Nonlinear Anal. RWA 87 (2026) 104450

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Higher differentiability of minimizers for non-autonomous orthotropic functionals

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ARTICLE INFO

Keywords:
Anisotropic growth
Sobolev regularity
Higher differentiability

ABSTRACT

We establish the higher differentiability for the minimizers of the following non-autonomous integral functionals

$$\mathcal{F}(u,\Omega) := \int_\Omega \sum_{i=1}^n a_i(x)|u_{x_i}|^{p_i}\,dx,$$

with exponents $p_i \geq 2$ and with coefficients $a_i(x)$ that satisfy a suitable Sobolev regularity. The main result is obtained, as usual, by imposing a gap bound on the exponents p_i, which depends on the dimension and on the degree of regularity of the coefficients $a_i(x)$.

1. Introduction

In this paper, we investigate the differentiability properties of local minimizers of convex functionals, exhibiting an orthotropic structure and depending also on the x−variable. More precisely, we consider

$$\mathcal{F}(u,\Omega) := \int_\Omega f(x,Du)\,dx, \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open subset, $n \geq 2$, $u : \Omega \to \mathbb{R}$, and $f : \Omega \times \mathbb{R}^n \to [0,+\infty)$ has the following form

$$f = f(x,\xi) = \sum_{i=1}^n a_i(x)|\xi_i|^{p_i}.$$

Here $a_i(x)$ are bounded measurable coefficients such that $0 < \lambda \leq a_i(x) \leq \Lambda$ for some constants λ, Λ, for every $i = 1,2,\dots,n$ and a.e. $x \in \Omega$. For the exponents $2 \leq p_1 \leq p_2 \leq \dots \leq p_n$, we shall denote by $\mathbf{p} = (p_1,p_2,\dots,p_n)$ and by

$$W^{1,\mathbf{p}}_{\mathrm{loc}}(\Omega) = \left\{ u \in W^{1,1}_{\mathrm{loc}}(\Omega) : u_{x_i} \in L^{p_i}_{\mathrm{loc}}(\Omega),\ i = 1,\dots,n \right\},$$

the corresponding anisotropic Sobolev space.

One can easily check that there exist positive constants L, l depending on λ, Λ, p_i such that

$$\langle D_\xi f(x,\xi) - D_\xi f(x,\eta), \xi - \eta\rangle \geq l \sum_{i=1}^n (|\xi_i|^2 + |\eta_i|^2)^{\frac{p_i-2}{2}} |\xi_i - \eta_i|^2 \tag{A1}$$

$$|D_\xi f(x,\xi) - D_\xi f(x,\eta)| \leq L \sum_{i=1}^n (|\xi_i|^2 + |\eta_i|^2)^{\frac{p_i-2}{2}} |\xi_i - \eta_i| \tag{A2}$$

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https://doi.org/10.1016/j.nonrwa.2025.104450
Received 10 March 2025; Received in revised form 7 June 2025; Accepted 10 June 2025
Available online 3 July 2025
1468-1218/

for a.e. $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^n$. Concerning the dependence on the x-variable, we assume that

$$a_i(x) \in W^{1,r}_{\mathrm{loc}}(\Omega) \qquad \forall i = 1, \dots, n,$$

so that, denoting by $g(x) = \max_i \{|Da_i(x)|\}$, we have that $g(x) \in L^r_{\mathrm{loc}}(\Omega)$ and the following inequality

$$|D_\xi f(x,\xi) - D_\xi f(y,\xi)| \le |x-y|\,(g(x)+g(y)) \sum_{i=1}^{n} |\xi_i|^{p_i-1} \tag{A3}$$

holds for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$.

Let us recall the following definition of local minimizer.

Definition 1.1. A function $u \in W^{1,\mathbf{p}}_{\mathrm{loc}}(\Omega)$ is a local minimizer of (1.1) if, for every open subset $\tilde{\Omega} \Subset \Omega$, we have $\mathcal{F}(u;\tilde{\Omega}) \le \mathcal{F}(\varphi;\tilde{\Omega})$ for all $\varphi \in u + W^{1,\mathbf{p}}_0(\tilde{\Omega})$.

The energy densities f that satisfy anisotropic growth condition fit into the wider context of those with (p,q)−growth condition as follows

$$|\xi|^p \le f(x,\xi) \le (1+|\xi|^2)^{\frac{q}{2}}, \qquad 1 < p < q,$$

where $p = c(p_i)$ and $q = \max\{p_i, i = 1, \dots, n\}$.

The regularity theory for functionals with non-standard growth conditions was first developed by P. Marcellini in the pioneering works [1,2]. From the beginning it has been clear that local minimizers of functionals like $\mathcal{F}$ may become unbounded when the ratio $\frac{q}{p}$ is too large. Indeed, the gap $\frac{q}{p}$ cannot differ to much from 1 and sufficient conditions to the regularity can be expressed as

$$\frac{q}{p} \le c(n) \to 1 \text{ as } \; n \to \infty,$$

as proved by the counterexamples in [1–4]. Since then, a huge number of papers dealing with the (p,q)−growth problem have been published (see, for example [5–20]). However, since it is impossible to provide an exhaustive list of references, we refer the interested reader to the recent survey [21,22].

However, compared to these papers, the difference with our work lies in the fact that the behavior of our functional at (1.1) depends on the components of the gradient and although they exhibit (p,q)-growth, they need to be treated with appropriate arguments.

Actually, the regularity of local minimizers of functionals with orthotropic structure is a well-studied problem: for the specific case of subquadratic nonstandard growth, some results can be found in [7,23–26], for $f = 0$. In the case of a right-hand side $f \neq 0$, we refer to [25,27,28].

It has been recently proved in [24] (sub-quadratic case) and in [23] (superquadratic case) that any local minimizer $U \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ is such that $DU \in L^\infty_{\mathrm{loc}}(\Omega, \mathbb{R}^n)$ and $|U_{x_i}|^{\frac{p_i-2}{2}} U_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega)$, for $i = 1, \dots, n$. One of the main difficulties of functionals with orthotropic structure is that they are degenerate.

In [25] they even consider the following widely degenerate functional

$$F(u;\Omega) = \sum_{i=1}^{n} \int_\Omega \frac{1}{p_i} \left(|u_{x_i}| - \delta_i\right)^{p_i} dx + \int_\Omega f\,u\,dx,$$

with $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty(\Omega)$, $f \in W^{1,p'}_{\mathrm{loc}}(\Omega)$, and they proved the higher differentiability of the local minimizers, that is

$$\left(|u_{x_i}| - \delta_i\right)_+^{\frac{p_i}{2}} \frac{u_{x_i}}{|u_{x_i}|} \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, n.$$

In particular, for local weak solutions of the anisotropic orthotropic p-Laplace equation (i.e. the case $\delta_i = 0$), they get

$$|u_{x_i}|^{\frac{p_i-2}{2}} u_{x_i} \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad i = 1, \dots, n.$$

Here we consider the case $\delta_i = 0$, but differently from [23]–[25], we allow the presence of some coefficients $a_i(x)$ and we are interested in the conditions that must be imposed on the coefficients in order to achieve analogous higher differentiability results for the solutions. Our motivation comes from the recent paper by P. Marcellini [20], in which is clearly stated that currently there are few known regularity results for problems of this type, in relation to the properties of the coefficients. Indeed, as far as we know, few regularity results are available for the local minimizers of functionals with an orthotropic structure in the so-called non-autonomous case (see [29]). We aim to fill this gap by trying to identify how the regularity of the functions $a_i(x)$ influences that of the minimizer.

Actually, our main result consists in proving that a suitable Sobolev regularity of the coefficients $a_i(x)$ transfers into a higher differentiability for the local minimizers of (1.1).

We recall that recent studies have shown that the weak differentiability of the map $D_\xi f$, whether of integer or fractional order with respect to the x-variable, is a sufficient condition to achieve higher differentiability (see [17,30–32] for the case of Sobolev spaces with integer order and [33–37] for the fractional one) both in case of standard and non standard growth.

In particular, the general requirement for obtaining higher differentiability for local minimizers is a Sobolev-type regularity for the coefficients with an exponent r that is greater than or equal to the dimension n. When dealing with bounded minimizers, the situation changes, and sufficient assumptions on the Sobolev regularity of the coefficients can be made independently of the dimension (see [7,17,30,38]). Our result follows this approach. Even though the order of summability of the coefficients depends on the dimension n, it does so only through the bound that allows us to handle locally bounded minimizers.

In the statement our main result, we shall use the auxiliary function $V_p(\xi)$ defined by

$$V_p(\xi) := |\xi|^{\frac{p-2}{2}}\xi, \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$, the harmonic mean of p_i defined by

$$\frac{1}{\bar{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}$$

as well as its Sobolev conjugate $\bar{p}^* = \frac{n\bar{p}}{n-\bar{p}}$.

Our aim is to prove the following

Theorem 1.2. *Let $u \in W^{1,\mathbf{p}}(\Omega)$ be a local minimizer of (1.1) under assumptions (A1)–(A3) with exponents $p_i \geq 2, \forall i = 1,\dots,n$ such that*

$$p_n < \begin{cases} \min\{\bar{p}^*, p_1+2\} & \text{if } p_n < \bar{p}^*, \\ p_1 + 2 & \text{otherwise}, \end{cases} \tag{1.3}$$

and with a function $g \in L^r_{\mathrm{loc}}(\Omega)$, with r such that

$$r > p_n + 2. \tag{1.4}$$

Then

$$V_{p_i}(u_{x_i}) \in W^{1,2}_{\mathrm{loc}}(\Omega), \qquad \forall i = 1,\dots,n \tag{1.5}$$

and the following estimates

$$\sum_{i=1}^{n}\left(\int_{B_{R/4}}|u_{x_i}|^{p_i+2}dx\right) \leq c\left(\sum_{i=1}^{n}||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)}\right)^{\sigma} \tag{1.6}$$

and

$$\sum_{i=1}^{n}\left(\int_{B_{\frac{R}{4}}}|u_{x_ix_j}|^2|u_{x_i}|^{p_i-2}dx\right) \leq c\left(\sum_{i=1}^{n}||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)}\right)^{\sigma}, \tag{1.7}$$

hold for every pair of concentric balls $B_{R/4} \subset B_R \Subset \Omega$, where $c = c(n, p_i, \lambda, \Lambda, R)$ and $\sigma = \sigma(n, p_i, q)$ are positive constants.

Now, we briefly describe the strategy of the proof, which, as usual, will consist of an approximation argument, a uniform a priori estimate, and a passage to the limit. It is well known that the a priori estimate needs to be established for sufficiently regular minimizers. Therefore, the first step is to establish higher differentiability for minimizers of functionals with Lipschitz continuous coefficients, since this result is not available in the literature. As usual the higher differentiability is achieved with the use of the well known difference quotient methods that is frequently used in this context (see the very recent paper [1,9,21]).

