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# On the hierarchical nature of partial preferences over lotteries

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**Abstract** In this work we consider preference relations that might not be total. Partial preferences may be helpful to represent those situations where, due to lack of information or vacillating desires, the decision maker would like to maintain different options "alive" and defer the final decision. In particular, we show that, when totality is relaxed, different axiomatizations of classical Decision Theory are no longer equivalent but form a hierarchy where some of them are more restrictive than others. We compare such axiomatizations with respect to theoretical aspects—such as their ability to propagate comparability/incomparability over lotteries and the induced topology—and to different preference elicitation methodologies that are applicable in concrete domains. We also provide a polynomial-time procedure based on the bipartite matching problem to determine whether one lottery is preferred to another.

**Keywords** Decision Theory · Partial preferences · Preference elicitation

## 1 Introduction

In recent years, economic paradigms have become a fruitful area of research in Multiagent Systems. Decision Theory and its axiomatic formalization of preferences play a foundational role in such paradigms, thus, when we *import* an economic paradigm into our field, we are tacitly importing such a model of agents' preferences.

In his pioneering work on Decision Theory [40], when delineating the very basic properties of a preference relation $\prec$, Savage makes the following point: given two potential outcomes $f$ and $g$, it cannot be the case that $f \prec g$ and $g \prec f$ at the same time. Clearly, this is logically equivalent to saying that either $f \not\prec g$ or $g \not\prec f$, which leads to three possible cases: *(i)* $f \not\prec g$ and $g \prec f$, *(ii)* $f \prec g$ and $g \not\prec f$, or *(iii)* $f \not\prec g$ and $g \not\prec f$. Then, he postulates that these three cases are the only possible judgements concerning $f$ and $g$. In particular, the last

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case ($f \not\prec g$ and $g \not\prec f$) allegedly implies that $f$ and $g$ are equivalent in the sense that in any situation wherein these are the only two possible options, the decision maker does not mind delegating to coin flipping. Consequently, in classical Decision Theory a very fundamental property of a preference relation is its totality.

From the theory's very start, this model of an *economic man* has raised some criticisms, among which Simon's was one of the most influential:

> This man is assumed to have knowledge of the relevant aspects of his environment which, if not absolutely complete, is at least impressively clear and voluminous. He is assumed also to have a well-organized and stable system of preferences, and a skill in computation that enables him to calculate, for the alternative courses of action that are available to him, which of these will permit him to reach the highest attainable point on his preference scale [44].

Based on his criticism, Simon moved towards a problem-solving perspective which was extremely influential in the field of Artificial Intelligence.[1] However, it would be wrong to look at classical Decision Theory and Problem Solving as Eteocles and Polynices.[2] To some extent, Simon's criticism was already considered by von Neumann and Morgenstern:

> It is conceivable and may even in a way be more realistic to allow for cases where the individual is neither able to state which of two alternatives he prefers nor that they are equally desirable. How real this possibility is, both for individuals and for organizations, seems to be an extremely interesting question, but it is a question of fact. It certainly deserves further study [46].

Several relevant application domains would advocate the foregoing considerations. Preferences often result from complex trade-offs between different attributes (functionalities, cost, Quality of Service, information disclosure risks, etc.) that a generic user may only have a vague idea of. Moreover, in some cases the decision maker is actually a collectivity whose internal debate does not always lead to a total preference [43]. Finally, sometimes our judgement is simply made hesitant by vacillating desires. In all these cases, possible lack of knowledge, multi-person preferences and introspective sensations of vacillation would naturally recommend a relaxation of the totality property, admitting that some pairs of outcomes may be incomparable. In this work we support this perspective, however, before undertaking our study, a preliminary step requires us to deeply understand why decision theorists have, for the most part, assumed totality [18,22]. We clarify this point with an example. Assume you are in a restaurant that offers exotic food. To make it simple, you can only choose between two dishes, say *pipi cruschi* and *stigghiola*, that you know nothing about. If someone asks *do you like them the same?* you would probably reply *How can I know?* This seems to suggest that, from an introspective viewpoint, you do not judge them as equivalent. However, if you are starving and we only look at how you actually behave, in this situation you would not mind delegating to coin flipping, which is just the same behavior you adopt in case of equivalence. In summary, classical Decision Theory does not claim to give an account of the entire introspective process leading to a decision, it only aims at justifying your behaviour *at the moment* a decision is made.

[1] Together with Newell, he won the Turing Award in 1975 for his contributions on "artificial intelligence, the psychology of human cognition, and list processing" (http://amturing.acm.org/award_winners/simon_1031467.cfm).

[2] In Greek mythology, Eteocles and Polynices, sons of Oedipus, contended for the city of Thebes and killed each other in battle.

