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TRIPLE PLANES WITH $p_g = q = 0$

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Abstract. We show that general triple planes with genus and irregularity zero belong to at most 12 families, that we call surfaces of type I to XII, and we prove that the corresponding Tschirnhausen bundle is a direct sum of two line bundles in cases I, II, III, whereas it is a rank 2 Steiner bundle in the remaining cases.

We also provide existence results and explicit descriptions for surfaces of type I to VII, recovering all classical examples and discovering several new ones. In particular, triple planes of type VII provide counterexamples to a wrong claim made in 1942 by Bronowski.

Finally, in the last part of the paper we discuss some moduli problems related to our constructions.

Contents

Introduction

A *triple plane* is a finite ramified cover $f\colon X \to \mathbb{P}^2$ of degree 3. Let $B \subset \mathbb{P}^2$ be the branch locus of f; then we say that f is a *general* triple plane if the following conditions are satisfied:

- **i)** f is unramified over $\mathbb{P}^2 \backslash B$;
- **ii)** $f^*B = 2R + R_0$, where R is irreducible and non-singular and R_0 is reduced;
- **iii)** $f_{|R}\colon R \to B$ coincides with the normalization map of B.

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The aim of this paper is to address the problem of classifying those smooth, projective surfaces X with $p_g(X) = q(X) = 0$ that arise as general triple planes. We incidentally remark that the corresponding classification problem for *double* planes is instead easy because, by the results of [BHPV04, §22], a smooth double cover $f\colon X \to \mathbb{P}^2$ with $p_g(X) = q(X) = 0$ has either a smooth branch locus of degree 2 (in which case X is isomorphic to a quadric surface $S_2 \subset \mathbb{P}^3$ and f is the projection from a point $p \notin S_2$), or a smooth branch locus of degree 4 (in which case X is the blow-up of a cubic surface $S_3 \subset \mathbb{P}^3$ at one point $p \in S_3$ and f is the resolution of the projection from p).

Some results toward the classification in the triple cover case were obtained by Du Val in [DV33, DV35], where he described those general triple planes whose branch curve has degree at most 14. Du Val's papers are written in the "classical", a bit old-fashioned (and sometimes difficult to read) language and make use of ad-hoc constructions based on synthetic projective geometry (see Remark 3.11 and Remark 3.16 for an outline on Du Val's work). The methods that we propose here are completely different; in fact, they are a mixture of adjunction theory and vector bundle techniques that allow us to treat the problem in a unified way.

The first cornerstone in our work is the general structure theorem for triple covers given in [Mir85, CE96]. More precisely, we relate the existence of a triple cover $f\colon X \to \mathbb{P}^2$ to the existence of a "sufficiently general" element $\eta \in H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$, where $\mathscr{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$ such that $f_*\mathscr{O}_X = \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{E}$. Such a bundle is called the *Tschirnhausen bundle* of the cover, and it turns out that the pair $(\mathscr{E}, \eta)$ completely encodes the geometry of f. Some of the invariants depend directly on $\mathscr{E}$, for instance, setting $b := -c_1(\mathscr{E})$ and $h := c_2(\mathscr{E})$, the branch curve B has degree $2b$ and contains $3h$ ordinary cusps as only singularities; see Proposition 2.4. However the X and f themselves also depend on η; we call η the *building section* of the cover.

So we can try to study general triple planes with $p_g = q = 0$ by analyzing their Tschirnhausen bundles together with the building sections. In fact, we show that these triple planes can be classified in (at most) 12 families, that we call surfaces of type I, II, ..., XII. We are also able to show that surfaces of type I, II, ..., VII actually exist. In the cases I, II, ..., VI we rediscover (in the modern language) the examples described by Du Val. On the other hand, not only are the triple planes of type VII (which have sectional genus equal to 6 and branch locus of degree 16) completely new, but they also provide explicit counterexamples to a wrong claim made by Bronowski in [Bro42]; see Remark 2.8.

A key point in our analysis is the fact that in cases I, II, III the bundle $\mathscr{E}$ splits as the sum of two line bundles, whereas in the remaining cases IV to XII it is indecomposable and it has a resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}^2}(1-b)^{b-4} \to \mathscr{O}_{\mathbb{P}^2}(2-b)^{b-2} \to \mathscr{E} \to 0.$$

This shows that $\mathscr{E}(b-2)$ is a so-called *Steiner bundle* (see §1.4 for more details on this topic), so we can use all the known results about Steiner bundles in order to get information on X. For instance, in cases VI and VII the geometry of the triple plane is tightly related to the existence of *unstable lines* for $\mathscr{E}$; see §§3.6, 3.7.

The second main ingredient in our classification procedure is adjunction theory; see [SV87, Fuj90]. For example, if we write $H = f^*L$, where $L \subset \mathbb{P}^2$ is a general line, we prove that the divisor $D = K_X + 2H$ is very ample (Proposition 2.9), so we consider the corresponding adjunction mapping

$$\varphi_{|K_X+D|} \colon X \to \mathbb{P}(H^0(X,\, \mathscr{O}_X(K_X + D))).$$

Iterating the adjunction process if necessary, we can achieve further information about the geometry of X. Furthermore, when $b \geq 7$ a more refined analysis of the adjunction map allows us to start the process with $D = H$; see Remark 2.18.

As a by-product of our classification, it turns out that general triple planes $f \colon X \to \mathbb{P}^2$ with sectional genus $0 \leq g(H) \leq 5$ (i.e., those of type I, ..., VI) can be realized via an embedding of X into $\mathbb{G}(1,\, \mathbb{P}^3)$ as a surface of bidegree $(3,\, n)$, such that the triple covering f is induced by the projection from a general element of the family of planes of $\mathbb{G}(1,\, \mathbb{P}^3)$ that are n-secant to X. In this way, we relate our work to the work of Gross [Gro93]; see Remarks 3.2, 3.4, 3.6, 3.8, 3.10, 3.15. On the other hand, this is not true for surfaces of type VII: here the only case where the triple cover is induced by an embedding in the Grassmannian is VII.7, where X is a *Reye congruence*, namely an Enriques surface having bidegree $(3,\, 7)$ in $\mathbb{G}(1,\, \mathbb{P}^3)$; see Remark 3.19.

We have not been able so far to use our method beyond case VII; thus the existence of surfaces of type VIII to XII is still an open problem. Furthermore, there are some interesting unsettled questions also in case VII, especially regarding the number of what we call the *unstable conics* for the Tschirnhausen bundle; see §3.7.2 for more details.

Let us explain now how this work is organized. In §1 we set up notation and terminology and we collect the background material which is needed in the sequel of the paper. In particular, we recall the theory of triple covers based on the study of the Tschirnhausen bundle (Theorems 1.2 and 1.3) and we state the main results on adjunction theory for surfaces (Theorem 1.4).

In §2 we start the analysis of general triple planes $f \colon X \to \mathbb{P}^2$ with $p_g(X) = q(X) = 0$. We compute the numerical invariants (degree of the branch locus, number of its cusps, K_X^2, sectional genus) for the surfaces in the 12 families I to XII (Proposition 2.11) and we describe their Tschirnhausen bundle (Theorem 2.12).

Next, §3 is devoted to the detailed description of surfaces of type I to VII. This description leads to a complete classification in cases I to VI (Propositions 3.1, 3.3, 3.5, 3.7, 3.9, 3.12) whereas in case VII we provide many examples, leaving only a few cases unsolved (Proposition 3.17).

Finally, in §4 we study some moduli problems related to our constructions.

Part of the computations in this paper were carried out by using the Computer Algebra System `Macaulay2`; see [GS]. The scripts are included in the Appendix.

1. Basic material

1.1. **Notation and conventions.** We work over the field $\mathbb{C}$ of complex numbers. Given a complex vector space V, we write $\mathbb{P}(V)$ for the projective space of 1-dimensional quotient spaces of V, and $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$. Similarly, if $\mathscr{E}$ is a locally free sheaf over a scheme, we use $\mathbb{P}(\mathscr{E})$ for the projective bundle of its quotients of rank 1. We write $\check{\mathbb{P}}(V)$ for $\mathbb{P}(V^\vee)$, so that $\check{\mathbb{P}}^n$ is the projective space of hyperplanes of $\mathbb{P}^n$. We put $\mathbb{G}(k, \mathbb{P}(V))$ for the Grassmannian of $(k+1)$-dimensional vector subspaces of V.

By "surface" we mean a projective, non-singular surface S, and for such a surface $\omega_S = \mathscr{O}_S(K_S)$ denotes the canonical class, $p_g(S) = h^0(S,\, K_S)$ is the geometric genus, $q(S) = h^1(S, K_S)$ is the irregularity and $\chi(\mathscr{O}_S) = 1 - q(S) + p_g(S)$ is the holomorphic Euler-Poincaré characteristic. We write $P_m(S) = h^0(S, mK_S)$ for the mth plurigenus of S.

If $k \leq n$ are non-negative integers we denote by $S(k, n)$ the rational normal scroll of type (k, n) in $\mathbb{P}^{k+n+1}$, i.e., the image of $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1}(k) \oplus \mathscr{O}_{\mathbb{P}^1}(n))$ by the linear system given by the tautological relatively ample line bundle (see [Har92, Lecture 8] for more details). A cone over a rational normal curve $C \subset \mathbb{P}^n$ of degree n may be thought of as the scroll $S(0, n) \subset \mathbb{P}^{n+1}$.

For $n \geq 1$, we write $\mathbb{F}_n$ for the Hirzebruch surface $\mathbb{P}(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(n))$; every divisor in $\mathrm{Pic}(\mathbb{F}_n)$ can be written as $a\mathfrak{c}_0 + b\mathfrak{f}$, where $\mathfrak{f}$ is the fiber of the $\mathbb{P}^1$-bundle map $\mathbb{F}_n \to \mathbb{P}^1$ and $\mathfrak{c}_0$ is the unique section with negative self-intersection, namely $\mathfrak{c}_0^2 = -n$. Note that the morphism $\mathbb{F}_n \to \mathbb{P}^{n+1}$ associated with the tautological linear system $|\mathfrak{c}_0 + n\mathfrak{f}|$ contracts $\mathfrak{c}_0$ to a point and is an isomorphism outside $\mathfrak{c}_0$, so its image is the cone $S(0, n)$.

For $n = 0$, the surface $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$; every divisor in $\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1)$ is written as $a_1L_1 + a_2L_2$, where the L_i are the two rulings.

The blow-up of $\mathbb{P}^2$ at the points $p_1, \ldots, p_k$ is denoted by $\mathbb{P}^2(p_1, \ldots, p_k)$. If $\sigma \colon \widetilde{X} \to X$ is the blow-up of a surface X at k points, with exceptional divisors $E_1, \ldots, E_k$, and L is a line bundle on X, we will write $L + \sum a_iE_i$ instead of $\sigma^*L + \sum a_iE_i$.

The Chern classes of coherent sheaves on $\mathbb{P}^2$ will usually be written as integers, namely for a sheaf $\mathscr{E}$ we write $c_i(\mathscr{E}) = d_i$, where $d_i \in \mathbb{Z}$ is such that $c_i(\mathscr{E}) = d_i(c_1(\mathscr{O}_{\mathbb{P}^2}(1)))^i$. If $\mathscr{E}$ is a vector bundle, its dual vector bundle is indicated by $\mathscr{E}^\vee$ and its kth symmetric power by $S^k\mathscr{E}$.

We will use basic material and terminology on vector bundles, more specifically on stable vector bundles on $\mathbb{P}^2$; we refer the reader to [OSS80].

1.2. **Triple covers and sections of vector bundles.** A *triple cover* is a finite flat morphism $f \colon X \to Y$ of degree 3. Our varieties X and Y will be smooth, irreducible projective manifolds. With a triple cover is associated an exact sequence

$$0 \to \mathscr{O}_Y \to f_*\mathscr{O}_X \to \mathscr{E} \to 0, \tag{1}$$

where $\mathscr{E}$ is a vector bundle of rank 2 on Y, called the *Tschirnhausen bundle* of f.

Proposition 1.1. *The following hold*:

- **i)** $f_*\mathscr{O}_X = \mathscr{O}_Y \oplus \mathscr{E}$.
- **ii)** $f_*\omega_X = \omega_Y \oplus (\mathscr{E}^\vee \otimes \omega_Y)$.
- **iii)** $f_*\omega_X^2 = S^2\mathscr{E}^\vee \otimes \omega_Y^2$.

Proof. The trace map yields a splitting of sequence (1), hence **i)** follows. Duality for finite flat morphisms implies $f_*\omega_X = (f_*\mathscr{O}_X)^\vee \otimes \omega_Y$, hence we obtain **ii)**. For **iii)** see [Par89, Lemma 8.2]. □

In order to reconstruct f from $\mathscr{E}$ we need an extra datum, namely the building section, which is a global section of $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}$. Moreover, we can naturally see X as sitting into the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathscr{E}^\vee)$ over Y. This is the content of the next two results; see [CE96, Theorem 1.5], [FS01, Proposition 4], [Mir85, Theorem 1.1].

Theorem 1.2. *Any triple cover $f \colon X \to Y$ is determined by a rank 2 vector bundle $\mathscr{E}$ on Y and a global section $\eta \in H^0(Y, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$, and conversely. Moreover, if*

$S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}$ is globally generated, a general global section η defines a triple cover $f\colon X \to Y$.

Theorem 1.3. *Let $f\colon X \to Y$ be a triple cover. Then there exists a unique embedding $i\colon X \to \mathbb{P}(\mathscr{E}^\vee)$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle \pi} & \\ & \mathbb{P}(\mathscr{E}^\vee) & \end{array}$$

According to Theorem 1.2, this embedding induces an isomorphism of X with the zero-scheme $D_0(\eta) \subset \mathbb{P}(\mathscr{E}^\vee)$ of a global section η of the line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E}^\vee)}(3) \otimes \pi^*(\wedge^2\mathscr{E})$.

1.3. **Adjunction theory.** We refer to [BS95, Chapter 10], [DES93, Theorem 1.10], [LP84, Theorem 2.5], [Som79, Proposition 1.5], [SV87, §0] for basic material on adjunction theory.

Theorem 1.4. *Let $X \subset \mathbb{P}^n$ be a smooth surface and D its hyperplane class. Then $|K_X + D|$ is non-special and has dimension $N = g(D) + p_g(X) - q(X) - 1$. Moreover:*

A) *$|K_X + D| = \emptyset$ if and only if*
- **1)** *$X \subset \mathbb{P}^n$ is a scroll over a curve of genus $g(D) = q(X)$ or*
- **2)** *$X = \mathbb{P}^2$, $D = \mathscr{O}_{\mathbb{P}^2}(1)$ or $D = \mathscr{O}_{\mathbb{P}^2}(2)$.*

B) *If $|K_X + D| \neq \emptyset$, then $|K_X + D|$ is base-point free. In this case $(K_X + D)^2 = 0$ if and only if*
- **3)** *X is a Del Pezzo surface and $D = -K_X$ (in particular X is rational) or*
- **4)** *$X \subset \mathbb{P}^n$ is a conic bundle.*

If $(K_X + D)^2 > 0$, then the adjunction map

$$\varphi_{|K_X+D|}\colon X \to X_1 \subset \mathbb{P}^N$$

defined by the complete linear system $|K_X + D|$ is birational onto a smooth surface X_1 of degree $(K_X + D)^2$ and blows down precisely the (-1)-curves E on X with $DE = 1$, unless
- **5)** *$X = \mathbb{P}^2(p_1, \dots, p_7)$, $\quad D = 6L - \sum_{i=1}^{7} 2E_i$,*
- **6)** *$X = \mathbb{P}^2(p_1, \dots, p_8)$, $\quad D = 6L - \sum_{i=1}^{7} 2E_i - E_8$,*
- **7)** *$X = \mathbb{P}^2(p_1, \dots, p_8)$, $\quad D = 9L - \sum_{i=1}^{8} 3E_i$,*
- **8)** *$X = \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is an indecomposable rank 2 vector bundle over an elliptic curve and $D = 3B$, where B is an effective divisor on X with $B^2 = 1$.*

We can apply Theorem 1.4 repeatedly, obtaining a sequence of surfaces and adjunction maps

$$X =: X_0 \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \longrightarrow \cdots \longrightarrow X_{n-1} \xrightarrow{\varphi_n} X_n \to \cdots .$$

At each step we must control the numerical data arising from the adjunction process. We have

$$(D_{i+1})^2 = (K_{X_i} + D_i)^2, \quad K_{X_{i+1}} D_{i+1} = (K_{X_i} + D_i) K_{X_i}.$$

For the computation of

$$(K_{X_{i+1}})^2 = (K_{X_i})^2 + \alpha_i$$

we also need to know the number α_i of *exceptional lines* on X_i, i.e., the number of smooth curves $E \subset X_i$ such that $K_{X_i}E = E^2 = -1,\ ED_i = 1$. Notice that by the Hodge Index Theorem (see [Har77, Exercise 1.9, p. 368]) we have

$$\det \begin{pmatrix} (D_i)^2 & K_{X_i}D_i \\ K_{X_i}D_i & (K_{X_i})^2 \end{pmatrix} \leq 0$$

and the equality holds if and only if K_{X_i} and D_i are numerically dependent.

Proposition 1.5. *Let $E \subset X_{n-1}$ be a curve contracted by the nth adjunction map $\varphi_n \colon X_{n-1} \to X_n$. Then, setting $\psi := \varphi_{n-1} \circ \varphi_{n-2} \circ \cdots \circ \varphi_1$ and $E^* := \psi^* E$, we have*

$$(E^*)^2 = -1, \quad K_X E^* = -1, \quad DE^* = n.$$

Proof. Since E is contracted by φ_n, we have $E^2 = -1$, $K_{X_{n-1}}E = -1$, $D_{n-1}E = 1$. The map ψ is birational, so $(E^*)^2 = E^2 = -1$. Moreover

$$\psi_* K_X = K_{X_{n-1}}, \quad \psi_* D = D_{n-1} - (n-1)K_{X_{n-1}}.$$

Applying the projection formula we obtain

$$\begin{aligned} K_X E^* &= (\psi_* K_X)E = K_{X_{n-1}}E = -1, \\ DE^* &= (\psi_* D)E = (D_{n-1} - (n-1)K_{X_{n-1}})E = n. \end{aligned}$$

This completes the proof. □

1.4. **Steiner bundles.** We collect here some material about coherent sheaves presented by matrices of linear forms, usually called Steiner sheaves, and more specifically about those that are locally free (Steiner bundles). We refer to [AO01] for basic results on this topic.

1.4.1. *Steiner sheaves and their projectivization.* Let U, V and W be finite-dimensional $\mathbb{C}$-vector spaces. Consider the projective spaces $\mathbb{P}(V)$ and $\mathbb{P}(U)$, and identify V and U with $H^0(\mathbb{P}(V), \mathscr{O}_{\mathbb{P}(V)}(1))$ and $H^0(\mathbb{P}(U), \mathscr{O}_{\mathbb{P}(U)}(1))$, respectively. Any element $\phi \in U \otimes V \otimes W$ gives rise to two maps

$$W^\vee \otimes \mathscr{O}_{\mathbb{P}(V)}(-1) \xrightarrow{M_\phi} U \otimes \mathscr{O}_{\mathbb{P}(V)}, \qquad W^\vee \otimes \mathscr{O}_{\mathbb{P}(U)}(-1) \xrightarrow{N_\phi} V \otimes \mathscr{O}_{\mathbb{P}(U)}. \tag{2}$$

Set $\mathscr{F} := \operatorname{coker} M_\phi$. We say that $\mathscr{F}$ is a *Steiner sheaf*, and we denote its projectivization by $\mathbb{P}(\mathscr{F})$; this is a projective bundle precisely when $\mathscr{F}$ is locally free (and in this case $\dim(U) \geq \dim(V) + \dim(W) - 1$). Let $\mathfrak{p} \colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}(V)$ be the bundle map and let $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)$ be the tautological, relatively ample line bundle on $\mathbb{P}(\mathscr{F})$, so that

$$H^0(\mathbb{P}(\mathscr{F}), \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)) \simeq H^0(\mathbb{P}(V), \mathscr{F}) \simeq U.$$

Since $\mathscr{F}$ is a quotient of $U \otimes \mathscr{O}_{\mathbb{P}(V)}$, we get a natural embedding

$$\mathbb{P}(\mathscr{F}) \subset \mathbb{P}(U \otimes \mathscr{O}_{\mathbb{P}(V)}) \simeq \mathbb{P}(V) \times \mathbb{P}(U).$$

The map $\mathfrak{q}$ associated with the linear system $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)|$ is just the restriction to $\mathbb{P}(\mathscr{F})$ of the second projection from $\mathbb{P}(V) \times \mathbb{P}(U)$. On the other hand, setting $\ell := \mathfrak{p}^*(\mathscr{O}_{\mathbb{P}(V)}(1))$, the linear system $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell)|$ is naturally associated with the map $\mathfrak{p}$. In this procedure the roles of U and V can be reversed. In other words, setting $\mathscr{G} = \operatorname{coker} N_\phi$, we get a second Steiner sheaf, this time on $\mathbb{P}(U)$, and a second projective bundle $\mathbb{P}(\mathscr{G})$ with maps $\mathfrak{p}'$ and $\mathfrak{q}'$ to $\mathbb{P}(U)$ and $\mathbb{P}(V)$, respectively. So we have two incidence diagrams

$$\begin{array}{ccc} & \mathbb{P}(\mathscr{F}) & \\ {}^{\mathfrak{p}}\swarrow & & \searrow^{\mathfrak{q}} \\ \mathbb{P}(V) & & \mathbb{P}(U), \end{array} \qquad \begin{array}{ccc} & \mathbb{P}(\mathscr{G}) & \\ {}^{\mathfrak{p}'}\swarrow & & \searrow^{\mathfrak{q}'} \\ \mathbb{P}(U) & & \mathbb{P}(V). \end{array}$$

The link between $\mathbb{P}(\mathscr{F})$ and $\mathbb{P}(\mathscr{G})$ is provided by the following result.

Proposition 1.6. *Let $\phi \in U \otimes V \otimes W$ and set $m = \dim W$. Then:*

- **i)** *The schemes $\mathbb{P}(\mathscr{F})$ and $\mathbb{P}(\mathscr{G})$ are both identified with the same m-fold linear section of $\mathbb{P}(V) \times \mathbb{P}(U)$. Moreover, under this identification, $\mathfrak{q} = \mathfrak{p}'$ and $\mathfrak{p} = \mathfrak{q}'$.*
- **ii)** *For any non-negative integer k, there are natural isomorphisms*

$$\mathfrak{p}_*\mathfrak{q}^*(\mathscr{O}_{\mathbb{P}(U)}(k)) \simeq S^k\mathscr{F}, \qquad \mathfrak{q}_*\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}(V)}(k)) \simeq S^k\mathscr{G}.$$

Proof. Set $M := M_\phi$. By construction, the scheme $\mathbb{P}(\mathscr{F})$ is defined as the set

$$\mathbb{P}(\mathscr{F}) = \{([v],\, [u]) \in \mathbb{P}(V) \times \mathbb{P}(U) \mid u \in \operatorname{coker} M(v)\},$$

where $v\colon V \to \mathbb{C}$ (resp., $u\colon U \to \mathbb{C}$) is a 1-dimensional quotient of V (resp., of U) and $M(v)\colon W^\vee \to U$ is the evaluation of M at the point $[v]$. Now, we get that u is defined on $\operatorname{coker} M(v)$ if and only if $u \circ M(v) = 0$. This clearly amounts to requiring $(u \circ M(v))(w) = 0$ for all $w \in W^\vee$, that is, $u \otimes v \otimes w(\phi) = 0$ for all $w \in W^\vee$. Summing up, we have

$$\mathbb{P}(\mathscr{F}) = \{([v],\, [u]) \mid u \otimes v \otimes w(\phi) = 0 \text{ for all } w \in W^\vee\}. \tag{3}$$

The same argument works for $\mathbb{P}(\mathscr{G})$ by interchanging the roles of v and u, hence $\mathbb{P}(\mathscr{F})$ and $\mathbb{P}(\mathscr{G})$ are both identified with the same subset of $\mathbb{P}(V) \times \mathbb{P}(U)$. Since each element w_i of a basis of $W^\vee$ gives a linear equation of the form $u \otimes v \otimes w_i(\phi) = 0$, we have that $\mathbb{P}(\mathscr{F})$ is an m-fold linear section (of codimension m or smaller) of $\mathbb{P}(V) \times \mathbb{P}(U)$.