More specifically, for a given $\varepsilon \geq 0$, we investigate the function

$$f_\varepsilon(x,\xi) = \sum_{i=1}^{n} b_i(x)|\xi_i|^{p_i} + \varepsilon\left(1 + |\xi|^{\frac{p_n}{2}}\right)^2$$

where the coefficients $b_i(x) : \Omega \to \mathbb{R}$ are non-negative and satisfy the following condition

$$L_i = \sup_{x,y\in\Omega, x\neq y}\frac{|b_i(x) - b_i(y)|}{|x-y|} < \infty.$$

The first goal is to analyze the regularity properties of minimizers of the following functional

$$\mathcal{F}_\varepsilon(u,\Omega) = \int_\Omega f_\varepsilon(x, Du)\,dx.$$

After, we aim to establish an uniform a priori estimate, which plays a crucial role in the proof of our main result. Finally the Theorem follows by proving that the a priori estimate is preserved in passing to the limit.

We briefly describe the structure of the paper. After summarizing some known results and fixing a few notations, we prove a lemma for approximating functionals to be used in the a priori estimates in Section 3. Eq. (4.3) is devoted to the proof of some a priori estimates for solutions to a family of approximating problems. Next, in Section 5, we pass to the limit in the approximating problems.

2. Preliminaries

In this section we introduce some notations and collect several results that we shall use to establish our main result. We denote by c or C a general constant that may vary on different occasions, even within the same line of estimates.

In what follows, $B(x,r) = B_r(x) = \{y \in \mathbb{R}^n : |y-x| < r\}$ will denote the ball centered at x of radius r. We will omit the indication of the center x when no confusion arises.

We recall here the well known Sobolev–Troisi inequalities [29,39].

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be a bounded open set and consider $u \in W_0^{1,\mathbf{p}}(E)$, $p_i \geq 1$ for all $i = 1,\dots,n$, and set*

$$\frac{1}{\overline{p}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}, \qquad \overline{p}^* = \frac{n\overline{p}}{n-\overline{p}}. \tag{2.1}$$

If $\overline{p} < n$, then there exists a positive constant γ depending only on the parameters $\{n, p_1, \dots, p_n\}$, such that

$$||u||_{\overline{p}*} \leq \gamma \sum_{i=1}^{n} ||u_{x_i}||_{p_i}.$$

If $\overline{p} = n$, for every $1 \leq r < \infty$ there exists a positive constant γ_1 depending only on $\{n, r, p_1, \dots, p_n\}$, such that

$$||u||_r \leq \gamma_1 \left[\sum_{i=1}^{n} ||u_{x_i}||_{p_i} + ||u||_1\right].$$

If $\overline{p} > n$, there exists a positive constant γ_2 depending only on $\{n, p_1, \dots, p_n\}$, such that

$$||u||_\infty \leq \gamma_2 \left[\sum_{i=1}^{n} ||u_{x_i}||_{p_i} + ||u||_1\right].$$

Now we recall a well-known iteration lemma (see [40, Lemma 6.1]).

Lemma 2.2. *Let $Z(t) : [\rho, R] \to \mathbb{R}$ be a bounded nonnegative function, where $R > 0$. Assume that for $\rho \leq t < s \leq R$ it holds*

$$Z(t) \leq \theta Z(s) + A(s-t)^{-\alpha} + B(s-t)^{-\beta} + C$$

where $\theta \in (0,1)$, A, B, $C \geq 0, \alpha > \beta > 0$ are constants. Then there exists a constant $c = c(\alpha,\theta,\beta)$ such that

$$Z(\rho) \leq c\bigg(A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C\bigg).$$

At this point, we recall a result that will become significant later (for the proof see [23, Proposition 4.3]).

Proposition 2.3. *Let $u \in W^{1,\mathbf{p}}_{\mathrm{loc}}(\Omega)$ and assume that there exists $k \in \{1,2,\dots,n\}$ such that*

$$|u_{x_k x_k}|^2 |u_{x_k}|^{p_k-2} \in L^1_{\mathrm{loc}}(\Omega).$$

If $u \in L^\infty_{\mathrm{loc}}(\Omega)$, then $u_{x_k} \in L^{p_k+2}_{\mathrm{loc}}(\Omega)$ and there exist constants $C = C(n,p_k) > 0$ and $\gamma = \gamma(p_k)$ such that for every pair of concentric balls $B_\rho \subset B_R \Subset \Omega$, it holds the following

$$\int_{B_\rho} |u_{x_k}|^{p_k+2} dx \leq C||u||^2_\infty \left(\int_{B_R} |u_{x_k x_k}|^2 |u_{x_k}|^{p_k-2} dx + \left(\frac{1}{R-\rho}\right)^\gamma \int_{B_R} |u_{x_k}|^{p_k} dx\right). \tag{2.2}$$

We shall use the following local boundedness result for minimizers of the anisotropic functional defined at (1.1), whose proof can be found in [41].

Theorem 2.4. *Let the integrand $f = f(x,\xi)$ in (1.1) satisfy the following*

$$f(x,\xi) = F(x, |\xi_1|, \dots, |\xi_i|, \dots, |\xi_n|), \tag{2.3}$$

$$f(x,\lambda\xi) \leq \lambda^\mu f(x,\xi), \text{ for some } \mu > 1 \text{ and for every } \lambda > 1.$$

If $p_n < \bar{p}^*$, where $\bar{p}^*$ is the Sobolev exponent of p ($\bar{p}$ defined at (2.1)), then every local minimizer u of (1.1) is locally bounded.

For the function $V_p(\xi)$, defined at (1.2), we recall the following estimate (see e.g. [40, Lemma 8.3]).

Lemma 2.5. *Let $1 < p < +\infty$. There exists a constants $c_1 = c(n,p) > 0$ and $c_2 = c(n,p) > 0$ such that*

$$c_1(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{|V_p(\xi) - V_p(\eta)|^2}{|\xi-\eta|^2} \leq c_2(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for any $\xi, \eta \in \mathbb{R}^m$, $\xi \neq \eta$.

For further needs we state the following Lemma for the function $Z_\delta(\xi) = (1+|\xi|^2)^{\frac{\delta}{2}}\xi$. the proof is obtained by combining [42, formula 2.1] and [43, Lemma 2.2].

Lemma 2.6. *Let $\delta \geq 0$. There exists a constant $c_1 = c_1(\delta) > 0$ such that*

$$c_1\langle Z_\delta(\xi) - Z_\delta(\eta), \xi - \eta\rangle \geq |\xi - \eta|^2(1 + |\xi|^2 + |\eta|^2)^{\frac{\delta}{2}}$$

We conclude this section with an algebraic inequality (see [40,44])

Lemma 2.7. *For any $\alpha > 0$, there exists a constant $c = c(\alpha)$ such that, for all $\eta, \zeta \in \mathbb{R}^n \setminus \{0\}$, $n \in \mathbb{N}$, we have*

$$\frac{1}{c}\left||\eta|^{\alpha-1}\eta - |\zeta|^{\alpha-1}\zeta\right| \leq (|\eta| + |\zeta|)^{\alpha-1}|\eta - \zeta| \leq c\left||\eta|^{\alpha-1}\eta - |\zeta|^{\alpha-1}\zeta\right|.$$

2.1. Difference quotient

In this section we recall some properties of the finite difference quotient operator that will be needed in the sequel.

Definition 2.8. Let F be a function defined in an open set $\Omega \subset \mathbb{R}^n$, let h be a real number, the finite difference operator $\tau_{s,h}F(x)$ is defined as follows

$$\tau_{s,h}F(x) := F(x + he_s) - F(x),$$

where e_s denotes the direction of the x_s axis.

The function $\tau_{s,h}F$ is defined in the set

$$\Omega_{|h|} := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > |h|\} = \{x \in \Omega : x + he_s \in \Omega\}.$$

We start with the description of some elementary properties that can be found, for example, in [40]. When no confusion can arise, we shall omit the index s, and we shall write simply τ_h instead of $\tau_{s,h}$

Proposition 2.9. *Let $F \in W^{1,p}(\Omega)$, with $p \geq 1$, and let $G : \Omega \to \mathbb{R}$ be a measurable function. Then*
(i) $\tau_h F \in W^{1,p}(\Omega_{|h|})$ and

$$D_i(\tau_h F) = \tau_h(D_i F).$$

(ii) If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then

$$\int_\Omega F\tau_h G\,dx = -\int_\Omega G\tau_{-h}F\,dx.$$

(iii) We have

$$\tau_h(FG)(x) = F(x + h)\tau_h G(x) + G(x)\tau_h F(x).$$

The next result about the finite difference operator is a kind of integral version of Lagrange Theorem.

Lemma 2.10. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$ and $F,\ D_sF \in L^p(B_R)$, then*

$$\int_{B_\rho} |\tau_{s,h}F(x)|^p dx \leq c(n,p)|h|^p \int_{B_R} |D_sF(x)|^p dx.$$

Moreover,

$$\int_{B_\rho} |F(x + h)|^p dx \leq \int_{B_R} |F(x)|^p dx.$$

We now state a lemma that will be needed in the sequel.

Lemma 2.11. *Let $F \in L^2(B_R)$. Suppose that there exist $\rho \in (0, R)$, and $M > 0$ such that*

$$\int_{B_\rho} |\tau_h F(x)|^2 dx \leq M^2|h|^2,$$

for every h such that $|h| < \frac{R-\rho}{2}$. Then $F \in L^{\frac{2n}{n-2}}(B_\rho)$ and

$$||F||_{L^{\frac{2n}{n-2}}(B_\rho)} \leq c(M + ||F||_{L^2(B_R)}),$$

with $c = c(n, R, \rho)$.

We conclude by recalling the following

Lemma 2.12. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < p < +\infty$ and $F \in L^p(B_R)$. If there exists a positive constant C such that*

$$\int_{B_\rho} |\tau_{s,h} F(x)|^p dx \leq C|h|^p, \tag{2.4}$$

for every h, then the distributional derivative $D_s F$ belongs to $L^p(B_\rho)$.

Moreover it holds

$$\int_{B_\rho} |DF(x)|^p dx \leq C. \tag{2.5}$$

3. A preliminary regularity result

In this section we prove a regularity result for more regular problems with respect to (1.1) that will be needed in the approximation procedure. More precisely, for $\varepsilon \geq 0$ we consider

$$f_\varepsilon(x,\xi) = \sum_{i=1}^n b_i(x)|\xi_i|^{p_i} + \varepsilon(1 + |\xi|^{\frac{p_n}{2}})^2 \tag{3.1}$$

with Lipschitz non-negative continuous coefficients $b_i(x) : \Omega \to \mathbb{R}$, $i = 1, \dots, n$, i.e., the following condition holds

$$L_i = \sup_{x,y\in\Omega, x\neq y} \frac{|b_i(x) - b_i(y)|}{|x-y|} < \infty. \tag{3.2}$$

Setting $\tilde{f}(x,\xi) = \sum_{i=1}^n b_i(x)|\xi_i|^{p_i}$, one can easily check that

$$|D_\xi \tilde{f}(x,\xi) - D_\xi \tilde{f}(y,\xi)| \leq K|x-y| \sum_{i=1}^n |\xi_i|^{p_i - 1} \tag{A3'}$$

for a.e. $x, y \in \Omega$ and every $\xi \in \mathbb{R}^n$, where $K = K(L_i, p_i)$.

We recall a Lipschitz regularity Theorem for the minimizers of the functional

$$\mathcal{F}_\varepsilon(x, Dv) = \int_\Omega f_\varepsilon(x, Dv),$$

that can be deduced by [45, Theorem 2.2] in the case $p = q = p_n$.