Nevertheless, this behaviouristic perspective, where preferences are modelled only on the base of observable choices, can be very restrictive in several contexts and does not provide an adequate explanation of some common phenomena. Consequently, a research history replaced "revealed" preferences with "multiperson" or "psychological" preferences that are assumed to be incomplete [3,15,25,29,43]. Moreover, there is a fundamental aspect which distinguishes the MAS perspective from Decision Theory. Unlike Decision Theory, where decision makers *own* their preferences, we generally aim at developing software agents acting *on behalf of* real users. For instance, in an electronic market, a software agent may apply an auction mechanism on behalf of a consumer, to filter the offers of the providers [7,8]. This clearly raises the issue of how user preferences can be effectively transferred to software agents, considering that web offers can be plenty and diversified. If the delegated agent only admits total preferences, a user would be forced to disambiguate all (possibly unexpected) offers and we have seen that one is introspectively reluctant to identify indecisions—either due to lack of knowledge or vacillating desires—with indifference. Consequently, as Simon suggests, a user is induced to acquire all the relevant information and dispel her vacillations to make a stable and accurate decision. Clearly, this can be a very hard and time-consuming task. In this case it would be probably more profitable if the software agent could use your still imperfect and vacillating preferences to filter out as many offers as possible so that you need to acquire further information only on what is left.

Furthermore, even if a user had no indecisions and conformed with a classical economic man, her preference could be so voluminous and irregular that she cannot be expected to have the patience to provide it entirely. In this respect, it is worth noting that the methodologies used to efficiently extract preference information from a user, in particular ceteris paribus (CP) nets and its extensions [9], generally do not force totality.

The foregoing arguments lead to the following consideration: if classical Decision Theory considers three possible options as sufficient (choose $a$, choose $b$, and flip a coin), we consider a fourth one for the delegated agent, namely, deferring the decision and asking the user again. This involves distinguishing between indifference and incomparability.

Besides foundational issues and motivations, you may think that, from a technical point of view, dealing with partial preferences is a straightforward task: it simply suffices to replace totality with reflexivity. This might be true when preferences range over certain outcomes [8,34], but in uncertain environments, where an outcome is actually a *lottery* of possible alternatives, we show that this is no longer the case. Indeed, classical Decision Theory can be modelled by several axiomatizations which turn out to be variants of the same theory. However, we show that their equivalence is inherently guaranteed by the totality axiom; as a consequence, by weakening totality in favour of reflexivity, the resulting axiomatizations characterize different notions of partial preferences. In this work we focus on six different axiomatizations, two of them were introduced in previous works [3,15] and the other ones derive from well known classical formulations [22,27]. Clearly, any debate concerning which of these theories should be adopted in a certain application domain would profit from a detailed comparison of their properties. Thus, we first characterize a hierarchical structure which shows that some axiomatizations are more restrictive than others. Then, we refine our investigation by analysing the "distance" from classical Decision Theory in terms of the ability to propagate comparability over mixtures of lotteries. This also emphasizes some relevant properties that distinguish one theory from another from a topological point of view.

These formal properties have a concrete impact when we integrate one of the considered axiomatizations with a preference elicitation methodology, such as the aforementioned CP nets. In particular, depending on which kind of preference statements a user can assert, some axiomatizations are safer, in the sense that they never conflict the user's statements, some others easily lead to an inconsistency, and yet others indirectly reintroduce the property of totality. Finally, we provide a procedure based on the bipartite matching problem to determine whether one lottery is preferred to another. In general, this procedure is correct and for a particular axiomatization and a large class of preference elicitation processes is also complete.

It is worth to note that the present work focuses on research problems that are somewhat different from the ones that are typically addressed in Decision Theory. In Decision Theory the main problem is how or at which extent we can gain back the classical utility representation of a partial preference. This analysis is highly theoretical and largely not constructive. Conversely, according to a MAS perspective, we are clearly interested on concrete methodologies which allow to algorithmically compare two lotteries. Moreover, since several elicitation methodologies are intrinsically not numerical [9,32,33], the utility representation problem can be a secondary issue.

In Sect. 2, we provide two scenarios that motivate partial preferences. In Sect. 3, we introduce four equivalent variants of classical Decision Theory and expose the mutual dependencies among their axioms. Section 4 shows the hierarchical nature of the different notions of partial preferences obtained from the aforementioned classical variants by replacing totality with reflexivity. Section 5 shows that some basic notions of dominance hold in all the considered theories, and conversely, Sects. 6 and 7 emphasize structural and topological differences. Section 8 focuses on different preference elicitation methodologies combined with the considered theories; it also analyses consistency, comparability and optimality problems. Related works and conclusions end the paper.

## 2 Motivating scenarios

In this section we introduce two scenarios where partial preferences (as opposed to total preferences) seem to provide a more natural way to describe a decision maker, or a software agent behaving on its behalf.

