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A THEORY OF THERMOELASTIC MATERIALS WITH VOIDS WITHOUT ENERGY DISSIPATION

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This article is concerned with the theory of thermoelastic materials with voids based on the concept of volume fraction [Goodman and Cowin, Arch. Rational Mech. Anal., vol. 44, pp. 249–266, 1972; Nunziato and Cowin, Arch. Rational Mech. Anal., vol. 72, pp. 175–201, 1979]. We use the results of Green and Naghdi [Proc. Roy Soc. London A, vol. 432, pp. 171–194, 1991; J. Elasticity, vol. 31, pp. 189–209, 1993] on thermomechanics of continua to derive a linear theory of thermoelastic materials with voids that does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. Then we establish a uniqueness result and the continuous dependence of solutions upon the initial data and body loads.

The concept of a distributed body introduced by Goodman and Cowin [1] in the context of granular and porous materials asserts that the mass density has the decomposition $\gamma\nu$, where γ is the density of the matrix material and ν is the volume fraction field. This representation introduces an additional degree of kinematic freedom and was employed by Nunziato and Cowin [2] to establish a nonlinear theory of elastic materials with voids. Capriz [3] showed that the theory of elastic materials with voids is a special case of the theory of affinely structured continua. The linear theory of elastic materials with voids was established by Cowin and Nunziato [4].

In contrast to the conventional thermoelasticity, nonclassical theories came into existence during the last two decades. These theories, referred to as generalized thermoelasticity, were introduced in the literature in an attempt to eliminate the shortcomings of the classical dynamical thermoelasticity. A survey article of representative theories in the range of generalized thermoelasticity is due to Hetnarski and Ignaczak [5].

In [6–8], Green and Naghdi presented a treatment of the thermomechanical theory of deformable media that differs from the usual one and, in particular, makes an entropy balance. Moreover, in comparison to the classical theory, the Fourier law is replaced by a heat flux rate-temperature gradient relation. A theory of

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thermoelastic bodies based on the new procedure was presented in [7]. The linearized theory of thermoelasticity established in [7] does not sustain energy dissipation and permits the transmission of heat as thermal waves at finite speed. Moreover, the heat flux vector is determined by the same potential function that determines the stress.

In this article we use the results of Green and Naghdi [6, 7] on the thermomechanics of continua to establish a linear theory of thermoelastic materials with voids without energy dissipation. The theory admits the possibility of "second sound" and leads to the result that the conduction tensor is symmetric. We find that, in contrast with the theory developed in [7], the field equations cannot be expressed in terms of mechanical variables and the temperature. We prove a uniqueness theorem, and we establish the continuous dependence of solutions upon the initial data and body loads.

THERMOELASTICITY

We consider a body that at time t_0 occupies the region B of Euclidean three-dimensional space and is bounded by the piecewise smooth surface ∂B. The motion of the body is referred to the reference configuration B and the fixed system of rectangular Cartesian axes Ox_k $(k = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions.

Green and Naghdi [6–8] presented a theory of thermomechanics of continua that uses the entropy balance

$$\int_V \rho_0 \dot{\eta}\, dV = \int_V \rho_0 (s + \xi)\, dV + \int_{\partial V} G\, dA \tag{1}$$

for every part V of B and every time. Here, ρ_0 is the density in the reference configuration, η is the entropy per unit mass and unit time, s is the external rate of supply of entropy per unit mass, ξ is the internal rate of production of entropy per unit mass, and G is the internal flux of entropy per unit mass. Following [6], from Eq. (1) we get

$$G = \Phi_i n_i \tag{2}$$

where Φ_i is the entropy flux vector and n_i is the outward unit normal at ∂V. In view of Eq. (2) the balance of entropy reduces to the local equation

$$\rho_0 \dot{\eta} = \rho_0 (s + \xi) + \Phi_{i,i} \tag{3}$$

Let Ω be an arbitrary material volume in the continuum, bounded by a surface $\partial\Omega$, at time t. We suppose that V is the corresponding region in the reference configuration B, bounded by a surface ∂V. Let q be the heat flux across the surface $\partial\Omega$ measured per unit area of ∂V. We denote by q_i the flux of heat associated with surfaces in the deformed body that were originally coordinate planes perpendicular to the x_i-axes through the point $\mathbf{x}$. Then

$$q = \theta G \qquad q_i = \theta \Phi_i \qquad q = q_i n_i \tag{4}$$

where θ is the absolute temperature. With the help of Eq. (4), Eq. (3) can be written in the form

$$\rho_0 \theta \dot{\eta} = \rho_0 \theta (s + \xi) + (\theta \Phi_i)_{,i} - \Phi_i \theta_{,i} \tag{5}$$