Note that, in view of the identification above, the map $\mathfrak{p}$ is just the projection from $\mathbb{P}(V) \times \mathbb{P}(U)$ onto $\mathbb{P}(V)$, restricted to the set given by (3). The same holds for $\mathfrak{q}'$, hence we are allowed to identify $\mathfrak{p}$ and $\mathfrak{q}'$. Analogously, both $\mathfrak{q}$ and $\mathfrak{p}'$ are given as projections onto the factor $\mathbb{P}(V)$. We have thus proved **i)**. Now let us check **ii)**. For any non-negative integer k we have

$$\begin{aligned} &\mathfrak{q}^*(\mathscr{O}_{\mathbb{P}(U)}(k)) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{F})}(k\xi), \\ &\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}(V)}(k)) \simeq (\mathfrak{q}')^*(\mathscr{O}_{\mathbb{P}(V)}(k)) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{G})}(k\xi'), \end{aligned}$$

where ξ' is the tautological relatively ample line bundle on $\mathbb{P}(\mathscr{G})$. Therefore the claim follows from the canonical isomorphisms

$$\mathfrak{p}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(k\xi)) \simeq S^k\mathscr{F}, \qquad \mathfrak{p}'_*(\mathscr{O}_{\mathbb{P}(\mathscr{G})}(k\xi')) \simeq S^k\mathscr{G}.$$

□

Remark 1.7. We can rephrase the content of Proposition 1.6 by using coordinates as follows. Take bases

$$\{z_i\},\ \{x_j\},\ \{y_k\}$$

for U, V, W, respectively. With respect to these bases, the tensor $\phi \in U \otimes V \otimes W$ will correspond to a trilinear form

$$\phi = \sum a_{ijk} z_i x_j y_k,$$

for a certain table of coefficients $a_{ijk} \in \mathbb{C}$. Write $\mathbb{V}$ and $\mathbb{U}$ for the symmetric algebras on V and U. Then ϕ induces two linear maps of graded vector spaces:

$$W^\vee \otimes \mathbb{V}(-1) \to U \otimes \mathbb{V}, \quad W^\vee \otimes \mathbb{U}(-1) \to V \otimes \mathbb{U},$$

both defined as

$$w \otimes \Psi \to \left(\sum a_{ijk} z_i x_j y_k(w) \right) \Psi,$$

where Ψ lies in $\mathbb{V}$ or in $\mathbb{U}$. The sheafification of these maps gives precisely the two maps of vector bundles M_ϕ and N_ϕ written in (2), whose defining matrices of linear forms are, respectively:

$$\left(\sum\nolimits_j a_{ijk} x_j \right)_{ik} \quad \text{and} \quad \left(\sum\nolimits_i a_{ijk} z_i \right)_{jk}.$$

An important observation is that $\mathbb{P}(\mathscr{F})$ and $\mathbb{P}(\mathscr{G})$ are both identified with the zero locus of the same set of m bilinear equations in $\mathbb{P}(V) \times \mathbb{P}(U)$, namely

$$\mathbb{P}(\mathscr{F}) = \mathbb{P}(\mathscr{G}) = \left\{ (x,\, z) \;\Big|\; \sum\nolimits_{i,\,j} a_{ij1} z_i x_j = \cdots = \sum\nolimits_{i,\,j} a_{ijm} z_i x_j = 0 \right\}.$$

This shows that $\mathbb{P}(\mathscr{F}) = \mathbb{P}(\mathscr{G})$ is the intersection of m divisors of bidegree $(1,1)$ in $\mathbb{P}(V) \times \mathbb{P}(U)$. We can thus write a presentation of the form:

$$\cdots \to W^\vee \otimes \mathscr{O}_{\mathbb{P}(V)\times\mathbb{P}(U)}(-1,-1) \to \mathscr{O}_{\mathbb{P}(V)\times\mathbb{P}(U)} \to \mathscr{O}_{\mathbb{P}(\mathscr{F})} \to 0. \tag{4}$$

We will mostly use this setup when $\mathbb{P}(V) = \mathbb{P}^2$, in order to study the geometry of a Steiner bundle $\mathscr{F}$ of rank 2 admitting the resolution

$$0 \to W^\vee \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \xrightarrow{M} U \otimes \mathscr{O}_{\mathbb{P}^2} \to \mathscr{F} \to 0, \tag{5}$$

and to compare it with the geometry of the sheaf $\mathscr{G}$ obtained by "flipping" the tensor ϕ as explained above and whose presentation is

$$W^\vee \otimes \mathscr{O}_{\mathbb{P}(U)}(-1) \xrightarrow{N} \mathscr{O}^3_{\mathbb{P}(U)} \to \mathscr{G} \to 0. \tag{6}$$

1.4.2. *Unstable lines.* Let us assume now $\dim V = 3$ and consider a Steiner bundle $\mathscr{F}$ of rank 2 on $\mathbb{P}^2 = \mathbb{P}(V)$. To be consistent with the notation that will appear later, we set $\dim U = b-2$ and $\dim W = b-4$, for $b \geq 4$, and we write $\mathscr{F}_b$ instead of $\mathscr{F}$. The sheafified minimal graded free resolution of $\mathscr{F}_b$ is then

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1)^{b-4} \xrightarrow{M} \mathscr{O}^{b-2}_{\mathbb{P}^2} \to \mathscr{F}_b \to 0, \tag{7}$$

where M is a $(b-2) \times (b-4)$ matrix of linear forms.

Given a line $L \subset \mathbb{P}^2$, there is an integer a such that

$$\mathscr{F}_b|_L = \mathscr{O}_L(a) \oplus \mathscr{O}_L(b-4-a).$$

Since $\mathscr{F}_b$ is globally generated, the same is true for $\mathscr{F}_b|_L$ and so

$$0 \leq a \leq b-4.$$

Definition 1.8. A line $L \subset \mathbb{P}^2$ is said to be *unstable* for $\mathscr{F}_b$ if $a = 0$, i.e.,

$$\mathscr{F}_b|_L \simeq \mathscr{O}_L \oplus \mathscr{O}_L(b-4).$$

Here are some useful characterizations of unstable lines.

Lemma 1.9. *The following are equivalent*:

i) *The line $L \subset \mathbb{P}^2$ is an unstable line for $\mathscr{F}_b$.*

ii) *The cohomology group $H^0(L,\, \mathscr{F}_b^\vee|_L)$ is non-zero.*

iii) *There is a non-zero global section of $\mathscr{F}_b$ whose vanishing locus contains $b-4$ points of L (counted with multiplicity).*

Proof. We first prove **i)** $\Leftrightarrow$ **ii)**. The restriction $\mathscr{F}_b|_L$ splits, so there is an integer a such that $\mathscr{F}_b|_L = \mathscr{O}_L(a) \oplus \mathscr{O}_L(b-4-a)$, and since $\mathscr{F}_b$ is globally generated we have $0 \leq a \leq b-4$. Condition **i)** corresponds to $a = 0$ or $a = b-4$, and this clearly implies **ii)**. Conversely, if **ii)** holds, then $a \leq 0$ or $a \geq b-4$; this implies either $a = 0$ or $a = b-4$, hence **i)** holds.

In order to check **ii)** $\Leftrightarrow$ **iii)**, we first claim that, given a line $L \subset \mathbb{P}^2$, the restriction map induces an isomorphism

$$H^0(\mathbb{P}^2, \mathscr{F}_b) \xrightarrow{\simeq} H^0(L, \mathscr{F}_b|_L). \tag{8}$$

Indeed, looking at (7), we see that we have

$$H^0(\mathbb{P}^2, \mathscr{F}_b(-1)) = H^1(\mathbb{P}^2, \mathscr{F}_b(-1)) = 0,$$

so our claim follows by taking cohomology in

$$0 \to \mathscr{F}_b(-1) \to \mathscr{F}_b \to \mathscr{F}_b|_L \to 0.$$

Now let us prove **ii)** $\Rightarrow$ **iii)**. Assuming **ii)**, we get a short exact sequence

$$0 \to \mathscr{O}_L \to \mathscr{F}_b^\vee|_L \to \mathscr{O}_L(4-b) \to 0,$$

so by dualizing we have

$$0 \to \mathscr{O}_L(b-4) \xrightarrow{\iota} \mathscr{F}_b|_L \to \mathscr{O}_L \to 0.$$

Composing ι with a non-zero map $\mathscr{O}_L \to \mathscr{O}_L(b-4)$, we obtain a global section of $\mathscr{F}_b|_L$ vanishing at $b-4$ points counted with multiplicity. Using the isomorphism (8) we can lift this section to a global section of $\mathscr{F}_b$ and we get **iii)**.

Conversely, assume that **iii)** holds. Then there is a global section s of $\mathscr{F}_b$ whose vanishing locus Z contains a subscheme of L of length $b-4$. Put $Z' = Z \cap L$, so that Z' has length $c \geq b-4$. Since $H^0(\mathbb{P}^2, \mathscr{F}_b(-1)) = 0$ it follows that Z contains no divisors, i.e., it has pure dimension 0, so we have an exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{F}_b \to \mathscr{I}_{Z/\mathbb{P}^2}(b-4) \to 0.$$

Applying $- \otimes_{\mathscr{O}_{\mathbb{P}^2}} \mathscr{O}_L$ to the exact sequence

$$0 \to \mathscr{I}_{Z/\mathbb{P}^2}(b-4) \to \mathscr{O}_{\mathbb{P}^2}(b-4) \to \mathscr{O}_Z \to 0$$

and using $\mathscr{T}or_1^{\mathscr{O}_{\mathbb{P}^2}}(\mathscr{O}_Z,\, \mathscr{O}_L) \simeq \mathscr{O}_{Z'}$, we get

$$0 \to \mathscr{O}_{Z'} \to \mathscr{I}_{Z/\mathbb{P}^2}(b-4)|_L \to \mathscr{O}_L(b-4) \to \mathscr{O}_{Z'} \to 0.$$

The image of the middle map is $\mathscr{I}_{Z'/L}(b-4) \simeq \mathscr{O}_L(b-c-4)$, so from the above sequence we obtain

$$0 \to \mathscr{O}_{Z'} \to \mathscr{I}_{Z/\mathbb{P}^2}(b-4)|_L \to \mathscr{O}_L(b-c-4) \to 0. \tag{9}$$

The scheme Z' is 0-dimensional, so we infer

$$\mathrm{Ext}^1(\mathscr{O}_L(b-4-c), \mathscr{O}_{Z'}) \simeq H^1(L, \mathscr{O}_{Z'} \otimes \mathscr{O}_L(c-b+4)) = 0$$

and this means that (9) splits, i.e.,

$$\mathscr{I}_{Z/\mathbb{P}^2}(b-4)|_L \simeq \mathscr{O}_L(b-c-4) \oplus \mathscr{O}_{Z'}. \tag{10}$$

Therefore, we have a surjection $\mathscr{F}_b|_L \to \mathscr{O}_L(b-c-4)$. Since $b-c-4 \le 0$, the dual of this surjection gives a non-zero global section of $\mathscr{F}_b^\vee|_L$ and the proof is finished. Note that, since we have now proved $\mathscr{F}_b|_L \simeq \mathscr{O}_L \oplus \mathscr{O}_L(b-4)$, the existence of a surjection $\mathscr{F}_b|_L \to \mathscr{O}_L(b-c-4)$ actually gives $c = b-4$, i.e., $Z' = Z \cap L$ has length precisely $b-4$. □

The set of unstable lines of $\mathscr{F}_b$ has a natural structure of subscheme of $\check{\mathbb{P}}^2$, given as follows. First define the point-line incidence $\mathbb{I}$ in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ by the condition that the point lies in the line. One has $\mathbb{I} \simeq \mathbb{P}(T_{\mathbb{P}^2}(-1))$ and $T_{\mathbb{P}^2}(-1)$ is a Steiner bundle. By Lemma 1.9, a line L is unstable for $\mathscr{F}$ if and only if $H^0(L, \mathscr{F}_b^\vee|_L) \neq 0$, i.e., by Serre duality, if and only if $H^1(L, \mathscr{F}_b(-2)|_L) \neq 0$, which happens if and only if L lies in the support of $R^1\mathfrak{q}_*(\mathfrak{p}^*\mathscr{F}_b(-2) \otimes \mathscr{O}_{\mathbb{I}})$. We denote the set of unstable lines, endowed with this scheme structure, by $\mathscr{W}(\mathscr{F}_b)$.

Let us now give a summary of the behavior of the unstable lines of $\mathscr{F}_b$ for small values of b.

- $b = 4$. We have $\mathscr{F}_4 \simeq \mathscr{O}_{\mathbb{P}^2}^2$, so $\mathscr{W}(\mathscr{F}_4)$ is empty.
- $b = 5$. There is an isomorphism $\mathscr{F}_5 \simeq T_{\mathbb{P}^2}(-1)$. Therefore $\mathscr{W}(\mathscr{F}_5) = \check{\mathbb{P}}^2$, because $T_{\mathbb{P}^2}$ is a uniform bundle of splitting type (1, 2); see [OSS80, §2].
- $b = 6$. The scheme $\mathscr{W}(\mathscr{F}_6)$ is a smooth conic in $\check{\mathbb{P}}^2$, and the unstable lines of $\mathscr{F}_6$ are the tangent lines to the dual conic; see [DK93, Proposition 6.8] and [Val00, Proposition 2.2].
- $b = 7$. The scheme $\mathscr{W}(\mathscr{F}_7)$ is either a set of six points in general linear position and contained in no conic or consists of a smooth conic in $\check{\mathbb{P}}^2$; see [Val00, Théorème 3.1]. The former case is the general one, and when it occurs $\mathscr{F}_7$ is a so-called *logarithmic bundle*. Instead, the latter case occurs if and only if $\mathscr{F}_7$ is a so-called *Schwarzenberger bundle*, whose matrix M, up to a linear change of coordinates, has the form

$$M = {}^{\mathrm{t}}\begin{pmatrix} x_0 & x_1 & x_2 & 0 & 0 \\ 0 & x_0 & x_1 & x_2 & 0 \\ 0 & 0 & x_0 & x_1 & x_2 \end{pmatrix}; \tag{11}$$

  see [FMV13, Theorem 3], [Val00, Théorème 3.1].
- $b \ge 8$. Unstable lines do not always exist in this range. The scheme $\mathscr{W}(\mathscr{F}_b)$ is either finite of length $\le b-1$ or consists of a smooth conic in $\check{\mathbb{P}}^2$. In the latter case, $\mathscr{F}_b$ is a Schwarzenberger bundle, whose matrix M, up to a linear change of coordinates, is a $(b-2) \times (b-4)$ matrix having the same form as (11). We can actually state a more precise result; see again [AO01, Proposition 3.11 and proof of Theorem 5.3].

  Proposition 1.10. *If $\mathscr{F}_b$ contains a finite number α_1 of unstable lines, then $0 \le \alpha_1 \le b-1$. More precisely, the following hold*:

  i) *If $0 \le \alpha_1 \le b-2$ then, up to a linear change of coordinates, the matrix M is of type*

$$M = {}^{\mathrm{t}}\left(\begin{array}{ccc|c} a_{1,1}H_1 & \cdots & a_{1,\alpha}H_\alpha & \\ \vdots & \vdots & \vdots & M' \\ a_{b-4,1}H_1 & \cdots & a_{b-4,\alpha}H_\alpha & \end{array}\right),$$

for some $(b-2-\alpha) \times (b-4)$ matrix M' of linear forms. In this case the unstable lines are given by

$$H_1 = 0, \quad H_2 = 0, \ldots, H_{\alpha_1} = 0.$$

ii) *If $\alpha_1 = b-1$, then $\mathscr{F}_b$ is a logarithmic bundle. In this case, the matrix M is of type*

$$M = {}^{\mathrm{t}}\begin{pmatrix} a_{1,1}H_1 & a_{1,2}H_2 & \cdots & a_{1,b-2}H_{b-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{b-4,1}H_1 & a_{b-4,2}H_2 & \cdots & a_{b-4,b-2}H_{b-2} \end{pmatrix},$$

where $H_1, \ldots, H_{b-2}$ are such that the linear form

$$H_{b-1} := \sum_{j=1}^{b-2} a_{i,j} H_j$$

does not depend on $i \in \{1, \ldots, b-4\}$. The unstable lines are given by

$$H_1 = 0, \quad H_2 = 0, \ldots, H_{b-1} = 0.$$

Remark 1.11. Using Proposition 1.10, we can give another proof of the implication **i)** $\Rightarrow$ **iii)** in Lemma 1.9. Indeed, we can take a basis $s_1, \ldots, s_{b-2}$ of $H^0(\mathbb{P}^2, \mathscr{F}_b)$ such that the homogeneous ideal I_k of the vanishing locus of s_k is defined by the maximal minors of the matrix obtained by deleting the kth row of M, namely by $b-3$ forms of degree $b-4$. Assume now that the unstable line L is defined by the equation $H_i = 0$. Then, if $k \neq i$, all the minors defining I_k are divisible by H_i, except the one obtained by deleting the kth and ith rows of M; so s_k vanishes at $b-4$ points on L.

Remark 1.12. In Proposition 1.10 we denoted the number of unstable lines of $\mathscr{F}_b$ by α_1. Further on, the notation α_1 will be reserved to the number of exceptional lines contracted by the first adjunction map $\varphi_{|K_X+D|} \colon X \to X_1$; see §1.3. The reason is that when we consider a general triple plane $f \colon X \to \mathbb{P}^2$ whose (twisted) Tschirnhausen bundle is isomorphic to $\mathscr{F}_b$, with $b \geq 7$, these two numbers are in fact the same (see §2.3.2, in particular Proposition 2.17).

1.5. **Criteria for a rank 2 vector bundle to be Steiner.** Here we present two simple criteria to check whether a vector bundle of rank 2 on $\mathbb{P}^2$ is a Steiner one. Both of them consist in fixing the numerical data and adding a single cohomology vanishing. In the second one, the condition is on a 0-dimensional subscheme from which the bundle is constructed via the Serre correspondence, provided that the Cayley-Bacharach property is satisfied.

To state the first result, fix an integer $b \geq 4$ and note that, if $\mathscr{F}$ is a Steiner bundle of type $\mathscr{F}_b$, then

$$c_1(\mathscr{F}) = b-4, \qquad c_2(\mathscr{F}) = \binom{b-3}{2} \tag{12}$$

and $H^i(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ for all i. Likewise, for $b \leq 2$ assume that $\mathscr{F}$ fits into

$$0 \to \mathscr{F} \to \mathscr{O}_{\mathbb{P}^2}(-1)^{4-b} \to \mathscr{O}_{\mathbb{P}^2}^{2-b} \to 0. \tag{13}$$

Then, using the standard convention on binomial coefficients with negative arguments, we see that (12) still holds; furthermore, we have $H^i(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ for all i. Note that $\mathscr{F}$ fits into (13) if and only if $\mathscr{F}^\vee(-1)$ is of type $\mathscr{F}_b$. One may extend the notation $\mathscr{F}_b$ to all b in $\mathbb{Z}$ as a bundle fitting into the long exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1)^{\max(b-4,0)} \to \mathscr{O}_{\mathbb{P}^2}^{\max(b-2,0)} \to \mathscr{F}_b \to \mathscr{O}_{\mathbb{P}^2}(-1)^{\max(4-b,0)} \to \mathscr{O}_{\mathbb{P}^2}^{\max(2-b,0)} \to 0,$$

where the value $b = 3$ corresponds to $\mathscr{F}_3 = \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}$.

Proposition 1.13. *Fix an integer $b \in \mathbb{Z}$ and let $\mathscr{F}$ be a vector bundle of rank 2 on $\mathbb{P}^2$ satisfying (12). Then the following hold:*

- **i)** *For $b \geq 4$, the bundle $\mathscr{F}$ is of type $\mathscr{F}_b$ if and only if $H^0(\mathbb{P}^2, \mathscr{F}(-1)) = 0$. If this happens, then $\mathscr{F}$ is stable for $b \geq 5$.*
- **ii)** *For $b \leq 2$, the bundle $\mathscr{F}^\vee(-1)$ is of type $\mathscr{F}_b$ if and only if $H^2(\mathbb{P}^2, \mathscr{F}(-1)) = 0$. If this happens, then $\mathscr{F}$ is stable for $b \leq 1$.*
- **iii)** *For $b = 3$, we have $\mathscr{F} \simeq \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}$ if and only if $H^0(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ or, equivalently, $H^2(\mathbb{P}^2, \mathscr{F}(-1)) = 0$.*

Proof. In each case, only one direction needs to be proved.

i) Let us assume $b \geq 4$ and $H^0(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ and let us show that $\mathscr{F}$ is of the form $\mathscr{F}_b$. First, since $\mathscr{F}$ is locally free of rank 2 and $c_1(\mathscr{F}) = b - 4$, there is the canonical isomorphism

$$\mathscr{F}^\vee \simeq \mathscr{F}(4 - b).$$

Then, for any integer $t \leq 2$, by Serre duality we have

$$h^2(\mathbb{P}^2, \mathscr{F}(-t)) = h^0(\mathbb{P}^2, \mathscr{F}^\vee(t-3)) = h^0(\mathbb{P}^2, \mathscr{F}(t-b+1)) = 0, \tag{14}$$

because by our assumptions $t - b + 1 \leq -1$ and already $h^0(\mathbb{P}^2, \mathscr{F}(-1)) = 0$.

Now, using (12) and the Riemann-Roch theorem we deduce $\chi(\mathbb{P}^2, \mathscr{F}(-1)) = 0$, so $h^1(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ because we know that $h^0(\mathbb{P}^2, \mathscr{F}(-1)) = h^2(\mathbb{P}^2, \mathscr{F}(-1)) = 0$. Again by Riemann-Roch, using (14) with $t = 2$ we obtain $h^1(\mathbb{P}^2, \mathscr{F}(-2)) = b - 4$.

Let us look at $h^i(\mathbb{P}^2, \mathscr{F})$. First, by using (14) with $t = 0$, we see that this vanishes for $i = 2$. Now take a line L in $\mathbb{P}^2$, tensor with $\mathscr{F}(t)$ the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2} \to \mathscr{O}_L \to 0 \tag{15}$$

and pass to cohomology. Since we proved that $h^1(\mathbb{P}^2, \mathscr{F}(-1)) = h^2(\mathbb{P}^2, \mathscr{F}(-2)) = 0$, we deduce $h^1(L, \mathscr{F}(-1)|_L) = 0$. Then, considering the short exact sequence

$$0 \to \mathscr{F}(t-1)|_L \to \mathscr{F}(t)|_L \to \mathscr{O}_x \oplus \mathscr{O}_x \to 0$$

and using induction on t, we obtain $h^1(L, \mathscr{F}(t)|_L) = 0$ for any $t \geq 0$. Therefore we get $h^1(\mathbb{P}^2, \mathscr{F}) = 0$, that in turn yields, again by Riemann-Roch, $h^0(\mathbb{P}^2, \mathscr{F}) = b - 2$.