In the first scenario, Alice's father has finally agreed to buy her a smartphone, and now she is browsing Ebay for possible offers. Unfortunately, the list is huge and patience is not Alice's forte. So, she would like to be assisted by a software agent to filter out undesired options. The software agent accepts constraints such as maximum cost and size, color restrictions, etc., and also a preference relation as a total order over offers. Then, according to the specified preference relation, the agent returns the best offer. Clearly, Alice's desires are influenced by several aspects such as operating system, color, weight, brand, and so on. For instance, she has a preference over operating systems in the following decreasing order: OS1, OS2, and OS3; over colors: blue, red, black, white; and over brands: Brand1, Brand2, Brand3. Furthermore, out of benevolence for her father, given a specific model, the cheaper the better. However, such preferences over single attributes do not constitute a total order. Moreover, Alice cannot establish a priority over attributes, for instance she prefers a Brand3 phone with operating system OS1 to a Brand2 with operating system OS2, but she prefers a Brand1 OS2 phone to a Brand2 OS1. Alice soon realizes that providing the software agent a total order is frustrating and requires about the same effort as comparing all the offers herself. This scenario reveals the following issue: in designing a software agent that behaves on behalf of real users, we have consider how users can *instruct* the agent about their own preferences. In electronic markets where the number of offers is huge, it could be unfeasible to transfer an exact representation of users' desires into a software agent. In this case, the agent should make do with an approximate representation of users' desires as a partial order and return a restricted list of choices from which the user can select the preferred one. As a further advantage, the user retains the ability to apply unforeseen, situation-specific knowledge and preferences that have not been formalized in advance.

In the second scenario, a molecular biology laboratory is willing to use a cloud computing infrastructure to speed up its on-going research on the human genome. The lab has to choose among an increasing number of IaaS providers, each of which proposes dozens of different solutions. Resource provisioning is often regulated by a Service Level Agreement (SLA), that is, a contract between customers and service providers which describes all relevant aspects of the agreement. Typically, a SLA specifies several functional and non-functional properties including performance features (CPU typology, number of cores, memory, storage etc.), security (how all stored and transmitted data are encrypted), availability (percentage of uptime per year), and legal aspects concerning, for instance, ineffectiveness of customer protection or conditions for rescission. Moreover, beside the usual fixed-price options, some providers (e.g. Amazon EC2) offer bid-based solutions where the price of a service is supply-demand based. Generally, to meet customers' needs, these providers disclose the price history of bid-based services so that it is possible to determine the probability to get a SLA at a given price. As in the previous example, it is hard to ponder all these heterogeneous factors. Furthermore, an accurate decision requires a laborious investigation of technical specifications, benchmarks, encryption algorithms, possible legal pitfalls and so on. However, some background knowledge and reasonable common sense rules would suffice to remarkably reduce the number of promising offers which deserve to be examined in depth. Therefore, automated service procurement can be effective only if preferences over SLAs are specified in a flexible (i.e. potentially partial) way. Furthermore, in presence of bid-based offers, a software agent has to be able to compare lotteries of SLAs, each of which specifies a certain price.

## 3 Preliminaries

In this section we provide some basic notions on lotteries and introduce six distinct formalizations of classical Decision Theory. Then, by proving the mutual equivalence of these formalizations, we study in detail the various interdependencies between their axioms. These results will be used in the next section where we weaken classical Decision Theory by allowing preferences to be partial.

Let $\mathcal{A}$ be a finite set of alternatives and $\Delta(\mathcal{A})$ the set of all probability distributions over $\mathcal{A}$ (that is, $\Delta(\mathcal{A})$ is the set of all functions $f : \mathcal{A} \to [0, 1]$ such that $\sum_{a\in\mathcal{A}} f(a) = 1$). In the Decision Theory jargon the elements of $\Delta(\mathcal{A})$ are also called *lotteries*.

Lotteries can be naturally embedded into the vector space $\mathbb{R}^N$, where $N$ is the cardinality of $\mathcal{A}$, provided with the usual sum and product operations $(f + g)(a) = f(a) + g(a)$ and $(\alpha f)(a) = \alpha f(a)$, for any real number $\alpha$ and alternative $a \in \mathcal{A}$. In particular, given a finite set of lotteries $F = \{f_1, \ldots, f_n\} \subset \Delta(\mathcal{A})$ and a sequence of positive real numbers $\alpha_1, \ldots, \alpha_n$, with $\alpha_1 + \cdots + \alpha_n = 1$, the lottery $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$ is called a *convex combination* (a.k.a. *mixture*) of $F$. Thereafter, by $\mathcal{H}_c(F)$ we denote the convex hull generated by $F$, that is, the set of all possible convex combinations of $F$.

We mean with $[a]$ the degenerate distribution that assigns probability 1 to $a$ and probability 0 to all the other alternatives. In particular, since each lottery $f \in \Delta(\mathcal{A})$ is equal to $\sum_{a\in\mathcal{A}} f(a)[a]$, $\Delta(\mathcal{A})$ is the convex hull generated by the set of degenerate lotteries

$[\mathcal{A}] = \{[a] \mid a \in \mathcal{A}\}$. Thereafter, for the sake of brevity, we write $\langle \alpha, f, g \rangle$ to indicate the mixture $\alpha f + (1 - \alpha)g$, with $0 < \alpha < 1$.