Theorem 3.1. *Let $\tilde{f}$ satisfy (A1)–(A3') and let $u_\varepsilon \in W^{1,p_n}(\Omega)$ be a local minimizer of*

$$\mathcal{F}_\varepsilon(u,\Omega) := \int_\Omega f_\varepsilon(x, Du)dx. \tag{3.3}$$

Then $u_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ and the following estimate

$$||Du_\varepsilon||_{L^\infty(B_\rho)} \leq C\left[1 + \int_{B_R} f_\varepsilon(x, Du_\varepsilon)dx\right]^{\frac{\sigma}{p_1}}, \tag{3.4}$$

holds for every pair of concentric balls $B_\rho \subset B_R \Subset \Omega$, with C and σ positive constants depending on n, p_n, ρ, ε and R.

We would like to notice that, since the function f_ε satisfies standard p_n-growth, in Theorem 3.1 we do not need to require that

$$\frac{p_n}{p_1} < 1 + \frac{1}{n},$$

as done in [45].

We shall use the higher differentiability of the minimizers u_ε of $\mathcal{F}_\varepsilon$ that, as far we know, is not available in literature and that is contained in the following

Lemma 3.2. *Let f satisfy (A1), (A2) and (A3') for exponents $p_i \geq 2, \forall i = 1, \dots, n$ and let $u_\varepsilon \in W^{1,p_n}_{\mathrm{loc}}(\Omega)$ be a local minimizer of the functional (3.3). Then we have*

$$(V_{p_i}((u_\varepsilon)_{x_j})) \in W^{1,2}_{\mathrm{loc}}(\Omega), \quad \forall i \in \{1,\dots,n\} \text{ and } \forall j \in \{1,\dots,n\}.$$

Proof. We start by observing that u_ε solves the following Euler–Lagrange system

$$\int_\Omega \langle f_\xi(x, Du_\varepsilon(x)) + \varepsilon(1 + |Du_\varepsilon|^2)^{\frac{p_n - 2}{2}} Du_\varepsilon, D\varphi\rangle\, dx = 0, \tag{3.5}$$

for every $\varphi \in W^{1,p_n}_0(\Omega, \mathbb{R}^n)$.

Fix a ball $B_R \Subset \Omega$ and consider radii $\rho < t < R$, a cut-off function $\eta \in C^\infty_0(B_t)$, with $\eta = 1$ on B_ρ, $0 \leq \eta \leq 1$, $|D\eta| \leq \frac{c}{t-\rho}$ and $|h| \leq \frac{R-t}{2}$. We test (3.5) with the function

$$\varphi = \tau_{j,-h}\left(\eta^2 \tau_{j,h}(u_\varepsilon)\right)$$

thus obtaining

$$\int_\Omega \langle f_\xi(x, Du_\varepsilon(x)) + \varepsilon(1+|Du_\varepsilon|^2)^{\frac{p_n-2}{2}} Du_\varepsilon, \tau_{j,-h} D\left(\eta^2 \tau_{j,h}(u_\varepsilon)\right)\rangle\, dx = 0,$$

and hence, by (ii) of Proposition 2.9,

$$\int_\Omega \langle \tau_{j,h} f_\xi(x, Du_\varepsilon(x)) + \varepsilon \tau_{j,h}(1+|Du_\varepsilon|^2)^{\frac{p_n-2}{2}} Du_\varepsilon, D\left(\eta^2 \tau_{j,h}(u_\varepsilon)\right)\rangle\, dx = 0. \tag{3.6}$$

Since

$$D(\eta^2 \tau_{j,h}(u_\varepsilon)) = 2\eta D\eta \tau_{j,h}(u_\varepsilon) + \eta^2 \tau_{j,h} Du_\varepsilon$$

and

$$\begin{aligned}\tau_{j,h} f_\xi(x, Du_\varepsilon(x)) + \varepsilon \tau_{j,h}\left((1+|Du_\varepsilon|^2)^{\frac{p_n-2}{2}} Du_\varepsilon\right) &= f_\xi(x+e_j h, Du_\varepsilon(x+e_j h)) - f_\xi(x+e_j h, Du_\varepsilon(x)) \\ &\quad + f_\xi(x+e_j h, Du_\varepsilon(x)) - f_\xi(x, Du_\varepsilon(x)) \\ &\quad + \varepsilon \tau_{j,h}\left((1+|Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} Du_\varepsilon(x)\right),\end{aligned}$$

we may rewrite equality (3.6) as follows

$$\begin{aligned}0 &= \int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x+e_j h)) - f_\xi(x+e_j h, Du_\varepsilon(x)), \eta^2 \tau_{j,h} Du_\varepsilon\rangle\, dx \\ &\quad + 2\int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x+e_j h)) - f_\xi(x+e_j h, Du_\varepsilon(x)), \eta D\eta \tau_{j,h}(u_\varepsilon)\rangle\, dx \\ &\quad + \int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x)) - f_\xi(x, Du_\varepsilon(x)), 2\eta D\eta \tau_{j,h}(u_\varepsilon)\rangle\, dx \\ &\quad + \int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x)) - f_\xi(x, Du_\varepsilon(x)), \eta^2 \tau_{j,h} Du_\varepsilon\rangle\, dx \\ &\quad + 2\varepsilon \int_\Omega \langle \tau_{j,h}\left((1+|Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} Du_\varepsilon(x)\right), \eta D\eta \tau_{j,h}(u_\varepsilon)\rangle\, dx \\ &\quad + \varepsilon \int_\Omega \langle \tau_{j,h}\left((1+|Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} Du_\varepsilon(x)\right), \eta^2 \tau_{j,h} Du_\varepsilon\rangle\, dx \\ &=: J_1 + J_2 + J_3 + J_4 + J_5 + J_6.\end{aligned} \tag{3.7}$$

Therefore

$$J_1 + J_6 \le |J_2| + |J_3| + |J_4| + |J_5|. \tag{3.8}$$

From (A1), we infer

$$\begin{aligned}J_1 &= \int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x+e_j h)) - f_\xi(x+e_j h, Du_\varepsilon(x)), \eta^2 \tau_{j,h} Du_\varepsilon\rangle\, dx \\ &\ge l \int_\Omega \eta^2 \sum_{i=1}^n \left(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)_{x_i}|^2 dx =: l\,\mathbf{RHS}\end{aligned} \tag{3.9}$$

The integral J_6 can be estimated by using Lemma 2.6 for $\delta = p_n - 2$, as follows

$$J_6 \ge \frac{\varepsilon}{c}\int_\Omega \eta^2 (1+|Du_\varepsilon(x+he_j)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} |\tau_{j,h}(Du_\varepsilon(x))|^2 =: \frac{\varepsilon}{c}\overline{\mathbf{RHS}}, \tag{3.10}$$

where $c = c(p_n)$ is the constant appearing in Lemma 2.6.

Now we consider the term $|J_2|$. By hypothesis (A2) and by Young's inequality, we get

$$\begin{aligned}|J_2| &= 2\left|\int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x+e_j h)) - f_\xi(x+e_j h, Du_\varepsilon(x)), \eta\, \eta_{x_i} \tau_{j,h}(u_\varepsilon)\rangle\, dx\right| \\ &\le \beta \underbrace{\int_\Omega \eta^2 \sum_{i=1}^n \left(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)_{x_i}|^2 dx}_{\mathbf{RHS}} \\ &\quad + c_\beta \int_\Omega \sum_{i=1}^n \left(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)|^2\, ||\eta_{x_i}||^2 dx \\ &\le \beta\,\mathbf{RHS} + \frac{c_\beta}{(t-\rho)^2}\int_{B_t} \sum_{i=1}^n |(u_\varepsilon)_{x_i}(x+e_j h) + (u_\varepsilon)_{x_i}(x)|^{p_i-2} ||\tau_{j,h}(u_\varepsilon)||^2 dx \\ &\le \beta\,\mathbf{RHS} + \frac{c_\beta(R)}{(t-\rho)^2}|h|^2 \sum_{i=1}^n ||(u_\varepsilon)_{x_i}||^{p_i-2}_{L^\infty(B_R)} \left(\int_{B_R} |(u_\varepsilon)_{x_j}(x)|^{p_j} dx\right)^{\frac{2}{p_j}},\end{aligned} \tag{3.11}$$

where to estimate the last integral in the right-hand side we used Theorem 3.1, Hölder's inequality and Lemma 2.10 .

Now, we turn our attention to $|J_3|$. From condition (A3') and the properties of η, we get

$$
\begin{aligned}
|J_3| &= \left|\int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x)) - f_\xi(x, Du_\varepsilon(x)), 2\eta D\eta \tau_{j,h}(u_\varepsilon)\rangle \, dx\right| \\
&\le |h|\, K \sum_{i=1}^n \int_\Omega 2\eta |\eta_{x_i}| |\tau_{j,h}(u_\varepsilon)|\, |(u_\varepsilon)_{x_i}|^{p_i-1} dx \\
&\le |h| \frac{c(K)}{t-\rho} \sum_{i=1}^n \left(\int_{B_t} |\tau_{j,h}(u_\varepsilon)|\, |(u_\varepsilon)_{x_i}|^{p_i-1} dx \right) \\
&\le |h|^2 \frac{c(K,R)}{t-\rho} \sum_{i=1}^n ||(u_\varepsilon)_{x_i}||^{p_i-1}_{L^\infty(B_R)} \left(\int_{B_R} |(u_\varepsilon)_{x_j}(x)|^{p_j} dx \right)^{\frac{1}{p_j}},
\end{aligned} \tag{3.12}
$$

where, as before, we used Theorem 3.1, Hölder's inequality and Lemma 2.10 .

Similarly as for the estimate of J_3, from condition (A3') and the properties of η, we obtain

$$
\begin{aligned}
|J_4| &= \left|\int_\Omega \langle f_\xi(x+e_j h, Du_\varepsilon(x)) - f_\xi(x, Du_\varepsilon(x)), \eta^2 \tau_{j,h} Du_\varepsilon \rangle \, dx\right| \\
&= \left|\int_\Omega \sum_{i=1}^n \eta^2 \Big(a_i(x+e_j h) - a_i(x) \Big) |(u_\varepsilon)_{x_i}|^{p_i-2} (u_\varepsilon)_{x_i} \, \tau_{j,h}(u_\varepsilon)_{x_i} dx\right| \\
&\le c|h|\, K \sum_{i=1}^n \int_\Omega \eta^2 |\tau_{j,h}(u_\varepsilon)_{x_i}|\, |(u_\varepsilon)_{x_i}|^{p_i-1} dx \\
&\le \beta \underbrace{\int_\Omega \eta^2 \sum_{i=1}^n \Big(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2 \Big)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)_{x_i}|^2 dx}_{\mathbf{RHS}} \\
&\quad + c_\beta |h|^2 \sum_{i=1}^n \int_{B_t} \eta^2 |(u_\varepsilon)_{x_i}|^{p_i} \, dx \\
&\le \beta \mathbf{RHS} + c_\beta |h|^2 \sum_{i=1}^n \int_{B_t} |(u_\varepsilon)_{x_i}|^{p_i} \, dx.
\end{aligned} \tag{3.13}
$$

