v_D + \langle x, \nabla v_D \rangle$ in D and boundary value $xv_D = 0$ on ∂D by (4.2), one has

$$0 = \int_{\partial D} \langle xv_D, \nu_{\partial D} \rangle d\sigma = \int_D \operatorname{div}(xv_D) dx = 2 \int_D v_D dx + \int_D \langle x, \nabla v_D \rangle dx,$$

and, by (2.7),

$$\int_D \langle x, \nabla v_D \rangle dx = -2 \int_D v_D dx = -2 \int_D |\nabla v_D|^2 dx.$$

From these identities, we obtain

$$\int_{\partial D} |\nabla v_D|^2 \langle x, \nu_{\partial D} \rangle d\sigma = 4 \int_D |\nabla v_D|^2 dx. \quad (4.30)$$

By (2.4) with $X(x) = x$, since $\operatorname{div}_{\partial D} x = 1$, we get

$$\int_{\partial D} H_{\partial D} \langle x, \nu_{\partial D} \rangle d\sigma = P(D),$$

and, from the divergence theorem,

$$\int_{\partial D} \langle x, \nu_{\partial D} \rangle d\sigma = \int_D \operatorname{div} x dx = \int_D 2 dx = 2|D|.$$

Hence (4.29) becomes

$$2\alpha_0^2 \int_D |\nabla v_D|^2 dx - \sigma_0 \beta \int_{\partial D} \frac{\langle x, \nu_{\partial D} \rangle}{|\nabla v_D|} d\sigma + \sigma_0 P(D) = 2c|D|.$$

Replacing the constant c with its formula in (4.27), we get the result. $\square$

There is also another way to determine the constants β and c .

Lemma 4.11. *Assume the hypotheses of Theorem 4.2. Then the constants β and c satisfy*

$$\sigma_0 \beta \left(\int_{\partial D} \frac{1}{|\nabla v_D|} d\sigma \right) + cP(D) = \frac{\alpha_0^2}{2} \left(\int_{\partial D} |\nabla v_D|^2 d\sigma \right) + 2\pi\sigma_0, \quad (4.31)$$

$$\sigma_0 \beta P(D) + c|D| = \frac{\alpha_0^2}{2} \left(\int_{\partial D} |\nabla v_D|^3 d\sigma \right) + \sigma_0 \left(\int_{\partial D} H_{\partial D} |\nabla v_D| d\sigma \right). \quad (4.32)$$

Moreover, the solution of system (4.31), (4.32) is unique if and only if D is not a disk.

Proof. The constants c and β are uniquely determined if D is not a disk. In fact, integrating (4.3) along ∂D , by (2.1) we get

$$\frac{\alpha_0^2}{2} \int_{\partial D} |\nabla v_D|^2 d\sigma - \sigma_0 \beta \int_{\partial D} \frac{1}{|\nabla v_D|} d\sigma + 2\pi\sigma_0 = cP(D).$$

Moreover, multiplying (4.3) by $|\nabla v_D|$ and integrating along ∂D , by (2.8) we have

$$\frac{\alpha_0^2}{2} \int_{\partial D} |\nabla v_D|^3 d\sigma - \sigma_0 \beta P(D) + \sigma_0 \int_{\partial D} H_{\partial D} |\nabla v_D| d\sigma = c \int_{\partial D} |\nabla v_D| d\sigma = c|D|.$$

Therefore β and c satisfy (4.31), (4.32). This linear system has a unique solution if and only if the corresponding matrix has nonzero determinant, i.e., if and only if the difference

$$\left(\int_{\partial D} \frac{1}{|\nabla v_D|} d\sigma \right) |D| - P^2(D)$$

is nonzero. By Hölder inequality,

$$P(D) = \int_{\partial D} 1 d\sigma = \int_{\partial D} \frac{|\nabla v_D|^{\frac{1}{2}}}{|\nabla v_D|^{\frac{1}{2}}} d\sigma \leq \left(\int_{\partial D} |\nabla v_D| d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial D} \frac{1}{|\nabla v_D|} d\sigma \right)^{\frac{1}{2}},$$

with equality if and only if $|\nabla v_D|$ is constant, i.e., by Serrin theorem, if and only if D is a disk. $\square$

Lemma 4.12. *Assume the hypotheses of Theorem 4.2. Then*

$$\begin{aligned} \sigma_0 \beta = \frac{1}{P(D)} & \left\{ \frac{\alpha_0^2}{2} \left(\int_{\partial D} |\nabla v_D|^3 d\sigma - 3 \int_D |\nabla v_D|^2 dx \right) \right. \\ & \left. + \sigma_0 \left(\int_{\partial D} H_{\partial D} |\nabla v_D| d\sigma - P(D) \right) \right\}. \end{aligned} \quad (4.33)$$

Proof. Plug (4.27) into (4.32). $\square$

4.3 Torsion function and rigidity

We now study in details the constant β . This will be the key to conclude our proof.