We can now use Beilinson's theorem; see for instance [OSS80, Chapter 2, §3.1.3]. The Beilinson table of $\mathscr{F}$, displaying the values of $h^j(\mathbb{P}^2, \mathscr{F}(-i))$, is

TABLE 1. The Beilinson table of $\mathscr{F}$

	$\mathscr{F}(-2)$	$\mathscr{F}(-1)$	$\mathscr{F}$
h^2	0	0	0
h^1	$b-4$	0	0
h^0	0	0	$b-2$

This gives in turn the resolution of $\mathscr{F}$,

$$0 \to H^1(\mathbb{P}^2,\, \mathscr{F}(-2)) \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \to H^0(\mathbb{P}^2,\, \mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}^2} \to \mathscr{F} \to 0, \tag{16}$$

which has the desired form. In fact, (16) becomes (5) if we set

$$W := H^1(\mathbb{P}^2,\, \mathscr{F}(-2))^\vee, \quad U := H^0(\mathbb{P}^2,\, \mathscr{F}), \quad \mathbb{P}^2 = \mathbb{P}(V). \tag{17}$$

Stability of $\mathscr{F}$ for $b \geq 5$ follows from Hoppe's criterion; see [Hop84, Lemma 2.6].

ii) Assume now $b \leq 2$. Set $\mathscr{F}' = \mathscr{F}^\vee(-1)$ and $b' = 6 - b$, so that $b' \geq 4$. The Chern classes of $\mathscr{F}'$ are

$$c_1(\mathscr{F}') = -c_1(\mathscr{F}) - 2 = b' - 4, \qquad c_2(\mathscr{F}') = c_2(\mathscr{F}) + c_1(\mathscr{F}) + 1 = \binom{b'-3}{2}.$$

Using the assumption $H^2(\mathbb{P}^2, \mathscr{F}(-1)) = 0$ and Serre duality, we get

$$H^0(\mathbb{P}^2, \mathscr{F}'(-1)) = H^0(\mathbb{P}^2, \mathscr{F}^\vee(-2)) \simeq H^2(\mathbb{P}^2, \mathscr{F}(-1))^\vee = 0,$$

so by part **i)** it follows that $\mathscr{F}'$ is a Steiner bundle of the form $\mathscr{F}_{b'}$.

iii) Finally, assume $b = 3$. From $H^0(\mathbb{P}^2,\, \mathscr{F}(-1)) = 0$ we deduce $H^2(\mathbb{P}^2,\, \mathscr{F}(-1)) = 0$ and conversely, because (14) still holds when $(t,\, b) = (1,\, 3)$. We can now conclude by applying [FV14, Lemma 3.3] to $\mathscr{F}$. □

Proposition 1.14. *Fix integers $b \geq 5$ and $t \geq 0$, and let $Z \subset \mathbb{P}^2$ be a 0-dimensional, local complete intersection subscheme of length l. Then the following hold*:

i) *A locally free sheaf $\mathscr{F}$ fitting into*

$$0 \to \mathscr{O}_{\mathbb{P}^2} \xrightarrow{s} \mathscr{F}(t) \to \mathscr{I}_{Z/\mathbb{P}^2}(2t + b - 4) \to 0 \tag{18}$$

exists if and only if Z satisfies the Cayley-Bacharach property with respect to $\mathscr{O}_{\mathbb{P}^2}(2t + b - 7)$, i.e., for any subscheme $Z' \subset Z$ of length $l - 1$ we have

$$h^0(\mathbb{P}^2,\, \mathscr{I}_{Z/\mathbb{P}^2}(2t + b - 7)) = h^0(\mathbb{P}^2,\, \mathscr{I}_{Z'/\mathbb{P}^2}(2t + b - 7)).$$

ii) *A locally free sheaf $\mathscr{F}$ as in* **i)** *is a Steiner bundle of the form $\mathscr{F}_b$ if and only if*

$$l = \binom{b-3}{2} + t(t + b - 4), \qquad H^0(\mathbb{P}^2,\, \mathscr{I}_{Z/\mathbb{P}^2}(t + b - 5)) = 0. \tag{19}$$

iii) *If* **i)** *and* **ii)** *are satisfied and in addition $h^1(\mathbb{P}^2, \mathscr{I}_{Z/\mathbb{P}^2}(t+b-7)) = 1$, then the extension* (18) *and the proportionality class of the global section s of $\mathscr{F}(t)$ vanishing at Z are uniquely determined by Z.*

Proof. The statement **i)** follows from [HL97, Part II, Theorem 5.1.1].

For **ii)**, let $\mathscr{F}$ be a Steiner bundle of the form $\mathscr{F}_b$. So $c_1(\mathscr{F}(t)) = 2t + b - 4$ and

$$l = c_2(\mathscr{F}(t)) = c_2(\mathscr{F}) + c_1(\mathscr{F}) + t^2 = \binom{b-3}{2} + t(t + b - 4).$$

Also, we have $H^0(\mathbb{P}^2,\, \mathscr{F}(-1)) = 0$, which yields $H^0(\mathbb{P}^2,\, \mathscr{I}_{Z/\mathbb{P}^2}(t + b - 5)) = 0$. Conversely, if Z satisfies (19), by Proposition 1.13 we see that $\mathscr{F}$ is of the form $\mathscr{F}_b$.

For **iii)**, by Serre duality we have

$$\begin{aligned} \mathrm{Ext}^1(\mathscr{I}_{Z/\mathbb{P}^2}(2t + b - 4),\, \mathscr{O}_{\mathbb{P}^2})^\vee &\simeq \mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^2},\, \mathscr{I}_{Z/\mathbb{P}^2}(2t + b - 7)) \\ &\simeq H^1(\mathbb{P}^2,\, \mathscr{I}_{Z/\mathbb{P}^2}(2t + b - 7)) \simeq \mathbb{C}. \end{aligned} \tag{20}$$

Since we are assuming that $\mathscr{F}$ is locally free, the extension (18) has to be non-trivial, and by (20) all such non-trivial extensions are equivalent up to a multiplicative scalar. □

2. General triple planes with $p_g = q = 0$

2.1. **General triple planes.** Given a triple plane $f\colon X \to \mathbb{P}^2$, we denote by H the pullback $H := f^*L$, where $L \subset \mathbb{P}^2$ is a line. The divisor H is ample, as L is ample and f is finite.

Recall that the Tschirnhausen bundle $\mathscr{E}$ of f is a rank 2 vector bundle on $\mathbb{P}^2$ such that $f_*\mathscr{O}_X \simeq \mathscr{O}_{\mathbb{P}^2} \oplus \mathscr{E}$. Proposition 1.1 allows us to relate the invariants of X and $\mathscr{E}$ as follows.

Proposition 2.1. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane with Tschirnhausen bundle $\mathscr{E}$. Then we have:*

$$\begin{aligned} p_g(X) &= h^0(\mathbb{P}^2,\, \mathscr{E}^\vee(-3)),\\ q(X) &= h^1(\mathbb{P}^2,\, \mathscr{E}^\vee(-3)),\\ P_2(X) &= h^0(X,\, 2K_X) = h^0(\mathbb{P}^2,\, S^2\mathscr{E}^\vee(-6)). \end{aligned}$$

Definition 2.2. Let $f\colon X \to \mathbb{P}^2$ be a triple plane and $B \subset \mathbb{P}^2$ its branch locus. We say that f is a *general triple plane* if the following conditions are satisfied:

- **i)** f is unramified over $\mathbb{P}^2 \setminus B$;
- **ii)** $f^*B = 2R + R_0$, where R is irreducible and non-singular and R_0 is reduced;
- **iii)** $f_{|R}\colon R \to B$ coincides with the normalization map of B.

A useful criterion to check that a triple plane is a general one is provided by the following.

Proposition 2.3. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane with X smooth. Then either f is general or f is a Galois cover. In the last case, f is totally ramified over a smooth branch locus.*

Proof. See [Tan02, Theorems 2.1 and 3.2]. □

Hence Theorem 1.2 shows that, if $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}$ is globally generated, the cover associated with a general section $\eta \in H^0(\mathbb{P}^2,\, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$ is a general triple plane as soon as it is not totally ramified.

Since the curve R is the ramification divisor of f and the ramification is simple, we have

$$K_X = f^*K_{\mathbb{P}^2} + R = -3H + R. \tag{21}$$

Moreover, by [Mir85, Proposition 4.7 and Lemma 4.1], we obtain the following proposition.

Proposition 2.4. *Let $f\colon X \to \mathbb{P}^2$ be a general triple plane with Tschirnhausen bundle $\mathscr{E}$ and define*

$$b := -c_1(\mathscr{E}), \quad h := c_2(\mathscr{E}).$$

Then the branch curve B has degree $2b$ and contains $3h$ ordinary cusps and no further singularities. Moreover the cusps are exactly the points where f is totally ramified.

Moreover, in view of [Mir85, Lemma 5.9] and [CE96, Corollary 2.2], we have the following information on R and R_0.

Proposition 2.5. *The curves R and R_0 are both smooth and isomorphic to the normalization of B. Furthermore, they are tangent at the preimages of the cusps of B and they do not meet elsewhere. Finally, the ramification divisor R is very ample on X.*

This allows us to compute the intersection numbers of R and R_0 as follows.

Proposition 2.6. *We have*

$$R^2 = 2b^2 - 3h, \quad RR_0 = 6h, \quad R_0^2 = 4b^2 - 12h. \tag{22}$$

Proof. The projection formula yields

$$R(2R + R_0) = R(f^*B) = (f_*R)B = B^2 = 4b^2.$$

By Proposition 2.5 it follows that $RR_0 = 6h$. So $2R^2 = 4b^2 - RR_0 = 4b^2 - 6h$, which gives the first equality. From $f^*B = 2R + R_0$ we deduce $(2R + R_0)^2 = 3B^2 = 12b^2$, so $R_0^2 = 12b^2 - 4R^2 - 4RR_0 = 4b^2 - 12h$. □

Corollary 2.7. *We have $3h \geq \frac{2}{3}b^2$.*

Proof. Since the divisor R is very ample, the Hodge Index Theorem implies $R^2R_0^2 \leq (RR_0)^2$ and the claim follows. □

Remark 2.8. Proposition 2.6 and Corollary 2.7 were already established by Bronowski in [Bro42]. Note that the (very) ampleness of R implies $R^2 > 0$, that is, $3h < 2b^2$. In [Bro42], it is also stated that the stronger inequality $3h \leq b^2$, or equivalently $R_0^2 \geq 0$, holds. This is actually false, and counterexamples will be provided by our surfaces of type VII; see §3.7. Bronowski's mistake is at page 28 of his paper, where he assumes that one can find a curve algebraically equivalent to R_0 and distinct from it; of course, when $R_0^2 < 0$ such a curve cannot exist.

Proposition 2.9. *Let $f\colon X \to \mathbb{P}^2$ be a general triple plane with $q(X) = 0$. If $K_X^2 \neq 8$, then $D := K_X + 2H$ is very ample.*

Proof. Since $(2H)^2 = 12$, by [Fuj90, Theorem 18.5] D is very ample, unless there exists an effective divisor Z such that $HZ{=}1$ and $Z^2 = 0$. By the projection formula we have

$$1 = HZ = (f^*L)Z = L(f_*Z),$$

hence $f_*Z \subset \mathbb{P}^2$ is a line. On the other hand, $HZ = 1$ implies that the restriction of f to Z is an isomorphism, so Z is a smooth and irreducible rational curve. Since $Z^2 = 0$, the surface X is birationally ruled and Z belongs to the ruling. Moreover, all the curves in the ruling are irreducible: in fact, if Z were algebraically equivalent to $Z_1 + Z_2$, then we would obtain

$$1 = HZ = HZ_1 + HZ_2,$$

contradicting the ampleness of H. Summing up, X is a minimal, geometrically ruled surface over a smooth curve; since $q(X) = 0$, this curve is isomorphic to $\mathbb{P}^1$, that is, X is isomorphic to $\mathbb{F}_n$ for some n and, in particular, $K_X^2 = 8$. □

When $D = K_X + 2H$ is very ample on X we can study the adjunction maps associated with D. Using Proposition 1.5, we obtain the following proposition.

Proposition 2.10. *Assume $q(X) = 0$ and $K_X^2 \neq 8$ and let $\varphi_n \colon X_{n-1} \to X_n$ be the nth adjunction map with respect to the very ample divisor $D = K_X + 2H$. Then φ_n is an isomorphism when n is even, whereas when n is odd φ_n contracts exactly the (-1)-curves $E \subset X$ such that $HE = (n+1)/2$.*

2.2. **The Tschirnhausen bundle in case $p_g = q = 0$.** Let $f \colon X \to \mathbb{P}^2$ be a general triple plane with Tschirnhausen bundle $\mathscr{E}$ and let B be the branch locus of f. Recall that, by Proposition 2.4, the curve B has degree $2b$ and contains $3h$ ordinary cusps as only singularities.

Proposition 2.11. *If $\chi(\mathscr{O}_X) = 1$, that is, $p_g(X) = q(X)$, then we have at most the following possibilities for the numerical invariants b, h, K_X^2, $g(H)$:*

TABLE 2. Possible numerical invariants for a general triple plane with $\chi(\mathscr{O}_X) = 1$

Case	b	h	K_X^2	$g(H)$
I	2	1	8	0
II	3	2	3	1
III	4	4	-1	2
IV	5	7	-4	3
V	6	11	-6	4
VI	7	16	-7	5
VII	8	22	-7	6
VIII	9	29	-6	7
IX	10	37	-4	8
X	11	46	-1	9
XI	12	56	3	10
XII	13	67	8	11

Proof. Using the projection formula we obtain

$$HR = (f^*L)R = L(f_*R) = LB = 2b. \tag{23}$$

Since $K_X = -3H + R$ and $H^2 = 3$ it follows that $K_X H = 2b - 9$, hence $g(H) = b - 2$. Using the *formule di corrispondenza* (cf. [Ive70, §V]) we infer

$$\begin{cases} 9h + 3 = 4b^2 - 6b + K_X^2, \\ 2h - 4 = b^2 - 3b. \end{cases}$$

Therefore $h = \frac{1}{2}(b^2 - 3b + 4)$ and $b^2 - 15b + 42 - 2K_X^2 = 0$. Imposing that the discriminant of this quadratic equation is non-negative, we get $K_X^2 \geq -7$; on the other hand, the Enriques-Kodaira classification and the Miyaoka-Yau inequality imply that any surface with $p_g = q$ satisfies $K_X^2 \leq 9$ (see [BHPV04, Chapter VII]), so $-7 \leq K_X^2 \leq 9$. Now a case-by-case analysis concludes the proof. □

Note that the previous proof shows that

$$c_1(\mathscr{E}) = -b, \quad c_2(\mathscr{E}) = \frac{1}{2}(b^2 - 3b + 4). \tag{24}$$

Moreover, using (21), (23) and the first equality in (22), we obtain

$$K_X R = -3HR + R^2 = 2b^2 - 6b - 3h. \tag{25}$$

From now on, we will restrict ourselves to the case $p_g(X) = q(X) = 0$, that is, in terms of the Tschirnhausen bundle $\mathscr{E}$, we suppose $h^1(\mathbb{P}^2, \mathscr{E}) = 0$ and $h^2(\mathbb{P}^2, \mathscr{E}) = 0$. Furthermore, we will use without further mention the natural isomorphism

$$\mathscr{E}^\vee \simeq \mathscr{E}(b).$$

Theorem 2.12. *Let $f\colon X \to \mathbb{P}^2$ be a general triple plane with $p_g = q = 0$ and let $\mathscr{E}$ be the corresponding Tschirnhausen bundle. With the notation of Proposition* 2.11*, the following hold*:

- **i)** *In case* I*, $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$.*
- **ii)** *In case* II*, $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$.*
- **iii)** *In case* III*, $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^2}(-2) \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$.*
- **iv)** *In cases* IV *to* XII*, the vector bundle $\mathscr{E}$ is stable and has a sheafified minimal graded free resolution of the form*

$$0 \to \mathscr{O}_{\mathbb{P}^2}(1-b)^{b-4} \to \mathscr{O}_{\mathbb{P}^2}(2-b)^{b-2} \to \mathscr{E} \to 0.$$

In particular, $\mathscr{E}(b-2)$ is a rank 2 *Steiner bundle on $\mathbb{P}^2$; see* §1.4.

Proof. Setting $\mathscr{F} := \mathscr{E}(b-2)$, by using (24) we obtain

$$c_1(\mathscr{F}) = b - 4, \quad c_2(\mathscr{F}) = \binom{b-3}{2}. \tag{26}$$

Now Proposition 2.1 allows us to calculate the cohomology groups of $\mathscr{F}(-i)$, for $i = 0, 1, 2$. We have

$$\begin{aligned} h^0(\mathbb{P}^2, \mathscr{F}(-1)) &= h^0(\mathbb{P}^2, \mathscr{E}(b-3)) = h^0(\mathbb{P}^2, \mathscr{E}^\vee(-3)) = p_g(X) = 0, \\ h^1(\mathbb{P}^2, \mathscr{F}(-1)) &= h^0(\mathbb{P}^2, \mathscr{E}(b-3)) = h^1(\mathbb{P}^2, \mathscr{E}^\vee(-3)) = q(X) = 0. \end{aligned} \tag{27}$$

Let us now check cases I to III. By (27), we can apply [FV14, Lemma 3.3] to $\mathscr{E}(1)$ in cases I and II, and to $\mathscr{E}(2)$ in case III. The result then follows.

In the cases IV to XII, the conditions (26) and (27) say that Proposition 1.13 applies, so $\mathscr{F}$ is a Steiner bundle of the form $\mathscr{F}_b$. This gives the desired resolution of $\mathscr{E}$. □

Corollary 2.13. *In cases* I *to* III*, general triple planes $f\colon X \to \mathbb{P}^2$ do exist and X is a rational surface.*

Proof. Let us consider case I. By Theorem 2.12 we have $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^2}(1)^4$ which is globally generated, so the triple cover exists by Theorem 1.2. Using Proposition 2.1 we obtain

$$P_2(X) = h^0(\mathbb{P}^2, S^2\mathscr{E}^\vee(-6)) = h^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(-4)^3) = 0,$$

hence Castelnuovo's Theorem (cf. [BHPV04, Chapter VI, §3]) implies that X is a rational surface. The argument in cases II and III is the same. □

2.3. **The projective bundle associated with a triple plane.**

2.3.1. *Triple planes and direct images.* Let $f\colon X \to \mathbb{P}^2$ be a general triple plane with $p_g = q = 0$ and Tschirnhausen bundle $\mathscr{E}$. We assume $b \geq 5$ and we write $\mathscr{F}$ as before in order to denote the bundle $\mathscr{E}(b-2)$. Sometimes, if we want to emphasize the role of b, we will use the notation $\mathscr{F}_b$ instead of $\mathscr{F}$. The rest of the notation in this subsection is borrowed from §1.4.

As shown in Theorem 2.12, $\mathscr{F}$ is a Steiner bundle of rank 2. Theorem 1.3 implies that X can be realized as a Cartier divisor in $\mathbb{P}(\mathscr{F})$, such that the restriction of $\mathfrak{p}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^2$ to X is our covering map f. More precisely, recall that we denote by ξ the tautological relatively ample line bundle on $\mathbb{P}(\mathscr{F})$ and by ℓ the pull-back to $\mathbb{P}(\mathscr{F})$ of a line in $\mathbb{P}^2$. Then the identification

$$S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E} \simeq S^3\mathscr{F} \otimes \mathscr{O}_{\mathbb{P}^2}(6-b) \tag{28}$$

shows that X lies in the complete linear system $|\mathscr{L}|$, with

$$\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi + (6-b)\ell). \tag{29}$$

Recall also the notation $U = H^0(\mathbb{P}^2, \mathscr{F})$, and consider the morphism $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}(U) \simeq \mathbb{P}^{b-3}$ associated with $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)| \simeq \mathbb{P}(U)$. Setting

$$\mathscr{R} := \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}((6-b)\ell)),$$

the projection formula yields natural identifications

$$\begin{aligned} H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}) &\simeq H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)) \\ &\simeq H^0(\mathbb{P}(\mathscr{F}),\, \mathscr{L}) \simeq H^0(\mathbb{P}^{b-3},\, \mathscr{R}(3)). \end{aligned} \tag{30}$$

In order to get information on the sheaf $\mathscr{R}$, it is useful to consider the Koszul resolution of $\mathbb{P}(\mathscr{F})$ in $\mathbb{P}(V) \times \mathbb{P}(U) \simeq \mathbb{P}^2 \times \mathbb{P}^{b-3}$, which is given taking exterior powers of (4). This reads

$$\wedge^\bullet(W^\vee \otimes \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-1,\,-1)) \to \mathscr{O}_{\mathbb{P}(\mathscr{F})} \to 0 \tag{31}$$

with $W^\vee = H^1(\mathbb{P}^2,\, \mathscr{F}(-2))$; see Proposition 1.6 and (17). We will write $\mathscr{K}_i$ for the image of the ith differential

$$d_i : (\wedge^i W^\vee) \otimes \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-i,\,-i) \to (\wedge^{i-1} W^\vee) \otimes \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(-i+1,\,-i+1)$$

of the complex (31). Moreover, we will often use the relation

$$R^i\mathfrak{q}_*(\mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^{b-3}}(n_1,\,n_2)) = H^i(\mathbb{P}^2,\, \mathscr{O}_{\mathbb{P}^2}(n_1)) \otimes \mathscr{O}_{\mathbb{P}^{b-3}}(n_2), \quad i \in \mathbb{N},\ n_1,\, n_2 \in \mathbb{Z}. \tag{32}$$

We finally define $Y \subset \mathbb{P}^{b-3}$ as the image of $\mathfrak{q}$; then the support of $\mathscr{R}$ is contained in Y. In §2.3.2 we shall see that, if $b \geq 6$, the morphism $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^{b-3}$ is generically injective, so $Y \subset \mathbb{P}^{b-3}$ is a (possibly singular) irreducible threefold which is generated by the 3-secant lines to the canonical curves of genus $g(H)$ representing in $\mathbb{P}^{b-3}$ the net $|H|$ inducing the triple cover. The threefold Y is defined by the 3×3 minors of the matrix N appearing in the resolution of $\mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell))$, namely

$$\mathscr{O}_{\mathbb{P}^{b-3}}(-1)^{b-4} \xrightarrow{N} \mathscr{O}^3_{\mathbb{P}^{b-3}} \to \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell)) \to 0.$$

2.3.2. *Adjunction maps and projective bundles.* We use the notation of §1.4.1. Recall that the canonical line bundle of $\mathbb{P}(\mathscr{F})$ is

$$\omega_{\mathbb{P}(\mathscr{F})} \simeq \mathscr{O}_{\mathbb{P}(\mathscr{F})}(-2\xi + (b-7)\ell); \tag{33}$$

see for instance [Har77, Ex. 8.4, p. 253]. The following result provides a link between the adjunction theory and the vector bundle techniques used in this paper.