Now, we take care of $|J_5|$. From Lemma 2.5, Young's and Hölder's inequalities with exponents $\frac{p_n}{2}$ and $\frac{p_n}{p_n-2}$ and the properties of η, we get

$$
\begin{aligned}
|J_5| &\le 2\varepsilon \int_\Omega |\tau_{j,h}((1+|Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} Du_\varepsilon(x))|\, |\eta| |D\eta| |\tau_{j,h}(u_\varepsilon)| \, dx \\
&\le c\varepsilon \int_\Omega (1+|Du_\varepsilon(x+he_j)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} |\tau_{j,h}(Du_\varepsilon(x))|\, \eta |D\eta| |\tau_{j,h}(u_\varepsilon)| \, dx \\
&\le \underbrace{\frac{\varepsilon}{2c} \int_\Omega \eta^2 (1+|Du_\varepsilon(x+he_j)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} |\tau_{j,h}(Du_\varepsilon(x))|^2 \, dx}_{=\overline{\mathbf{RHS}}} \\
&\quad + \frac{c\varepsilon}{(t-\rho)^2} \int_{B_t} (1+|Du_\varepsilon(x+he_j)|^2 + |Du_\varepsilon(x)|^2)^{\frac{p_n-2}{2}} |\tau_{j,h}(u_\varepsilon)|^2 \, dx \\
&\le \frac{\varepsilon}{2c} \overline{\mathbf{RHS}} + \frac{c\varepsilon}{(t-\rho)^2} \left(\int_{B_t} (1+|Du_\varepsilon(x+he_j)| + |Du_\varepsilon(x)|)^{p_n} dx \right)^{\frac{p_n-2}{p_n}} \left(\int_{B_t} |\tau_{j,h}(u_\varepsilon)|^{p_n} dx \right)^{\frac{2}{p_n}} \\
&\le \frac{\varepsilon}{2c} \overline{\mathbf{RHS}} + \frac{c\varepsilon}{(t-\rho)^2} |h|^2 \left(\int_{B_R} (1+|Du_\varepsilon(x)|)^{p_n} dx \right)^{\frac{p_n-2}{p_n}} \left(\int_{B_t} |Du_\varepsilon|^{p_n} dx \right)^{\frac{2}{p_n}} \\
&\le \frac{\varepsilon}{2c} \overline{\mathbf{RHS}} + \frac{c\varepsilon}{(t-\rho)^2} |h|^2 \int_{B_R} (1+|Du_\varepsilon(x)|)^{p_n} dx,
\end{aligned} \tag{3.14}
$$

where we used Lemma 2.10.

Inserting (3.9), (3.10), (3.11), (3.12), (3.13), (3.14) in (3.8) and reabsorbing the term $\overline{\mathbf{RHS}}$, we obtain

$$
\begin{aligned}
l \int_\Omega \eta^2 \sum_{i=1}^n \Big(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2 \Big)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)_{x_i}|^2 dx + \frac{\varepsilon}{2c} \overline{\mathbf{RHS}} \\
\le 2\beta \int_\Omega \eta^2 \sum_{i=1}^n \Big(|(u_\varepsilon)_{x_i}(x+e_j h)|^2 + |(u_\varepsilon)_{x_i}(x)|^2 \Big)^{\frac{p_i-2}{2}} |\tau_{j,h}(u_\varepsilon)_{x_i}|^2 dx
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{c_\beta}{(t-\rho)^2}|h|^2\sum_{i=1}^n ||u_{x_i}||^{p_i-2}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{2}{p_j}}\\
&+\frac{c(K)}{(t-\rho)}|h|^2\sum_{i=1}^n ||u_{x_i}||^{p_i-1}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{1}{p_j}}\\
&+c_\beta|h|^2\sum_{i=1}^n\int_{B_t}|(u_\varepsilon)_{x_i}|^{p_i}\,dx\\
&+\frac{c\varepsilon}{(t-\rho)^2}|h|^2\int_{B_t}(1+|Du_\varepsilon(x)|)^{p_n}\,dx. \qquad (3.15)
\end{aligned}
$$

Since $\overline{\mathbf{RHS}}$ is non-negative, we obviously have

$$
\begin{aligned}
&\frac{l}{2}\int_\Omega \eta^2\sum_{i=1}^n\left(|(u_\varepsilon)_{x_i}(x+e_jh)|^2+|(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}}|\tau_{j,h}(u_\varepsilon)_{x_i}|^2dx\\
&\le\frac{l}{2}\int_\Omega \eta^2\sum_{i=1}^n\left(|(u_\varepsilon)_{x_i}(x+e_jh)|^2+|(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}}|\tau_{j,h}(u_\varepsilon)_{x_i}|^2dx+\frac{\varepsilon}{2c}\overline{\mathbf{RHS}}. \qquad (3.16)
\end{aligned}
$$

Hence, from (3.15) and (3.16), choosing $\beta=\frac{l}{4}$ and reabsorbing the first term in the right-hand side by the left-hand side, we get

$$
\begin{aligned}
&\frac{l}{2}\int_\Omega \eta^2\sum_{i=1}^n\left(|(u_\varepsilon)_{x_i}(x+e_jh)|^2+|(u_\varepsilon)_{x_i}(x)|^2\right)^{\frac{p_i-2}{2}}|\tau_{j,h}(u_\varepsilon)_{x_i}|^2dx\\
&\le\frac{c}{(t-\rho)^2}|h|^2\sum_{i=1}^n||(u_\varepsilon)_{x_i}||^{p_i-2}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{2}{p_j}}\\
&\quad+\frac{c}{(t-\rho)}|h|^2\sum_{i=1}^n||(u_\varepsilon)_{x_i}||^{p_i-1}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{1}{p_j}}\\
&\quad+c|h|^2\sum_{i=1}^n\int_{B_t}|(u_\varepsilon)_{x_i}|^{p_i}\,dx+\frac{c\varepsilon}{(t-\rho)^2}|h|^2\int_{B_t}(1+|Du_\varepsilon(x)|)^{p_n}\,dx, \qquad (3.17)
\end{aligned}
$$

with a constant $c=c(l,K,R)$.

Applying Lemma 2.5 in the left-hand side of (3.17) and using that $\eta=1$ on B_ρ, we get

$$
\begin{aligned}
\sum_{i=1}^n\int_{B_\rho}|\tau_{j,h}V_{p_i}((u_\varepsilon)_{x_i})|^2dx\le|h|^2\Bigg\{&\frac{c}{(t-\rho)^2}\sum_{i=1}^n||(u_\varepsilon)_{x_i}||^{p_i-2}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{2}{p_j}}\\
&+\frac{c}{(t-\rho)}\sum_{i=1}^n||(u_\varepsilon)_{x_i}||^{p_i-1}_{L^\infty(B_R)}\left(\int_{B_R}|(u_\varepsilon)_{x_j}(x)|^{p_j}\,dx\right)^{\frac{1}{p_j}}\\
&+c\sum_{i=1}^n\int_{B_t}|(u_\varepsilon)_{x_i}|^{p_i}\,dx\\
&+\frac{c\varepsilon}{(t-\rho)^2}\int_{B_R}(1+|Du_\varepsilon(x)|)^{p_n}\,dx\Bigg\}.
\end{aligned}
$$

and so

$$
\sum_{i=1}^n\int_{B_\rho}|\tau_{j,h}V_{p_i}((u_\varepsilon)_{x_i})|^2dx\le c|h|^2\left[\sum_{i=1}^n||(u_\varepsilon)_{x_i}||_{L^{p_i}(B_R)}+\varepsilon\int_{B_R}(1+|Du_\varepsilon(x)|)^{p_n}\,dx\right]^\gamma, \qquad (3.18)
$$

for positive constants $c=c(n,p_i,\rho,R,\lambda,K,||Du||_\infty)$. Then we have that

$$
V_{p_i}((u_\varepsilon)_{x_i})\in W^{1,2}_{\mathrm{loc}}(\Omega),\qquad \forall i=1,\dots,n
$$

which implies in particular that

$$
|(u_\varepsilon)_{x_ix_j}|^2|(u_\varepsilon)_{x_i}|^{p_i-2}\in L^1_{\mathrm{loc}}(\Omega),\qquad \forall i=1,\dots,n \text{ and } \forall j=1,\dots,n. \qquad \square
$$

Remark 3.3. For further needs, we record that, by virtue of Lemma 3.2 it holds in particular that $|(u_\varepsilon)_{x_ix_i}|^2|(u_\varepsilon)_{x_i}|^{p_i-2}\in L^1_{\mathrm{loc}}(\Omega)$. Therefore, since by Theorem 3.1 we have that $u_\varepsilon\in L^\infty_{\mathrm{loc}}(\Omega)$, we are legitimate to use Proposition 2.3 to deduce that

$$
(u_\varepsilon)_{x_i}\in L^{p_i+2}_{\mathrm{loc}}(\Omega),\qquad \forall i=1,\dots,n.
$$

4. A priori estimates

The main aim of this section is to establish the following a priori estimate which is the main step in the proof of our main result.

Theorem 4.1. *Let $u \in W^{1,\mathbf{p}}(\Omega)$ be a local minimizer of (1.1) under assumptions (A1)–(A3) with exponents $p_i \geq 2, \forall i = 1, \dots, n$ such that*

$$p_n < \begin{cases} \min\{\bar{p}^*, p_1 + 2\} & \text{if } p_n < \bar{p}^*, \\ p_1 + 2 & \text{otherwise,} \end{cases} \tag{4.1}$$

and with a function $g \in L^r_{\mathrm{loc}}(\Omega)$, with r such that

$$r > p_n + 2. \tag{4.2}$$

If we assume that

$$V_{p_i}(u_{x_i}) \in W^{1,2}_{\mathrm{loc}}(\Omega), \qquad \forall i = 1, \dots, n \tag{4.3}$$

then the following estimates

$$\sum_{i=1}^n \left(\int_{B_{R/4}} |u_{x_i}|^{p_i+2} dx \right) \leq c \left(\sum_{i=1}^n ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^\sigma \tag{4.4}$$

and

$$\bar{c} \sum_{i=1}^n \left(\int_{B_{\frac{R}{4}}} |u_{x_i x_j}|^2 |u_{x_i}|^{p_i-2} dx \right) \leq c \left(\sum_{i=1}^n ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^\sigma, \tag{4.5}$$

hold for every pair of concentric balls $B_{R/4} \subset B_R \Subset \Omega$, where $c = c(n, p_i, \lambda, \Lambda, R)$ and $\sigma = \sigma(n, p_i, q)$ are positive constants.