By (4.28), we see immediately that $\beta < 0$, because the scalar product $\langle x, \nu_{\partial D} \rangle$ is positive for all $x \in \partial D$; in fact, this property holds for all star-shaped sets with respect to the origin. To obtain a stronger bound on β , we use (4.33), starting with analysing the terms with coefficient σ_0 . We use Reilly formula (2.5).

Lemma 4.13. *Let $D \subset \mathbb{R}^2$ be a bounded open convex set with C^2 boundary, and let v_D be its torsion function. Then*

$$\int_{\partial D} H_{\partial D} |\nabla v_D| d\sigma \leq \frac{P(D)}{2}. \quad (4.34)$$

Proof. To lighten the notation, in this proof we denote v the torsion function v_D of the set D , ν the normal vector $\nu_{\partial D}$, and H the mean curvature $H_{\partial D}$. Reilly formula (2.5) applied to the set D and the torsion function v is

$$\int_{\partial D} H \langle \nabla v, \nu \rangle^2 d\sigma = \int_D \left((\Delta v)^2 - |\nabla^2 v|^2 \right) dx,$$

where $\nabla^2 v$ is the Hessian matrix of v . By Cauchy-Schwarz inequality, one has the pointwise inequality for the Hessian matrix and its trace

$$\frac{(\Delta v)^2}{2} \leq |\nabla^2 v|^2.$$

Therefore, by (2.6),

$$\int_{\partial D} H |\nabla v|^2 d\sigma = \int_{\partial D} H \langle \nabla v, \nu \rangle^2 d\sigma \leq \frac{1}{2} \int_D (\Delta v)^2 dx = \frac{|D|}{2}.$$

Since D is convex, $H \geq 0$. Hence, by Hölder inequality, Gauss-Bonnet formula (2.1), and the isoperimetric inequality, we have

$$\int_{\partial D} H |\nabla v| d\sigma \leq \left(\int_{\partial D} H |\nabla v|^2 d\sigma \right)^{\frac{1}{2}} \left(\int_{\partial D} H d\sigma \right)^{\frac{1}{2}} \leq \sqrt{\pi |D|} \leq \frac{P(D)}{2}. \quad \square$$

Now we study the terms with coefficient α_0^2 in (4.33).

Lemma 4.14. *Let D, v_D be like in Lemma 4.13. Then*

$$2 \int_D v_D \det(\nabla^2 v_D) dx = \int_D |\nabla v_D|^2 dx + \int_D \langle (\nabla^2 v_D) \nabla v_D, \nabla v_D \rangle dx, \quad (4.35)$$

$$\int_{\partial D} |\nabla v_D|^3 d\sigma = 3 \int_D |\nabla v_D|^2 dx - 4 \int_D v_D \det(\nabla^2 v_D) dx, \quad (4.36)$$

where $\nabla^2 v_D$ is the Hessian matrix of v_D .

Proof. To lighten the notation, in this proof we denote v the torsion function v_D , $v_j = \partial_{x_j} v$, $v_{jk} = \partial_{x_j x_k} v$ its partial derivatives, and ν the normal $\nu_{\partial D}$. By (2.6) and the divergence theorem,

$$\begin{aligned} \int_{\partial D} |\nabla v|^3 d\sigma &= - \int_{\partial D} |\nabla v|^2 \langle \nabla v, \nu \rangle d\sigma \\ &= - \int_D \operatorname{div}\{|\nabla v|^2 \nabla v\} dx \\ &= - \int_D |\nabla v|^2 \Delta v dx - 2 \int_D \langle (\nabla^2 v) \nabla v, \nabla v \rangle dx. \end{aligned} \quad (4.37)$$

One has

$$\int_D |\nabla v|^2 \Delta v dx = - \int_D |\nabla v|^2 dx$$

because $\Delta v = -1$ in D . For the other integral, we apply the divergence theorem to the vector field $F = (\nabla^2 v)(\nabla v)v$, and we get

$$\int_D \operatorname{div}\{(\nabla^2 v)(\nabla v)v\} dx = \int_D \operatorname{div} F dx = \int_{\partial D} \langle F, \nu \rangle d\sigma = 0$$

because $v = 0$ on ∂D , whence also $F = 0$ on ∂D . We calculate

$$\operatorname{div}\{(\nabla^2 v)(\nabla v)v\} = \operatorname{div}\{(\nabla^2 v)(\nabla v)\}v + \langle (\nabla^2 v)(\nabla v), \nabla v \rangle$$