A (classical) preference relation $\preceq$ is a binary relation over lotteries in $\Delta(\mathcal{A})$ which satisfies the following axioms:

1. $f \preceq g$ or $g \preceq f$;
2. if $f \preceq g$ and $g \preceq h$, then $f \preceq h$;
3. for all $0 < \alpha < 1$ and $h$, $f \preceq g$ iff $\langle \alpha, f, h \rangle \preceq \langle \alpha, g, h \rangle$;
4. if $h \prec \langle \alpha, f, g \rangle$ (resp. $\langle \alpha, f, g \rangle \prec h$), for all $0 < \alpha < 1$, then $f \not\prec h$ (resp. $h \not\prec g$).

where as usual, $f \prec g$ means that $f \preceq g$ and $g \not\preceq f$, whereas $f \sim g$ means that $f \preceq g$ and $g \preceq f$.

The first two axioms force $\preceq$ to be a total and transitive relation, that is a total preorder.[3] Axiom 3 states that the preference between $f$ and $g$ is preserved by "diluition" with $h$ and, conversely, if we have a certain preference between diluited lotteries, the same preference holds between the undiluited lotteries $f$ and $g$. Finally, Axiom 4 is a very weak form of continuity, it just prevents the case where the strict preference between a lottery $h$ and the segment of all convex combinations involving $f$ and $g$ goes in one direction for all internal points and in precisely the opposite direction for one end-point.

Axioms 1–4 are not the only way to define classical preference relations. Different equivalent axiomatizations have been proposed. For instance in [27] Myerson replaces 3 with axioms

5. if $f_1 \preceq g_1$, $f_2 \preceq g_2$, and $0 < \alpha < 1$, then $\langle \alpha, f_1, f_2 \rangle \preceq \langle \alpha, g_1, g_2 \rangle$;
6. if $f_1 \prec g_1$, $f_2 \preceq g_2$, and $0 < \alpha < 1$, then $\langle \alpha, f_1, f_2 \rangle \prec \langle \alpha, g_1, g_2 \rangle$;

and Axiom 4 with the axiom

7. if $h \prec f$ and $f \prec g$, then there exists some $0 < \alpha < 1$ such that $f \sim \langle \alpha, g, h \rangle$.

Axiom 5 asserts a substitution principle: given a lottery between $f_1$ and $f_2$, if we substitute each of them with a better lottery (i.e. $g_1$ and $g_2$, respectively), then we obtain a better lottery. Axiom 6 is a strict version of 5. Axiom 7 is a different notion of continuity assuring that the convex combinations of two lotteries $f$ and $g$ encompass all possible gradations of the preference relation between $f$ and $g$.

Finally, in [15] continuity is formalized in terms of convergent sequences: given two sequences of lotteries $\{f_n\}_{n\in\mathbb{N}} \longrightarrow f$ and $\{g_n\}_{n\in\mathbb{N}} \longrightarrow g$ which converge to $f$ and $g$ according to the standard euclidean distance in $\mathbb{R}^N$:

8. if for all $n \in \mathbb{N}$ $f_n \preceq g_n$, then $f \preceq g$.

In general, it is easy to see that Axiom 8 implies the notion of continuity used in [42]: for all $f_1, f_2, g_1, g_2 \in \Delta(\mathcal{A})$, the set $\{\alpha \in [0, 1] \mid \langle \alpha, f_1, f_2 \rangle \preceq \langle \alpha, g_1, g_2 \rangle\}$ is closed in $[0, 1]$. However, it is proved in [15] that if $\mathcal{A}$ is finite, as in our settings, then the two notions of continuity are equivalent.

Clearly, we can also consider mixed cases where only Axiom 3 is replaced with Axioms 5–6 (respectively, Axiom 4 with 7 or 8). Doing so, we obtain six different axiomatizations, namely $\mathrm{CDT}_0$–$\mathrm{CDT}_5$ (see Table 1).

Generally speaking, we say that a set of axioms $T_2$ *entails* another set of axioms $T_1$, $T_1 \leq T_2$, if and only if all the preference relations satisfying $T_2$ satisfy $T_1$ as well. We show that $\mathrm{CDT}_0$–$\mathrm{CDT}_5$ are equivalent in the sense that they characterize the same set of preference

[3] A preorder is a reflexive and transitive relation, clearly Axiom 1 implies reflexivity.

**Table 1** The six axiomatizations of classical preference relations $CDT_0$, $CDT_1$, $CDT_2$, $CDT_3$, $CDT_4$, and $CDT_5$

| | Axioms 1, 2 | Axiom 3 | Axioms 5 and 6 | Axiom 4 | Axiom 7 | Axiom 8 |
|---|---|---|---|---|---|---|
| $CDT_0$ | ✓ | | ✓ | ✓ | | |
| $CDT_1$ | ✓ | ✓ | | ✓ | | |
| $CDT_2$ | ✓ | | ✓ | | ✓ | |
| $CDT_3$ | ✓ | ✓ | | | ✓ | |
| $CDT_4$ | ✓ | | ✓ | | | ✓ |
| $CDT_5$ | ✓ | ✓ | | | | ✓ |

relations. First, we need some preliminary results that point out the interdependencies among the axioms introduced above. In particular, we will take care to use a minimal set of axioms that imply another axiom. In so doing, we will also use these results in Sect. 4 to show the hierarchical nature of partial preferences.[4] Lemma 1 shows that to obtain Axioms 5 and 6 from Axiom 3, totality is not needed. In particular, any transitive relation which satisfies Axiom 3 also satisfies Axioms 5 and 6.