Proof. Fix a ball $B_R \Subset \Omega$ and consider radii $\frac{R}{4} < \rho < s < t < t' < r < R$, a cut-off function $\eta \in C^\infty_0(B_t)$, with $\eta = 1$ on B_s, $0 \leq \eta \leq 1$, $|D\eta| \leq \frac{c}{t-s}$ and $|h| \leq \frac{t'-t}{2}$. We test the Euler–Lagrange equation of (1.1) with the function

$$\varphi = \tau_{j,-h}(\eta^2 \tau_{j,h} u),$$

thus getting

$$\begin{aligned} 0 = &\int_\Omega \langle f_\xi(x + e_j h, Du(x + e_j h)) - f_\xi(x + e_j h, Du(x)), \eta^2 \tau_{j,h} Du \rangle \, dx \\ &+ \int_\Omega \langle f_\xi(x + e_j h, Du(x + e_j h)) - f_\xi(x + e_j h, Du(x)), 2\eta D\eta \tau_{j,h} u \rangle \, dx \\ &+ \int_\Omega \langle f_\xi(x + e_j h, Du(x)) - f_\xi(x, Du(x)), 2\eta D\eta \tau_{j,h} u \rangle \, dx \\ &+ \int_\Omega \langle f_\xi(x + e_j h, Du(x)) - f_\xi(x, Du(x)), \eta^2 \tau_{j,h} Du \rangle \, dx \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

From the previous equality, we deduce that

$$I_1 \leq |I_2| + |I_3| + |I_4|. \tag{4.6}$$

By virtue of assumption (A1), we infer

$$\begin{aligned} I_1 &= \int_\Omega \langle f_\xi(x + e_j h, Du(x + e_j h)) - f_\xi(x + e_j h, Du(x)), \eta^2 \tau_{j,h} Du \rangle \, dx \\ &\geq l \int_\Omega \eta^2 \sum_{i=1}^n \left(|u_{x_i}(x + e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx =: l\mathbf{RHS}. \end{aligned} \tag{4.7}$$

Now we consider the term $|I_2|$. By using hypothesis (A2), Young's inequality, we get

$$\begin{aligned} |I_2| &= \left| \int_\Omega \langle f_\xi(x + e_j h, Du(x + e_j h)) - f_\xi(x + e_j h, Du(x)), 2\eta \, \eta_{x_i} \tau_{j,h} u \rangle \, dx \right| \\ &\leq 2L \int_\Omega \eta \sum_{i=1}^n \left(|u_{x_i}(x + e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}| |\tau_{j,h} u| |\eta_{x_i}| dx \\ &\leq 2\varepsilon \underbrace{\int_\Omega \eta^2 \sum_{i=1}^n \left(|u_{x_i}(x + e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx}_{\mathbf{RHS}} \\ &\quad + c_\varepsilon \int_\Omega \sum_{i=1}^n \left(|u_{x_i}(x + e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u|^2 \, |\eta_{x_i}|^2 dx \\ &\leq 2\varepsilon \, \mathbf{RHS} + \frac{c_\varepsilon}{(t-s)^2} \int_{B_t} \sum_{i=1}^n \left(|u_{x_i}(x + e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u|^2 dx \end{aligned}$$

$$\leq 2\varepsilon\,\mathbf{RHS} + \frac{c_\varepsilon}{(t-s)^2}\sum_{i=1}^{n}\Big(\int_{B_t}\Big(|u_{x_i}(x+e_jh)|^2+|u_{x_i}(x)|^2\Big)^{\frac{p_i}{2}}dx\Big)^{\frac{p_i-2}{p_i}}\cdot$$
$$\cdot\Big(\int_{B_t}|\tau_{j,h}u|^{p_i}dx\Big)^{\frac{2}{p_i}}$$
$$\leq 2\varepsilon\,\mathbf{RHS} + \frac{c_\varepsilon}{(t-s)^2}\sum_{i=1}^{n}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{p_i-2}{p_i}}\cdot\Big(\int_{B_t}|\tau_{j,h}u|^{p_i}dx\Big)^{\frac{2}{p_i}}, \tag{4.8}$$

where in the last two inequalities we used that $|D\eta|\leq\frac{c}{t-s}$, Hölder's inequality and Lemma 2.10. For the estimate of the last term in (4.8), we have to distinguish between two cases, as done in [25].

Case 1: $\mathbf{i\leq j\leq n}$. In this case we have that $p_i\leq p_j$ and this allows us to Hölder's inequality as follows

$$\int_{B_t}|\tau_{j,h}u|^{p_i}dx\leq c(R,p_i,p_j)\left(\int_{B_t}|\tau_{j,h}u|^{p_j}dx\right)^{\frac{p_i}{p_j}}\leq c(R,p_i,p_j)|h|^{p_i}\left(\int_{B_{t'}}|u_{x_j}|^{p_j}dx\right)^{\frac{p_i}{p_j}}. \tag{4.9}$$

Case 2: $\mathbf{j+1\leq i\leq n}$. Now, we have that $p_i\geq p_j$ but, by assumption (4.1), we have $p_i\leq p_n\leq p_1+2\leq p_j+2$ therefore by Hölder's inequality we have

$$\int_{B_t}|\tau_{j,h}u|^{p_i}dx\leq c(R,p_i,p_j)\left(\int_{B_t}|\tau_{j,h}u|^{p_j+2}\right)^{\frac{p_i}{p_j+2}}\leq c(R,p_i,p_j)|h|^{p_i}\left(\int_{B_{t'}}|u_{x_j}|^{p_j+2}\right)^{\frac{p_i}{p_j+2}}. \tag{4.10}$$

Let us explicitly remark that, by assumption (4.3) and Theorem 2.4, last integral in (4.10) is finite by the interpolation inequality of Proposition 2.3.

Next, inserting (4.9) and (4.10) in (4.8), we infer

$$|I_2|\leq 2\varepsilon\,\mathbf{RHS}+\frac{c_\varepsilon}{(t-s)^2}|h|^2\sum_{i=1}^{j}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{p_i-2}{p_i}}\cdot\left(\int_{B_{t'}}|u_{x_j}|^{p_j}dx\right)^{\frac{2}{p_j}}$$
$$+\frac{c_\varepsilon}{(t-s)^2}|h|^2\sum_{i=j+1}^{n}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{p_i-2}{p_i}}\cdot\left(\int_{B_{t'}}|u_{x_j}|^{p_j+2}\right)^{\frac{2}{p_j+2}}$$
$$\leq 2\varepsilon\,\mathbf{RHS}+\frac{c_\varepsilon}{(t-s)^2}|h|^2\sum_{i=1}^{j}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{p_i-2}{p_i}}\cdot\left(\int_{B_{t'}}|u_{x_j}|^{p_j}dx\right)^{\frac{2}{p_j}}$$
$$+\frac{c_{\varepsilon,\beta}}{(t-s)^2}|h|^2\sum_{i=j+1}^{n}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{(p_i-2)(p_j+2)}{p_ip_j}}+\beta|h|^2\left(\int_{B_{t'}}|u_{x_j}|^{p_j+2}\right),$$

where, in the last line, we used Young's inequality. Therefore, we conclude that

$$|I_2|\leq 2\varepsilon\,\mathbf{RHS}+\frac{c_\varepsilon}{(t-s)^2}|h|^2\sum_{i=1}^{j}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{p_i-2}{p_i}}\cdot\left(\int_{B_{t'}}|u_{x_j}|^{p_j}dx\right)^{\frac{2}{p_j}}$$
$$+\frac{c_{\varepsilon,\beta}}{(t-s)^2}|h|^2\sum_{i=j+1}^{n}\Big(\int_{B_{t'}}|u_{x_i}(x)|^{p_i}dx\Big)^{\frac{(p_i-2)(p_j+2)}{p_ip_j}}+\beta|h|^2\left(\int_{B_{t'}}|u_{x_j}|^{p_j+2}\right). \tag{4.11}$$

From condition (A3), Lemma 2.10, the properties of η and Hölder's inequality, we derive

$$|I_3|=\left|\int_\Omega\langle f_\xi(x+e_jh,Du(x))-f_\xi(x,Du(x)),2\eta D\eta\tau_{j,h}u\rangle\,dx\right|$$
$$\leq 2|h|\sum_{i=1}^{n}\int_\Omega\eta|\eta_{x_i}||\tau_{j,h}u|\Big(g(x+h)+g(x)\Big)|u_{x_i}|^{p_i-1}dx$$
$$\leq\frac{c}{t-s}|h|\sum_{i=1}^{n}\int_{B_t}|\tau_{j,h}u|\Big(g(x+h)+g(x)\Big)|u_{x_i}|^{p_i-1}dx$$
$$\leq\frac{c}{t-s}|h|\left(\int_{B_R}g(x)^rdx\right)^{\frac{1}{r}}\underbrace{\sum_{i=1}^{n}\left(\int_{B_t}|\tau_{j,h}u|^{\frac{r}{r-1}}|u_{x_i}|^{\frac{r(p_i-1)}{r-1}}dx\right)^{\frac{r-1}{r}}}_{J}. \tag{4.12}$$

Notice that

$$\frac{1}{p_j+2}\frac{r}{r-1}<1$$

if and only if

$$r>\frac{p_j+2}{p_j+1}. \tag{4.13}$$

Inequality (4.13) is satisfied by virtue of assumption (4.2), once we observe that

$$r > p_n + 2 \geq p_j + 2 > \frac{p_j + 2}{p_j + 1}.$$

Thanks to Hölder's inequality and Lemma 2.10, we get

$$J \leq \left(\int_{B_t} |\tau_{j,h} u|^{p_j+2} \, dx \right)^{\frac{1}{p_j+2}} \sum_{i=1}^{n} \left(\int_{B_t} |u_{x_i}|^{\frac{(p_j+2)(p_i-1)r}{(r-1)(p_j+2)-r}} \, dx \right)^{\frac{(r-1)(p_j+2)-r}{r(p_j+2)}}$$

Notice that

$$\frac{(p_j+2)(p_i-1)r}{(r-1)(p_j+2)-r} \leq p_i + 2$$

if and only if

$$r \geq \frac{(p_i+2)(p_j+2)}{3p_j - p_i + 4}. \tag{4.14}$$

Inequality (4.14) is satisfied by virtue of assumption (4.2) and since

$$\frac{(p_i+2)(p_j+2)}{3p_j - p_i + 4} < p_i + 2,$$

once we observe that

$$p_i < 2p_j + 2$$

holds true by assumption (4.1).

Hence, we can apply Hölder's inequality and Lemma 2.10 to infer

$$\begin{aligned} J &\leq c(R) \left(\int_{B_t} |\tau_{j,h} u|^{p_j+2} \, dx \right)^{\frac{1}{p_j+2}} \sum_{i=1}^{n} \left(\int_{B_t} |u_{x_i}|^{p_i+2} \, dx \right)^{\frac{p_i-1}{p_i+2}} \\ &\leq c(R)|h| \left(\int_{B_t} |u_{x_j}|^{p_j+2} \, dx \right)^{\frac{1}{p_j+2}} \sum_{i=1}^{n} \left(\int_{B_t} |u_{x_i}|^{p_i+2} \, dx \right)^{\frac{p_i-1}{p_i+2}} \end{aligned} \tag{4.15}$$

Now inserting (4.15) in (4.12), using twice Young's inequality and denoting by $\alpha = \frac{p_i+2}{3}$ the conjugate of the exponent $\frac{p_i+2}{p_i-1}$ and δ the conjugate of $\frac{p_j+2}{\alpha} > 1$, we have

$$\begin{aligned} |I_3| &\leq \sum_{i=1}^{n} \frac{c_\beta}{t-s} |h|^2 \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha}{r}} \left(\int_{B_t} |u_{x_j}|^{p_j+2} \, dx \right)^{\frac{\alpha}{p_j+2}} \\ &\quad + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \, dx \right) \\ &\leq \frac{\bar{c}_\beta}{t-s} |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha\delta}{r}} + |h|^2 \beta \int_{B_t} |u_{x_j}|^{p_j+2} \, dx \\ &\quad + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \, dx \right) \\ &\leq \frac{\bar{c}_\beta}{t-s} |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha\delta}{r}} + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_t} |u_{x_i}|^{p_i+2} \, dx \right) \\ &\quad + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \, dx \right) \\ &\leq \sum_{i=1}^{n} \frac{\bar{c}_\beta}{t-s} |h|^2 \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha\delta}{r}} \\ &\quad + |h|^2 2\beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \, dx \right). \end{aligned} \tag{4.16}$$