and

$$\operatorname{div}\{(\nabla^2 v)(\nabla v)\} = \sum_{k,j} \partial_{x_k} (v_{kj} v_j) = \sum_{k,j} v_{kkj} v_j + \sum_{k,j} v_{kj}^2 = \langle \nabla(\Delta v), \nabla v \rangle + |\nabla^2 v|^2 = |\nabla^2 v|^2,$$

because $\nabla(\Delta v) = \nabla(-1) = 0$. Moreover,

$$|\nabla^2 v|^2 = v_{11}^2 + 2v_{12}^2 + v_{22}^2 = (v_{11} + v_{22})^2 - 2v_{11}v_{22} + 2v_{12}^2 = (\Delta v)^2 - 2\det(\nabla^2 v).$$

Since $(\Delta v)^2 = 1$, we obtain

$$\begin{aligned} \int_D \langle (\nabla^2 v) \nabla v, \nabla v \rangle dx &= - \int_D v \operatorname{div}\{(\nabla^2 v)(\nabla v)\} dx \\ &= - \int_D v |\nabla^2 v|^2 dx \\ &= - \int_D v \{1 - 2\det(\nabla^2 v)\} dx \\ &= - \int_D v dx + 2 \int_D v \det(\nabla^2 v) dx, \end{aligned}$$

and, recalling (2.7), we get (4.35). Then (4.36) trivially follows from (4.37) and (4.35). $\square$

When D is an ellipse, the torsion function v_D is concave (see [14]), and therefore the eigenvalues of the Hessian matrix $\nabla^2 v_D$ are both ≤ 0 ; this implies that $\det(\nabla^2 v_D) \geq 0$, and hence, by (4.36),

$$\int_{\partial D} |\nabla v_D|^3 d\sigma \leq 3 \int_D |\nabla v_D|^2 d\sigma. \quad (4.38)$$

For a more general convex set D , the torsion function v_D is $\frac{1}{2}$ -concave, that is, $\sqrt{v_D}$ is concave, but, in general, v_D itself is not concave, so that we cannot use the inequality $v_D \det(\nabla^2 v_D) \geq 0$ *pointwise*. Nonetheless, in the next lemma we show that inequality (4.38) holds for general convex sets.

Lemma 4.15. *Let D, v_D be like in Lemma 4.13. Then*

$$\int_D v_D \det(D^2 v_D) dx \geq 0, \quad (4.39)$$

and (4.38) holds.

Proof. To lighten the notation, in this proof we denote v the torsion function v_D , $v_j = \partial_{x_j} v$, $v_{jk} = \partial_{x_j x_k} v$ its partial derivatives, and ν the normal $\nu_{\partial D}$. From [25], we know that v is $\frac{1}{2}$ -concave, that is, the Hessian matrix of $v^{\frac{1}{2}}$ is a nonpositive definite matrix. Hence its determinant (product of its two eigenvalues) is ≥ 0 . We calculate the partial derivatives

$$\partial_{ij}(v^{\frac{1}{2}}) = \frac{2vv_{ij} - v_i v_j}{4v^{3/2}},$$

and

$$\begin{aligned} 0 \leq \det[\nabla^2(v^{\frac{1}{2}})] &= \frac{1}{16v^3} \{(2vv_{11} - v_1^2)(2vv_{22} - v_2^2) - (2vv_{12} - v_1 v_2)^2\} \\ &= \frac{\det(\nabla^2 v)}{4v} - \frac{v_1^2 v_{22} + v_2^2 v_{11} - 2v_1 v_2 v_{12}}{8v^2}. \end{aligned}$$

Multiplying by $8v^2$,

$$2v \det(\nabla^2 v) \geq v_1^2 v_{22} + v_2^2 v_{11} - 2v_1 v_2 v_{12}.$$

Since $v_{11} + v_{22} = \Delta v = -1$, we have

$$\begin{aligned} v_1^2 v_{22} + v_2^2 v_{11} - 2v_1 v_2 v_{12} &= v_1^2(-1 - v_{11}) + v_2^2(-1 - v_{22}) - 2v_1 v_2 v_{12} \\ &= -|\nabla v|^2 - \langle (\nabla^2 v) \nabla v, \nabla v \rangle. \end{aligned}$$

Integrating over D , and recalling (4.35), we obtain

$$2 \int_D v \det(\nabla^2 v) dx \geq - \int_D |\nabla v|^2 dx - \int_D \langle (\nabla^2 v) \nabla v, \nabla v \rangle dx = -2 \int_D v \det(\nabla^2 v) dx,$$

which gives (4.39). Then (4.38) follows trivially from (4.39) and (4.36). $\square$

Now we have all the ingredients to prove Theorem 1.7.