Lemma 2.14. *Let $f\colon X \to \mathbb{P}^2$ be a general triple plane with $p_g(X) = q(X) = 0$. Then $\mathfrak{q}|_X$ coincides with the first adjoint map $\varphi_{|K_X+H|}\colon X \to \mathbb{P}^{b-3}$ associated with the ample divisor H.*

Proof. Since H is ample, by the Kodaira vanishing theorem we have $h^1(X,\, K_X + H) = h^2(X,\, K_X + H) = 0$, so the Riemann-Roch theorem gives $h^0(X,\, K_X + H) = g(H) = b - 2$. Therefore it suffices to show that

$$\omega_X \otimes \mathscr{O}_X(H) \simeq \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)|_X.$$

The adjunction formula, together with (29)and (33), yields

$$\omega_X = (\omega_{\mathbb{P}(\mathscr{F})} \otimes \mathscr{L})|_X \simeq \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi - \ell)|_X.$$

Since $\ell|_X = \mathscr{O}_X(H)$, the claim follows. □

Lemma 2.15. *The morphism $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^{b-3}$ contracts precisely the negative sections of the Hirzebruch surfaces of the form $\mathbb{P}(\mathscr{F}|_L)$, where L is an unstable line of $\mathscr{F}$. Moreover, if $b \geq 6$, then $\mathfrak{q}$ is birational onto its image $Y \subseteq \mathbb{P}^{b-3}$, which is a birationally ruled threefold of degree $\binom{b-4}{2}$.*

Proof. We first show that $\mathfrak{q}$ contracts the negative sections. If L is an unstable line of $\mathscr{F}$, then $\mathscr{F}|_L \simeq \mathscr{O}_L \oplus \mathscr{O}_L(b-4)$, so $\mathbb{P}(\mathscr{F}|_L)$ is isomorphic to the Hirzebruch surface $\mathbb{F}_{b-4}$. The divisor $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)$ cuts on $\mathbb{P}(\mathscr{F}|_L)$ the complete linear system $|\mathfrak{c}_0 + (b-4)\mathfrak{f}|$; therefore $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi) \cdot \mathfrak{c}_0 = 0$, that is, $\mathfrak{q}$ contracts $\mathfrak{c}_0$. In particular, this means that the image of $\mathbb{P}(\mathscr{F}|_L)$ via $\mathfrak{q}$ is a cone $S(0,\, b-4) \subset \mathbb{P}^{b-3}$.

Conversely, we now show that $\mathfrak{q}$ is injective on the complement of the set of negative sections over unstable lines. More precisely, assuming that x_1 and x_2 are points of $\mathbb{P}(\mathscr{F})$ not separated by $\mathfrak{q}$, we will prove that x_1 and x_2 lie in one of such sections. In fact, since $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)$ is very ample when restricted to the fibers of $\mathfrak{p}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^2$, the points $\mathfrak{p}(x_1)$ and $\mathfrak{p}(x_2)$ are distinct. Let L be the unique line through $\mathfrak{p}(x_1)$ and $\mathfrak{p}(x_2)$ and let us restrict $\mathfrak{q}$ to $\mathbb{P}(\mathscr{F}|_L)$. If L were not unstable for $\mathscr{F}$, then $\mathscr{F}|_L \simeq \mathscr{O}_L(a) \oplus \mathscr{O}_L(b-4-a)$ with $a > 0$ and $b - 4 - a > 0$ (cf. the proof of Lemma 1.9), and in this situation the restriction of $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi)$ to $\mathbb{P}(\mathscr{F}|_L)$ would be very ample, hence $\mathfrak{q}$ would separate x_1 and x_2, a contradiction. This shows that L is necessarily an unstable line for $\mathscr{F}$ and that moreover x_1 and x_2 must both lie on the unique negative section of $\mathbb{P}(\mathscr{F}|_L) \simeq \mathbb{F}_{b-4}$. The same argument also works if x_1 and x_2 are infinitely near, and this ends the proof of the first statement.

Regarding the second statement, the subscheme $\mathscr{W}(\mathscr{F}_b)$ of unstable lines has positive codimension in $\check{\mathbb{P}}^2$ for $b \geq 6$; see §1.4.2. Then $\mathfrak{q}$ is birational onto its image $Y \subset \mathbb{P}^{b-3}$, and this in particular says that Y is a birationally ruled threefold in $\mathbb{P}^{b-3}$ (of course for $b = 6$ the image is the whole $\mathbb{P}^3$).

We can now use (26) and the Chern equation for $\mathbb{P}(\mathscr{F}_b)$ in order to compute the degree of Y, obtaining

$$\deg Y = \xi^3 = \mathfrak{p}^*(c_1(\mathscr{F}_b)^2 - c_2(\mathscr{F}_b))\xi = (b-4)^2 - \binom{b-3}{2} = \binom{b-4}{2}. \qquad \square$$

Lemma 2.16. *Let $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi + (6-b)\ell)$ and let $\mathfrak{c}_0$ be the negative section of the Hirzebruch surface $\mathbb{P}(\mathscr{F}|_L)$, where L is an unstable line for $\mathscr{F}$. If $b \geq 7$, then $\mathfrak{c}_0$ is contained in the base locus of $|\mathscr{L}|$.*

Proof. By restricting any element of $|\mathscr{L}|$ to $\mathbb{P}(\mathscr{F}|_L)$ we obtain a divisor $\mathscr{L}'$ linearly equivalent to

$$3(\mathfrak{c}_0 + (b-4)\mathfrak{f}) + (6-b)\mathfrak{f} = 3\mathfrak{c}_0 + (2b-6)\mathfrak{f}.$$

We have $\mathscr{L}'\mathfrak{c}_0 = 3(4-b) + (2b-6) = 6-b$, so if $b \geq 7$ we have $\mathscr{L}'\mathfrak{c}_0 < 0$ and this in turn implies that $\mathfrak{c}_0$ is a component of $\mathscr{L}'$. Hence $\mathfrak{c}_0$ is contained in every element of the linear system $|\mathscr{L}|$. □

Let us come back now to our general triple planes $f\colon X \to \mathbb{P}^2$.

Proposition 2.17. *If $b \geq 7$, then the first adjoint map $\varphi_{|K_X+H|}\colon X \to \mathbb{P}^{b-3}$ is a birational morphism onto its image $X_1 \subset \mathbb{P}^{b-3}$. Furthermore, X_1 is a smooth surface and $\varphi_{|K_X+H|}$ contracts precisely the (-1)-curves E in X such that $HE = 1$. There is one, and only one, curve with this property for each unstable line of $\mathscr{F}$.*

Proof. By Lemma 2.15 the map $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^{b-3}$ is birational onto its image and contracts precisely the negative sections of $\mathbb{P}(\mathscr{F}|_L)$, where L is an unstable line of $\mathscr{F}$; let E be one of these sections. In view of Lemma 2.14 we have $\varphi_{|K_X+H|} = \mathfrak{q}|_X$, and moreover by Lemma 2.16 the curve E is contained in X, because $X \in |\mathscr{L}|$ by construction (see §2.3). We have $f = \mathfrak{p}|_X$, hence $f|_E = \mathfrak{p}|_E$ and, since $\mathfrak{p}|_E\colon E \to L$ is an isomorphism, by the projection formula we obtain

$$HE = f^*L \cdot E = L \cdot f_*E = L^2 = 1.$$

Finally, each Hirzebruch surface $\mathbb{P}(\mathscr{F}|_L)$ contains precisely one negative section, so we are done. □

Remark 2.18. When $b \geq 7$, Proposition 2.17 will allow us to apply the iterated adjunction process described in §1.3 starting from $D = H$, even if H is ample but *not* very ample.

Remark 2.19. Proposition 3.9 will show that $\varphi_{|K_X+H|}$ is birational also for $b = 6$: more precisely, in this case X is the blow-up at nine points of a cubic surface $S \subset \mathbb{P}^3$, and $\varphi_{|K_X+H|}$ is the blow-down morphism. In fact, $\mathscr{W}(\mathscr{F}_6)$ is a smooth conic in $\check{\mathbb{P}}^2$; cf. §1.4.2. If L is an unstable line of $\mathscr{F}_6$, namely a line tangent to this conic, we have $\mathbb{P}(\mathscr{F}_6|_L) \simeq \mathbb{F}_2$ and $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^3$ contracts the unique negative section of this Hirzebruch surface to a point. The locus of points in $\mathbb{P}^3$ constructed in this way is a twisted cubic C, the map $\mathfrak{q}$ is the blow-up of $\mathbb{P}^3$ at C and the nine points that we blow-up in S consist of the subset $S \cap C$.

3. The classification in cases I to VII

Since all the triple planes considered in the sequel are general, for the sake of brevity the word *general* will be from now on omitted.

3.1. Triple planes of type I.

In this case the invariants are

$$K_X^2 = 8, \quad b = 2, \quad h = 1, \quad g(H) = 0$$

and the Tschirnhausen bundle splits as $\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-1)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.1 below provides their complete classification.

Proposition 3.1. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane of type* I. *Then X is isomorphic to the cubic scroll $S(1,\,2) \subset \mathbb{P}^4$ and f is the projection of this scroll from a general line of $\mathbb{P}^4$.*

Proof. By Proposition 2.5 we know that R is very ample, and by (25) we have $K_X R = -7$. Therefore no multiple of K_X can be effective and X is a rational surface, as predicted by Corollary 2.13. The curve R is the normalization of B (Proposition 2.5), which is a tricuspidal quartic curve (Proposition 2.4), hence $g(R) = 0$. Then by the first statement in Theorem 1.4 we get

$$\dim |K_X + R| = g(R) + p_g(X) - q(X) - 1 = -1,$$

that is, $|K_X + R| = \emptyset$. The condition $K_X^2 = 8$ implies that the X is not isomorphic to $\mathbb{P}^2$ so, again by Theorem 1.4, part A), it must be a rational normal scroll, with the scroll structure arising from the embedding given by $|R|$. By the first equality in (22) we have $R^2 = 5$, and there are two different kinds of smooth rational normal scrolls of dimension 2 and degree 5, namely

- $S(1,\,4)$, that is, $\mathbb{F}_3$ embedded in $\mathbb{P}^6$ via $|\mathfrak{c}_0 + 4\mathfrak{f}|$;
- $S(2,\,3)$, that is, $\mathbb{F}_1$ embedded in $\mathbb{P}^6$ via $|\mathfrak{c}_0 + 3\mathfrak{f}|$.

In the former case, using (21) we obtain $H = \mathfrak{c}_0 + 3\mathfrak{f}$, which is not ample on $\mathbb{F}_3$; so this case cannot occur. In the latter case we have $H = \mathfrak{c}_0 + 2\mathfrak{f}$, that is very ample and embeds $\mathbb{F}_1$ in $\mathbb{P}^4$ as a cubic scroll $S(1,\,2)$. The triple plane is now obtained by taking the morphism to $\mathbb{P}^2$ associated with a general net of curves inside $|H|$, which corresponds to the projection of $S(1,\,2)$ from a general line of $\mathbb{P}^4$. □

Remark 3.2. Another description of triple planes of type I is the following. Let X' be the Veronese surface, embedded in the Grassmannian $\mathbb{G}(1,\,\mathbb{P}^3)$ as a surface of bidegree (3, 1); see [Gro93, Theorem 4.1 (*a*)]. There is a family of 1-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type I (in fact, X is the blow-up of X' at one point).

3.2. **Triple planes of type** II. In this case the invariants are

$$K_X^2 = 3, \quad b = 3, \quad h = 2, \quad g(H) = 1$$

and the Tschirnhausen bundle splits as $\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.3 below provides their complete classification.

Proposition 3.3. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane of type* II. *Then X is isomorphic to a smooth cubic surface $S \subset \mathbb{P}^3$ and f is the projection of S from a general point of $\mathbb{P}^3$. The branch locus B is a sextic plane curve with six cusps lying on a conic.*

Proof. By Proposition 2.9, the divisor $D := K_X + 2H$ is very ample. Using $K_X H = 2b - 9 = -3$ (see the proof of Proposition 2.11), we obtain

$$D^2 = (K_X + 2H)^2 = K_X^2 + 4K_X H + 4H^2 = 3 - 12 + 12 = 3,$$

hence the map $\varphi_{|D|}\colon X \to \mathbb{P}^3$ is an isomorphism onto a smooth cubic surface S. The statement about the position of the cusps in the branch locus is a well-known classical result; see [Zar29, p. 320]. □

Remark 3.4. Other descriptions of triple planes of type II are the following.

- Let X' be a smooth Del Pezzo surface of degree 5, embedded in $\mathbb{G}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 2); see [Gro93, Theorem 4.1 (*b*)]. There is a family of 2-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type II (in fact, X is the blow-up of X' at two points).
- Let X' be a smooth Del Pezzo surface of degree 6, embedded in $\mathbb{G}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 3); see [Gro93, Theorem 4.1 (*d*)]. There is a family of 3-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type II (in fact, X is the blow-up of X' at three points).

3.3. **Triple planes of type** III. In this case the invariants are

$$K_X^2 = -1, \quad b = 4, \quad h = 4, \quad g(H) = 2$$

and the Tschirnhausen bundle splits as $\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}(-2) \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$. The existence of these triple planes follows from Corollary 2.13, whereas Proposition 3.5 below provides their complete classification.

Proposition 3.5. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane of type* III. *Then X is a blow-up at nine points $\sigma\colon X \to \mathbb{F}_n$ of a Hirzebruch surface $\mathbb{F}_n$, with $n \in \{0, 1, 2, 3\}$, and*

$$H = 2\mathfrak{c}_0 + (n+3)\mathfrak{f} - \sum_{i=1}^{9} E_i. \tag{34}$$

Proof. By Proposition 2.9, the divisor $D := K_X + 2H$ is very ample. We have

$$\begin{pmatrix} D^2 & K_X D \\ K_X D & K_X^2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & -1 \end{pmatrix},$$

in particular $K_X D < 0$ shows that X is a rational surface. By Serre duality and the Kodaira vanishing theorem we have $h^1(X, D) = h^1(X, -2H) = 0$, and analogously $h^2(X, D) = h^0(X, -2H) = 0$, so by the Riemann-Roch theorem we obtain

$$h^0(X, D) = \chi(X, D) = \frac{D(D-K_X)}{2} + \chi(\mathscr{O}_X) = 6.$$

The morphism $\varphi_{|D|}\colon X \to X_1 \subset \mathbb{P}^5$ is an isomorphism of X onto its image X_1, which is a surface of degree 7 with $K_{X_1}^2 = -1$. Embedded projective varieties of degree at most 7 are classified in [Ion84]; in particular, the table on page 148 of that paper shows that X_1 is a blow-up at nine points $\sigma\colon X_1 \to \mathbb{F}_n$, with $n \in \{0, 1, 2, 3\}$, and that

$$D = 2\mathfrak{c}_0 + (n+4)\mathfrak{f} - \sum_{i=1}^{9} E_i.$$

Using $2H = D - K_X$, we obtain (34). □

Remark 3.6. When $n = 0$, the surface X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at nine points and a birational model of the triple plane $f\colon X \to \mathbb{P}^2$ is obtained by using the curves of bidegree (2, 3) passing through these points, since (34) becomes $H = 2L_1 + 3L_2 - \sum_{i=1}^{9} E_i$.

When $n = 1$, since $\mathbb{F}_1$ is the blow-up of the plane at one point, we see from (34) that X can be also seen as the blow-up of $\mathbb{P}^2$ at 10 points and that $H = 4L - 2E_{10} - \sum_{i=1}^{9} E_i$.

Another description of triple planes of type III is the following. Let X' be a Castelnuovo surface with $K_{X'}^2 = 2$, embedded in $\mathbb{G}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 3); see [Gro93, Theorem 4.1 (e)]. There is a family of 3-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type III (in fact, X is the blow-up of X' at three points).

3.4. **Triple planes of type** IV**.** In this case the invariants are

$$K_X^2 = -4, \quad b = 5, \quad h = 7, \quad g(H) = 3.$$

By Theorem 2.12, the resolution of $\mathscr{F} = \mathscr{E}(3)$ is

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to \mathscr{O}_{\mathbb{P}^2}^3 \to \mathscr{F} \to 0,$$

hence $\mathscr{F} \simeq T_{\mathbb{P}^2}(-1)$ and (28) implies that $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}$ is isomorphic to $S^3(T_{\mathbb{P}^2}(-1)) \otimes \mathscr{O}_{\mathbb{P}^2}(1)$, which is globally generated. By Theorem 1.2 this ensures the existence of triple planes of type IV, whereas Proposition 3.7 below provides their complete classification.

Proposition 3.7. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane of type* IV*. Then*:

i) *The surface X is isomorphic to the blow-up of the plane at a subset Z of* 13 *points imposing only* 12 *conditions on quartic curves, and $|H|$ is the complete linear system of quartics passing through Z.*

ii) *Z can be naturally identified with a* 0*-dimensional subscheme of $\check{\mathbb{P}}^2$, that we call again Z, arising as the zero locus of a global section of $T_{\check{\mathbb{P}}^2}(2)$ canonically associated with the building section $\eta \in H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$ of the triple plane. Furthermore, the subscheme $Z \subset \check{\mathbb{P}}^2$ determines η up to a multiplicative constant.*

Proof. Let us show **i)**. By Proposition 2.9 the divisor $D := K_X + 2H$ is very ample. Therefore, the first adjunction map

$$\varphi_1 := \varphi_{|K_X+D|}\colon X \to X_1 \subset \mathbb{P}^5$$

is a birational morphism onto a smooth surface X_1. Moreover, the intersection matrix of X_1 is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & -4+\alpha_1 \end{pmatrix},$$

where D_1 and α_1 are defined in §1.3. In particular $K_{X_1}D_1 < 0$ shows that X_1 (and so X) is a rational surface. We have $g(D_1) = 0$, thus by Theorem 1.4 the adjoint linear system $|K_{X_1} + D_1|$ has dimension -1, i.e., it is empty. By the same result, it follows that the surface X_1 is either a rational normal scroll (and in this case $\alpha_1 = 12$) or $\mathbb{P}^2$ (and in this case $\alpha_1 = 13$). Let us exclude the former case. There are two types of smooth quartic rational normal scroll surfaces: $S(2, 2)$, namely $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^5$ by $|L_1 + 2L_2|$, and $S(1, 3)$, namely $\mathbb{F}_2$ embedded in $\mathbb{P}^5$ by $|\mathfrak{c}_0 + 3\mathfrak{f}|$. The equality $D_1 = 2K_X + 2H$ implies that if $X_1 = \mathbb{P}^1 \times \mathbb{P}^1$ we have

$$2H = 5L_1 + 6L_2 - \sum_{i=1}^{12} 2E_i,$$

whereas if $X_1 = \mathbb{F}_2$ we have

$$2H = 5\mathfrak{c}_0 + 11\mathfrak{f} - \sum_{i=1}^{12} 2E_i.$$

In both cases we obtain a contradiction, since H must be a divisor with integer coefficients.

It follows that $(X_1, D_1) = (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(2))$, hence $\alpha_1 = 13$ and φ_1 contracts exactly 13 exceptional lines, i.e., X is isomorphic to the blow-up of $\mathbb{P}^2$ at 13 points. Therefore we get

$$X = \mathbb{P}^2(p_1, \ldots, p_{13}), \quad D = 5L - \sum_{i=1}^{13} E_i,$$

which implies $H = 4L - \sum_{i=1}^{13} E_i$. Since $h^0(X, \mathscr{O}_X(H)) = 3$, the points in the set $Z := \{p_1, \ldots, p_{13}\}$ impose only 12 conditions on plane quartic curves.

We now prove **ii)**. We use the notation of §1.4.1, so that the vector bundle $\mathscr{F} \simeq T_{\mathbb{P}^2}(-1)$ has a resolution of the form (5), with the 3-dimensional vector space $U = H^0(\mathbb{P}^2, \mathscr{F})$ being naturally identified with $V^\vee$. By the results in §2.3, in this case $\mathbb{P}(\mathscr{F})$ is the point-line incidence correspondence in $\mathbb{P}^2 \times \check{\mathbb{P}}^2$, namely a smooth hyperplane section of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$, so we have

$$0 \to \mathscr{O}_{\mathbb{P}^2 \times \check{\mathbb{P}}^2}(-1, -1) \to \mathscr{O}_{\mathbb{P}^2 \times \check{\mathbb{P}}^2} \to \mathscr{O}_{\mathbb{P}(\mathscr{F})} \to 0. \tag{35}$$

Twisting (35) by $\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}^2}(1)) = \mathscr{O}_{\mathbb{P}^2 \times \check{\mathbb{P}}^2}(1, 0)$, applying the functor $\mathfrak{q}_*$ and using (32) we obtain

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1) \to H^0(\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1)) \otimes \mathscr{O}_{\check{\mathbb{P}}^2} \to \mathfrak{q}_*\big(\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}^2}(1)) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{F})}\big) \to 0,$$

so the Euler sequence yields

$$\mathscr{R} = \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell)) = \mathfrak{q}_*\big(\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}^2}(1)) \otimes \mathscr{O}_{\mathbb{P}(\mathscr{F})}\big) \simeq T_{\check{\mathbb{P}}^2}(-1)$$

and equality (30) implies

$$H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}) = H^0(\check{\mathbb{P}}^2, \mathscr{R}(3)) = H^0(\check{\mathbb{P}}^2, T_{\check{\mathbb{P}}^2}(2)).$$

This shows that the building section η of our triple plane is naturally associated with a global section of $T_{\check{\mathbb{P}}^2}(2)$ that we call η, too, and whose vanishing locus will be denoted by $Z = D_0(\eta)$. Note that Z is a 0-dimensional subscheme of $\check{\mathbb{P}}^2$ such that $\mathrm{length}(Z) = c_2(T_{\check{\mathbb{P}}^2}(2)) = 13$.

Furthermore we have $\mathscr{R}(3) = \mathfrak{q}_*\mathscr{L}$, where $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi + \ell)$, and our triple plane X is a smooth divisor in the complete linear system $|\mathscr{L}|$; see (29). Since a global section of $\mathscr{L}$ corresponds to a non-zero morphism $\mathscr{O}_{\mathbb{P}(\mathscr{F})} \to \mathscr{L}$, we obtain a short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(-3\xi) \to \mathscr{L}(-3\xi) \to \mathscr{O}_X(H) \to 0, \tag{36}$$

and so, taking the direct image via $\mathfrak{q}$, we get

$$0 \to \mathscr{O}_{\check{\mathbb{P}}^2}(-3) \to T_{\check{\mathbb{P}}^2}(-1) \to \mathscr{I}_{Z/\check{\mathbb{P}}^2}(4) \to 0. \tag{37}$$

The inclusion $X \simeq \mathbb{P}(\mathscr{O}_X(H)) \hookrightarrow \mathbb{P}(\mathscr{F})$ corresponds to the surjection $\mathscr{L}(-3\xi) \to \mathscr{O}_X(H)$ in (36); then (37) shows that X can be identified with $\mathbb{P}(\mathscr{I}_{Z/\check{\mathbb{P}}^2}(4))$, embedded in $\mathbb{P}(T_{\check{\mathbb{P}}^2}(-1))$ via the surjection $T_{\check{\mathbb{P}}^2}(-1) \to \mathscr{I}_{Z/\check{\mathbb{P}}^2}(4)$. Hence a model of the triple cover map $f\colon X \to \mathbb{P}^2$ is the rational map $\check{\mathbb{P}}^2 \dashrightarrow \mathbb{P}^2$ given by the linear system of (dual) quartics through Z. This identifies X with the blow-up of $\check{\mathbb{P}}^2$ at Z.