Similarly as for the estimate of I_3, we obtain

$$|I_4| = \left| \int_{\Omega} \langle f_\xi(x + e_j h, Du(x)) - f_\xi(x, Du(x)), \eta^2 \tau_{j,h} Du \rangle \, dx \right|$$

$$
\begin{aligned}
&\le c|h| \sum_{i=1}^{n} \int_{\Omega} \eta^2 |\tau_{j,h} u_{x_i}| \left(g(x+h)+g(x)\right) |u_{x_i}|^{p_i-1} dx \\
&\le \varepsilon \underbrace{\int_{\Omega} \eta^2 \sum_{i=1}^{n} \left(|u_{x_i}(x+e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx}_{\textbf{RHS}} \\
&\quad + c_\varepsilon |h|^2 \sum_{i=1}^{n} \int_{\Omega} \eta^2 |u_{x_i}|^{p_i} \left(g(x+h)+g(x)\right)^2 dx \\
&\le \varepsilon \textbf{RHS} + c_\varepsilon |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r dx \right)^{\frac{2}{r}} \left(\int_{B_t} |u_{x_i}|^{\frac{p_i r}{r-2}} dx \right)^{\frac{r-2}{r}}.
\end{aligned} \tag{4.17}
$$

We note that

$$
\frac{(p_i+2)}{p_i r}(r-2) > 1 \iff r > p_i + 2
$$

which is true by assumption (4.2). Therefore applying Hölder's inequality in (4.17), we infer

$$
\begin{aligned}
|I_4| &\le \varepsilon \textbf{RHS} + c_\varepsilon |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r dx \right)^{\frac{2}{r}} \left(\int_{B_t} |u_{x_i}|^{p_i+2} dx \right)^{\frac{p_i}{p_i+2}} \\
&\le \varepsilon \textbf{RHS} + c_{\varepsilon,\beta} |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{p_i+2}{r}} + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} dx \right) \\
&\le \varepsilon \textbf{RHS} + c_{\varepsilon,\beta} |h|^2 \left(\int_{B_R} g(x)^r \, dx \right)^{\gamma} + |h|^2 \beta \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} dx \right),
\end{aligned} \tag{4.18}
$$

where we used Young's inequality with exponents $\frac{p_i+2}{p_i}$ and $\frac{p_i+2}{2}$ and $\gamma = \gamma(p_i)$.

Inserting (4.7), (4.11), (4.16) and (4.18) in (4.6), we get

$$
\begin{aligned}
&l \int_{\Omega} \eta^2 \sum_{i=1}^{n} \left(|u_{x_i}(x+e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx \\
&\quad \le 3\varepsilon \int_{\Omega} \eta^2 \sum_{i=1}^{n} \left(|u_{x_i}(x+e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx \\
&\quad + \frac{c_\varepsilon}{(t-s)^2} |h|^2 \sum_{i=1}^{j} \left(\int_{B_{t'}} |u_{x_i}(x)|^{p_i} \, dx \right)^{\frac{p_i-2}{p_i}} \cdot \left(\int_{B_{t'}} |u_{x_j}|^{p_j} \, dx \right)^{\frac{2}{p_j}} \\
&\quad + \frac{c_{\varepsilon,\beta}}{(t-s)^2} |h|^2 \sum_{i=j+1}^{n} \left(\int_{B_{t'}} |u_{x_i}(x)|^{p_i} \, dx \right)^{\frac{(p_i-2)(p_j+2)}{p_i p_j}} + 4\beta |h|^2 \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \right) \\
&\quad + \frac{\bar{c}_\beta}{t-s} |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha\delta}{r}} \\
&\quad + c_{\varepsilon,\beta} |h|^2 \left(\int_{B_R} g(x)^r \, dx \right)^{\gamma}.
\end{aligned} \tag{4.19}
$$

Choosing $\varepsilon = \frac{l}{6}$, we can reabsorb the first integral in the right-hand side of (4.19) by the left-hand side thus getting

$$
\begin{aligned}
&\int_{\Omega} \eta^2 \sum_{i=1}^{n} \left(|u_{x_i}(x+e_j h)|^2 + |u_{x_i}(x)|^2 \right)^{\frac{p_i-2}{2}} |\tau_{j,h} u_{x_i}|^2 dx \\
&\quad \le \frac{c}{(t-s)^2} |h|^2 \sum_{i=1}^{j} \left(\int_{B_{t'}} |u_{x_i}(x)|^{p_i} \, dx \right)^{\frac{p_i-2}{p_i}} \cdot \left(\int_{B_{t'}} |u_{x_j}|^{p_j} \, dx \right)^{\frac{2}{p_j}} \\
&\quad + \frac{c_\beta}{(t-s)^2} |h|^2 \sum_{i=j+1}^{n} \left(\int_{B_{t'}} |u_{x_i}(x)|^{p_i} \, dx \right)^{\frac{(p_i-2)(p_j+2)}{p_i p_j}} + 4\beta |h|^2 \sum_{i=1}^{n} \left(\int_{B_{t'}} |u_{x_i}|^{p_i+2} \right) \\
&\quad + \frac{\bar{c}_\beta}{t-s} |h|^2 \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r \, dx \right)^{\frac{\alpha\delta}{r}} \\
&\quad + c_{\varepsilon,\beta} |h|^2 \left(\int_{B_R} g(x)^r \, dx \right)^{\gamma}.
\end{aligned} \tag{4.20}
$$

Then, by Lemmas 2.5 and 2.12, letting $|h| \to 0$, we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\int_{B_s}|u_{x_ix_j}|^2|u_{x_i}|^{p_i-2}dx\right)\\
&\quad\le \frac{c}{(t-s)^2}\sum_{i=1}^{j}\left(\int_{B_R}|u_{x_i}(x)|^{p_i}\,dx\right)^{\frac{p_i-2}{p_i}}\cdot\left(\int_{B_R}|u_{x_j}|^{p_j}\,dx\right)^{\frac{2}{p_j}}\\
&\quad+\frac{c}{(t-s)^2}\sum_{i=j+1}^{n}\left(\int_{B_R}|u_{x_i}(x)|^{p_i}\,dx\right)^{\frac{(p_i-2)(p_j+2)}{p_i\,p_j}}\\
&\quad+\frac{\bar{c}_\beta}{t-s}\sum_{i=1}^{n}\left(\int_{B_R}g(x)^r\,dx\right)^{\frac{\alpha\,\delta}{r}}\\
&\quad+c_{\varepsilon,\beta}\left(\int_{B_R}g(x)^r\,dx\right)^{\gamma}+4\beta\sum_{i=1}^{n}\left(\int_{B_{t'}}|u_{x_i}|^{p_i+2}\right),
\end{aligned}\tag{4.21}$$

which in particular yields for every $i=1,\dots,n$

$$\begin{aligned}
\int_{B_s}|u_{x_ix_i}|^2|u_{x_i}|^{p_i-2}dx &\le \frac{c}{(t-s)^2}\sum_{h=1}^{i}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{p_h-2}{p_h}}\cdot\left(\int_{B_R}|u_{x_i}|^{p_i}\,dx\right)^{\frac{2}{p_i}}\\
&\quad+\frac{c}{(t-s)^2}\sum_{h=i+1}^{n}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{(p_h-2)(p_i+2)}{p_h\,p_i}}\\
&\quad+\frac{\bar{c}_\beta}{t-s}\sum_{i=1}^{n}\left(\int_{B_R}g(x)^r\,dx\right)^{\frac{\alpha\,\delta}{r}}\\
&\quad+c_{\varepsilon,\beta}\left(\int_{B_R}g(x)^r\,dx\right)^{\gamma}+4\beta\sum_{h=1}^{n}\left(\int_{B_{t'}}|u_{x_h}|^{p_h+2}\right).
\end{aligned}\tag{4.22}$$

Summing over $i=1,\dots,n$, the previous inequality, we infer

$$\begin{aligned}
&\sum_{i=1}^{n}\left(\int_{B_s}|u_{x_ix_i}|^2|u_{x_i}|^{p_i-2}dx\right)\\
&\quad\le \frac{c}{(t-s)^2}\sum_{i=1}^{n}\sum_{h=1}^{i}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{p_h-2}{p_h}}\cdot\left(\int_{B_R}|u_{x_i}(x)|^{p_i}\,dx\right)^{\frac{2}{p_i}}\\
&\qquad+\frac{c}{(t-s)^2}\sum_{i=1}^{n}\sum_{h=i+1}^{n}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{(p_h-2)(p_i+2)}{p_h\,p_i}}\\
&\qquad+n\left[\frac{\bar{c}_\beta}{t-s}\sum_{h=1}^{n}\left(\int_{B_R}g(x)^r\,dx\right)^{\frac{\alpha\,\delta}{r}}\right.\\
&\qquad\left.+c_{\varepsilon,\beta}\left(\int_{B_R}g(x)^r\,dx\right)^{\gamma}+4\beta\sum_{h=1}^{n}\left(\int_{B_{t'}}|u_{x_h}|^{p_h+2}\right)\right]\\
&\quad\le \frac{c}{(t-s)^2}\sum_{i=1}^{n}\left(1+\sum_{h=1}^{n}\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{p_i-2}{p_i}}\cdot\sum_{i=1}^{n}\left(1+\sum_{h=1}^{n}\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{2}{p_i}}\\
&\qquad+\frac{c}{(t-s)^2}\sum_{i=1}^{n}\sum_{h=i+1}^{n}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{(p_h-2)(p_i+2)}{p_h\,p_i}}\\
&\qquad+n\left[\frac{\bar{c}_\beta}{t-s}\sum_{i=1}^{n}\left(\int_{B_R}g(x)^r\,dx\right)^{\frac{\alpha\,\delta}{r}}\right.\\
&\qquad\left.+c_{\varepsilon,\beta}\left(\int_{B_R}g(x)^r\,dx\right)^{\gamma}+4\beta\sum_{h=1}^{n}\left(\int_{B_{t'}}|u_{x_h}|^{p_h+2}\right)\right]\\
&\quad\le \frac{c}{(t-s)^2}\sum_{i=1}^{n}\left(1+\sum_{h=1}^{n}\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\tilde{\alpha}}\cdot\sum_{i=1}^{n}\left(1+\sum_{h=1}^{n}\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\tilde{\beta}}\\
&\qquad+\frac{c}{(t-s)^2}\sum_{i=1}^{n}\sum_{h=i+1}^{n}\left(\int_{B_R}|u_{x_h}(x)|^{p_h}\,dx\right)^{\frac{(p_h-2)(p_i+2)}{p_h\,p_i}}\\
&\qquad+n\left[\frac{\bar{c}_\beta}{t-s}\sum_{h=1}^{n}\left(\int_{B_R}g(x)^r\,dx\right)^{\frac{\alpha\,\delta}{r}}\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + c_\beta \left(\int_{B_R} g(x)^r\, dx \right)^{\gamma} + 4\beta \sum_{h=1}^{n} \left(\int_{B_{t'}} |u_{x_h}|^{p_h+2} \right) \Bigg] \\
&\le \frac{c}{(t-s)^2}\, n \left(1 + \sum_{h=1}^{n} \int_{B_R} |u_{x_h}(x)|^{p_h}\, dx \right)^{\tilde{\alpha}} \cdot n \left(1 + \sum_{h=1}^{n} \int_{B_R} |u_{x_h}(x)|^{p_h}\, dx \right)^{\tilde{\beta}} \\
&\quad + \frac{c}{(t-s)^2} \sum_{i=1}^{n} \sum_{h=i+1}^{n} \left(\int_{B_R} |u_{x_h}(x)|^{p_h}\, dx \right)^{\frac{(p_h-2)(p_i+2)}{p_h\, p_i}} \\
&\quad + n \Bigg[\frac{\bar{c}_\beta}{t-s} \sum_{h=1}^{n} \left(\int_{B_R} g(x)^r\, dx \right)^{\frac{\alpha\,\delta}{r}} \\
&\quad + c_{\varepsilon,\beta} |h|^2 \left(\int_{B_R} g(x)^r\, dx \right)^{\gamma} + 4\beta \sum_{h=1}^{n} \left(\int_{B_{t'}} |u_{x_h}|^{p_h+2} \right) \Bigg] \\
&\le \frac{n^2 c}{(t-s)^2} \left(1 + \sum_{h=1}^{n} \int_{B_R} |u_{x_h}(x)|^{p_h}\, dx \right)^{\tilde{\alpha}+\tilde{\beta}} \\
&\quad + \frac{c}{(t-s)^2} \sum_{i=1}^{n} \sum_{h=i+1}^{n} \left(\int_{B_R} |u_{x_h}(x)|^{p_h}\, dx \right)^{\frac{(p_h-2)(p_i+2)}{p_h\, p_i}} \\
&\quad + n \Bigg[\frac{\bar{c}_\beta}{t-s} \sum_{h=1}^{n} \left(\int_{B_R} g(x)^r\, dx \right)^{\frac{\alpha\,\delta}{r}} \\
&\quad + c_{\varepsilon,\beta} |h|^2 \left(\int_{B_R} g(x)^r\, dx \right)^{\gamma} + 4\beta \sum_{h=1}^{n} \left(\int_{B_{t'}} |u_{x_h}|^{p_h+2} \right) \Bigg],
\end{aligned} \tag{4.23}
$$

