PHENOMENOLOGICAL APPROACH TO FOSTERING PHYSICAL MODELLING AND MATHEMATICAL FORMALISATION

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Modelling physical phenomena and acquiring mathematical formalisation can be challenging in physics and math education at all grades. It is recommended that interpretative and mathematical models be developed gradually from an early age, starting from kindergarten. Although innate visuospatial representation and numeracy skills are present in individuals from an early age, numerous socio-cultural factors encountered during the educational journey impede the realisation of an individual's full potential. This study explored how to harness the transformative power of transduction and measurement in mathematics education. The key connection between these activities is the shift from using the entire body for large, macro movements to manipulating small objects at a micro level. This progression from bodily transformations to fine motor skills helps students link significant movements with precise tasks.

Keywords: modeling, phenomenological approach, transduction processes.

THEORETICAL BACKGROUND

The human mind possesses innate capacities for perceiving numerical quantities and basic aspects of spatial orientation (Changeux, 1997, 2002; Dehaene, 2007, 2009). However, formal calculation is not a natural ability—it is a more advanced cognitive skill that requires targeted educational strategies to develop (Dehaene, 2011, 2020). Research into learning difficulties in mathematics and physics (Bollen et al., 2017; Mullis et al., 2020; OECD, 2023) highlights a persistent issue: these disciplines are often taught as separate subjects, resulting in a fragmented understanding. Integrating mathematical formalisation and physical modelling is therefore essential for curriculum reform, as it helps bridge gaps between these fields and supports students in grasping the unified cognitive processes underlying conceptual understanding (Burkholder et al., 2021; White Brahmia et al., 2021).

This integration, however, introduces complexities for education policy(Levrini et al., 2008), teacher development (Guskey, 2003; Coppola et al., 2012; van Driel et al., 2023), and the recognition of diverse learning profiles (Kress et al., 2001; Brown, 2020), challenging traditional views that see learning as primarily dependent on language. Instead, meaning in science and mathematics is constructed across multiple modes—verbal, visual, symbolic, and physical—both separately and through their interactions (Kress et al., 2001). Thus, a multimodal approach, valuing drawings, symbolic notations, and bodily gestures equally, is imperative for analyzing and evaluating learning processes.

Accordingly, it is crucial to design educational activities tailored to diverse learning styles, starting from the earliest years. Choosing suitable artefacts is essential in experimental teaching approaches that support students' meaning-making. The literature underscores the value of incorporating the body and tangible models as cultural artefacts—tools that foster multimodal reasoning and enhance problem-solving (Guidoni et al., 2005b; Bartolini Bussi & Mariotti, 2008; Yeo & Nielsen, 2020). The educational paradigm of embodied cognition positions body-based processes as fundamental to didactic mediation (Merleau-Ponty,1978). Studies show that embodied approaches in teaching mathematics and physics, including the manipulation of tangible objects like geometry models, measuring instruments, and mathematical machines, clarify disciplinary concepts through physical interaction and multimodal representation (Castelnuovo, 1967; Bartolini Bussi & Maschietto, 2006; Arzarello et al., 2009; Sabena et al., 2016; Volkwyn et al., 2019; Crippa & Milici, 2023).

This paper hypothesises that well-designed activities, integrating embodied cognition and hands-on manipulation, facilitate students' mastery of vital cross-cutting concepts—specifically, transduction and measurement.

The transduction and measurement processes

The interplay between mathematics and physics is especially evident in the interconnected concepts of transduction and measurement. "Transduction" here refers, first, to the movement of meaning across different semiotic systems within a multimodal framework (Bezemer & Kress, 2008; Halliday, 1978, 1994, 2004). For example, translating a mathematical function into a graph may involve using gestures to illustrate features, helping learners bridge the gap between abstract and concrete representations (Svesson & Eriksson, 2020). The "transductive link" marks the moment a learner connects two semiotic systems, forming a chain of knowledge building across representations. Second, in the context of scientific instrumentation, transduction refers to the process by which devices convert phenomena (such as X-rays) into accessible signs (for example, graphical output), enabling targeted investigation (Arcà & Guidoni, 1987; Volkwyn et al., 2019; Kapodistrias & Airey, 2025). At the core, our sensorimotor system mediates interactions with the world (Euler et al., 2019; Kontra et al., 2015). Hands, as tools for writing or manipulating, bridge internal cognitive processes and external forms of communication (Arcà & Guidoni, 1987). In mathematics education, learning often requires integrating verbal, numeric, algebraic, and visual representations; manipulating physical models can link hands-on exploration and graphical forms, facilitating meaning-making (Duval, 2006; Bartolini Bussi & Maschietto, 2006).