In the procedure of Green and Naghdi [8], the reduced energy equation is regarded as an identity for all thermodynamical processes and will place restrictions on the functional dependence of the constitutive equations.

Following [2, 6] we postulate an energy balance in the form

$$\int_V (\rho_0 \ddot{u}_i \dot{u}_i + \chi \ddot{v} \dot{v} + \rho_0 \dot{e})\, dV = \int_V \rho_0 (F_i \dot{u}_i + L\dot{v} + s\theta)\, dV + \int_{\partial V} (t_i \dot{u}_i + h\dot{v} + G\theta)\, dA \tag{6}$$

for all regions V of B and every time. In Eq. (6) we used the notation: $\mathbf{u}$ is the displacement vector, v is the volume fraction field, $\mathbf{F}$ is the external body force per unit mass, L is the external equilibrated body force, $\mathbf{t}$ is the stress vector, h is the equilibrated stress, χ is the equilibrated inertia, and e is the internal energy per unit mass. Using invariance requirements under superposed rigid body motions [9], from Eq. (6) we get

$$\int_V \rho_0 \ddot{\mathbf{u}}\, dV = \int_V \rho_0 \mathbf{F}\, dV + \int_{\partial V} \mathbf{t}\, dA \tag{7}$$

for every part V of B and every time. From Eq. (7) we obtain

$$t_i = t_{ji} n_j \tag{8}$$

where t_{ij} is the stress tensor. The field equation for momentum balance is

$$t_{ji,j} + \rho_0 F_i = \rho_0 \ddot{u}_i \tag{9}$$

Taking into account Eqs. (4), (8), and (9), from Eq. (6) we obtain

$$\int_V (\rho_0 \dot{e} + \chi \ddot{v} \dot{v})\, dV = \int_V [t_{ji} \dot{u}_{i,j} + \rho_0 L\dot{v} + \rho_0 s\theta + (\theta \Phi_i)_{,i}]\, dV + \int_{\partial V} h\dot{v}\, dA \tag{10}$$

for all regions V of B and any time. With an argument similar to that used in obtaining Eq. (8), from Eq. (10) we find that

$$h = h_i n_i \tag{11}$$

where h_i is the equilibrated stress associated with surfaces in the deformed body that were originally coordinate planes perpendicular to the x_i-axes, measured per unit area of these planes. Using Eq. (11) and the divergence theorem, from Eq. (10) we obtain the local form of energy balance

$$\rho_0 \dot{e} = t_{ji}\dot{u}_{i,j} + h_i \dot{v}_{,i} - g\dot{v} + \rho_0 s\theta + (\theta\Phi_i)_{,i} \tag{12}$$

where

$$g = \chi\ddot{v} - h_{i,i} - \rho_0 L \tag{13}$$

We now consider a motion of the continua that differs from the given motion only by a superposed uniform rigid body angular velocity. Using the procedure presented in [9], from Eq. (12) we obtain

$$t_{ij} = t_{ji} \tag{14}$$

If we introduce the Helmholtz energy A by

$$A = e - \eta\theta \tag{15}$$

and take into account the relations (5) and (14), then Eq. (12) becomes

$$\rho_0 \dot{A} = t_{ij}\dot{e}_{ij} + h_i \dot{v}_{,i} - g\dot{v} + \rho_0 \eta\dot{\theta} - \rho_0 \theta\xi + \Phi_i \theta_{,i} \tag{16}$$