Proof of Theorem 1.7. From (4.33), (4.34), (4.38) (see Lemma 4.15), it follows that

$$\beta \leq -\frac{1}{2}. \quad (4.40)$$

By Theorem 4.2, the torsion function v_D satisfies (4.3), that is,

$$F_1(|\nabla v_D|) = c - \sigma_0 H_{\partial D} \quad \text{on } \partial D,$$

where F_1 is the real function

$$F_1 : (0, \infty) \rightarrow \mathbb{R}, \quad F_1(t) := \frac{\alpha_0^2}{2} t^2 - \frac{\sigma_0 \beta}{t} = \frac{\alpha_0^2}{2} t^2 + \frac{\sigma_0 |\beta|}{t}.$$

The function F_1 is decreasing in the interval $(0, t_1]$, where

$$t_1 := \left(\frac{\sigma_0 |\beta|}{\alpha_0^2} \right)^{\frac{1}{3}}.$$

By (4.40), one has $|\beta| \geq \frac{1}{2}$, whence

$$t_1 \geq \left(\frac{\sigma_0}{2\alpha_0^2} \right)^{\frac{1}{3}}.$$

By Theorem 2.4,

$$\|\nabla v_D\|_{L^\infty(D)} < \sqrt{\frac{|D|}{2\pi}}.$$

Therefore, if

$$\sqrt{\frac{|D|}{2\pi}} \leq t_1, \quad (4.41)$$

then $|\nabla v_D| \in (0, t_1)$. Note that inequality (1.6) implies (4.41). Since F_1 is invertible in $(0, t_1]$, with decreasing inverse F_1^{-1} , we have

$$\frac{\partial v_D}{\partial \nu_{\partial D}} = \langle \nabla v_D, \nu_{\partial D} \rangle = -|\nabla v_D| = -F_1^{-1}(c - \sigma_0 H_{\partial D}) =: f(H_{\partial D}), \quad (4.42)$$

and f is decreasing. To apply Serrin Theorem 2.2 to the set D , we just need to show that its boundary ∂D is of class C^3 . This is achieved by a simple bootstrap argument based on Schauder regularity theory (see [13], Chapter 6, and [12]). By assumption, the boundary $\partial\Omega$ of the fluid domain $\Omega \subset \mathbb{R}^3$ is of class C^2 , hence the boundary ∂D of the planar set $D \subset \mathbb{R}^2$ in (4.1) is also of class C^2 . Thus, in particular, ∂D is of class $C^{1,\alpha}$ for all $\alpha \in (0, 1)$, and therefore (see [12], page 69, at the end of section 2.6), the torsion function v_D , which is the solution of (4.2), is $C^{1,\alpha}(\overline{D})$ for any $\alpha \in (0, 1)$. Since ∇v_D is nonzero on ∂D , with lower and upper bound in Lemma 4.7, from identity (4.3) we obtain that

$$H_{\partial D} \in C^{0,\alpha}(\partial D). \quad (4.43)$$

Since we already know that the curve ∂D is of class C^2 , (4.43) implies that ∂D is of class $C^{2,\alpha}$. The proof is classical (standard frozen coefficients argument for linear elliptic differential operators). Now, since ∂D is $C^{2,\alpha}$, the torsion function v_D is $C^{2,\alpha}(\overline{D})$ (see Theorem 2.35 in [12], page 69). Hence $H_{\partial D} \in C^{1,\alpha}(\partial D)$, which implies that ∂D is of class $C^{3,\alpha}$. Since ∂D is C^3 , we apply Theorem 2.2, so that D is a disk. Thus, Remark 4.4 applies, and the proof of Theorem 1.7 is complete. $\square$

Remark 4.16. Theorem 1.7 holds also assuming Ω convex, instead of strictly convex. The only change is that $\partial\Omega$ may also contain a cylindrical part between the two graphical parts. The arguments in section 4 and 4.3 that lead to prove that D is a disk are still valid (the notation becomes a bit heavier but the proofs work). Finally, when computing the exact solution following Remark 4.4, one can show that Ω is necessarily strictly convex. Indeed, $\partial\Omega$ would contain a cylindrical part if the mean curvature of $\partial\Omega$ at the curve joining the graphical part and the cylindrical part would not match, hence one would have a discontinuous mean curvature, in contradiction with the assumption that $\partial\Omega$ is of class C^2 . Thus, we use strictly convexity as a simplifying assumption, but in fact it is not necessary to show our rigidity result. $\square$

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