Transduction is closely linked with measurement. Students use counting strategies to quantify variables, but frequently, the functional relationship between variable changes—covariation—remains implicit (Antonini et al., 2020). The act of transduction can illuminate this relationship, aiding understanding of patterns, causality, and how functions model dynamic situations (Carlson et al., 2002).

No single representation provides complete conceptual understanding; synthesizing insights from various semiotic systems offers a fuller picture. Collectively, these modes broaden students' capacity to construct meaning.

This study examines activities designed and implemented in various educational settings, including primary schools and university courses, grounded in theories of embodied cognition and transduction. It demonstrates that by engaging the body and manipulating objects, students develop socio-semiotic representations that connect phenomenological experiences with the mastery of mathematical modelling and formalisation. The findings support the idea that this approach emphasises the relationship between key concepts shared by mathematics and physics, such as transduction and measurement. The hypothesis suggests that these two scientific practices create a valuable educational partnership, helping students overcome the challenges associated with the formalism of mathematical entities and physics modelling.

CONCEPTUAL FRAMEWORK AND METHODOLOGY

The research paradigm is based on a theoretical framework developed by the Physics and Math Education Research (PMER) group at the University of Naples "Federico II" over the years (Balzano et al., 2007; Hoffmann & Guidoni, 1979). It is identified four interconnected components:

- A model of cognitive dynamics. The model is based on the resonance between external request and internal evocation-response through a constantly evolving cognitive system (Guidoni et al., 2005a, 2005b);
- A phenomenological approach to modelling scientific processes. Modeling is a fundamental scientific practice that systematises experience and connects the individual's structure with the discipline's supporting structures (Lehrer & Schauble, 2015; Lehrer & Schauble, 2004);
- A comprehensive review of the disciplines' contents reveals the importance of organizing school and academic subjects. When used to design and experiment with vertical teaching paths, this organisation forms the basis for a mathematical formalism of physics systems (Ahn et al., 2021; Balzano et al., 2005).
- A systematic interaction with different educational contexts (formal- informal, class, adult education, basic cultural exchange) in which at the same time, the action of 'didactic mediation' is direct and reflective interaction with the educational system where not only more resources but also different approaches in the way of understanding didactic re-search would be needed (Artiano & Balzano, 2023; Lo Sapio et al., 2022).

The heart of the conceptual framework is the Cognitive Resonance Model (CRM), which is based on the dynamics of learning and teaching, partly on innate characteristics (Dehaene, 2020; Rizzolatti & Sinigaglia, 2023), and partly on cultural transmission (Cini, 1991; Cini et al., 1971). The CRM is designed to resonate with and advance the dynamics of physics and math teaching and learning, taking into account students' subjective and sociocultural dimensions, while highlighting the characteristics of specific contexts (i.e., formal and informal). This approach emphasises the relationship between our internal body elements (sensory-motor perception, emotions, language, thinking) and the external environment, allowing for a comparison between pre-constructed and pre-organised schematisation based on sensory data. In the CRM we achieve consciousness of our thought processes by recognising when an innate pre-representation resonates with external reality. To facilitate this awareness, particularly in abstract subjects such as mathematics and physics, various representations (linguistic, iconic, formal) systems can be employed, as pointed out by Duval (Duval, 1995).

This qualitative study builds upon the theoretical framework established by the PMER group over the years. For us, doing qualitative research in physics and math education means that the inquiry is a product of specific epistemic cultures (Sandkuehler, 2014), which define membership through shared ways of acting, speaking, and being. This vision establish norms and expectations for viewing and interpreting each other, our artefacts, and discourse and evolve through concerted activity and redefinition of vocabulary, metaphors, and ways of taking action. Qualitative inquiry thus constructs collective awareness through understanding, empathy, and curiosity. Understanding and building a web of beliefs, as well as breaking down the distinction between participant and researcher, helps us comprehend the other, whether they are struggling students, overworked school leaders, or individuals facing challenges (Kelly, 2023). Cultivating understanding and empathy can foster ways of being together that enhance educational practices and contribute to creating a culture of mathematics and science in education.

This frame raises several open research questions: What types of semiotic resources develop when students engage with mathematics and physics using tangible objects and their bodies? How do they transduce different semiotic registers—such as graphic, symbolic, verbal, operational, and digital—to build meaning and understanding of concepts?