where

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{17}$$

Following Green and Naghdi [6, 7], we introduce the thermal displacement α by $\dot{\alpha} = \theta$. We require constitutive equations for $A, t_{ij}, h_i, g, \eta, \Phi_i$, and ξ and assume that these are functions of the variables $(e_{ij}, v_{,i}, v, \theta, \alpha_{,i}) = \Pi$. As in [7], for simplicity, we regard the material to be homogeneous. Introduction of constitutive equations of the form

$$A = \hat{A}(\Pi) \qquad t_{ij} = \hat{t}_{ij}(\Pi) \qquad ,\ldots, \qquad \xi = \hat{\xi}(\Pi)$$

into the reduced energy equation (16) yields

$$\left(\frac{\partial U}{\partial \theta} + \rho_0 \eta\right)\dot{\theta} + \left(\frac{\partial U}{\partial e_{ij}} - t_{ij}\right)\dot{e}_{ij} + \left(\frac{\partial U}{\partial v_{,i}} - h_i\right)\dot{v}_{,i} + \left(\frac{\partial U}{\partial v} + g\right)\dot{v} + \left(\frac{\partial U}{\partial \alpha_{,i}} - \Phi_i\right)\theta_{,i} + \rho_0 \theta\xi = 0 \tag{18}$$

where $U = \rho_0 \hat{A}$. We find that the necessary and sufficient relations for Eq. (18) to be satisfied under these constitutive assumptions are

$$t_{ij} = \frac{\partial U}{\partial e_{ij}} \qquad h_i = \frac{\partial U}{\partial v_{,i}} \qquad g = -\frac{\partial U}{\partial v}$$
$$\Phi_i = \frac{\partial U}{\partial \alpha_{,i}} \qquad \rho_0 \eta = -\frac{\partial U}{\partial \theta} \qquad \xi = 0 \tag{19}$$

As in [1], we assume that there exists the reference time t_0 such that

$$\theta(\mathbf{x}, t_0) = T_0 \quad \text{and} \quad \alpha(\mathbf{x}, t_0) = \alpha_0 \qquad \mathbf{x} \in \bar{B} \tag{20}$$

where T_0 and α_0 are constants. If we denote

$$T = \theta - T_0 \qquad \psi = \int_{t_0}^{t} T\, dt \tag{21}$$

then

$$\alpha = \psi + T_0(t - t_0) + \alpha_0 \qquad \alpha_{,i} = \psi_{,i} \qquad \dot{\psi} = T \tag{22}$$

We introduce the notation

$$\varphi = v - v_0 \tag{23}$$

where v_0 is the volume fraction for the reference configuration. We assume that v_0 is a given constant.

In the linear theory we have

$$\mathbf{u} = \varepsilon \mathbf{u}' \qquad \varphi = \varepsilon \varphi' \qquad T = \varepsilon T' \tag{24}$$

where ε is a constant small enough for squares and higher powers to be neglected and $\mathbf{u}', \varphi'$, and T' are independent of ε. For a linearized theory we assume that U is a quadratic function of the variables $e_{ij}, \varphi_{,i}, \varphi, T$, and $\psi_{,i}$. Thus, for a body with a center of symmetry, we have

$$U = \frac{1}{2} C_{ijrs} e_{ij} e_{rs} + A_{ij} e_{ij} \varphi - \beta_{ij} e_{ij} T + \frac{1}{2} B_{ij} \varphi_{,i} \varphi_{,j} + M_{ij} \varphi_{,i} \psi_{,j}$$
$$+ \frac{1}{2} \zeta \varphi^2 - d T \varphi - \frac{1}{2} a T^2 + \frac{1}{2} k_{ij} \psi_{,i} \psi_{,j} \tag{25}$$

where the constitutive coefficients have the symmetries

$$C_{ijrs} = C_{rsij} = C_{jirs} \qquad B_{ij} = B_{ji} \qquad k_{ij} = k_{ji} \tag{26}$$

In view of Eq. (19), we obtain

$$
\begin{aligned}
t_{ij} &= C_{jirs}e_{rs} + A_{ij}\varphi - \beta_{ij}T \\
h_i &= B_{ij}\varphi_{,j} + M_{ij}\psi_{,j} \\
g &= -A_{ij}e_{ij} - \zeta\varphi + dT \\
\rho_0\eta &= \beta_{ij}e_{ij} + d\varphi + aT \\
\Phi_i &= M_{ji}\varphi_{,j} + k_{ij}\psi_{,j}
\end{aligned}
\tag{27}
$$

where $T = \dot{\psi}$.