Finally, let us show that the subscheme Z determines $\eta \in H^0(\check{\mathbb{P}}^2, T_{\check{\mathbb{P}}^2}(2))$ up to a multiplicative constant. To this purpose, we use Proposition 1.14 with $t = 0$, so we only have to check that $h^1(\mathbb{P}^2, \mathscr{I}_{Z/\mathbb{P}^2}(4)) = 1$. But this is clear since $\chi(\mathbb{P}^2, \mathscr{I}_{Z/\mathbb{P}^2}(4)) = -2$ and $h^0(\mathbb{P}^2, \mathscr{I}_{Z/\mathbb{P}^2}(4)) = 3$. □

Remark 3.8. A *Bordiga surface* is a smooth surface of degree 6 in $\mathbb{P}^4$, given by the blow-up of $\mathbb{P}^2$ at 10 points embedded by the linear system of plane quartics through them; see [Ott95, Capitolo 5]. Then Proposition 3.7 shows that a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type IV can be realized as the projection of a Bordiga surface from a 3-secant line.

Furthermore, contracting one of the exceptional divisors in the Bordiga surface, we obtain a rational surface X' with $K^2_{X'} = 0$ that can be embedded in $\mathbb{G}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 4); see [Gro93, Theorem 4.1 (f)]. There is a family of 4-secant planes to X'; projecting from one of these planes, we obtain another birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type IV (in fact, X is the blow-up of X' at four points).

3.5. **Triple planes of type** V**.** In this case the invariants are

$$K^2_X = -6, \quad b = 6, \quad h = 11, \quad g(H) = 4$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathscr{F}$ has a resolution of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1)^2 \xrightarrow{M} \mathscr{O}^4_{\mathbb{P}^2} \longrightarrow \mathscr{F} \longrightarrow 0. \tag{38}$$

Since $\mathscr{F}$ is globally generated, it follows that $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E} = S^3\mathscr{F}$ is globally generated, too. Hence triple planes $f\colon X \to \mathbb{P}^2$ of type V do exist by Theorem 1.2. The next result provides their classification.

Proposition 3.9. *Let $f\colon X \to \mathbb{P}^2$ be a triple plane of type* V*. Then:*

- **i)** *The surface X is isomorphic to the blow-up $\mathbb{P}^2(p_1, \ldots, p_{15})$ of $\mathbb{P}^2$ at 15 points and the triple plane map is induced by the linear system of plane sextics singular at $p_1, \ldots, p_6$ and passing through $p_7, \ldots, p_{15}$.*
- **ii)** *The nine points $p_7, \ldots, p_{15}$ consist of the intersection $S \cap C$, where $S = \mathbb{P}^2(p_1, \ldots, p_6)$ is a cubic surface in $\mathbb{P}^3$, naturally associated with the building section $\eta \in H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$, whereas C is a twisted cubic such that $\mathbb{P}(\mathscr{F})$ is the blow-up of $\mathbb{P}^3$ at C.*

Proof. Let us show **i)**. By Proposition 2.9 the divisor $D := K_X + 2H$ is very ample. We have $K_X H = 2b - 9 = 3$, and the genus formula yields $g(D) = 10$, so by Theorem 1.4 we deduce that the first adjoint system $|K_X + D|$ has dimension 9. Therefore the first adjunction map

$$\varphi_1 = \varphi_{|K_X+D|}\colon X \to X_1 \subset \mathbb{P}^9$$

is birational onto its image X_1, whose intersection matrix is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ -6 & -6+\alpha_1 \end{pmatrix}.$$

In particular $K_{X_1}D_1 < 0$ shows that X_1 (and so X) is a rational surface. Now we consider the second adjunction map $\varphi_2\colon X_1 \to X_2 \subset \mathbb{P}^3$, which is an isomorphism onto its image X_2 (Proposition 2.10), whose intersection matrix is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} -6+\alpha_1 & -12+\alpha_1 \\ -12+\alpha_1 & -6+\alpha_1 \end{pmatrix}.$$

This shows that X_2 is a non-degenerate, smooth rational surface in $\mathbb{P}^3$, hence it is either a quadric surface or a cubic surface. If X_2 were a quadric, then $(D_2)^2 = 2$, hence $\alpha_1 = 8$ and the intersection matrix would give $(K_{X_2})^2 = 2$, which is a contradiction. Therefore X_2 is a cubic surface S, hence $\alpha_1 = 9$. Moreover X_1 is isomorphic to X_2, so X is the blow-up of S at nine points. It follows that

$$X = \mathbb{P}^2(p_1, \dots, p_{15}), \quad D = 9L - \sum_{i=1}^{6} 3E_i - \sum_{i=7}^{15} E_j,$$

which implies $H = 6L - \sum_{i=1}^{6} 2E_i - \sum_{i=7}^{15} E_j$.

We turn to **ii)**. Here we use the approach developed in §1.4.1, in particular we consider again the resolution (5), where in this case $U = H^0(\mathbb{P}^2, \mathscr{F})$ is a 4-dimensional vector space. Set $\mathbb{P}^3 = \mathbb{P}(U)$. By Proposition 1.6, the projective bundle $\mathbb{P}(\mathscr{F})$ is the complete intersection of two divisors of bidegree (1, 1) in $\mathbb{P}^2 \times \mathbb{P}^3$, so the corresponding Koszul resolution is

$$(39) \qquad 0 \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-2,\,-2) \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-1,\,-1)^2 \xrightarrow{d_1} \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3} \to \mathscr{O}_{\mathbb{P}(\mathscr{F})} \to 0.$$

Twisting (39) by $\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}^2}(1)) = \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,\,0)$ and splitting it into short exact sequences, we get

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(-1,\,-2) \longrightarrow \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(0,\,-1)^2 \longrightarrow \widetilde{\mathscr{K}_1} \longrightarrow 0,$$
$$0 \to \widetilde{\mathscr{K}_1} \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,\,0) \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell) \to 0,$$

where $\widetilde{\mathscr{K}_1} := \mathscr{K}_1 \otimes \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^3}(1,\,0)$ and $\mathscr{K}_1$ is the image of the first differential d_1 of the Koszul complex; see §2.3.1. Applying the functor $\mathfrak{q}_*$ and using (32), we infer

$$\mathfrak{q}_*\widetilde{\mathscr{K}_1} = \mathscr{O}_{\mathbb{P}^3}(-1)^2, \quad R^1\mathfrak{q}_*\widetilde{\mathscr{K}_1} = 0,$$

obtaining

$$(40) \qquad 0 \longrightarrow \mathscr{O}_{\mathbb{P}^3}(-1)^2 \xrightarrow{N} \mathscr{O}_{\mathbb{P}^3}^3 \longrightarrow \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell)) \longrightarrow 0.$$

Hence we can identify $\mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell))$ with $\mathscr{I}_{C/\mathbb{P}^3}(2)$, the ideal sheaf of quadrics in $\mathbb{P}^3$ containing a twisted cubic C, which is precisely the image in $\mathbb{P}^3$ of the conic parametrizing the unstable lines of $\mathscr{F}$ (Remark 2.19). Note that C is given by the vanishing of the three 2×2 minors of the matrix of linear forms N appearing in (40); this matrix coincides with the one obtained by "flipping" the matrix M in (38) as explained in §1.4.1; see in particular Remark 1.7. Then $\mathscr{G} = \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell))$, and by Proposition 1.6 we infer

$$\mathbb{P}(\mathscr{F}) \simeq \mathbb{P}(\mathscr{G}) \simeq \mathbb{P}(\mathscr{I}_{C/\mathbb{P}^3}(2)),$$

that is, $\mathbb{P}(\mathscr{F})$ is isomorphic to the blow-up of $\mathbb{P}^3$ along the twisted cubic C and the morphism $\mathfrak{p}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^2$ is induced by the net $|\mathscr{I}_{C/\mathbb{P}^3}(2)|$.

We also get $\mathscr{R} \simeq \mathfrak{q}_*\mathscr{O}_{\mathbb{P}(\mathscr{F})} \simeq \mathscr{O}_{\mathbb{P}^3}$, so (30) yields

$$H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}) = H^0(\mathbb{P}^3,\, \mathscr{R}(3)) = H^0(\mathbb{P}^3,\, \mathscr{O}_{\mathbb{P}^3}(3)).$$

This means that the choice of the (proportionality class of the) building section η in Theorem 1.2 is given by the choice of a cubic surface $S \subset \mathbb{P}^3$. Moreover, from the exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(-3\xi + \ell) \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell) \to \mathscr{O}_X(H) \to 0$$

it follows that $X \simeq \mathbb{P}(\mathscr{O}_X(H))$ is the strict transform of S in $\mathbb{P}(\mathscr{F})$. Also, the triple cover map $f\colon X \to \mathbb{P}^2$ is associated with $|\mathscr{O}_X(H)|$, so that it is induced on S by the linear system of quadrics that contain the intersection $S \cap C$. This intersection consists of nine points $p_7, \dots, p_{15}$. Identifying S with $\mathbb{P}^2(p_1, \dots, p_6)$ with exceptional divisors $E_1, \dots, E_6$, we get thus nine exceptional divisors $E_7, \dots, E_{15}$ on X corresponding to this intersection, and

$$H = 2H_S - \sum_{j=7}^{15} E_j = 6L - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{15} E_j.$$

This identifies the sets $\{p_1, \dots, p_6\}$ and $\{p_7, \dots, p_{15}\}$ with those in part **i)**. □

Remark 3.10. A birational model of the triple plane $f\colon X \to \mathbb{P}^2$ is the projection of a hyperplane section T of a Palatini scroll from a 4-secant line. In fact, T is a surface of degree 7 in $\mathbb{P}^4$ and with $K_T^2 = -2$ (see [Ott95, Capitolo 5]), which is isomorphic to $\mathbb{P}^2$ blown-up at 11 points and embedded in $\mathbb{P}^4$ by the complete linear system $|6L - \sum_{i=1}^{6} 2E_i - \sum_{j=7}^{11} E_j|$. Actually, this is the unique non-degenerate, rational surface of degree 7 in $\mathbb{P}^4$; see [Oko84, Theorems 4 and 6].

Contracting one of the exceptional divisors E_j in T, we obtain a rational surface X' with $K_{X'}^2 = -1$ that can be embedded in $\mathbb{G}(1, \mathbb{P}^3)$ as a surface of bidegree (3, 5); see [Gro93, Theorem 4.1 (g)]. So there is a family of 5-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type V (in fact, X is the blow-up of X' at five points).

Remark 3.11. Triple planes of type I to V were previously considered via "classical" methods by Du Val in [DV33]. For the reader's convenience, let us shortly describe in modern language and using our notation Du Val's nice geometric constructions. They use part of the mass of results on particular rational surfaces proven by nineteenth century algebraic geometers; the classical, a bit old-fashioned monograph on the subject (in Italian) is [Con45], for a modern exposition see [Dol12].

I) We have $g(H) = 0$, and from this one sees that the net $|H|$ is the pull-back of the net of lines $|L|$ in $\mathbb{P}^2$ via the projection of the cubic scroll $S(1, 2) \subset \mathbb{P}^4$ from a general line. The generators of the scroll become an ∞^1 family of lines of index 3 in $\mathbb{P}^2$, i.e., such that for a general point of the plane pass three lines of the family. The envelop of this family is a tricuspidal quartic curve, namely the branch locus B of the triple plane.

II) This time $g(H) = 1$, so that the surface X is either rational or ruled. When $p_g(X) = q(X) = 0$ we are in the first case, and the only possibility for the triple plane is the projection of a smooth cubic surface $S_3 \subset \mathbb{P}^3$ from an external point p. Then the ramification locus R is given by the intersection of S_3 with the polar hypersurface $P_p(S_3)$, which is a quadric Q. Hence R is a smooth curve of degree 6 and genus 4 in $\mathbb{P}^3$, and the six cusps of the branch locus B arise from the intersection of R with the second polar of p, which is a plane Π. In particular, the cusps of B are contained in the projections of both the curves $Q \cap \Pi$ and $S_3 \cap \Pi$, namely they are the complete intersection of a conic and a plane cubic.

III) In this case $g(H) = 2$, and a surface X with a net of genus 2 curves is either a double plane with a branch curve of order 6 (i.e., a $K3$ surface) or a rational surface. In the last case, a detailed analysis of the possible linear systems representing X on $\mathbb{P}^2$ shows that the only possibility in order to have a net $|H|$ inducing a triple plane is that X is the blow-up of $\mathbb{P}^2$ at 10 points, so that the curves of $|H|$ correspond to quartics with one double and nine simple base points. We recovered by modern methods this result; see Remark 3.6 (since Du Val only works with representative linear systems on $\mathbb{P}^2$, he does not consider the birational models of these triple planes arising from linear systems on $\mathbb{F}_n$). It can be observed that this construction corresponds to the projection to $\mathbb{P}^2$ of a quartic surface $S_4 \subset \mathbb{P}^3$, having a double line, from a general point $p \in S_4$. In fact, S_4 is represented on the plane by quartic curves with one double and eight simple base points. On the surface S_4 there is a pencil of conics, corresponding to the pencil of lines on $\mathbb{P}^2$ through the double base point; in the triple plane representation, this becomes a family ∞^1 of conics of index 3, whose envelop is a curve B of degree 8 with 12 cusps, which is precisely the branch locus of our triple plane.

IV) In this case we have $g(H) = 3$, and a detailed analysis of the linear systems $|H|$ and $|K_X + H|$ shows that a birational model of the triple plane is given from the projection of a quintic surface $S_5 \subset \mathbb{P}^2$ having a double twisted cubic from a point of the double curve. From this fact one recovers the plane representation of the linear system $|H|$ as a net of quartics with 13 simple base points, and the representation of the branch curve B as the Jacobian curve of this net. According to Proposition 3.7, the base points are not in general position. In fact, 11 of them, say $p_1, \ldots, p_{11}$, can be taken at random, whereas the remaining two must belong to the g^1_2 of the unique hyperelliptic curve of degree 7 having nodes at $p_1, \ldots, p_{11}$.

V) In this case $g(H) = 4$, and the assumption $p_g(X) = q(X) = 0$ shows that the adjoint linear system $|K_X + H|$ cuts on the general curve of the net $|H|$ the complete canonical system $|K_X|$. Then the image of $|H|$ via the first adjoint map $\varphi_{|K_X+H|} \colon X \to \mathbb{P}^3$ is a net of canonical curves of genus 4 and degree 6. So there is precisely one quadric surface containing each of these curves, and one system of generators of each of these quadrics traces a system of ∞^2 trisecant lines to the image of X, that together define a degree 3 "involution" (Du Val, like his contemporaries, use this term also when dealing with finite covers of degree > 2) which gives a birational model of our triple plane. Pushing this analysis further, it is possible to show that such a system of trisecant lines is actually the system of chords of a twisted cubic C, and this implies that the net $|H|$ can be represented on a cubic surface $S \subset \mathbb{P}^3$ by means of sections by quadrics passing through C. Correspondingly, X is a rational surface that can be represented on the plane by sextic curves with six double and nine simple base points, the latter corresponding to the intersections of S with C. Part **ii)** of Proposition 3.9 is a modern rephrasing of this argument that uses completely different techniques based on vector bundles. Finally, by using envelops one computes that the branch locus B of the triple plane has degree 12; its cusps arise from the chords of C that are also inflectional tangents of S, and a Schubert calculus computation shows that their number equals 33.

3.6. **Triple planes of type** VI. In this case the invariants are

$$K^2 = -7, \quad b = 7, \quad h = 16, \quad g(H) = 5$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathscr{F}$ has a resolution of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1)^3 \xrightarrow{M} \mathscr{O}_{\mathbb{P}^2}^5 \longrightarrow \mathscr{F} \longrightarrow 0. \tag{41}$$

The existence and classification of triple planes of type VI are established in Proposition 3.12 below.

Proposition 3.12. *Let* $f\colon X \to \mathbb{P}^2$ *be a triple plane of type* VI. *Then the following hold*:

- **i)** *The vector bundle* $\mathscr{F}$ *is a logarithmic bundle associated with six lines in general position in* $\mathbb{P}^2$.
- **ii)** *The morphism* $\mathfrak{q}\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^4$ *is birational onto its image, which is a determinantal cubic threefold* $Y \subset \mathbb{P}^4$, *which has exactly six nodes as singularities.*
- **iii)** *The surface* X *is the blow-up of a Bordiga surface* $X_1 \subset Y$ *at the six nodes of* Y *that belong to* X_1. *So* X *is the blow-up* $\mathbb{P}^2$ *at* 16 *points and the net* $|H|$ *defining the triple cover* f *is given by*

$$H = 7L - \sum_{i=1}^{10} 2E_i - \sum_{j=11}^{16} E_j. \tag{42}$$

Proof of **ii)**. We use again the approach and notation of §1.4. We look at the exact sequence (5) and we consider the projective space $\mathbb{P}^4 = \mathbb{P}(U)$ that coincides with the space of global sections of the Steiner bundle $\mathscr{F}$. By (6), the 5×3 matrix M of linear forms presenting $\mathscr{F}$ is naturally associated with a 3×3 matrix N, generically of maximal rank, defining a Steiner sheaf $\mathscr{G}$ over $\mathbb{P}^4$, namely

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^4}(-1)^3 \xrightarrow{N} \mathscr{O}_{\mathbb{P}^4}^3 \longrightarrow \mathscr{G} \longrightarrow 0. \tag{43}$$

Now recall that the morphism $\mathfrak{q}$ is birational onto its image by Lemma 2.15, and that $\mathbb{P}(\mathscr{G}) \simeq \mathbb{P}(\mathscr{F})$ by Proposition 1.6, so that $\mathfrak{q}$ maps $\mathbb{P}(\mathscr{F})$ to the support of $\mathscr{G}$, which is the determinantal hypersurface $Y \subset \mathbb{P}^4$ defined by $\det(N) = 0$. Note that the Porteous formula says that the threefold Y is singular, expectedly at six points; see [ACGH85, Chapter II]. □

Claim 3.13. *The surface* $X_1 \subset \mathbb{P}^4$, *the image of the first adjunction map* $\varphi_{|K_X+H|}\colon X \to \mathbb{P}^4$, *is a Bordiga surface of degree* 6. *It is defined by the vanishing of the maximal minors of a* 3×4 *matrix obtained by stacking a row to the transpose of* N.

Proof. By the results of §2.3.1, the surface X corresponds to a global section

$$\eta \in H^0(\mathbb{P}(\mathscr{F}),\, \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi - \ell)) \simeq H^0(\mathbb{P}^4,\, \mathscr{R}(3)),$$

where $\mathscr{R} = \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(-\ell))$. The idea is to directly relate $\mathscr{R}$ to the sheaf $\mathscr{G}$ appearing in (43) or, equivalently, to the matrix N.

By Proposition 1.6 the projective bundle $\mathbb{P}(\mathscr{F})$ is a 3-fold linear section of $\mathbb{P}^2 \times \mathbb{P}^4$, i.e., the complete intersection of three divisors of bidegree (1, 1) in $\mathbb{P}^2 \times \mathbb{P}^4$. Tensoring the Koszul resolution (31) of $\mathscr{O}_{\mathbb{P}(\mathscr{F})}$ inside $\mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}$ with $\mathfrak{p}^*(\mathscr{O}_{\mathbb{P}^2}(-1)) = \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$ and splitting it into short exact sequences, we obtain

$$0 \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-4, -3) \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-3, -2)^3 \to \widetilde{\mathscr{K}_2} \to 0, \tag{44}$$

$$0 \to \widetilde{\mathscr{K}_2} \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-2, -1)^3 \to \widetilde{\mathscr{K}_1} \to 0, \tag{45}$$

$$0 \to \widetilde{\mathscr{K}_1} \to \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0) \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(-\ell) \to 0, \tag{46}$$

where $\widetilde{\mathscr{K}_i} := \mathscr{K}_i \otimes \mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$ and $\mathscr{K}_i$ denotes the image of the ith differential of the Koszul complex; see §2.3. Applying the functor $\mathfrak{q}_*$ to (44) and using (32), we deduce $\mathfrak{q}_*\widetilde{\mathscr{K}_2} = 0$ and we get

$$0 \to R^1\mathfrak{q}_*\widetilde{\mathscr{K}_2} \to \mathscr{O}_{\mathbb{P}^4}(-3)^3 \to \mathscr{O}_{\mathbb{P}^4}(-2)^3 \to R^2\mathfrak{q}_*\widetilde{\mathscr{K}_2} \to 0. \tag{47}$$

By (46) the sheaf $\widetilde{\mathscr{K}_1}$ injects into $\mathscr{O}_{\mathbb{P}^2\times\mathbb{P}^4}(-1, 0)$, so we have $\mathfrak{q}_*\widetilde{\mathscr{K}_1} = 0$. Therefore, applying $\mathfrak{q}_*$ to (45), we get

$$R^1\mathfrak{q}_*\widetilde{\mathscr{K}_2} = 0, \quad R^1\mathfrak{q}_*\widetilde{\mathscr{K}_1} \simeq R^2\mathfrak{q}_*\widetilde{\mathscr{K}_2}. \tag{48}$$

Finally, applying the functor $\mathfrak{q}_*$ to (46) we infer

$$\mathscr{R} = \mathfrak{q}_*(\mathscr{O}_{\mathbb{P}(\mathscr{F})}(-\ell)) \simeq R^1\mathfrak{q}_*\widetilde{\mathscr{K}_1}. \tag{49}$$

Using (48) and (49), the exact sequence (47) becomes

$$0 \to \mathscr{O}_{\mathbb{P}^4}(-3)^3 \to \mathscr{O}_{\mathbb{P}^4}(-2)^3 \to \mathscr{R} \to 0,$$

that can be rewritten as

$$0 \to \mathscr{O}_{\mathbb{P}^4}^3 \xrightarrow{{}^tN} \mathscr{O}_{\mathbb{P}^4}(1)^3 \to \mathscr{R}(3) \to 0. \tag{50}$$

Indeed, the self-duality of the Koszul complex implies

$$\mathscr{R} \simeq \mathscr{G}^\vee \simeq \mathscr{E}xt^1_{\mathscr{O}_{\mathbb{P}^4}}(\mathscr{G}(3), \mathscr{O}_{\mathbb{P}^4}),$$

where the second isomorphism is Grothendieck duality; see [Har66, Chapter III, Proposition 7.2].

Let us consider now a non-zero global section $\eta\colon \mathscr{O}_{\mathbb{P}^4} \to \mathscr{R}(3)$ of $\mathscr{R}(3)$, whose cokernel we denote by $\mathscr{H}$. The section η lifts to a map $\mathscr{O}_{\mathbb{P}^4} \to \mathscr{O}_{\mathbb{P}^4}(1)^3$, so by (50) we get an exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^4}(-3) \longrightarrow \mathscr{O}_{\mathbb{P}^4}^4 \xrightarrow{{}^t(N,\eta)} \mathscr{O}_{\mathbb{P}^4}(1)^3 \to \mathscr{H} \longrightarrow 0. \tag{51}$$

The sheaf $\mathscr{H}$ is supported on the surface $X_1 \subset \mathbb{P}^4$. More precisely, this surface is defined by the vanishing of the 3×3 minors of the 3×4 matrix ${}^t(N, \eta)$ of linear forms appearing in (51), hence it is a Bordiga surface of degree 6; see [Ott95, Capitolo 5]. □

By the results of §1.4.2 it follows that the bundle $\mathscr{F}$ has either six or infinitely many unstable lines. Let us give the proof of **iii)** in the former case.