**Lemma 1** *Axioms 2 and 3 imply Axioms 5 and 6.*

*Proof* Assume that $f_1 \preceq g_1$, $f_2 \preceq g_2$, and $0 < \alpha < 1$. Due to Axiom 3 and $f_1 \preceq g_1$, $\langle \alpha, f_1, f_2 \rangle \preceq \langle \alpha, g_1, f_2 \rangle$. Analogously, Axiom 3 and $f_2 \preceq g_2$ imply that $\langle \alpha, g_1, f_2 \rangle \preceq \langle \alpha, g_1, g_2 \rangle$. Finally, by transitivity $\langle \alpha, f_1, f_2 \rangle \preceq \langle \alpha, g_1, g_2 \rangle$, hence Axiom 5 holds.

Assume that $f_1 \prec g_1$, $f_2 \preceq g_2$, and $0 \leq \alpha \leq 1$. By definition, $f_1 \prec g_1$ means that $f_1 \preceq g_1$ and $g_1 \not\preceq f_1$. Then, by applying Axiom 3 in both directions we have that $\langle \alpha, f_1, f_2 \rangle \prec \langle \alpha, g_1, f_2 \rangle$. As in the previous case, it also holds that $\langle \alpha, g_1, f_2 \rangle \preceq \langle \alpha, g_1, g_2 \rangle$, hence, by transitivity, $\langle \alpha, f_1, f_2 \rangle \prec \langle \alpha, g_1, g_2 \rangle$. □

The following lemma goes in the opposite direction, however to obtain Axiom 3 from 5 and 6 we require $\preceq$ to be a total relation.

**Lemma 2** *Axioms 1, 5, and 6 imply Axiom 3.*

*Proof* Let $h$ be a lottery and $0 < \alpha < 1$. Let $f_1 = f$, $g_1 = g$ and $f_2 = g_2 = h$, if $f \preceq g$, then by Axiom 5 we have that $\langle \alpha, f, h \rangle \preceq \langle \alpha, g, h \rangle$.

Conversely, assume that for some lottery $h$ and $0 < \alpha < 1$, $\langle \alpha, f, h \rangle \preceq \langle \alpha, g, h \rangle$. We apply the contrapositive of Axiom 6, from $\langle \alpha, g, h \rangle \not\prec \langle \alpha, f, h \rangle$ we obtain $g \not\prec f$. Then, since $\preceq$ is a total relation, it holds that $f \preceq g$. □

The following two lemmas concern Axioms 4 and 7. As expected, Axiom 7 defines a stronger notion of continuity which alone suffices to derive Axiom 4. Conversely, we can get Axiom 7 back from 4 with the help of totality and the strict substitution principle.

**Lemma 3** *Axiom 7 implies Axiom 4.*

*Proof* By contradiction, assume that $f \prec h$ and for all $0 < \alpha' < 1$, $h \prec \langle \alpha', f, g \rangle$. Let $k = \langle \alpha, f, g \rangle$, for some $0 < \alpha < 1$. Since $f \prec h$ and $h \prec k$, by Axiom 7 there

[4] To the best of our knowledge, such an extensive analysis of axioms' interdependencies has not been shown elsewhere.

exists a $\beta \in (0, 1)$ such that $h \sim \langle \beta, f, k \rangle$. Therefore, we have that $h \sim \langle \gamma, f, g \rangle$, where $\gamma = \beta + (1 - \beta)\alpha$ belongs to the interval $(\alpha, 1)$. This is a contradiction because for all $\alpha' \in (\alpha, 1)$ $h \prec \langle \alpha', f, g \rangle$.

Analogously, it can be proved that if for all $0 < \alpha' < 1$, $\langle \alpha', f, g \rangle \prec h$, then $h \not\prec g$. Consequently, Axiom 4 holds. □

**Lemma 4** *Axioms 1, 6 and 4 imply Axiom 7.*

*Proof* Assume $h \prec f \prec g$, Let $A = \{\alpha \in [0, 1] \mid f \prec \langle \alpha, g, h \rangle\}$ and $B = \{\alpha \in [0, 1] \mid \langle \alpha, g, h \rangle \prec f\}$. Clearly, $A \cap B = \emptyset$, $1 \in A$ and $0 \in B$.