In the case of homogeneous and isotropic solids, the constitutive equations reduce to

$$
\begin{aligned}
t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + \gamma\varphi\delta_{ij} - \beta\dot{\psi}\delta_{ij} \\
h_i &= b\varphi_{,i} + m\psi_{,i} \\
g &= -\gamma e_{\rho\rho} - \zeta\varphi + d\dot{\psi} \\
\rho_0\eta &= \beta e_{\rho\rho} + d\varphi + a\dot{\psi} \\
\Phi_i &= m\varphi_{,i} + k\psi_{,i}
\end{aligned}
\tag{28}
$$

where $\lambda, \mu, \beta, \gamma, b, m, \zeta, d, a, m$, and k are constitutive coefficients.

The equation of entropy (3) reduces to

$$
\rho_0\dot{\eta} = \Phi_{i,i} + \rho_0 s \tag{29}
$$

The equation (13) can be written in the form

$$
h_{i,i} + g + \rho_0 L = \chi\ddot{\varphi} \tag{30}
$$

where χ is a given constant.

Thus, the basic equations of the theory consist of the equations of motion (9) and (30), the equation of entropy (29), the constitutive equations (27), and the geometrical equations (17). To these equations we adjoin boundary conditions and initial conditions. In the case of the mixed boundary value problem the boundary conditions are

$$
\begin{aligned}
&u_i = \tilde{u}_i \quad \text{on } \bar{S}_1 \times I \quad && \varphi = \tilde{\varphi} \quad \text{on } \bar{S}_3 \times I \quad && \psi = \tilde{\psi} \quad \text{on } \bar{S}_5 \times I \\
&t_{ji}n_j = \tilde{t}_i \quad \text{on } S_2 \times I \quad && h_in_i = \tilde{h} \quad \text{on } S_4 \times I \quad && \Phi_in_i = \tilde{\Phi} \quad \text{on } S_6 \times I
\end{aligned}
\tag{31}
$$

where S_r $(r = 1, 2, \ldots, 6)$ are subsets of ∂B such that $\bar{S}_1 \cup S_2 = \bar{S}_3 \cup S_4 = \bar{S}_5 \cup S_6 = \partial B$, $S_1 \cap S_2 = S_3 \cap S_4 = S_5 \cap S_6 = \emptyset$, $I = (0, \infty)$, and $\tilde{u}_i, \tilde{\varphi}, \tilde{\chi}, \tilde{t}_i, \tilde{h}$, and $\tilde{\Phi}$ are prescribed functions. The initial conditions are

$$
\begin{aligned}
&u_i(\mathbf{x}, 0) = u_i^0(\mathbf{x}) \qquad && \dot{u}_i(\mathbf{x}, 0) = v_i^0(\mathbf{x}) \qquad && \varphi(\mathbf{x}, 0) = \varphi^0(\mathbf{x}) \\
&\dot{\varphi}(\mathbf{x}, 0) = \nu^0(\mathbf{x}) \qquad && \psi(\mathbf{x}, 0) = \psi^0(\mathbf{x}) \qquad && \dot{\psi}(\mathbf{x}, 0) = \chi^0(\mathbf{x}) \qquad \mathbf{x} \in \bar{B}
\end{aligned}
\tag{32}
$$

where $\mathbf{u}^0, \mathbf{v}^0, \varphi^0, \nu^0, \psi^0$, and χ^0 are given.

The field equations can be expressed in terms of the functions u_i, φ, and ψ. In the case of homogeneous and isotropic bodies these equations are

$$\begin{aligned} \mu\Delta\mathbf{u} + (\lambda+\mu)\text{ grad div }\mathbf{u} + \gamma\text{grad }\varphi - \beta\text{grad }\dot{\psi} + \rho_0\mathbf{F} &= \rho_0\ddot{\mathbf{u}} \\ b\Delta\varphi + m\Delta\psi - \gamma\text{div }\mathbf{u} - \zeta\varphi + d\dot{\psi} + \rho_0 L &= \chi\ddot{\varphi} \\ k\Delta\psi + m\Delta\varphi - \beta\text{div }\dot{\mathbf{u}} - d\dot{\varphi} - a\dot{\psi} &= -\rho_0 s \end{aligned} \tag{33}$$

on $B \times I$.