Proof of **iii)**. We assume that $\mathscr{F}$ has six unstable lines. Using Claim 3.13 and Remark 3.8, we can see X_1 as the blow-up of $\mathbb{P}^2$ at 10 points, with exceptional divisors $E_1, \ldots, E_{10}$, embedded in $\mathbb{P}^4$ by the linear system $|4L - \sum_{i=1}^{10} E_i|$. On the other hand, by Proposition 2.17 the first adjoint map $\varphi := \varphi_{|K_X+H|}\colon X \to X_1$ is a birational morphism, contracting precisely the six exceptional divisors $E_{11}, \dots, E_{16}$ on X coming from the blow-up of X_1 at the six nodes of Y. Hence we obtain

$$K_X = \varphi^* K_{X_1} + \sum_{j=11}^{16} E_j = \varphi^* \bigg(-3L + \sum_{i=1}^{10} E_i \bigg) + \sum_{j=11}^{16} E_j \quad \text{and}$$

$$K_X + H = \varphi^* \mathscr{O}_{X_1}(1) = \varphi^* \bigg(4L - \sum_{i=1}^{10} E_i \bigg),$$

so (42) follows. □

If, instead, $\mathscr{F}$ has infinitely many unstable lines, then it is of Schwarzenberger type. The next result shows that this case cannot occur, proving **i)** and so completing the proof of Proposition 3.12.

Claim 3.14. *If $\mathscr{F}$ is a Schwarzenberger bundle, then the vanishing locus of any non-zero global section $\eta \in H^0(\mathbb{P}(\mathscr{F}), \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi - \ell))$ is a reducible surface. As a consequence, if $f \colon X \to \mathbb{P}^2$ is a triple plane of type* VI*, then its Tschirnhausen bundle is a logarithmic one.*

Proof. If $\mathscr{F}$ is a Schwarzenberger bundle, then, up to a change of coordinates, the matrix M defining it is given by (11) and so, using Remark 1.7, one easily finds that the matrix N is

$$N = \begin{pmatrix} z_0 & z_1 & z_2 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_4 \end{pmatrix}.$$

The singular locus of Y is the determinantal variety given by the vanishing of the 2×2 minors of N, and this is a rational normal curve of degree four, $C_4 \subset \mathbb{P}^4$. This curve is also the base locus of the net $|T_L|$ generated by the three determinantal surfaces T_i defined by the 2×2 minors of the matrix N_i obtained from N by removing the ith line. By [Val00, Proposition 1.2] we have

$$h^0(\mathbb{P}^2, S^2\mathscr{F}(-2)) = 1. \tag{52}$$

This global section gives a relative quadric Q in $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(2\xi - 2\ell)|$ over $\mathbb{P}(\mathscr{F})$. The morphism $\mathfrak{q} \colon \mathbb{P}(\mathscr{F}) \to Y$ is the blow-up along C_4, and Q is its exceptional divisor.

The divisor $Q \in |\mathscr{O}_{\mathbb{P}(\mathscr{F})}(2\xi - 2\ell)|$ gives a sheaf map $\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\xi + \ell) \to \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi - \ell)$, which is injective on global sections. Since $h^0(\mathbb{P}^2, S^2\mathscr{F}(-2)) = 1$, this gives an inclusion

$$H^0(\mathbb{P}^2, S^2\mathscr{F}(-2)) \otimes H^0(\mathbb{P}^2, \mathscr{F}(1)) \subseteq H^0(\mathbb{P}^2, S^3\mathscr{F}(-1)). \tag{53}$$

On the other hand, we can compute

$$h^0(\mathbb{P}^2, \mathscr{F}(1)) = 12, \quad h^0(\mathbb{P}^2, S^3\mathscr{F}(-1)) = 12. \tag{54}$$

Indeed, the first equality in (54) is just obtained by twisting (41) by $\mathscr{O}_{\mathbb{P}^2}(1)$ and taking global sections. For the second equality, we tensor the third symmetric power of the exact sequence (41) with $\mathscr{O}_{\mathbb{P}^2}(-1)$, obtaining

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-4) \to \mathscr{O}_{\mathbb{P}^2}(-3)^{15} \to \mathscr{O}_{\mathbb{P}^2}(-2)^{45} \xrightarrow{r_1} \mathscr{O}_{\mathbb{P}^2}(-1)^{35} \xrightarrow{r_0} S^3\mathscr{F}(-1) \to 0.$$

Taking cohomology, we get

$$H^i(\mathbb{P}^2, S^3\mathscr{F}(-1)) \simeq H^{i+1}(\mathbb{P}^2, \ker r_0) \simeq H^{i+2}(\mathbb{P}^2, \ker r_1)$$

for all i, which implies $H^i(\mathbb{P}^2, S^3\mathscr{F}(-1)) = 0$ for $i > 0$. Then

$$h^0(\mathbb{P}^2,\, S^3\mathscr{F}(-1)) = \chi(\mathbb{P}^2,\, S^3\mathscr{F}(-1)) = 12.$$

By (52) and (54) it follows that the inclusion in (53) is actually an equality. Geometrically, this means that any non-zero global section of $S^3\mathscr{F}(-1)$ vanishes along the relative quadric Q, that is, its vanishing locus is the union of this relative quadric and a relative plane. This proves Claim 3.14. □

Remark 3.15. Another way to describe triple planes of type VI is the following. Let X' be the blow-up of $\mathbb{P}^2$ at 10 points, embedded in $\mathbb{G}(1,\, \mathbb{P}^3)$ as a surface of bidegree (3, 6) via the complete linear system $|7L - \sum_{i=1}^{10} 2E_i|$; see [Gro93, Theorem 4.2 (i)]. There is a family of 6-secant planes to X'; projecting from one of these planes, we obtain a birational model of a triple plane $f\colon X \to \mathbb{P}^2$ of type VI (in fact, X is the blow-up of X' at six points).

Remark 3.16. Triple planes of type VI were previously considered (using methods of synthetic projective geometry) by Du Val in [DV35, p. 72]. Let us give a short description of his construction.

We have $g(H) = 5$, and the assumption $p_g(X) = q(X) = 0$ shows that the adjoint linear system $|K_X + H|$ cuts on the general curve of the net $|H|$ the complete canonical system $|K_X|$. Then the image of $|H|$ via the first adjoint map $\varphi_{|K_X+H|}\colon X \to \mathbb{P}^4$ is a net of canonical curves of genus 5 and degree 10. There is an ∞^2 system of trisecant lines to these curves that together give a degree 3 "involution" on the image of X. Such trisecant lines generate a threefold $Y \subset \mathbb{P}^3$ that Du Val recognizes as a determinantal cubic threefold. At this point, the triple cover is constructed by blowing up a Bordiga surface $X_1 \subset Y$ at the six nodes of Y that belong to X_1. Part **iii)** of Proposition 3.12 is a modern rephrasing of this argument that uses completely different techniques based on vector bundles. By using his remarkable knowledge of "classical" algebraic geometry, at the end of his analysis Du Val is also able to identify X as a congruence of type (3, 6) inside $\mathbb{G}(1,\, \mathbb{P}^3)$; see Remark 3.15.

3.7. **Triple planes of type** VII. In this case we have

$$K_X^2 = -7, \quad b = 8, \quad h = 22, \quad g(H) = 6$$

and by Theorem 2.12 the twisted Tschirnhausen bundle $\mathscr{F}$ has a resolution of the form

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^2}(-1)^4 \xrightarrow{M} \mathscr{O}_{\mathbb{P}^2}^6 \longrightarrow \mathscr{F} \longrightarrow 0. \tag{55}$$

By Remark 2.18, we can start the adjunction process on X by using the first adjoint divisor $K_X + H$. According to §1.3, we denote by α_n the number of exceptional curves contracted by the nth adjunction map $\varphi_n\colon X_{n-1} \to X_n$. Recall that α_1, the number of lines contracted by the first adjunction map, is precisely the number of unstable lines of the twisted Tschirnhausen bundle $\mathscr{F}$; see Proposition 2.17.

3.7.1. *The occurrences for triple planes of type VII.*

Proposition 3.17. *If $f\colon X \to \mathbb{P}^2$ is a triple plane of type* VII*, then X belongs to the following list. The cases marked with* $(*)$ *do actually exist.*

(**VII.1a**) $\alpha_1 = 1,\, \alpha_2 = 14 :$ *X is the blow-up at 15 points of a Hirzebruch surface $\mathbb{F}_n$, with $n \in \{0,\, 2\}$, and*

$$H = 5\mathfrak{c}_0 + \left(\frac{5}{2}n + 6\right)\mathfrak{f} - \sum_{i=1}^{14} 2E_i - E_{15};$$

(**VII.1b**)(∗) $\alpha_1 = 1,\, \alpha_2 = 15 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 8L - \sum_{i=1}^{15} 2E_i - E_{16};$$

(**VII.2**)(∗) $\alpha_1 = 2 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 9L - \sum_{i=1}^{4} 3E_i - \sum_{j=5}^{14} 2E_j - \sum_{k=15}^{16} E_k;$$

(**VII.3**)(∗) $\alpha_1 = 3 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 10L - 4E_1 - \sum_{i=2}^{7} 3E_i - \sum_{j=8}^{13} 2E_j - \sum_{k=14}^{16} E_k;$$

(**VII.4a**) $\alpha_1 = 4,\, \alpha_2 = 2 :$ *X is the blow-up of $\mathbb{F}_n$ (with $n \in \{0,\, 1,\, 2,\, 3\}$) at 15 points and*

$$H = 6\mathfrak{c}_0 + (3n + 8)\mathfrak{f} - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{11} 2E_j - \sum_{k=12}^{15} E_k;$$

(**VII.4b**)(∗) $\alpha_1 = 4,\, \alpha_2 = 3 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 10L - \sum_{i=1}^{9} 3E_i - \sum_{j=10}^{12} 2E_j - \sum_{k=13}^{16} E_k;$$

(**VII.4c**) $\alpha_1 = 4,\, \alpha_2 = 4 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 12L - \sum_{i=1}^{7} 4E_i - 3E_8 - \sum_{j=9}^{12} 2E_j - \sum_{k=13}^{16} E_k;$$

(**VII.5a**) $\alpha_1 = 5,\, \alpha_2 = 0 :$ *X is the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at 15 points, and*

$$H = 7L_1 + 7L_2 - \sum_{i=1}^{10} 3E_i - \sum_{j=11}^{15} E_j;$$

(**VII.5b**)(∗) $\alpha_1 = 5,\, \alpha_2 = 1 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 12L - \sum_{i=1}^{6} 4E_i - \sum_{j=7}^{10} 3E_j - 2E_{11} - \sum_{k=12}^{16} E_k;$$

(**VII.6**)(∗) $\alpha_1 = 6 :$ *X is the blow-up of $\mathbb{P}^2$ at 16 points and*

$$H = 13L - \sum_{i=1}^{10} 4E_i - \sum_{j=11}^{16} E_j;$$

(**VII.7**)(∗) $\alpha_1 = 7 :$ *X is the blow-up of an Enriques surface at seven points.*

Proof. We have a birational morphism

$$\varphi_{|K_X+H|}\colon X \to X_1 \subset \mathbb{P}^5$$

and an intersection matrix

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & -7+\alpha_1 \end{pmatrix}.$$

By the Hodge Index Theorem we infer $0 \leq \alpha_1 \leq 7$. Let us consider separately the different cases.

• $\boldsymbol{\alpha_1 = 0}$. The second adjunction map gives a pair $(X_2,\, D_2)$, such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 3 & -7 \\ -7 & -7+\alpha_2 \end{pmatrix}.$$

This gives a contradiction, since a smooth surface of degree 3 in $\mathbb{P}^5$ is necessarily contained in a hyperplane. Hence the case $\alpha_1 = 0$ cannot occur.

• $\boldsymbol{\alpha_1 = 1}$. The second adjunction map gives a pair $(X_2,\, D_2)$, such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 4 & -6 \\ -6 & -6+\alpha_2 \end{pmatrix}.$$

A smooth, linearly normal surface of degree 4 in $\mathbb{P}^5$ is either a rational scroll or the Veronese surface. In the former case we have $(K_{X_2})^2 = 8$, hence $\alpha_2 = 14$ and, using the classification of rational scrolls in $\mathbb{P}^5$ (see the proof of Proposition 3.7), we get (**VII.1a**). In the latter case we have $(K_{X_2})^2 = 9$, hence $\alpha_2 = 15$. This gives (**VII.1b**).

• $\boldsymbol{\alpha_1 = 2}$. The second adjunction map gives a pair $(X_2,\, D_2)$, such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 5 & -5 \\ -5 & -5+\alpha_2 \end{pmatrix}.$$

In particular X_2 has degree 5, hence it must be a Del Pezzo surface. So $(K_{X_2})^2 = 5$, that is, $\alpha_2 = 10$. This gives (**VII.2**).

• $\boldsymbol{\alpha_1 = 3}$. The second adjunction map gives a pair $(X_2,\, D_2)$, such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ -4 & -4+\alpha_2 \end{pmatrix}.$$

The Hodge Index Theorem implies $\alpha_2 \leq 6$. On the other hand, Theorem 1.4 implies $(K_{X_2}+D_2)^2 \geq 0$, hence $\alpha_2 \geq 6$. It follows that $\alpha_2 = 6$, hence $(K_{X_2}+D_2)^2 = 0$. So X_2 is a conic bundle of degree 6 and sectional genus 2 in $\mathbb{P}^5$, containing precisely six reducible fibers because $(K_{X_2})^2 = 2$. It turns out that X_2 is the blow-up of $\mathbb{P}^2$ at seven points, embedded in $\mathbb{P}^5$ via the linear system

$$D_2 = 4L - 2E_1 - \sum_{i=2}^{7} E_i;$$

see [Ion81]. This is case (**VII.3**).

• $\alpha_1 = 4$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -3 & -3+\alpha_2 \end{pmatrix}.$$

The Hodge Index Theorem implies $\alpha_2 \leq 4$, whereas the condition $(K_{X_2}+D_2)^2 \geq 0$ gives $\alpha_2 \geq 2$; then $2 \leq \alpha_2 \leq 4$.

⋄ If $\alpha_2 = 2$, then by [Ion84, p. 148] it follows that X_2 is the blow-up at nine points of $\mathbb{F}_n$, with $n \in \{0, 1, 2, 3\}$, and that

$$D_2 = 2\mathfrak{c}_0 + (n+4)\mathfrak{f} - \sum_{i=1}^{9} E_i.$$

This is case (**VII.4a**).

⋄ If $\alpha_2 = 3$, then the third adjunction map gives a pair (X_3, D_3) whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & \alpha_3 \end{pmatrix}.$$

This implies $(X_3, D_3) = (\mathbb{P}^2, \mathscr{O}_{\mathbb{P}^2}(1))$, so $\alpha_3 = 9$. This is case (**VII.4b**).

⋄ If $\alpha_2 = 4$, then (X_2, D_2) is as in case 6) of Theorem 1.4. This is (**VII.4c**).

• $\alpha_1 = 5$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 8 & -2 \\ -2 & -2+\alpha_2 \end{pmatrix}.$$

Then the Hodge Index Theorem implies $0 \leq \alpha_2 \leq 2$.

⋄ If $\alpha_2 = 0$, then the third adjunction map gives a pair (X_3, D_3), where $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 2 & -4 \\ -4 & -2+\alpha_3 \end{pmatrix}.$$

Hence $(X_3, D_3) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathscr{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1, 1))$, so in particular $\alpha_3 = 10$. This is case (**VII.5a**).

⋄ If $\alpha_2 = 1$, then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ -3 & -1+\alpha_3 \end{pmatrix}.$$

Therefore $X_3 = \mathbb{P}^2(p_1, \ldots, p_6)$ is a smooth cubic surface, in particular $\alpha_3 = 4$ and $D_3 = 3L - \sum_{i=1}^6 E_i$. This is case (**VII.5b**).

⋄ If $\alpha_2 = 2$, then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^3$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ -2 & \alpha_3 \end{pmatrix}.$$

Therefore X_3 is a smooth quartic surface, a contradiction because we are assuming $p_g(X) = 0$. This case cannot occur.

• $\alpha_1 = 6$. The second adjunction map gives a pair (X_2, D_2), such that the intersection matrix on the surface $X_2 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_2)^2 & K_{X_2}D_2 \\ K_{X_2}D_2 & (K_{X_2})^2 \end{pmatrix} = \begin{pmatrix} 9 & -1 \\ -1 & -1+\alpha_2 \end{pmatrix}.$$

Then the Hodge Index Theorem implies $0 \leq \alpha_2 \leq 1$.

⋄ If $\alpha_2 = 0$, then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^4$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & -1+\alpha_3 \end{pmatrix}.$$

Then X_3 is a smooth surface of degree 6 and sectional genus 3 in $\mathbb{P}^4$. Looking at the classification given in [Ion81] we see that X_3 is a Bordiga surface (see Remark 3.8), so $\alpha_3 = 0$ and

$$D_3 = 4L - \sum_{i=1}^{10} E_i.$$

This gives case **(VII.6)**.

⋄ If $\alpha_2 = 1$, then the third adjunction map gives a pair (X_3, D_3), with $X_3 \subset \mathbb{P}^4$ and whose intersection matrix is

$$\begin{pmatrix} (D_3)^2 & K_{X_3}D_3 \\ K_{X_3}D_3 & (K_{X_3})^2 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ -1 & \alpha_3 \end{pmatrix}.$$

By the Hodge Index Theorem we obtain $\alpha_3 = 0$, hence $(K_{X_3})^2 = 0$. This is a contradiction, because the unique non-degenerate, smooth rational surface of degree 7 in $\mathbb{P}^4$ has $K^2 = -2$; see Remark 3.10. So this case does not occur.

• $\alpha_1 = 7$. In this case the intersection matrix on the surface $X_1 \subset \mathbb{P}^5$ is

$$\begin{pmatrix} (D_1)^2 & K_{X_1}D_1 \\ K_{X_1}D_1 & (K_{X_1})^2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix}.$$

The Hodge Index Theorem implies that K_{X_1} is numerically trivial. So X_1 is a minimal Enriques surface, and X is the blow-up of X_1 at seven points. This yields **(VII.7)**.

The proof of the existence for the cases marked with $(*)$ goes as follows. We first choose $\alpha_1 \in \{1, \ldots, 7\}$. According to Proposition 2.17, we need a rank 2 Steiner bundle $\mathscr{F}$ on $\mathbb{P}^2$ with a resolution like (55) and having precisely α_1 distinct unstable lines. Bundles with these properties are described in Proposition 1.10.

Then, we take $\mathbb{P}(\mathscr{F})$ and we choose a sufficiently general global section η of $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\mathscr{F})}(3\xi - 2\ell)$. We do this by looking directly at the image Y of $\mathfrak{q} \colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^5$, namely we consider η as a global section of $\mathscr{R}(3)$ via the natural identification given by (30). In this setting, Y is a scroll of degree 6 in $\mathbb{P}^5$ defined by the minors of order 3 of the 3×4 matrix of linear forms N over $\mathbb{P}^5$ obtained via the construction of §2.3, i.e.,

$$\mathscr{O}_{\mathbb{P}^5}(-1)^4 \xrightarrow{N} \mathscr{O}_{\mathbb{P}^5}^3,$$

and the zero locus of η is a cubic hypersurface of $\mathbb{P}^5$ containing the union of two surfaces S_1 and S_2 in Y, both obtained as the image via $\mathfrak{q}$ of a divisor belonging to $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(\ell)|$.

Concretely, S_1 and S_2 lie in the net generated by the rows of N, i.e., they can be defined by the 2×2 minors of 4×2 matrices obtained by taking random linear combinations of these rows.

Now we compute the resolution of the homogeneous ideal defining $S_1 \cup S_2$ in $\mathbb{P}^5$; we take a general cubic in this ideal and we consider the residual surface X_1 in Y. The image of the first adjunction map

$$\varphi_{|K_X+H|}\colon X \to \mathbb{P}^5$$

is precisely X_1, so that X is the blow-up of X_1 at α_1 points.

It remains to compute α_2, or equivalently $(K_{X_2})^2$. To do this, we observe that the second adjunction map of X is defined by the restriction to X_1 of the linear system $|\mathscr{O}_Y(2\xi - \ell)|$, and this in turn coincides with the restriction to X_1 of the linear system generated by the six quadrics in the ideal defining S_1.

The image of X_1 via this linear system is the surface X_2, hence we compute $(K_{X_2})^2$ by taking the dual of the resolution of the homogeneous ideal of X_2 in the target $\mathbb{P}^5$. All this, together with the verification that X_1 (and hence X) is smooth, is done with the help of `Macaulay2`. In the Appendix at the end of the paper we explain in detail how this computer-aided construction is performed. $\square$

Remark 3.18. In [Ale88], Alexander showed the existence of a non-special, linearly normal surface of degree 9 in $\mathbb{P}^4$, obtained by embedding the blow-up of $\mathbb{P}^2$ at 10 general points via the very ample complete linear system

$$\left| 13L - \sum_{i=1}^{10} 4E_i \right|.$$

By using the LeBarz formula (see [LB90, Théorème 5]), we can see that Alexander's surface has precisely one 6-secant line. Projecting from this line to $\mathbb{P}^2$, one obtains a birational model of a general triple cover; it is immediate to see that this corresponds to case **(VII.6)** in Proposition 3.17.

Remark 3.19. Let us say something more about case (VII.7). Since $\alpha_1 = 7$, we deduce that $\mathscr{F}$ has seven unstable lines, hence it is a logarithmic bundle (see Proposition 1.10). In this situation, the surface X_1 is a smooth Enriques surface of degree 10 and sectional genus 6 in $\mathbb{P}^5$, that is, a so-called *Fano model*. Actually, one can check that X_1 is contained into the Grassmannian $\mathbb{G}(1, \mathbb{P}^3)$ as a *Reye congruence*, i.e., a 2-dimensional cycle of bidegree (3, 7); see [Gro93, Theorem 4.3]. In particular, X_1 admits a family of 7-secant planes, and the projection from one of these planes provides a birational model of the triple cover $f\colon X \to \mathbb{P}^2$ (in fact, X is the blow-up of X_1 at seven points).

For more details about Fano and Reye models, see [Cos83, CV93].