Assume that $\beta_1$ and $\beta_2$ are in $A$ and $\beta_1 < \beta_2$, this means that $f \prec h_1$ and $f \prec h_2$ where $h_1 = \langle \beta_1, g, h \rangle$ and $h_2 = \langle \beta_2, g, h \rangle$. It is easy to verify that for each $\beta_1 < \beta < \beta_2$, there exists an $0 < \alpha' < 1$ such that $\langle \beta, g, h \rangle = \langle \alpha', h_1, h_2 \rangle$. Then, by applying Axiom 6 with $f_1 = f_2 = f$, $g_1 = h_1$, and $g_2 = h_2$, it holds that $f \prec \langle \alpha', h_1, h_2 \rangle$. Consequently, $A$ is an interval of $[0, 1]$. Similarly, it can be shown that $B$ is an interval as well.

Let $\delta$ be the greatest lower bound of $A$ and $\gamma$ the least upper bound of $B$—clearly, the least upper bound of $A$ is 1 and the greatest lower bound of $B$ is 0. In case $\gamma < \delta$, by the totality axiom we would have that for all $\lambda \in (\gamma, \delta)$ $f \sim \langle \lambda, g, h \rangle$ and hence the thesis. Assume then that $\gamma = \delta$. Let $k = \langle \gamma, g, h \rangle$, notice that for each $0 < \alpha' < 1$, $\langle \alpha', k, h \rangle \prec f$ and $f \prec \langle \alpha', g, k \rangle$, then by Axiom 4 $f \not\prec k$ and $k \not\prec f$. By totality, $f \not\prec k$ implies $k \preceq f$ and $k \not\prec f$ implies $f \preceq k$, consequently $f \sim k$ and hence the thesis. □

The following lemmas show that also Axiom 8 implies Axiom 4. On the contrary, to obtain Axiom 8 we consider the whole theory $\mathrm{CDT}_2$ and use the well-known expected utility theorem [27]. A utility $u : \mathcal{A} \to \mathbb{R}$ is a function from the set of alternatives $\mathcal{A}$ to real numbers. Since both a utility function $u$ and a lottery $f$ can be seen as vectors of $\mathbb{R}^N$, where the set of degenerate lotteries $[\mathcal{A}]$ is a basis, we define as usual the scalar product $f \cdot u = \sum_{a \in \mathcal{A}} f(a)u(a)$. Note that the scalar product $f \cdot u$ is nothing more than the expected utility of $u$ according to the probability distribution $f$.

**Lemma 5** *Axiom 8 implies Axiom 4.*

*Proof* Assume that for all $0 < \alpha < 1$ $h \prec \langle \alpha, f, g \rangle$. Then, let $h_n = h$ and $f_n = \langle 1 - \frac{1}{n}, f, g \rangle$, with $n \in \mathbb{N}$. Clearly, $\{h_n\}_{n \in \mathbb{N}}$ and $\{f_n\}_{n \in \mathbb{N}}$ converge to $h$ and $f$, respectively. Moreover, since for all $n \in \mathbb{N}$ $h_n \prec f_n$, by Axiom 8 $h \preceq f$ and hence a fortiori $f \not\prec h$. □

**Lemma 6** *$\mathit{CDT}_2$ implies Axiom 8.*

*Proof* Let $\{f_n\}_{n \in \mathbb{N}}$ and $\{g_n\}_{n \in \mathbb{N}}$ be two sequences of lotteries which converge to $f$ and $g$, respectively, and assume that $f_n \preceq g_n$, for all $n \in \mathbb{N}$. By applying the expected utility theorem, we have that the exists a $u \in \Delta(\mathcal{A})$ such that $f_n \cdot u \leq g_n \cdot u$. Then, by continuity of the scalar product $f_n \cdot u$ converges to $f \cdot u$ and $g_n \cdot u$ converges to $g \cdot u$. Finally, since for all $n \in \mathbb{N}$ $f_n \cdot u \leq g_n \cdot u$, we also have that $f \cdot u \leq g \cdot u$ and hence by applying again the expected utility theorem in the other direction $f \preceq g$. □

As final result, we use the previous lemmas to prove that, as expected, $\mathrm{CDT}_0$–$\mathrm{CDT}_5$ model the same class of relations.

**Theorem 1** *Theories $\mathit{CDT}_0$–$\mathit{CDT}_5$ are equivalent.*

*Proof* First, we show that $\mathrm{CDT}_0$, $\mathrm{CDT}_1$, $\mathrm{CDT}_2$, and $\mathrm{CDT}_3$ are equivalent. Let $\preceq$ be a relation satisfying $\mathrm{CDT}_0$. By the definition of $\mathrm{CDT}_0$ and Lemma 2, $\preceq$ satisfies Axiom 3 and hence

$\mathrm{CDT}_1$. Assume now that $\preceq$ satisfies $\mathrm{CDT}_1$, by Lemma 1 $\preceq$ satisfies Axioms 5 and 6. Therefore, due to Lemma 4, $\preceq$ also satisfies Axiom 7 and hence $\mathrm{CDT}_3$. In turn, if $\preceq$ satisfies $\mathrm{CDT}_3$, by applying Lemma 1, it satisfies $\mathrm{CDT}_2$ as well. Finally, Lemma 3 assures that each $\preceq$ satisfying $\mathrm{CDT}_2$ satisfies $\mathrm{CDT}_0$. In summary, it holds that

$$\mathrm{CDT}_0 \geq \mathrm{CDT}_1 \geq \mathrm{CDT}_3 \geq \mathrm{CDT}_2 \geq \mathrm{CDT}_0.$$

Consequently, $\mathrm{CDT}_0$–$\mathrm{CDT}_3$ are equivalent.