Remark. In contrast to the theory developed in [7], in our case the field equations cannot be expressed in terms of mechanical variables and the temperature T. The differentiation with respect to the time of field equations (e.g., Eq. $(33)_2$) leads to some complications.

UNIQUENESS THEOREM

In this section we establish a uniqueness theorem in the linear theory of thermoelastic materials with voids. We introduce the notation

$$\begin{aligned} 2W &= C_{ijrs}e_{ij}e_{rs} + 2A_{ij}e_{ij}\varphi + B_{ij}\varphi_{,i}\varphi_{,j} + 2M_{ij}\varphi_{,i}\psi_{,j} + \zeta\varphi^2 + k_{ij}\psi_{,i}\psi_{,j} \\ V &= \frac{1}{2}\int_B (\rho_0\dot{\mathbf{u}}^2 + \chi\dot{\varphi}^2 + aT^2 + 2W)\, dv \end{aligned} \tag{34}$$

Theorem 1. *Assume that*

i. ρ_0, χ, *and* a *are strictly positive;*
ii. W *is a positive semi-definite quadratic form.*

Then the boundary initial value problem of the linear theory has at most one solution.

Proof. With the help of Eq. (27) we obtain

$$t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \rho_0\dot{\eta}T + \Phi_i T_{,i} = \frac{1}{2}\frac{\partial}{\partial t}(2W + aT^2 + k_{ij}\psi_{,i}\psi_{,j}) \tag{35}$$

On the other hand, from Eqs. (9), (29), (30), and (17) we find that

$$\begin{aligned} &t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \rho_0\dot{\eta}T + \Phi_i T_{,i} \\ &\quad = (t_{ij}\dot{u}_i + h_j\dot{\varphi} + \Phi_j T)_{,j} + \rho_0 F_i\dot{u}_i + \rho_0 L\dot{\varphi} + \rho_0 sT - \rho_0\ddot{u}_i\dot{u}_i - \chi\ddot{\varphi}\dot{\varphi} \end{aligned} \tag{36}$$

By the divergence theorem and Eqs. (35) and (36) we find that

$$\dot{V} = \int_{\partial B}(t_{ji}n_j\dot{u}_i + h_i n_i\dot{\varphi} + \Phi_i n_i T)\, da + \int_B \rho_0(F_i\dot{u}_i + L\dot{\varphi} + s\dot{\psi})\, dv \tag{37}$$

Suppose that there are two solutions $\{u_i^{(\alpha)}, \varphi^{(\alpha)}, \psi^{(\alpha)}\}$ $(\alpha = 1, 2)$. Then their difference $\Xi = \{u_i^*, \varphi^*, \psi^*\}$, $u_i^* = u_i^{(1)} - u_i^{(2)}$, $\varphi^* = \varphi^{(1)} - \varphi^{(2)}$, $\psi^* = \psi^{(1)} - \psi^{(2)}$ corresponds to null data. If V^* is the function associated with Ξ and defined by Eq. $(34)_2$, then from Eq. (37) we obtain $\dot{V}^* = 0$ on I. The initial conditions imply $V^*(0) = 0$, and we conclude that $V^* = 0$ on I. With the help of the hypotheses of the theorem we find that $\dot{\mathbf{u}}^* = 0$, $\dot{\varphi}^* = 0$, and $\psi^* = 0$ on $B \times I$. But $\mathbf{u}^*$, φ^*, ψ^* vanish initially, so $\mathbf{u}^* = 0$, $\varphi^* = 0$, and $\psi^* = 0$ on $B \times I$. The proof is complete.