3.7.2. *Some further considerations on triple planes of type* VII. We mentioned in the previous subsection that we are able to construct many, but not all, cases of triple planes of type VII (see Proposition 3.17). We conjecture that the remaining cases do not exist. More precisely, our expectation is that the values of α_1 and α_2 should necessarily satisfy the rule

$$\alpha_2 = \binom{7-\alpha_1}{2}.$$

Let us explain now what the geometric evidence is beyond our conjecture. The second adjunction map $\varphi_2\colon X_1 \to X_2 \subset \mathbb{P}^5$ can be lifted to the map $\zeta\colon \mathbb{P}(\mathscr{F}) \to \mathbb{P}^5$ associated with the linear system $|\mathscr{O}_{\mathbb{P}(\mathscr{F})}(2\xi - \ell)|$. Note that

$$H^0(\mathbb{P}(\mathscr{F}), \mathscr{O}_{\mathbb{P}(\mathscr{F})}(2\xi - \ell)) \simeq H^0(\mathbb{P}^2, S^2\mathscr{F}(-1)) \simeq \wedge^2 W^\vee,$$

where the last isomorphism is obtained by taking global sections in the second symmetric power of the short exact sequence

$$0 \to W^\vee \otimes \mathscr{O}_{\mathbb{P}(V)}(-1) \to U \otimes \mathscr{O}_{\mathbb{P}(V)} \to \mathscr{F} \to 0$$

defining $\mathscr{F}$ (see (2)), namely

$$0 \to \wedge^2 W^\vee \otimes \mathscr{O}_{\mathbb{P}^2}(-3) \to W^\vee \otimes U \otimes \mathscr{O}_{\mathbb{P}^2}(-2) \to S^2U \otimes \mathscr{O}_{\mathbb{P}^2}(-1) \to S^2\mathscr{F}(-1) \to 0.$$

One can show that the projective closure Y' of the image of the map $\zeta \colon \mathbb{P}(\mathscr{F}) \dashrightarrow \mathbb{P}(\wedge^2 W^\vee)$ is contained in the Plücker quadric $\mathbb{G} = \mathbb{G}(1, \mathbb{P}(W^\vee))$ and that Y' is the degeneracy locus of a map on $\mathbb{G}$ defined by the tensor $\phi \in U \otimes V \otimes W$ considered in §1.4.1. More precisely, denoting by $\mathscr{U}$ the tautological rank 2 subbundle on $\mathbb{G}$, once noted that $H^0(\mathbb{G}, \mathscr{U}^\vee) = W$ we see that ϕ gives a morphism

$$V^\vee \otimes \mathscr{U} \to U \otimes \mathscr{O}_{\mathbb{G}}.$$

The variety Y' is the vanishing locus of the determinant of this morphism, so that Y' can be expressed as a complete intersection of the Plücker quadric and a cubic hypersurface in $\mathbb{P}^5$.

The locus where this morphism has rank ≤ 4 is contained in the singular locus of Y' and coincides with it for a general choice of $\mathscr{F}$. By Porteous' formula, for such a general choice we expect that Y' has 21 singular points. One can see that these points are precisely the images of the sections of negative self-intersection of the Hirzebruch surfaces in $\mathbb{P}(\mathscr{F})$ lying above the smooth conics in $\mathbb{P}^2$ where $\mathscr{F}$ splits as $\mathscr{O}_{\mathbb{P}^1}(1) \oplus \mathscr{O}_{\mathbb{P}^1}(7)$, once chosen an isomorphism to $\mathbb{P}^1$ (it would be natural to call these conics *unstable conics*, and the argument above shows that there are in general 21 of them).

Also, the indeterminacy locus of ζ is exactly the union of the sections of negative self-intersection on the Hirzebruch surfaces lying above the unstable lines of $\mathscr{F}$. So, α_1 and α_2 should depend only on $\mathscr{F}$ and not on X, and moreover α_1 should determine α_2. However, it is not clear yet how the number of unstable lines determines the precise number of unstable conics.

4. Moduli spaces

In this section we describe some moduli problems related to our triple planes. For $b \in \{2, 3, 4\}$ we set

$$\mathscr{E}_b := \begin{cases} \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-1) & \text{if } b = 2, \\ \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}(-2) & \text{if } b = 3, \\ \mathscr{O}_{\mathbb{P}^2}(-2) \oplus \mathscr{O}_{\mathbb{P}^2}(-2) & \text{if } b = 4, \end{cases}$$

whereas for $b \in \{5, 6, 7, 8\}$ we denote by $\mathscr{F}_b = \mathscr{E}_b(b-2)$ a rank 2 Steiner bundle on $\mathbb{P}^2$ having sheafified minimal graded free resolution of the form

$$0 \to \mathscr{O}_{\mathbb{P}^2}(1-b)^{b-4} \to \mathscr{O}_{\mathbb{P}^2}(2-b)^{b-2} \to \mathscr{E}_b \to 0.$$

Then, for any $b \in \{2, \ldots, 8\}$, we define two spaces $\mathfrak{N}_b$ and $\mathfrak{M}_b$ as follows:

$$\mathfrak{N}_b = \left\{ (\mathscr{E}_b, \eta) \,\middle|\, \begin{array}{l} \eta \in \mathbb{P}(H^0(\mathbb{P}^2, S^3\mathscr{E}_b^\vee \otimes \wedge^2 \mathscr{E}_b)) \text{ is the building section} \\ \text{of a general triple plane with } p_g = q = 0 \\ \text{and Tschirnhausen bundle } \mathscr{E}_b \end{array} \right\} / \simeq,$$

$$\mathfrak{M}_b = \left\{ (\mathscr{E}_b,\, \eta) \;\middle|\; \begin{array}{l} \eta \in \mathbb{P}H^0(\mathbb{P}^2,\, S^3\mathscr{E}_b^\vee \otimes \wedge^2\mathscr{E}_b) \text{ is the building section} \\ \text{of a general triple plane with } p_g = q = 0 \\ \text{and Tschirnhausen bundle } \mathscr{E}_b \end{array} \right\} / \sim,$$

where we set $(\mathscr{E}_b,\, \eta) \simeq (\mathscr{E}_b',\, \eta')$ if and only if there is an isomorphism $\Psi\colon \mathscr{E} \to \mathscr{E}'$ such that $\Psi^*\eta' = \eta$ and the diagram

$$\begin{array}{ccc} \mathscr{E}_b & \xrightarrow{\Psi} & \mathscr{E}_b' \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{id} & \mathbb{P}^2 \end{array}$$

commutes, whereas $(\mathscr{E}_b,\, \eta) \sim (\mathscr{E}_b',\, \eta')$ if and only if there is an isomorphism $\Psi\colon \mathscr{E} \to \mathscr{E}'$ and an automorphism $\lambda\colon \mathbb{P}^2 \to \mathbb{P}^2$ such that $\Psi^*\eta' = \eta$ and the diagram

$$\begin{array}{ccc} \mathscr{E}_b & \xrightarrow{\Psi} & \mathscr{E}_b' \\ \downarrow & & \downarrow \\ \mathbb{P}^2 & \xrightarrow{\lambda} & \mathbb{P}^2 \end{array}$$

commutes. We have $\mathfrak{M}_b = \mathfrak{N}_b/\mathrm{PGL}_3(\mathbb{C})$, because the equivalence $(\mathscr{E}_b,\, \eta) \simeq (\mathscr{E}_b',\, \eta')$ is obtained from $(\mathscr{E}_b,\, \eta) \sim (\mathscr{E}_b',\, \eta')$ via the natural $\mathrm{PGL}_3(\mathbb{C})$-action on the base. Note that, with the terminology of [HL97, Chapter 4], the pair $(\mathscr{E}_b,\, \eta)$ consisting of the Tschirnhausen bundle and of the building section is a *framed sheaf*.

Given a general triple plane $f\colon X \to \mathbb{P}^2$ branched over a curve of degree $2b$, by Theorems 1.2 and 2.12 we can functorially associate with $(X,\, f)$ a framed sheaf $(\mathscr{E}_b,\, \eta)$, and conversely. In other words, considering the set of framed sheaves $(\mathscr{E}_b,\, \eta)$ up to the equivalence relation $\simeq$ or $\sim$ defined above actually amounts to considering the set of pairs $(X,\, f)$ up to the corresponding equivalence relation.

Thus, from this point of view, $\mathfrak{M}_b$ can be identified with the moduli space of the pairs $(X,\, f)$ up to isomorphisms, and $\mathfrak{N}_b$ with the moduli space of the pairs $(X,\, f)$ up to *cover* isomorphisms.

In the sequel, we will use interchangeably the above notation $\mathfrak{N}_b$ and $\mathfrak{M}_b$, with $b \in \{2, \ldots, 8\}$, and $\mathfrak{N}_i$ and $\mathfrak{M}_i$, with $i \in \{\mathrm{I}, \ldots, \mathrm{VII}\}$. In each case, the moduli space $\mathfrak{N}_b$ can be constructed as follows:

- take the versal deformation space $\mathrm{Def}(\mathscr{E}_b)$ of $\mathscr{E}_b$;
- stratify $\mathrm{Def}(\mathscr{E}_b)$ in such a way that $H^0(\mathbb{P}^2,\, S^3\mathscr{E}_b^\vee \otimes \wedge^2\mathscr{E}_b)$ has constant rank and consider the locally trivial projective bundle over each stratum whose fibers are given by $\mathbb{P}H^0(\mathbb{P}^2,\, S^3\mathscr{E}_b^\vee \otimes \wedge^2\mathscr{E}_b)$;
- consider the quotient of this projective bundle by the natural action of the group $\mathrm{Aut}(\mathscr{E}_b)$.

In order to obtain $\mathfrak{M}_b$, we must further take the quotient of the above moduli space by the natural action of $\mathrm{PGL}_3(\mathbb{C})$. In particular, the expected dimensions of $\mathfrak{N}_b$ and $\mathfrak{M}_b$ will be given by

$$\begin{aligned} &\text{exp-dim}\, \mathfrak{N}_b = \dim \mathrm{Def}(\mathscr{E}_b) + h^0(\mathbb{P}^2,\, S^3\mathscr{E}_b^\vee \otimes \wedge^2\mathscr{E}_b) - \dim \mathrm{Aut}(\mathscr{E}_b), \\ &\text{exp-dim}\, \mathfrak{M}_b = \dim \mathrm{Def}(\mathscr{E}_b) + h^0(\mathbb{P}^2,\, S^3\mathscr{E}_b^\vee \otimes \wedge^2\mathscr{E}_b) - \dim \mathrm{Aut}(\mathscr{E}_b) - 8. \end{aligned} \tag{56}$$

From now on, we will simply write $\mathscr{E}$ instead of $\mathscr{E}_b$ if no confusion can arise.

4.1. **Moduli of triple planes with decomposable Tschirnhausen bundle.** Let us first consider cases I, II, III. Here $\mathscr{E}$ splits as a sum of two line bundles and it is rigid.

Theorem 4.1. *The following hold*:

- **i)** *The moduli space* $\mathfrak{M}_{\mathrm{I}}$ *consists of a single point.*
- **ii)** *The moduli space* $\mathfrak{M}_{\mathrm{II}}$ *is unirational of dimension* 7.
- **iii)** *The moduli space* $\mathfrak{M}_{\mathrm{III}}$ *is unirational of dimension* 12.

Proof. As a preliminary step, note that in all these cases the bundle $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}$ is globally generated. Therefore, Theorem 1.2 applies and shows that the moduli spaces $\mathfrak{M}_{\mathrm{I}}$, $\mathfrak{M}_{\mathrm{II}}$ and $\mathfrak{M}_{\mathrm{III}}$ are obtained as a quotient of a Zariski dense open subset of $H^0(\mathbb{P}^2, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E})$ by the action of some linear group, so that all of them are irreducible, unirational varieties.

Let us check **i)**. In this case, the branch curve $B \subset \mathbb{P}^2$ is a tricuspidal plane quartic curve, which is unique up to projective transformations. By a topological monodromy argument (see [ST80, §58]) and the Grauert-Remmert extension theorem (see [Gro63, XII.5.4]) this implies that the number of triple planes of type I up to isomorphisms equals the number of group epimorphisms

$$\varrho\colon \pi_1(\mathbb{P}^2 - B) \to \mathfrak{S}_3$$

up to conjugation in $\mathfrak{S}_3$. Now, it is well known that

$$\pi_1(\mathbb{P}^2 - B) = \mathrm{B}_3(\mathbb{P}^1) = \langle \alpha,\, \beta \mid \alpha^3 = \beta^2 = (\beta\alpha)^2 \rangle$$

(see [Dim92, Chapter 4, Proposition 4.8]), and this group has a unique epimorphism ϱ to $\mathfrak{S}_3$ up to conjugation. In fact, $\varrho(\alpha)$ must be a 3-cycle whereas $\varrho(\beta)$ must be a transposition, so we may assume

$$\varrho(\alpha) = (1\,2\,3), \quad \varrho(\beta) = (1\,2).$$

This proves that $\mathfrak{M}_{\mathrm{I}}$ consists of a single point.

Let us now analyze **ii)**. Recall that in this case the branch locus $B \subset \mathbb{P}^2$ is a plane sextic curve with six cusps lying on the same conic. Each of these curves can be written as

$$(f_2)^3 + (f_3)^2 = 0, \tag{57}$$

where f_k denotes a homogeneous form of degree k, and the construction depends on

$$6 + 10 - 1 - \dim \mathrm{PGL}_3(\mathbb{C}) = 7$$

parameters. The same monodromy argument used in part **i)** shows that this also computes the effective dimension $\dim \mathfrak{M}_{\mathrm{II}}$. More precisely, we can see that every fixed curve B of equation (57) is the branch locus of a unique triple cover up to isomorphisms, namely the one whose birational model is provided by the hypersurface

$$z^3 + bz + c = 0,$$

where $b = -f_2/\sqrt[3]{4}$ and $c = f_3/\sqrt{-27}$. In fact, we have

$$\pi_1(\mathbb{P}^2 - B) = (\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/3\mathbb{Z}) = \langle \alpha,\, \beta \mid \alpha^3 = \beta^2 = 1 \rangle$$

(see [Dim92, Chapter 4, Proposition 4.16]), and this group has a unique epimorphism to $\mathfrak{S}_3$ up to conjugation.

We finally look at **iii)**, where $\mathscr{E} = \mathscr{O}_{\mathbb{P}^2}(-2) \oplus \mathscr{O}_{\mathbb{P}^2}(-2)$. The automorphism group of $\mathscr{E}$ is isomorphic to $\mathrm{GL}_2(\mathbb{C})$. Moreover

$$h^0(\mathbb{P}^2,\, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}) = h^0\big(\mathbb{P}^2,\, \mathscr{O}_{\mathbb{P}^2}(2)^4\big) = 24,$$

hence (56) implies

$$\text{exp-dim}\, \mathfrak{M}_{\mathrm{III}} = 24 - 4 - 8 = 12.$$

This number coincides with the effective dimension $\dim\, \mathfrak{M}_{\mathrm{III}}$. In fact, in this case X is the blow-up at nine points of $\mathbb{F}_n$, with $n \in \{0,\, 1,\, 2,\, 3\}$. The stratum of maximal dimension corresponds to the value of n such that $\mathrm{Aut}(\mathbb{F}_n) = H^0(\mathbb{F}_n,\, T_{\mathbb{F}_n})$ has minimal dimension, namely to $n = 0$ for which we have

$$\dim\, \mathfrak{M}_{\mathrm{III}} = 2 \cdot 9 - \dim\, \mathrm{Aut}(\mathbb{F}_n) = 18 - 6 = 12. \qquad \square$$

4.2. **Moduli of triple planes with stable Tschirnhausen bundle.** We now start the analysis of the cases IV, ..., VII, where $\mathscr{E}$ is indecomposable. Using the notation introduced in §2, we will write $\mathscr{F} = \mathscr{E}(b-2)$, so that $\mathscr{F}$ fits into the short exact sequence

$$0 \to \mathscr{O}_{\mathbb{P}^2}(-1)^{b-4} \to \mathscr{O}_{\mathbb{P}^2}^{b-2} \to \mathscr{F} \to 0.$$

Thus $\mathrm{Def}(\mathscr{E}) = \mathrm{Def}(\mathscr{F})$ and

$$H^0(\mathbb{P}^2,\, S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E}) = H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)).$$

The vector bundle $\mathscr{F}$ is stable (Theorem 2.12), so $\mathrm{Aut}(\mathscr{F}) = \mathbb{C}^*$; its deformation space $\mathrm{Def}(\mathscr{F})$ is described for instance in [Cas02, Introduction], and we have

$$\dim\, \mathrm{Def}(\mathscr{F}) = 3(b-4)(b-2) - 1 = (b-1)(b-5).$$

Then (56) yields

$$\begin{aligned} \dim\, \mathfrak{N}_b = \text{exp-dim}\, \mathfrak{N}_b &= (b-1)(b-5) + h^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)) - 1, \\ \text{exp-dim}\, \mathfrak{M}_b &= (b-1)(b-5) + h^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)) - 9. \end{aligned} \tag{58}$$

Furthermore, the equality $\text{exp-dim}\, \mathfrak{M}_b = \dim\, \mathfrak{M}_b$ holds if and only if $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathfrak{N}_b$ with generically finite stabilizer.

Theorem 4.2. *For $i \in \{\mathrm{IV}, \mathrm{V}, \mathrm{VI}\}$ the moduli space $\mathfrak{N}_i$ is rational and irreducible, while $\mathfrak{M}_i$ is unirational of dimension* $\dim\, \mathfrak{N}_i - 8$, *where*

- **i)** $\dim\, \mathfrak{N}_{\mathrm{IV}} = 23$;
- **ii)** $\dim\, \mathfrak{N}_{\mathrm{V}} = 24$;
- **iii)** $\dim\, \mathfrak{N}_{\mathrm{VI}} = 23$.

Moreover the moduli space $\mathfrak{N}_{\mathrm{VII}}$ has at least seven irreducible components, all unirational of dimension 20, *that are distinguished by the number $\alpha_1 \in \{1, \dots, 7\}$ of unstable lines for $\mathscr{F}$.*

First of all we note that, as in the proof of Theorem 4.1, in cases IV and V the bundle $S^3\mathscr{E}^\vee \otimes \wedge^2\mathscr{E} \simeq S^3\mathscr{F}(6-b)$ is globally generated. Indeed, in these cases $b \leq 6$ and $\mathscr{F}$ is globally generated, so the same is true for $S^3\mathscr{F}$ and for $S^3\mathscr{F}(6-b)$. Therefore, the spaces $\mathfrak{M}_i$ and $\mathfrak{N}_i$ are irreducible as soon as the parameter space of the bundle $\mathscr{E}$, or equivalently of $\mathscr{F}$, is irreducible. Moreover, since $\mathfrak{N}_i$ is an open subset of a projective bundle over such parameter space, rationality of the latter will imply rationality of the former, and also unirationality of $\mathfrak{M}_i$.

The proof of Theorem 4.2 is based on a case-by-case analysis that will be done in §§4.2.1, 4.2.2, 4.2.3, 4.2.4 below. Our strategy is to compute $\dim \mathfrak{N}_i$ and to show that $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathfrak{N}_i$ with generically finite stabilizers for all $i \in \{\mathrm{IV}, \mathrm{V}, \mathrm{VI}, \mathrm{VII}\}$, to prove that $\mathfrak{N}_i$ is rational and irreducible for $i \in \{\mathrm{IV}, \mathrm{V}, \mathrm{VI}\}$, and finally to find at least seven irreducible unirational components of $\mathfrak{N}_{\mathrm{VII}}$.

4.2.1. *Moduli of triple planes of type* IV.

Proposition 4.3. *The moduli space $\mathfrak{N}_{\mathrm{IV}}$ is an open dense subset of $\mathbb{P}^{23}$, in particular it is irreducible and rational. The space $\mathfrak{M}_{\mathrm{IV}}$ has dimension* 15.

Proof. Case IV, i.e., $b = 5$, was analyzed in Proposition 3.7. We have $\mathscr{F} = T_{\mathbb{P}^2}(-1)$ and a natural identification

$$H^0(\mathbb{P}^2,\, S^3\mathscr{F}(1)) = H^0(\check{\mathbb{P}}^2,\, T_{\check{\mathbb{P}}^2}(2)) = \mathbb{C}^{24}.$$

Set $\mathbb{P}^{23} = \mathbb{P}H^0(\check{\mathbb{P}}^2,\, T_{\check{\mathbb{P}}^2}(2))$ and observe that the bundle $\mathscr{F}$ is rigid, stable and unobstructed, so the moduli space consists of a single, reduced point. Consequently, the triple cover $f\colon X \to \mathbb{P}^2$ only depends on the section $\eta \in H^0(\check{\mathbb{P}}^2,\, T_{\check{\mathbb{P}}^2}(2))$ or, better, on its proportionality class $[\eta]$, that lies in a Zariski dense open subset of $\mathbb{P}^{23}$.

By (58) we have $\operatorname{exp-dim}\mathfrak{M}_{\mathrm{IV}} = 15$. It remains to show that $\operatorname{exp-dim}\mathfrak{M}_{\mathrm{IV}} = \dim \mathfrak{M}_{\mathrm{IV}}$ or, equivalently, that $\mathrm{PGL}_3(\mathbb{C})$ acts on $\mathbb{P}^{23} = \mathbb{P}H^0(\check{\mathbb{P}}^2,\, T_{\check{\mathbb{P}}^2}(2))$ with generically finite stabilizer. Take a generic element $\eta \in \mathbb{P}^{23}$ and let $Z = D_0(\eta) \subset \check{\mathbb{P}}^2$ be its vanishing locus and $G = G_\eta \subset \mathrm{PGL}_3(\mathbb{C})$ its stabilizer. So Z consists of 13 reduced points and we want to show that G is finite. Every homography in G must preserve Z and hence permute its 13 points, so we obtain a group homomorphism

$$\psi\colon G \to \mathfrak{S}_{13}.$$

If $L \subset \mathbb{P}^2$ is a line, we have

$$T_{\check{\mathbb{P}}^2}(2)|_L = \mathscr{O}_L(3) \oplus \mathscr{O}_L(4). \tag{59}$$

Now set $Z' := Z \cap L$ and $c := \operatorname{length}(Z')$. Arguing as in part **iii)** of Lemma 1.9, we deduce the existence of a surjection $T_{\check{\mathbb{P}}^2}(2)|_L \to \mathscr{O}_L(7 - c)$, and using (59) this yields $c \leq 4$. So there are no more than four points of Z on a single line, hence the support of Z contains at least four points in general linear position.

Now, a homography in $\ker\psi$ must fix the subscheme Z pointwise. Since a homography of the plane fixing at least four points in general position is the identity, we have that ψ is injective. So G is a subgroup of $\mathfrak{S}_{13}$, hence a finite group. □

4.2.2. *Moduli of triple planes of type* V.

Proposition 4.4. *The moduli space $\mathfrak{N}_{\mathrm{V}}$ is a Zariski open dense subset of a $\mathbb{P}^{19}$-bundle over $\mathbb{P}^5$, in particular it is rational and irreducible of dimension* 24. *The space $\mathfrak{M}_{\mathrm{V}}$ has dimension* 16.

Proof. Case V, i.e., $b = 6$, was analyzed in Proposition 3.9. The bundle $\mathscr{F} = \mathscr{F}_6$ is determined by its set of unstable lines, which form a smooth conic $\mathscr{W}(\mathscr{F}) \subset \check{\mathbb{P}}^2$, so we can identify the moduli space of $\mathscr{F}$ with the open subset $\mathscr{U} \subset \mathbb{P}^5$ consisting of smooth conics via the Veronese embedding. This is the base of our $\mathbb{P}^{19}$-bundle.

Proposition 3.9 (cf. also §1.4.1) shows that, once the Tschirnhausen bundle $\mathscr{F}$ is chosen, we have a 4-dimensional space $U = H^0(\mathbb{P}^2, \mathscr{F})$ and a corresponding projective space $\mathbb{P}^3 = \mathbb{P}(U)$, together with a fixed twisted cubic $C \subset \mathbb{P}^3$ such that $\mathbb{P}(\mathscr{F})$ is the blow-up of $\mathbb{P}^3$ at C. Moreover, the building sections η of the triple plane are in bijection with an open dense subset of the space of cubic surfaces, in view of the identification

$$H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)) = H^0(\mathbb{P}^3,\, \mathscr{O}_{\mathbb{P}^3}(3)) = \mathbb{C}^{20}, \tag{60}$$

so their proportionality classes belong to an open dense subset of $\mathbb{P}^{19} = \mathbb{P}H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b))$, and our claim about $\mathfrak{N}_{\mathrm{V}}$ is proven.