Furthermore, by Lemmas 5 and 6, we have that $\mathrm{CDT}_0 \leq \mathrm{CDT}_4 \leq \mathrm{CDT}_2$ and, by Lemmas 1 and 2, $\mathrm{CDT}_5 \leq \mathrm{CDT}_4 \leq \mathrm{CDT}_5$. Consequently, also $\mathrm{CDT}_4$ and $\mathrm{CDT}_5$ are equivalent to the other theories. □

## 4 Axiomatizing partial preferences

In contrast to classical Decision Theory, in this section we consider the possibility that some lotteries are incomparable. As discussed in the introduction, different aspects may induce incomparability: vacillating desires, lack of knowledge, multi-person utilities. In these cases a user might not feel comfortable completely delegating her choice to a software agent, more likely she wants to use the software agent as a filter to eliminate dominated options. As shown in Sect. 2, providing a total preference can be a very long and involved task. Consequently, the methodologies that have been designed so far to inject the user's preference into a software agent, such as the CP nets and their extensions, generally do not force preferences to be total.

*Example 1* Consider the cloud computing scenario described in Sect. 2. For the sake of simplicity, we consider simplified SLAs represented by triples $(m; s; p)$ where $m$ indicates the allocated memory (8/16 GB), $s$ the storage (250/500 GB), and $p$ is the price (0.20/0.30/0.40 dollars per hour). We start with a minimal CP net stating that, ceteris paribus (all things being equal), the cheaper the better, and the more the memory/storage is granted, the better it is. This induces a preference over SLAs—or, more precisely, over the corresponding degenerate distributions—which is depicted in Fig. 1.

Then, assume that a user tells the agent that, for any given price, memory is more important than storage. This means that, given two SLAs $l_1 = (m_1; s_1; p_1)$ and $l_2 = (m_2; s_2; p_2)$, if $p_1 = p_2$ and $m_1 < m_2$, then $[l_1] \prec [l_2]$. The corresponding preference is shown in Fig. 2. Note that both the presented preferences can be easily formalized through Tradeoff-enhanced CP nets—we refer to [10] for formal details. Under a market-price policy, each hardware configuration is associated with a distribution of prices. Therefore, an offer is in general a lottery $f = \alpha_1[m; s; 0.20] + \alpha_2[m; s; 0.40] + \alpha_3[m; s; 0.60]$. □

Since lotteries may be incomparable, preference relations are modelled as (possibly partial) preorders. For this reason, we weaken totality in favour of reflexivity:

1'. $f \preceq f$;

and we focus on the theories $\mathrm{PDT}_0$–$\mathrm{PDT}_5$ obtained by replacing Axiom 1 with 1' in $\mathrm{CDT}_0$–$\mathrm{CDT}_5$, respectively. We will also consider a theory, $\mathrm{PDT}_6$, that is obtained by adding Axiom 8 to $\mathrm{PDT}_3$. As we will show in Sect. 7, $\mathrm{PDT}_6$ satisfies a particularly suited generalization of the expected utility theorem.

It is worth noting that the proof of $\mathrm{CDT}_0$–$\mathrm{CDT}_5$ equivalence makes use of Axiom 1 in Lemmas 2, 4, and 6. We show that, by weakening totality with reflexivity, $\mathrm{PDT}_4$ and $\mathrm{PDT}_5$ are still equivalent whereas the other theories are not.