A CONTINUOUS DEPENDENCE RESULT

In what follows we study the continuous dependence of solutions upon initial data and body loads. We assume that the potential W is a positive definite quadratic form. In the case of isotropic bodies this fact implies that

$$3\lambda + 2\mu > 0 \qquad \mu > 0 \qquad b > 0 \qquad \zeta > 0 \qquad k > 0 \tag{38}$$

As in the preceding section we assume that

$$\rho_0 > 0 \qquad \chi > 0 \qquad a > 0 \tag{39}$$

We restrict our attention to isotropic bodies and introduce the dimensionless variables

$$\begin{aligned} x_i' &= \frac{1}{l}x_i \qquad t' = \frac{c_1}{l}t \qquad u_i' = \frac{1}{l}u_i \\ \varphi' &= \varphi \qquad \psi' = \frac{c_1}{lT_0}\psi \end{aligned} \tag{40}$$

where l is a standard length and $c_1 = [(\lambda + 2\mu)/\rho_0]^{1/2}$. Introducing Eq. (38) into Eq. (33) and suppressing primes, we obtain

$$\begin{aligned} \tau\Delta u_i + (1-\tau)u_{j,ji} + a_1\varphi_{,i} - a_2\dot{\psi}_{,i} + H_i &= \ddot{u}_i \\ \alpha\Delta\varphi + b_1\Delta\psi - a_1 u_{j,j} - b_2\varphi + b_3\dot{\psi} + P &= r\ddot{\varphi} \\ \sigma\Delta\psi + b_1\Delta\varphi - a_2\dot{u}_{j,j} - b_3\dot{\varphi} - p\ddot{\psi} &= -Q \end{aligned} \tag{41}$$

where

$$\begin{aligned} &\tau = (c_2/c_1)^2 && c_2 = (\mu/\rho_0)^{1/2} && a_1 = \gamma/(\rho_0 c_1^2) \\ &a_2 = \beta T_0/(\rho_0 c_1^2) && \alpha = b/(\rho_0 l^2 c_1^2) && b_1 = mT_0/(\rho_0 l c_1^3) \\ &b_2 = \zeta/(\rho_0 c_1^2) && b_3 = dT_0/(\rho_0 c_1^2) && r = \chi/(l^2 c_1^2) \\ &\sigma = k/(\rho_0 c_1^4) && p = aT_0^2/(\rho_0 c_1^2) && H_i = \frac{l}{c_1^2}F_i \\ &P = \frac{1}{c_1^2}L && Q = \frac{l}{c_1^3}T_0 s && \end{aligned} \tag{42}$$

Throughout this section we study the behavior of the continuum on the finite time interval $[0, t_1]$. To Eqs. (41) we adjoin the initial conditions

$$\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}) \qquad \dot{\mathbf{u}}(\mathbf{x},0)=\mathbf{v}_0(\mathbf{x}) \qquad \varphi(\mathbf{x},0)=\varphi_0(\mathbf{x})$$
$$\dot{\varphi}(\mathbf{x},0)=\nu_0(\mathbf{x}) \qquad \psi(\mathbf{x},0)=\psi_0(\mathbf{x}) \qquad \dot{\psi}(\mathbf{x},0)=\chi_0(\mathbf{x}) \qquad \mathbf{x}\in\bar{B} \tag{43}$$

and the boundary conditions

$$\mathbf{u}=\tilde{\mathbf{u}} \qquad \varphi=\tilde{\varphi} \qquad \psi=\tilde{\psi} \qquad \text{on } \partial B\times[0,t_1] \tag{44}$$

We assume that: (i) the prescribed functions $\mathbf{u}_0$, $\mathbf{v}_0$, φ_0, ν_0, ψ_0, χ_0 are continuous on $\bar{B}$; (ii) the functions $\tilde{\mathbf{u}}$, $\tilde{\varphi}$, and $\tilde{\psi}$ are continuous on $\partial B\times[0,t_1]$; and (iii) $\mathbf{H}$, P, and Q are continuous on $\bar{B}\times[0,t_1]$.