where $\tilde{\alpha} = \max\{ \frac{p_i-2}{p_i}, i = 1, \dots, n\}$ and $\tilde{\beta} = \max\{ \frac{2}{p_i}, i = 1, \dots, n\}$.

From the hypothesis

$$V_{p_i}(u_{x_i}) \in W^{1,2}_{\mathrm{loc}}(\Omega)$$

and Proposition 2.3 together Theorem with 2.4, we have

$$u_{x_i} \in L^{p_i+2}_{\mathrm{loc}}(\Omega), \quad \forall i = 1, \dots, n$$

and the following estimate holds for every $i = 1, \dots, n$

$$
\begin{aligned}
\int_{B_{t'}} |u_{x_i}|^{p_i+2} dx &\le c ||u||_\infty \left[\left(\int_{B_r} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) + \frac{1}{(r-t')^\gamma} \left(\int_{B_r} |u_{x_i}|^{p_i} dx \right)^{\gamma} \right] \\
&\le c ||u||_\infty \left[\left(\int_{B_r} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) + \frac{1}{(r-t')^\gamma} \left(\int_{B_R} |u_{x_i}|^{p_i} dx \right)^{\gamma} \right],
\end{aligned} \tag{4.24}
$$

with $c = c(n, p_i)$ and $\gamma = \gamma(p_i)$.

Summing the inequality (4.24) over $i = 1, \dots, n$ and inserting the resulting inequality in (4.23), we find that

$$
\begin{aligned}
&\sum_{i=1}^{n} \left(\int_{B_s} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) \\
&\le \frac{n^2 c}{(t-s)^2} \left(1 + \sum_{i=1}^{n} \int_{B_R} |u_{x_i}(x)|^{p_i}\, dx \right)^{\tilde{\alpha}+\tilde{\beta}} \\
&\quad + \frac{c_2}{(t-s)^2} \sum_{j=1}^{n} \sum_{i=j+1}^{n} \left(\int_{B_R} |u_{x_i}(x)|^{p_i}\, dx \right)^{\frac{(p_i-2)(p_j+2)}{p_i\, p_j}} \\
&\quad + \frac{c_1}{t-s} \sum_{i=1}^{n} \left(\int_{B_R} g(x)^r\, dx \right)^{\frac{\alpha\,\delta}{r}} \\
&\quad + c_3 \left(\int_{B_R} g(x)^r\, dx \right)^{\frac{p_n+2}{r}} \\
&\quad + \beta\, \tilde{c} ||u||_\infty \sum_{i=1}^{n} \left[\left(\int_{B_r} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) + \frac{c_4}{(r-t')^\gamma} \left(\int_{B_R} |u_{x_i}|^{p_i} dx \right)^{\gamma} \right].
\end{aligned} \tag{4.25}
$$

Setting

$$\phi(\rho) = \sum_{i=1}^{n} \left(\int_{B_\rho} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) dx,$$

inequality (4.25) can be written as follows

$$\phi(\rho) \le \beta\tilde{c}\|u\|_\infty \phi(r) + c_3 + \frac{c_1}{t-s} + \frac{c_2}{(t-s)^2} + \frac{c_4}{(r-t')^\gamma} \tag{4.26}$$

Choosing $\beta > 0$ such that $\theta = \beta\tilde{c}\|u\|_\infty < 1$ and since (4.26) holds true for every radii $\rho < s < t < t' < r$, we may select radii selecting radii

$$s = \rho + \frac{r-\rho}{4},\ t = \rho + \frac{r-\rho}{2},\ t' = \rho + \frac{3(r-\rho)}{4},$$

so that $t - s = r - t' = \frac{r-\rho}{4}$. Hence, (4.26) becomes

$$\phi(\rho) \le \theta\phi(r) + c_3 + \frac{\tilde{c}_1}{r-\rho} + \frac{\tilde{c}_2}{(r-\rho)^2} + \frac{\tilde{c}_4}{(r-\rho)^\gamma}. \tag{4.27}$$

Since (4.27) holds true for every $\frac{R}{4} < \rho < r < R$, we are legitimate to apply Lemma 2.2 to the function $\phi : [\frac{R}{4}, R] \to \mathbb{R}$, thus getting

$$\phi\left(\frac{R}{4}\right) \le c_3 + \frac{\tilde{c}_1}{R} + \frac{\tilde{c}_2}{R^2} + \frac{\tilde{c}_4}{R^\gamma}. \tag{4.28}$$

Recalling the definition of $\phi(t)$, (4.28) yields that

$$\sum_{i=1}^n \left(\int_{B_{\frac{R}{4}}} |u_{x_i x_i}|^2 |u_{x_i}|^{p_i-2} dx \right) \le c \left(\sum_{i=1}^n ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^\sigma, \tag{4.29}$$

where $c = c(p_i, n, l, L, R)$ and $\sigma = \sigma(n, p_i)$ Putting (4.29) in (4.24), we derive also

$$\sum_{i=1}^n \int_{B_{\frac{R}{4}}} |u_{x_i}|^{p_i+2} dx \le c \left(\sum_{i=1}^n ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^\sigma, \tag{4.30}$$

for positive constants $c = c(n, p_i, \lambda, \Lambda, R)$ and $\sigma = \sigma(n, p_i)$.

Now, inserting (4.30) in (4.21), we infer for every $i = 1, \dots, n$ and for every $j = 1, \dots, n$.

$$\sum_{i=1}^n \left(\int_{B_{\frac{R}{4}}} |u_{x_i x_j}|^2 |u_{x_i}|^{p_i-2} dx \right) \le c \left(\sum_{i=1}^n ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^\sigma. \tag{4.31}$$

which concludes the proof. □

5. Approximation

In order to perform the approximation argument, let us fix a non-negative smooth kernel $\phi \in C_0^\infty(B_1(0))$ such that $\int_{B_1(0)} \phi = 1$ and consider the corresponding family of mollifiers $(\phi_k)_{k>0}$. We set, for a given ball $B_R \Subset \Omega$,

$$a_i^k = a_i * \phi_k,$$

$$g_k(x) = \max_i \{|Da_i^k(x)|\} \tag{5.1}$$

and, for $x \in B_R$, we define

$$f^k(x, \xi) = \sum_{i=1}^n a_i^k(x) |\xi|^{p_i}, \tag{5.2}$$

for every $k < \text{dist}(B_R, \Omega)$. We shall use the following

$$\mathcal{F}^k(u, B_R) := \int_{B_R} f^k(x, \xi) dx$$

and

$$\mathcal{F}_\varepsilon^k(u, B_R) := \int_{B_R} \left(f^k(x, \xi) + \varepsilon(1 + |\xi|^2)^{\frac{p_n}{2}} \right) dx.$$

One can easily check that $f^k(x, \xi)$ satisfies assumptions (A1)–(A2) and (A3'), with $K = \sup_{B_R} |g_k(x)|$ that is finite since $a_i^k(x) \in W^{1,\infty}(B_R)$ and then $g_k \in L^\infty(B_R)$.

We consider the variational problems

$$\inf \left\{ \int_{B_R} \left(f^k(x, Dv) + \varepsilon(1 + |Dv|^2)^{\frac{p_n}{2}} \right) dx \ : \ v \in W_0^{1,p_n}(B_R) + u^\eta \right\}, \tag{5.3}$$

where

$$u^\eta = u * \varsigma_\eta$$

is the mollification of the local minimizer u of (1.1), for a sequence of mollifier ς_n.

It is well known that, by the direct methods of the calculus of variations, there exists a unique solution $u^{\eta}_{k,\varepsilon} \in W^{1,p_n}_0(B_R) + u^{\eta}$ of the problem (5.3). Since the integrand $f^k(x, Dv) + \varepsilon(1 + |Dv|^2)^{\frac{p_n}{2}}$ satisfies the assumptions of Theorem 3.1, we have that

$$V_{p_i}((u^{\eta}_{k,\varepsilon})_{x_i}) \in W^{1,2}_{\mathrm{loc}}(B_R).$$

Hence we are legitimate to use estimate (4.4) of Theorem 4.1, to obtain that

$$\begin{aligned}
\sum_{i=1}^{n}\left(\int_{B_{R/4}} |(u^{\eta}_{k,\varepsilon})_{x_i}|^{p_i+2} dx\right) &\le c\left(\sum_{i=1}^{n}\int_{B_R} |(u^{\eta}_{k,\varepsilon})_{x_i}|^{p_i}\, dx + ||g_k||_{L^r(B_R)}\right)^{\sigma}\\
&\le c(\lambda)\left(\sum_{i=1}^{n}\int_{B_R} a^k_i(x)\,|(u^{\eta}_{k,\varepsilon})_{x_i}|^{p_i}\, dx + ||g_k||_{L^r(B_R)}\right)^{\sigma}\\
&= c\left(\sum_{i=1}^{n}\int_{B_R} f^k(x, Du^{\eta}_{k,\varepsilon}) dx + ||g_k||_{L^r(B_R)}\right)^{\sigma}\\
&\le c\left(\sum_{i=1}^{n}\int_{B_R} \left(f^k(x, Du^{\eta}_{k,\varepsilon}) + \varepsilon(1 + |Du^{\eta}_{k,\varepsilon}|^2)^{\frac{p_n}{2}}\right) dx + ||g_k||_{L^r(B_R)}\right)^{\sigma},
\end{aligned}$$

where c is a positive constant that is independent on ε, k and η and where we used that $a^k_i(x) \ge \lambda, \forall i, k$. Using the minimality of $u^{\eta}_{k,\varepsilon}$ in the right-hand side of previous estimate, we have that

$$\begin{aligned}
\sum_{i=1}^{n}\left(\int_{B_{R/4}} |(u^{\eta}_{k,\varepsilon})_{x_i}|^{p_i+2} dx\right) &\le c\left(\sum_{i=1}^{n}\int_{B_R} \left(f^k(x, Du^{\eta}_{k,\varepsilon}) + \varepsilon(1 + |Du^{\eta}_{k,\varepsilon}|^2)^{\frac{p_n}{2}}\right) dx + ||g_k||_{L^r(B_R)}\right)^{\sigma}\\
&\le c\left(\sum_{i=1}^{n}\int_{B_R} \left(f^k(x, Du^{\eta}) + \varepsilon(1 + |Du^{\eta}|^2)^{\frac{p_n}{2}}\right) dx + ||g_k||_{L^r(B_R)}\right)^{\sigma},
\end{aligned} \tag{5.4}$$

where c is a positive constant that does not depend on η, k and ε.