DESIGNING RESEARCH AND METHODS

Analysis of a case study

To understand the reasons and methods behind the impact of certain experimental activities, a case study approach was employed. The study analysed the conditions under which specific activities were carried out by investigating the relationship between the proposed approach and the context in which it was implemented. The case study is structured as follows:

- The title and the synopsis: include a brief introduction and the objectives of the case study.

- Paradigmatic activities are examples chosen from several experiments that better help describe and support the discussion of the case study.

Embodied Experiences and Mathematical Machines: A Case Study on Geometric Transformations

This case study explores the effect of a didactic activity on understanding geometric transformations and constructions. It features two experiments that examine the link between manipulating mathematical tools and the body's role in formalising theoretical ideas. These experiments were conducted in two distinct educational environments: one with primary school pupils and another with undergraduate mathematics students. Both activities were conducted in collaboration with the students' teacher.

Choosing these two contexts—representing the extremes of the educational system—aims to compare learning processes at both the initial and advanced stages. This method provides valuable insights into the evolution of teaching and learning in mathematics and physics over time. Moreover, adopting a broader perspective helps gain a comprehensive view of the entire system.

The goal is to gather insights on how studying transformations can support the development of effective teaching and learning strategies in mathematics and geometry, providing a unified approach that activates diverse representations of core concepts. The analysis highlights both vertical and transversal elements of the approach. Specifically, it investigates how primary and university students engage cognitive processes aligned with sociosemiotic multimodality and Duval's ideas of conversion and treatment. Data collection involved direct observations, field notes, video and audio recordings of student voices, as well as inscriptions created by students, which were scanned. A selection of data focused on activities related to multimodality and aimed to individualise the transduction chain. The results will be discussed in the last section of this paper.

At school

According to the theoretical background and adopted framework, the activity begins activating the sensorimotor system of the students (Euler et al., 2019; Kontra et al., 2015). With a group of 20 primary school children, we asked them to hold hands and form a circle. But how do we determine that the shape we are arranged in is indeed a circle? We start to discuss the characteristics of a circle. After a few exchanges, referring to the radii of a bicycle wheel or the sun, it quickly becomes clear that this figure we are trying to reproduce has a centre. But how is the centre found? Several suggestions come from the children: some propose counting the steps from the edge to the hypothetical centre, while others, standing at a point inside the group, stretch out their arms and spin around, giving the others a new position on the edge. After finding a certain 'roundish' configuration, we asked to move together to transform the circle into a rectangle. This action involves very important cognitive dynamics; let's analyse them: firstly, each child has their own idea of a rectangle, which may differ from another's image in terms of size and orientation in space; moreover, transforming a circle into a rectangle may seem like a simple problem ('flatten' or 'unroll' it), but from a mathematical perspective, it hides several complexities. The circle has constant positive curvature; the rectangle has zero curvature and sharp angles. A transformation that preserves curvature does not exist; it inevitably deforms. From a topological point of view, the circle is the boundary of a disc, the rectangle is the boundary of a quadrilateral; moving from one to the other involves changing the topology of the curvature lines. At the same time, it is impossible to 'unroll' a circle without distortion, so an isometric transformation cannot be performed. The choice of what to preserve and what to sacrifice opens up various mathematical approaches; however, in the context of this work, it leads to numerous attempts at solutions proposed by the children. In this sense, proposing complex mathematical questions to children is a pedagogical richness because it allows them to argue, measure, count, solve problems, or, in other words, to activate multimodal learning channels. These initial activities can be repeated by inserting a rope: instead of holding hands, the children hold the rope, which helps them visualise the shape they are seeking. One step forward involves building and operating tangible objects, which are perforated wood rods designed to create mathematical mechanisms.

The “Mathematics Machines” activities were inspired by Emma Castelnuovo (1967), Conti & Giusti (1993), and Bartolini Bussi & Maschietto (2006). In mathematics, mechanical drawing machines are valuable tools for studying remarkable curves such as conic sections. In many cases, the design of the drawing machine mirrors the defining properties of the curves, which are thus perceived dynamically. A historical digression: In fact, Isaac Newton invented calculus (and dynamics) through a kinematic study of geometrical curves. So, in some sense, the mechanical study of curves was a cornerstone for the development of modern calculus. In the 17th century, this approach was known as geometria organica, where “organica” is derived from the Greek and means “mechanical.” But before Newton, a critical scholar who approached the mechanical study of curves was R. Descartes (Bos, 1981, 2001). Descartes places himself in a period straddling two opposing visions of Geometry: one linked to geometric constructions and the other to algebraic and symbolic manipulation. Descartes' work had suggested a standard for distinguishing between geometric curves, that is, those that are intuitive and analytically manageable, and mechanical ones, that is, in modern terms, transcendental curves (Milici, 2015). In his study of algebraic curves, he introduced some mathematical machines. Among the most significant are the hyperbolograph, used to draw hyperbolas, consisting of a rod constrained to slide along a straight groove and which drags with it three other rods, two grooved and one of fixed length, described in detail in the second book of La Géométrie, and the trisector, a tool made up of eight hinged rods designed to divide an angle into equal parts.