Now (58) yields $\text{exp-dim}\, \mathfrak{M}_{\mathrm{V}} = 16$, so it only remains to show that $\mathrm{PGL}_3(\mathbb{C})$ acts on the set of pairs $(\mathscr{F},\, \eta)$ with generically finite stabilizer. Let $G = G_{(\mathscr{F},\, \eta)} \subset \mathrm{PGL}_3(\mathbb{C})$ be the stabilizer of the pair $(\mathscr{F},\, \eta)$. Then every element $g \in G$ must fix $\mathscr{F}$, and hence the conic $\mathscr{W}(\mathscr{F})$. By [FH91, p. 154], the subgroup of automorphisms of $\mathbb{P}^n$ that preserve a rational normal curve C_n is precisely $\mathrm{PGL}_2(\mathbb{C})$, so G is a subgroup of a copy of $\mathrm{PGL}_2(\mathbb{C})$ inside $\mathrm{PGL}_3(\mathbb{C})$. On the other hand, g fixes $\eta \in H^0(\mathbb{P}^3,\, \mathscr{O}_{\mathbb{P}^3}(3))$, hence it fixes the cubic surface $S \subset \mathbb{P}^3$.

Next, we have seen in Lemma 2.15 (cf. also Remark 2.19) that the image in $\mathbb{P}^3$ of the negative sections lying above the lines of $\mathscr{W}(\mathscr{F})$ is precisely the twisted cubic C. The whole construction is therefore g-invariant, so g must preserve the intersection $S \cap C$.

Furthermore, the construction giving rise to the 2×3 matrix N whose 2×2 minors define C (cf. (40)) can be reversed in order to give back the matrix M presenting $\mathscr{F}$ (cf. (38)). Since M is generic, this implies that N and C are generic. In addition, by (60) we also know that the cubic S corresponding to the building section η can be chosen generically. In particular, the intersection $S \cap C$ is reduced for a general choice of our data, i.e., it consists of nine distinct points.

Summing up, we get a group homomorphism

$$\psi \colon G \to \mathfrak{S}_9$$

that must be injective since an element of $\mathrm{PGL}_2(\mathbb{C})$ fixing at least three distinct points is necessarily the identity. So G is a subgroup of $\mathfrak{S}_9$, hence a finite group. □

4.2.3. *Moduli of triple planes of type* VI. We denote by $\mathrm{Hilb}_d(\check{\mathbb{P}}^2)$ the Hilbert scheme of 0-dimensional subschemes of length d of $\check{\mathbb{P}}^2$.

Proposition 4.5. *The moduli space $\mathfrak{N}_{\mathrm{VI}}$ is a Zariski open dense subset of a $\mathbb{P}^{11}$-bundle over $\mathrm{Hilb}_6(\check{\mathbb{P}}^2)$, in particular it is a rational variety of dimension* 23*. The moduli space $\mathfrak{M}_{\mathrm{VI}}$ has dimension* 15*.*

Proof. Case VI was analyzed in Proposition 3.12. We mentioned in §1.4.2 (cf. case $b = 7$ before Proposition 1.10) that $\mathscr{F} = \mathscr{F}_7$ is a logarithmic bundle, i.e., it has six unstable lines which are in general linear position, and that these six lines in turn uniquely determine $\mathscr{F}$. This identifies the moduli space of Steiner bundles of type $\mathscr{F}_7$ as an open dense subset $\mathscr{U}$ of the Hilbert scheme of six points of $\check{\mathbb{P}}^2$.

We have a direct image sheaf $\mathscr{R}(3)$, fitting into (50), and a natural identification

$$H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b)) = H^0(\mathbb{P}^4,\, \mathscr{R}(3)) = \mathbb{C}^{12};$$

see the proof of Claim 3.13. The sheaf $\mathscr{R}(3)$ is supported on a determinantal cubic threefold $Y \subset \mathbb{P}^4$. In addition, the vanishing locus of a general global section of $\mathscr{R}(3)$ is a Bordiga surface $X_1 \subset \mathbb{P}^4$ and, moreover, the divisor $X = D_0(\eta) \subset \mathbb{P}(\mathscr{F})$ is the blow-up of X_1 at the six nodes of Y. Summing up, the proportionality classes $[\eta]$ of building sections of triple covers of type VI lie in a dense open subset of $\mathbb{P}^{11} = \mathbb{P}H^0(\mathbb{P}^2,\, S^3\mathscr{F}(6-b))$, and this proves our claim about $\mathfrak{N}_{\mathrm{VI}}$.

We now consider the moduli space $\mathfrak{M}_{\mathrm{VI}}$. First, (58) implies $\dim \mathfrak{M}_{\mathrm{VI}} = 15$. In order to conclude the proof, we must show that $\mathrm{PGL}_3(\mathbb{C})$ acts on the set of pairs $(\mathscr{F}, \eta)$ with generically finite stabilizer. Let $G = G_{(\mathscr{F}, \eta)} \subset \mathrm{PGL}_3(\mathbb{C})$ be the stabilizer of the pair $(\mathscr{F}, \eta)$. Then every element $g \in G$ must fix $\mathscr{F}$, and hence the set of its six unstable lines. Consequently, g permutes the corresponding six points in $\check{\mathbb{P}}^2$, which are in general position. This in turn defines a group homomorphism

$$\psi \colon G \to \mathfrak{S}_6,$$

which must be injective since a homography of the plane that fixes at least four points in general position is the identity. So G is a subgroup of $\mathfrak{S}_6$, hence a finite group. □

4.2.4. *Moduli of triple planes of type* VII. Let us finally consider case VII, i.e., $b = 8$. We need the following preliminary result.

Proposition 4.6. *Assume $b = 8$ and let $\mathscr{F} := \mathscr{F}_8$ be a Steiner bundle with α_1 unstable lines. Then*

$$h^0(\mathbb{P}^2, S^3\mathscr{F}(-2)) \geq \alpha_1. \tag{61}$$

Proof. Let $L_1, \ldots, L_{\alpha_1}$ be the unstable lines of $\mathscr{F}$. We can perform the reduction of $\mathscr{F}$ along such unstable lines, i.e., a sequence of elementary transformations of $\mathscr{F}$ along the L_i; see [DK93, §§2.7–2.8] and [Val00, Proposition 2.1]. This gives an exact sequence

$$0 \to \mathscr{K} \to \mathscr{F} \to \bigoplus_{i=1}^{\alpha_1} \mathscr{O}_{L_i} \to 0, \tag{62}$$

where $\mathscr{K}$ is a vector bundle of rank 2. From (62) we get $H^i(\mathbb{P}^2, \mathscr{K}(-1)) = 0$ for all i. Computing Chern classes and applying Proposition 1.13 to $\mathscr{K}$, we see that $\mathscr{K}$ behaves according to the following table:

(63)

α_1	1, 2, 3	4	5	6	7
$\mathscr{K}$	$\mathscr{F}_{8-\alpha_1}$	$\mathscr{O}_{\mathbb{P}^2}^2$	$\mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}$	$\mathscr{O}_{\mathbb{P}^2}(-1)^2$	$\Omega^1_{\mathbb{P}^2}$

Indeed, $\mathscr{K}$ is a Steiner bundle for $\alpha_1 = 1, 2, 3, 4$ (corresponding to the cases $b = 7, 6, 5, 4$ in Proposition 1.13). For $\alpha_1 = 5$ (the case $b = 3$ in Proposition 1.13) we have $\mathscr{K} \simeq \mathscr{O}_{\mathbb{P}^2}(-1) \oplus \mathscr{O}_{\mathbb{P}^2}$. Finally, for $\alpha_1 = 6, 7$ (the cases $b = 2, 1$ in Proposition 1.13) we have that $\mathscr{K}^\vee(-1)$ is a Steiner bundle respectively of the form $\mathscr{O}_{\mathbb{P}^2}^2$ for $\alpha_1 = 6$ or $T_{\mathbb{P}^2}(-1)$ for $\alpha_1 = 7$, and hence $\mathscr{K} \simeq \mathscr{O}_{\mathbb{P}^2}(-1)^2$ or $\mathscr{K} \simeq \Omega^1_{\mathbb{P}^2}$.

From Pieri's formulas (cf. [Wey03, Corollary 2.3.5, p. 62]) we obtain

$$\mathscr{F} \otimes S^2\mathscr{F}(-2) \simeq S^3\mathscr{F}(-2) \oplus \wedge^2\mathscr{F} \otimes \mathscr{F}(-2) \simeq S^3\mathscr{F}(-2) \oplus \mathscr{F}(2). \tag{64}$$

Also, the fact that L_i is unstable implies

$$S^2\mathscr{F}(-2)|_{L_i} \simeq \mathscr{O}_{L_i}(-2) \oplus \mathscr{O}_{L_i}(2) \oplus \mathscr{O}_{L_i}(6). \tag{65}$$

So, tensoring (62) with $S^2\mathscr{F}(-2)$ we get

$$0 \to \mathscr{K} \otimes S^2\mathscr{F}(-2) \to \begin{array}{c} S^3\mathscr{F}(-2) \\ \oplus \\ \mathscr{F}(2) \end{array} \to \bigoplus_{i=1}^{\alpha_1} \begin{array}{c} \mathscr{O}_{L_i}(-2) \\ \oplus \\ \mathscr{O}_{L_i}(2) \\ \oplus \\ \mathscr{O}_{L_i}(6) \end{array} \to 0. \tag{66}$$

Twisting (62) by $\mathscr{O}_{\mathbb{P}^2}(2)$ and taking cohomology we get $H^1(\mathbb{P}^2, \mathscr{F}(2)) = 0$. Now, since we are in characteristic 0, the stability of $\mathscr{F}$ implies that $S^2\mathscr{F}^\vee(-1)$ is semistable, of slope -5. On the other hand, by table (63), each summand of $\mathscr{K}^\vee$ is semistable (and $\mathscr{K}$ is even stable for $\alpha_1 \neq 3, 4, 5$) of slope between $-3/2$ (for $\alpha_1 = 1$) and $3/2$ (for $\alpha_1 = 7$). In any case, all summands of $\mathscr{K}^\vee \otimes S^2\mathscr{F}^\vee(-1)$ are semistable of strictly negative slope, so using Serre duality we get

$$H^2(\mathbb{P}^2, \mathscr{K} \otimes S^2\mathscr{F}(-2)) \simeq H^0(\mathbb{P}^2, \mathscr{K}^\vee \otimes S^2\mathscr{F}^\vee(-1))^\vee = 0.$$

Therefore, taking cohomology in (66) we obtain $H^2(\mathbb{P}^2, S^3\mathscr{F}(-2)) = 0$ and a surjection

$$H^1(\mathbb{P}^2, S^3\mathscr{F}(-2)) \to \bigoplus_{i=1}^{\alpha_1} H^1(L_i, \mathscr{O}_{L_i}(-2)) \to 0,$$

which in turn implies $h^1(\mathbb{P}^2, S^3\mathscr{F}(-2)) \geq \alpha_1$. By the Riemann-Roch theorem we have $\chi(\mathbb{P}^2, S^3\mathscr{F}(-2)) = 0$, hence $h^0(\mathbb{P}^2, S^3\mathscr{F}(-2)) \geq \alpha_1$, which is (61). □

Let us now state the result concluding the proof of Theorem 4.2.

Proposition 4.7. *The moduli space $\mathfrak{N}_{\mathrm{VII}}$ has at least seven connected, irreducible, unirational components, all of dimension* 20*, that are distinguished by the number $\alpha_1 \in \{1, \dots, 7\}$ of unstable lines for $\mathscr{F}$.*

Proof. Proposition 3.17 shows the existence of seven families

$$\mathfrak{N}^1_{\mathrm{VII}}, \dots, \mathfrak{N}^7_{\mathrm{VII}}$$

of triple planes, one for each value of the number $\alpha_1 \in \{1, \dots, 7\}$ of unstable lines of $\mathscr{F}$. Such families are pairwise disjoint subsets of $\mathfrak{N}_{\mathrm{VII}}$, because α_1 coincides with the number of lines contracted by the first adjunction map of X, and this number is an invariant of the triple cover. Moreover, all the cases missing the star in Proposition 3.17 have different values of α_2 than the covers belonging to the $\mathfrak{N}^{\alpha_1}_{\mathrm{VII}}$. Since also α_2 is an invariant of the triple cover, the connected components of $\mathfrak{N}_{\mathrm{VII}}$ possibly containing the missing cases are necessarily disjoint from all the $\mathfrak{N}^{\alpha_1}_{\mathrm{VII}}$. This shows that our seven families actually are seven connected components of $\mathfrak{N}_{\mathrm{VII}}$.

Let us show now that such connected components are also irreducible and unirational. Consider the 21-dimensional (rational) moduli space $\mathrm{M}_{\mathbb{P}^2}(2, 4, 10)$ of rank 2 stable bundles on $\mathbb{P}^2$ with Chern classes (4, 10) and having a Steiner-type resolution, and let $\mathscr{U}^{\alpha_1} \subset \mathrm{M}_{\mathbb{P}^2}(2, 4, 10)$ be the stratum corresponding to vector bundles having α_1 unstable lines. These strata are irreducible and unirational and their codimension is precisely α_1; see [AO01, Theorem 5.6].

Our computations with `Macaulay2` (cf. Appendix) show that there exist examples of bundles $\mathscr{F}$ with α_1 unstable lines and satisfying

$$h^0(\mathbb{P}^2, S^3\mathscr{F}(-2)) = \alpha_1. \tag{67}$$

So, by Proposition 4.6 and semicontinuity, equality (67) holds for the general member of the stratum $\mathscr{U}^{\alpha_1}$. Each $\mathfrak{N}^{\alpha_1}_{\mathrm{VII}}$ has an open dense subset which is an open dense piece of a $\mathbb{P}^{\alpha_1 - 1}$-bundle over $\mathscr{U}^{\alpha_1}$, and as such it is an irreducible, unirational variety. For every $\alpha_1 \in \{1, \dots, 7\}$, using (67) we obtain

$$\dim \mathfrak{N}^{\alpha_1}_{\mathrm{VII}} = \dim \mathscr{U}^{\alpha_1} + h^0(\mathbb{P}^2, S^3\mathscr{F}(-2)) - 1 = (21 - \alpha_1) + \alpha_1 - 1 = 20.$$

Summing up, every $\mathfrak{N}^{\alpha_1}_{\mathrm{VII}}$ is a connected, irreducible, unirational 20-dimensional component of $\mathfrak{N}_{\mathrm{VII}}$. □

Remark 4.8. We could also give a geometric interpretation of the equality dim $\mathfrak{N}^{\alpha_1}_{\mathrm{VII}}$ $= 20$ by using in each case the explicit description of the surface X provided by Proposition 3.17. We will not develop this point here, and we will limit ourselves to discussing as an example the case $\alpha_1 = 6$. In this situation, we know that X is isomorphic to the blow-up at six points of an Alexander surface of degree 9 in $\mathbb{P}^4$; see Remark 3.18. Such points are the intersection of the Alexander surface with its unique 6-secant line, and they completely determine the triple cover map $f\colon X \to \mathbb{P}^2$. So the dimension of the component $\mathfrak{N}^{6}_{\mathrm{VII}}$ equals the dimension of an open, dense subset of $S^{10}(\mathbb{P}^2)$, which is 20.

Appendix: The computer-aided construction of triple planes

Here we explain how we can use the Computer Algebra System `Macaulay2` in order to show the existence of general triple planes in the cases marked with $(*)$ in Proposition 3.17. The computation can be performed either over $\mathbb{Q}$ or over a prime field (the latter being considerably faster).

The setup for adjunction. Define the coordinate ring of $\mathbb{P}^2$ and of $\mathbb{P}^{b-3} = \mathbb{P}^5$ needed for the first adjunction map, together with a second $\mathbb{P}^5$ (the projectivization of the 6-dimensional polynomial ring `V`) that will be the target space for the second adjunction.

```
b = 8;
k = QQ;
T = k[x_0..x_2];
S = k[y_0..y_(b-3)];
R = T**S;
V = k[t_0..t_5];
```

The command `fliptensor` takes as input the matrix M and gives as output the matrix N; cf. §2.3.1.

```
fliptensor := M->(Q = substitute(vars S,R) * (substitute(M,R));
    sub((coefficients(Q,(Variables=>{x_0,x_1,x_2})))_1,S));
```

The 3-fold scroll $Y \subset \mathbb{P}^5$ is defined by the 3×3 minors of N. The command `twosections` gives back the ideal of the union of two surface sections S_1 and S_2 of the scroll Y, with $S_i \in |\mathscr{O}_Y(\ell)|$ and $\mathscr{O}_Y(\ell) = \mathfrak{p}^*\mathscr{O}_{\mathbb{P}^2}(1)$; cf. §1.4.1 and §2.3.1. Each of them is defined by the 2×2 minors of a random submatrix of N, obtained by composing N with a random matrix of scalars.

```
twosections := N->(A = random(S^{3:0},S^{3:0});
Nrandom = (transpose(N)*A);
N1 = submatrix(Nrandom, {0,1});
N2 = submatrix(Nrandom, {0,2});
IS1 = minors(2, N1);
IS2 = minors(2, N2);
I12 = intersect(IS1,IS2));
```

The command `cubicgenerator` takes a random cubic in the ideal of cubics of Y through $S_1 \cup S_2$, and we call X_1 the residual surface. This surface is precisely the image of the first adjunction map $\varphi_1\colon X \to X_1 \subset \mathbb{P}^5$; see the last part of the proof of Proposition 3.17.

```
cubicgenerator := I12 -> (SU = super basis(3,I12);
cubic = SU*random(S^{rank(source(SU)):0},S^{1:0});
ideal(cubic));
```

The cases according to the number of unstable lines. Here we define the Steiner bundle $\mathscr{F}$ by giving its presentation matrix M. More precisely, for any $\alpha_1 \in \{1, \ldots, 7\}$ we define a random Steiner bundle with α_1 unstable lines.

The cases $1 \leq \alpha_1 \leq 6$. For $1 \leq \alpha_1 \leq 6$, we put random coefficients in the layout of Proposition 1.10 in order to define $\mathscr{F}$. The command `GenM` takes an integer `a`, picks `a` random linear forms, multiplies each of them by a column matrix of size 4 of random scalars, and stacks them together with a random matrix of linear forms in order to obtain a matrix M of size 4×6, given as output.

```
use T
GenM:=(a)->(
    for j from 0 to a-1 do
        M_j=((random(T^{1},T^{0}))_(0,0))*random (T^{4:0},T^{1:0});
    Mcu = transpose M_0;
    for j from 1 to a-1 do Mcu=(Mcu||transpose(M_j));
    Mco = (random(T^{6-a:0},T^{4:-1}));
    ((transpose Mcu) | (transpose Mco)))
```

We choose α_1, define the Steiner sheaf as cokernel of M and check that it is locally free of rank 2.

```
for a from 1 to 6 do F_a = coker (
   transpose map(T^{b-4:1},T^{b-2:0},GenM(a)))
for a from 1 to 6 do print dim (minors(4,presentation F_a))
```

The output of this is 0 in all seven cases, so the sheaves are locally free.

The case $\alpha_1 = 7$. In this case $\mathscr{F}$ is a logarithmic bundle, so its dual appears as the first syzygy of the Jacobian map ∇f of partial derivatives of the product f of the seven linear forms that define the seven unstable lines. In other words, we have an exact sequence

$$0 \to \mathscr{F}^\vee \to \mathscr{O}_{\mathbb{P}^2}(1)^3 \xrightarrow{\nabla f} \mathscr{O}_{\mathbb{P}^2}(7);$$

cf. for instance [FMV13, (1.10)]. We choose these seven lines randomly and define $\mathscr{F}$ as the dual of $\ker(\nabla f)$.

```
f = 1_T; for j from 1 to 7 do f=f*(random(T^{1},T^{0}))_(0,0)
M=transpose ((res ker diff(vars T,f))).dd_1
MM = map(T^{b-4:1},T^{b-2:0},M);
dim minors(4,MM) == 0
F_7 = coker transpose MM;
```

We check incidentally that the vanishing $H^0(\mathbb{P}^2, S^2\mathscr{F}(-2)) = 0$ and the equality $h^0(\mathbb{P}^2, S^3\mathscr{F}(-2)) = \alpha_1$ hold true for all values of α_1 (this fact was needed in the proof of Proposition 4.7.

```
for a from 1 to 7 do print(
    HH^0((sheaf (symmetricPower(2,F_a)))(-2)),
    rank HH^0((sheaf (symmetricPower(3,F_a)))(-2)))
```

The output is $(0,\, a)$ with $a = \alpha_1 \in \{1,\, \ldots,\, 7\}$.

Construction of the triple plane. We take $\mathscr{F}$ and extract the two matrices of linear forms M and N.

```
for a from 1 to 7 do NN_a = fliptensor(presentation (F_a));
for a from 1 to 7 do IY_a = minors(rank target NN_a,NN_a);
for a from 1 to 7 do singY_a = ideal singularLocus variety IY_a;
```

Singularity test: the only singular points of Y are α_1 points of multiplicity 6. They all come from the locus where the matrix N defining Y has rank at most 1.

```
for a from 1 to 7 do I2Y_a = minors(rank (target NN_a)-1,NN_a);
for a from 1 to 7 do print (dim singY_a, degree singY_a)
for a from 1 to 7 do print (dim(singY_a:I2Y_a),degree(singY_a:I2Y_a))
```

The output of the last command is $(1, a)$, where $a = \alpha_1$ goes from 1 to 7 in the seven cases, and means that Y is singular precisely at the a double points coming from the a unstable lines. Define now X_1 as a random cubic in the ideal of the union of two surface sections of Y from $|\mathscr{O}_Y(\ell)|$. Perform a degree, genus and singularity test.

```
for a from 1 to 7 do II12_a = twosections(NN_a);
for a from 1 to 7 do IC3_a = cubicgenerator(II12_a);
for a from 1 to 7 do IX1_a = ((IC3_a + IY_a):II12_a);
for a from 1 to 7 do X1_a = variety(IX1_a);
for a from 1 to 7 do print(dim X1_a, degree X1_a,genera X1_a)
for a from 1 to 7 do (dim singularLocus X1_a)
```

In all seven cases, the output of the penultimate command is `(2, 10, {0, 6, 9})`, which means that X is a surface of degree 10 with sectional genera $(0, 6, 9)$. The output of the last command is $-\infty$, i.e., X is smooth. This takes about 15 minutes on a laptop if performed on a prime field.

The second adjunction map of $\varphi_2\colon X_1 \to X_2 \subset \mathbb{P}^5$ is defined by the restriction to X_1 of the linear system $|\mathscr{O}_Y(2\xi - \ell)|$, and this in turn coincides with the restriction to X_1 of the linear system generated by the six quadrics in the ideal defining S_1; see again the proof of Proposition 3.17.

Having this in mind, we can finally compute the ideal of X_2 and its canonical sheaf $\omega_{X_2} = \mathscr{O}_{X_2}(K_{X_2})$ in order to find $K^2_{X_2}$.

```
for a from 1 to 7 do X2_a = (ker map(S/(IX1_a),V,
        gens  minors(2,random(S^{2:0},S^{3:0})*NN_a)))
for a from 1 to 7 do omegaX2_a = (Ext^2(X2_a,V^{1:-6}))**(V/(X2_a));
for a from 1 to 7 do print(euler(dual omegaX2_a)-1)
```

Here is the output of the last command, providing the value of $K^2_{X_2}$ for all $\alpha_1 = a \in \{1, \dots, 7\}$:

```
9, 5, 2, 0, -1, -1, 0
```

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