Let $(\mathbf{u}^{(1)},\varphi^{(1)},\psi^{(1)})$ and $(\mathbf{u}^{(2)},\varphi^{(2)},\psi^{(2)})$ be solutions corresponding to the external data systems $I^{(1)}=\{\mathbf{H}^{(1)}P^{(1)},Q^{(1)},\mathbf{u}_0^{(1)},\mathbf{v}_0^{(1)},\varphi_0^{(1)},\nu_0^{(1)},\psi_0^{(1)},\chi_0^{(1)},\tilde{\mathbf{u}}^{(1)},\tilde{\varphi}^{(1)},\tilde{\psi}^{(1)}\}$ and $I^{(2)}=\{\mathbf{H}^{(2)}P^{(2)},Q^{(2)},\mathbf{u}_0^{(2)},\mathbf{v}_0^{(2)},\varphi_0^{(2)},\nu_0^{(2)},\psi_0^{(2)},\chi_0^{(2)},\tilde{\mathbf{u}}^{(1)},\tilde{\varphi}^{(1)},\tilde{\psi}^{(1)}\}$ respectively. If we define $\mathbf{u}=\mathbf{u}^{(1)}-\mathbf{u}^{(2)}$, $\varphi=\varphi^{(1)}-\varphi^{(2)}$, $\psi=\psi^{(1)}-\psi^{(2)}$, then $(\mathbf{u},\varphi,\psi)$ is a solution of the boundary initial value problem corresponding to the external data system $I=\{\mathbf{H},P,Q,\mathbf{u}_0,\mathbf{v}_0,\varphi_0.\nu_0,\psi_0,\chi_0,\mathbf{0},0,0\}$, where $\mathbf{H}=\mathbf{H}^{(1)}-\mathbf{H}^{(2)}$, $P=P^{(1)}-P^{(2)}$, $Q=Q^{(1)}-Q^{(2)}$, $\mathbf{u}_0=\mathbf{u}_0^{(1)}-\mathbf{u}_0^{(2)}$, $\mathbf{v}_0=\mathbf{v}_0^{(1)}-\nu_0^{(2)}$, $\varphi_0=\varphi_0^{(1)}-\varphi_0^{(2)}$, $\nu_0=\nu_0^{(1)}-\nu_0^{(2)}$, $\psi_0=\psi_0^{(1)}-\psi_0^{(2)}$, and $\chi_0=\chi_0^{(1)}-\chi_0^{(2)}$. We denote this problem by (A) and define the "distance" between the solutions $(\mathbf{u}^{(1)},\varphi^{(1)},\psi^{(1)})$ and $(\mathbf{u}^{(2)},\varphi^{(2)},\psi^{(2)})$ by

$$\Gamma=\frac{1}{2}\int_B(2E+\dot{u}_i\dot{u}_i+r\dot{\varphi}^2+p\dot{\psi}^2)\,dv \tag{45}$$

where

$$\begin{aligned}2E&=(1-2\tau)e_{rr}e_{ss}+2\tau e_{ij}e_{ij}+2a_1e_{rr}\varphi+\alpha\varphi_{,i}\varphi_{,i}+2b_1\varphi_{,i}\psi_{,i}\\&\quad+\sigma\psi_{,i}\psi_{,i}+b_2\varphi^2\\2e_{ij}&=u_{i,j}+u_{j,i}\end{aligned} \tag{46}$$

It follows from the positive definiteness assumption on W that E is a positive definite quadratic form in the variables $e_{ij},\varphi,\varphi_{,i}$, and $\psi_{,i}$. Thus, there exist the positive constants k_1 and k_2 such that

$$k_1(e_{ij}e_{ij}+\varphi_{,i}\varphi_{,i}+\varphi^2+\psi_{,i}\psi_{,i})\le E\le k_2(e_{ij}e_{ij}+\varphi_{,i}\varphi_{,i}+\varphi^2+\psi_{,i}\psi_{,i}) \tag{47}$$

Lemma 1. *Let* $(\mathbf{u},\varphi,\psi)$ *be a solution of the problem* (A). *Then*

$$\dot{\Gamma}=\int_B(H_i\dot{u}_i+P\dot{\varphi}+Q\dot{\psi})\,dv \tag{48}$$

Proof. If we denote

$$\begin{aligned}&\pi_{ij}=(1-2\tau)e_{rr}\delta_{ij}+2\tau e_{ij}+a_1\varphi\delta_{ij}-a_2\dot{\psi}\delta_{ij}\\&s_i=\alpha\varphi_{,i}+b_1\psi_{,i} \qquad Q=-a_1e_{rr}-b_2\varphi+b_3\dot{\psi}\\&f=a_2e_{rr}+b_3\varphi+p\dot{\psi} \qquad g_i=\sigma\psi_{,i}+b_1\varphi_{,i}\end{aligned} \tag{49}$$

then Eqs. (41) can be written in the form

$$\pi_{ji,j} + H_i = \ddot{u}_i, \qquad s_{i,i} + Q + P = r\ddot{\varphi} \qquad \dot{f} = g_{i,i} + Q \tag{50}$$