In the mathematical machines activities, primary students work in groups to design and build a dynamic mechanism that draws a sign on paper as it moves. They analyse which parts of the articulated mechanism change — such as areas and angles — and which stay constant, like the perimeter. All observations and reasoning are documented in a diary. The available materials include perforated wooden rods with five holes each, a paper fastener, a sheet of paper, and pencils. Each group starts by constructing a simple mathematical machine by attaching two rods to a sample holder, creating a device with a single degree of freedom that can draw a circle. They can vary the circle size by choosing different holes. Next, students are encouraged to create more complex machines by combining multiple rods, increasing the degrees of freedom. By placing the pencil into a hole and moving the mechanism, they observe the resulting curves, discovering how different arrangements of rods produce unique, more intricate curves. They realise that specific configurations enable them to design machines that generate distinct curves (Kempe, 1876).

At university

Bachelor students were encouraged to adopt the role of arpedonaptis, the Ancient Egyptian specialists tasked with re-establishing property boundaries after Nile floods. They used a geodetic rope and applied early geometric principles. Highly esteemed, they were considered expert surveyors and geometers (L. Russo et al., 2019; Conti & Giusti, 1993).

The activity involved delimiting an area with an ellipse. Participants used a rope and their body movements to implement the gardener's technique. They positioned themselves at the vertices of a triangle and connected these points with a string. By fixing two vertices and moving the third, they created a series of triangles equal in number to the steps taken. Similar to primary school exercises, this setup highlights the variables and constants involved in the transformation. The triangles formed are all isoperimetric; however, the area changes when transitioning from one triangle to another. To draw an ellipse, the moving vertex must continue in the same direction and return to its starting point. Adjusting the distance between the fixed vertices alters the ellipse's focal length. During this process, two degenerate cases occur when all three points are collinear. A similar experience can be achieved by working with a pen, a string and paper.

In the desk dimension, the students have the opportunity to reproduce the transformation and to measure the areas and perimeters of the triangles. From the measurements, some students deduced a property that characterises all the points on the curve: the sum of the distances from each point to the two foci remains constant. The difference between the two activities lies in the interface system. In the first, the entire body is involved, whereas in the second, ocular-manual abilities and coordination are solicited through gestures (Robutti et al., 2022). The hand plays a crucial role in writing and drawing, as it serves as an interface structure that connects our internal states, such as thoughts, to external states, including language and movement (Arcà & Guidoni, 1987). By observing the person's hand movements, we can gain insight into their character. The hand's movement can be fluid, flowing, awkward, or hesitant, and studying its movements can reveal valuable information about the individual. This is true regardless of the individual's age, whether they are a child, a teenager, or an adult.

DISCUSSION

This study explored how to harness the transformative power of transduction and measurement in mathematics education. The key connection between these activities is the shift from using the entire body for large, macro movements to manipulating small objects at a micro level. This progression from bodily transformations to fine motor skills helps students link big movements with precise tasks. Experiencing transformations physically and then relating them to hand manipulation deepens students' awareness of their bodies, reasoning, and understanding. Moreover, demonstrating geometric properties through collective body movements and then replicating them with a pen on paper enhances comprehension of the relationship between modelling, formalisation, and phenomenology. These practices help bridge the gap between intuitive, qualitative, and experiential aspects of reality and formal mathematical principles. The measurement process is essential in this regard, as it verifies the invariant or constant characteristics of objects.

While students are generally comfortable using their bodies, university students studying mathematics often find it challenging to incorporate bodily activities into their learning. Considering that many may later teach math, and the body plays a crucial mediating role, can they adopt methods that enable students to utilise their bodies in learning? Achieving this requires collaboration among physics and mathematics education researchers to develop innovative teaching strategies and communication methods, ensuring all students feel engaged and supported. By training teachers, researchers, and educators, these practical applications can inspire new strategies and practices, fostering a deeper understanding of phenomenology and mathematical objects.

In conclusion, adopting a transduction perspective in science and math education involves reexamining the fundamental and epistemological aspects of mathematics and physics to craft new teaching and learning approaches. Viewing phenomena through a transduction lens requires revisiting the core epistemological foundations of both physics and mathematics.

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