Since $u^{\eta} \in W^{1,p_n}(B_R)$, the right-hand side of the previous estimate is finite, and therefore, the left-hand side of the previous estimate is uniformly bounded with respect to ε for $\varepsilon \in (0, 1)$. Therefore there exists v^{η}_k such that, up to a subsequence,

$$u^{\eta}_{k,\varepsilon} \rightharpoonup v^{\eta}_k \text{ weakly in } W^{1,\mathbf{p}+2}(B_R) \text{ as } \varepsilon \to 0.$$

By the weak lower semicontinuity of the norm, we get

$$\begin{aligned}
\sum_{i=1}^{n}\left(\int_{B_{R/4}} |(v^{\eta}_k)_{x_i}|^{p_i+2} dx\right) &\le \liminf_{\varepsilon} \sum_{i=1}^{n}\left(\int_{B_{R/4}} |(u^{\eta}_{k,\varepsilon})_{x_i}|^{p_i+2} dx\right)\\
&= c\left(\sum_{i=1}^{n}\int_{B_R} f^k(x, Du^{\eta}) dx + ||g_k||_{L^r(B_R)}\right)^{\sigma},
\end{aligned} \tag{5.5}$$

where we used (5.4) and that, since $u^{\eta} \in W^{1,p_n}(\Omega)$, it holds

$$\liminf_{\varepsilon\to 0} \varepsilon \int_{B_R} (1 + |Du^{\eta}|^2)^{\frac{p_n}{2}} dx = 0. \tag{5.6}$$

Using the definition of f_k at (5.2), we have that

$$\left|\int_{B_R} f^k(x, Du^{\eta}) - f(x, Du^{\eta}) dx\right| \le \sum_{i=1}^{n}\int_{B_R} |a^k_i(x) - a_i(x)|\,|u^{\eta}_{x_i}|^{p_i} dx. \tag{5.7}$$

So the fact that

$$a^k_i(x) \to a_i(x) \quad \text{uniformly on compact sets}$$

gives that

$$\lim_{k\to 0^+} \sum_{i=1}^{n} ||a^k_i(x) - a_i(x)||_{\infty} \int_{B_R} |u^{\eta}_{x_i}|^{p_i} dx = 0.$$

Therefore, passing to the limit as $k \to 0^+$ in (5.7), yields

$$\int_{B_R} f^k(x, Du^{\eta}) dx \to \int_{B_R} f(x, Du^{\eta}) dx. \tag{5.8}$$

By the properties of mollification it holds $g_k \to g$ in $L^r(B_R)$, where g_k has been found at (5.1) and g is the function appearing in (A3). So, by (5.8), the right-hand side of (5.5) is uniformly bounded w.r.t. to k. Therefore there exists v^{η} such that, up to a subsequence,

$$v^{\eta}_k \rightharpoonup v^{\eta} \text{ weakly in } W^{1,\mathbf{p}+2}(B_R) \text{ as } k \to 0.$$

Again by the weak lower semicontinuity of the norm, (5.5) and (5.8), we get

$$\sum_{i=1}^{n}\left(\int_{B_{R/4}}|(v^\eta)_{x_i}|^{p_i+2}dx\right)\le \liminf_k \sum_{i=1}^{n}\left(\int_{B_{R/4}}|(v_k^\eta)_{x_i}|^{p_i+2}dx\right)$$
$$\le c\left(\sum_{i=1}^{n}\int_{B_R} f(x,Du^\eta)dx+||g||_{L^r(B_R)}\right)^\sigma . \tag{5.9}$$

Now, we note that

$$\left|\int_{B_R} f(x,Du^\eta)-f(x,Du)dx\right|\le\left|\sum_{i=1}^{n}\int_{B_R}\left(a_i(x)\,|u^\eta_{x_i}|^{p_i}-a_i(x)|u_{x_i}|^{p_i}\right)dx\right|$$
$$\le \sum_{i=1}^{n}\int_{B_R} a_i(x)\left||u^\eta_{x_i}|^{p_i}-|u_{x_i}|^{p_i}\right|dx$$
$$\le \bar c\sum_{i=1}^{n}\int_{B_R} a_i(x)\,\left(|u^\eta_{x_i}|+|u_{x_i}|\right)^{p_i-1}\,|u^\eta_{x_i}-u_{x_i}|dx$$
$$\le \sum_{i=1}^{n} c(\Lambda,p_i)\int_{B_R}\left(|u^\eta_{x_i}|+|u_{x_i}|\right)^{p_i-1}\,|u^\eta_{x_i}-u_{x_i}|dx, \tag{5.10}$$

where we used the left inequality in Lemma 2.7, with $\xi=|u_{x_i}|$ and $\eta=|u^\eta_{x_i}|$ and $\alpha=p_i$.

Since, by property of mollification, it holds

$$u^\eta_{x_i}\to u_{x_i}\ \text{strongly in } L^{p_i},$$

passing to the limit as $\eta\to 0^+$ in (5.10), we have

$$\int_{B_R} f(x,Du^\eta)\to\int_{B_R} f(x,Du). \tag{5.11}$$

By virtue of (5.11), we observe that the right-hand side of (5.9) is uniformly bounded w.r.t. η. Therefore there exists v such that, up to a subsequence,

$$v^\eta \rightharpoonup v \text{ weakly in } W^{1,\mathbf{p}+2}(B_R)\ \text{ as } \eta\to 0.$$

Once again, by the weak lower semicontinuity of the norm, we derive

$$\sum_{i=1}^{n}\left(\int_{B_{R/4}}|v_{x_i}|^{p_i+2}dx\right)\le \liminf_\eta \sum_{i=1}^{n}\left(\int_{B_{R/4}}|(v^\eta)_{x_i}|^{p_i+2}dx\right)$$
$$\le \lim_\eta c\left(\sum_{i=1}^{n}\int_{B_R} f(x,Du^\eta)dx+||g||_{L^r(B_R)}\right)^\sigma$$
$$= c\left(\sum_{i=1}^{n}\int_{B_R} f(x,Du)dx+||g||_{L^r(B_R)}\right)^\sigma . \tag{5.12}$$

In order to conclude the proof, it suffices to show that $u=v$ a.e. in B_R.

By the weak lower semicontinuity of the functionals $\mathcal{F}$ and $\mathcal{F}^k$ in $W^{1,\mathbf{p}}(B_R)$, the weak convergence of $v^\eta\rightharpoonup v$ in $W^{1,\mathbf{p}+2}(B_R)$, (5.8) and $v^\eta_k\rightharpoonup v^\eta$ in $W^{1,\mathbf{p}+2}(B_R)$, we get

$$\int_{B_R} f(x,Dv)dx\le\liminf_\eta\int_{B_R} f(x,Dv^\eta)dx\le\liminf_\eta\liminf_k\int_{B_R} f^k(x,Dv^\eta)dx$$
$$\le\liminf_\eta\liminf_k\int_{B_R} f^k(x,Dv^\eta_k)dx$$
$$\le\liminf_\eta\liminf_k\int_{B_R}\left(f^k(x,Dv^\eta_k))+\varepsilon(1+|Du^\eta_{k,\varepsilon}|^2)^{\frac{p_n}{2}}\right)dx,$$

where the last inequality is trivial, by the non-negativity of the last term.

Thanks to the weak convergence of $u^\eta_{k,\varepsilon}\rightharpoonup v^\eta_k$ in $W^{1,\mathbf{p}+2}(B_R)$ and again the weak lower semicontinuity of functional $\mathcal{F}^k_\varepsilon$, we have

$$\int_{B_R} f(x,Dv)dx\le\liminf_\eta\liminf_k\liminf_\varepsilon\int_{B_R}\left(f^k(x,Du^\eta_{k,\varepsilon})+\varepsilon(1+|Du^\eta_{k,\varepsilon}|^2)^{\frac{p_n}{2}}\right)dx$$
$$\le\liminf_\eta\liminf_k\liminf_\varepsilon\int_{B_R}\left(f^k(x,Du^\eta)+\varepsilon(1+|Du^\eta|^2)^{\frac{p_n}{2}}\right)dx$$
$$=\liminf_\eta\liminf_k\int_{B_R} f^k(x,Du^\eta)dx, \tag{5.13}$$

where we used the minimility of $u^{\eta}_{k,\varepsilon}$ and (5.6). By (5.13), (5.8) and (5.11), we derive

$$\int_{B_R} f(x, Dv)dx \le \liminf_{\eta} \int_{B_R} f(x, Du^{\eta})dx \le \int_{B_R} f(x, Du)dx, \tag{5.14}$$

and then, by minimality of u, we infer

$$\int_{B_R} f(x, Dv)dx = \int_{B_R} f(x, Du)dx.$$

By the strict convexity of $\xi \to f(x,\xi)$, we deduce that $u = v$. Then $u \in W^{1,\mathbf{p}+2}$ and so we can argue as in the proof of Theorem 4.1, thus obtaining

$$\sum_{i=1}^{n} \left(\int_{B_{\frac{R}{4}}} |u_{x_i x_j}|^2 |u_{x_i}|^{p_i-2} dx \right) \le c \left(\sum_{i=1}^{n} ||u_{x_i}||_{L^{p_i}(B_R)} + ||g||_{L^r(B_R)} \right)^{\sigma}, \tag{5.15}$$

and this conclude the proof.

Acknowledgments

The author wishes to express her heartfelt gratitude to Prof. A. Passarelli Di Napoli for proposing the problem, for many fruitful discussions, for her careful reading of the manuscript, and for the generous time dedicated throughout this work.
The author is also grateful to the anonymous referee for her/his valuable suggestions and constructive comments.
The author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). She has been supported through the project: Regolarita' di soluzioni di equazioni paraboliche a crescita nonstandard degeneri - "INdAM - GNAMPA Project", codice CUP $\#E5324001950001\#$. In addition she has also been supported through the project: Sustainable Mobility Center (Centro Nazionale per la Mobilità Sostenibile – CNMS) - SPOKE 10. This manuscript reflects only the author's views and opinions, and the Ministry cannot be considered responsible for them.

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