From Eqs. (49) and (46) we obtain

$$\pi_{ij}\dot{e}_{ij} + s_i\dot{\varphi}_{,i} - Q\dot{\varphi} + f\dot{\psi} + g_i\dot{\psi}_{,i} = \frac{\partial}{\partial t}\left(E + \frac{1}{2}p\dot{\psi}^2\right) \tag{51}$$

In view of Eqs. (50) we get

$$\begin{aligned}\pi_{ij}\dot{e}_{ij} + s_i\dot{\varphi}_{,i} - Q\dot{\varphi} + f\dot{\psi} + g_i\dot{\psi}_{,i} &= (\pi_{ij}\dot{u}_i + s_j\dot{\varphi} + g_j\dot{\psi})_{,j} \\ &\quad + H_i\dot{u}_i + P\dot{\varphi} + Q\dot{\psi} - \frac{1}{2}\frac{\partial}{\partial t}(\dot{\mathbf{u}}^2 + r\dot{\varphi}^2)\end{aligned} \tag{52}$$

If we integrate Eq. (52) over B and use the divergence theorem, the boundary conditions, and Eq. (51), then we obtain the desired result.

We introduce the functions y and Λ on $[0, t_1]$ by

$$\begin{aligned} y &= \left\{\int_B (e_{ij}e_{ij} + \varphi^2 + \varphi_{,i}\varphi_{,i} + \psi_{,i}\psi_{,i} + \dot{u}_i\dot{u}_i + \dot{\varphi}^2 + \dot{\psi}^2)\, dv\right\}^{1/2} \\ \Lambda &= \left\{\int_B (\mathbf{H}^2 + P^2 + Q^2)\, dv\right\}^{1/2} \end{aligned} \tag{53}$$

The next result establishes the continuous dependence of solutions upon initial data and body loads.

Theorem 2. *Let $(\mathbf{u}, \varphi, \psi)$ be a solution of the problem (A). Then there exist the positive constants M and N such that*

$$y(t) \le My(0) + N\int_0^t \Lambda(s)\, ds \qquad t \in [0, t_1] \tag{54}$$

Proof. Lemma 1 and the Schwarz inequality imply that

$$\dot{\Gamma} \le \Lambda\left\{\int_B (\dot{\mathbf{u}}^2 + \dot{\varphi}^2 + \dot{\psi}^2)\, dv\right\}^{1/2} \tag{55}$$

From Eqs. (53) and (55) we get

$$\dot{\Gamma} \le \Lambda y$$

so

$$\Gamma(t) \le \Gamma(0) + \int_0^t \Lambda(s)y(s)\, ds \qquad t \in [0, t_1] \tag{56}$$

In view of Eqs. (45) and (47) we obtain

$$\Gamma(t) \geq m_1 y^2(t) \quad \text{and} \quad \Gamma(0) \leq m_2 y^2(0) \qquad t \in [0, t_1] \tag{57}$$

where

$$m_1 = \frac{1}{2}\min\{1, 2k_1, r, p\} \qquad m_2 = \frac{1}{2}\max\{1, 2k_2, r, p\} \tag{58}$$

It follows from Eqs. (56) and (57) that

$$y^2(t) \leq M^2 y^2(0) + 2N \int_0^t \Lambda(s) y(s)\, ds \qquad t \in [0, t_1] \tag{59}$$

where

$$M = (m_2/m_1)^{1/2} \qquad N = 1/(2m_1)$$

The relation (59) and the Gronwall inequality imply the desired result.

Theorem 2 can be extended for anisotropic bodies and other kinds of boundary conditions.

The classical theory of thermoelastic materials with voids was presented in [10].

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