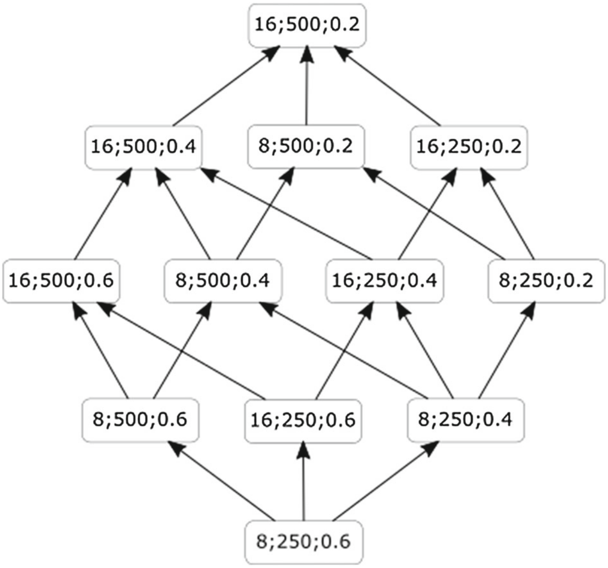



**Fig. 1** A preference relation in Example 1, *arrows* represent strict preference between SLAs

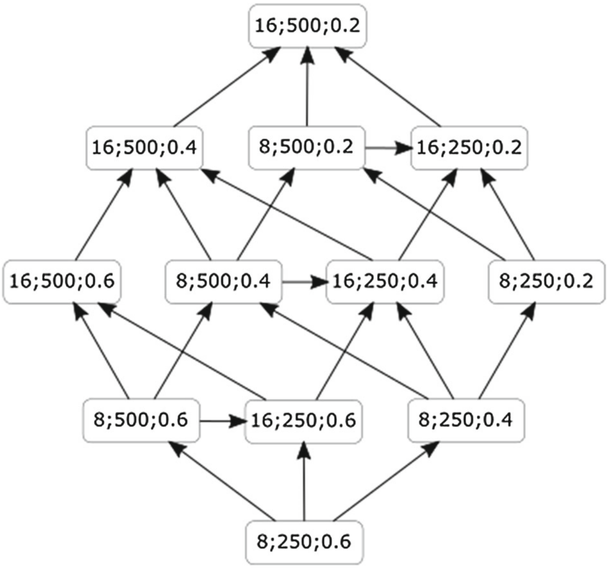



**Fig. 2** A preference relation in Example 1, *arrows* represent strict preference between SLAs

**Theorem 2** *$PDT_4$ and $PDT_5$ are equivalent.*

*Proof* That $\mathrm{PDT}_4 \leq \mathrm{PDT}_5$ derives from Lemma 1 whereas $\mathrm{PDT}_5 \leq \mathrm{PDT}_4$ is a direct consequence of Lemma 1 and Proposition 1 in [15]. □

Thereafter, since $\mathrm{PDT}_4$ and $\mathrm{PDT}_5$ are equivalent, for the sake of brevity we will denote them with $\mathrm{PDT}_{4/5}$. Table 2 summarizes the axiomatizations $\mathrm{PDT}_0$–$\mathrm{PDT}_6$.

**Table 2** The six axiomatizations of partial preference relations $PDT_0$, $PDT_1$, $PDT_2$, $PDT_3$, $PDT_{4/5}$, and $PDT_6$

| | Axioms 1', 2 | Axiom 3 | Axioms 5 and 6 | Axiom 4 | Axiom 7 | Axiom 8 |
|---|---|---|---|---|---|---|
| $PDT_0$ | ✓ | | ✓ | ✓ | | |
| $PDT_1$ | ✓ | ✓ | | ✓ | | |
| $PDT_2$ | ✓ | | ✓ | | ✓ | |
| $PDT_3$ | ✓ | ✓ | | | ✓ | |
| $PDT_{4/5}$ | ✓ | ✓ | | | | ✓ |
| $PDT_6$ | ✓ | ✓ | | | ✓ | ✓ |

The following lemmas will be used to show that $PDT_0$–$PDT_6$ lie in a hierarchy of strict entailments.

**Lemma 7** *There exists a preference relation that satisfies $PDT_{4/5}$ and does not satisfy Axiom 7.*

*Proof* Let $\mathcal{A} = \{a, b, c\}$ and $\preceq$ be the preference relation $\preceq$ satisfying the following two conditions:

$$f \sim g \text{ iff } f = g \tag{1}$$
$$f \prec g \ \text{ iff either } f(a) < g(a)\,, f(b) \leq g(b) \text{ or } f(a) \leq g(a)\,, f(b) < g(b). \tag{2}$$

First, we show that that $\preceq$ satisfies $PDT_5$. Reflexivity and transitivity are immediate. With respect to Axiom 3, we distinguish two cases. The first case concerns condition (1). For all $0 < \alpha < 1$ and $\forall h \in \Delta(\mathcal{A})$, the following chain of implications holds:

$$f \sim g \Leftrightarrow f = g$$
$$f = g \Leftrightarrow \langle \alpha, f, h\rangle = \langle \alpha, g, h\rangle$$
$$\langle \alpha, f, h\rangle = \langle \alpha, g, h\rangle \Leftrightarrow \langle \alpha, f, h\rangle \sim \langle \alpha, g, h\rangle\,.$$

The second case takes into account condition (2). Assume that $f \prec g$ and, in particular, $f(a) < g(a)$ and $f(b) \leq g(b)$. Then, for all $0 < \alpha < 1$ and $h \in \Delta(\mathcal{A})$ we have that $f(a) < g(a)$ and $f(b) \leq g(b)$ iff $\alpha f(a) + (1-\alpha)h(a) < \alpha g(a) + (1-\alpha)h(a)$ and $\alpha f(b) + (1-\alpha)h(b) \leq \alpha g(b) + (1-\alpha)h(b)$. The other case, $f(a) \leq g(a)$ and $f(b) < g(b)$, is symmetrical. Consequently, $f \prec g$ iff $\langle \alpha, f, h\rangle \prec \langle \alpha, g, h\rangle$.

With respect to Axiom 8, let $\{f_n\}_{n\in\mathbb{N}} \longrightarrow f$ and $\{g_n\}_{n\in\mathbb{N}} \longrightarrow g$ and assume that $f_n \preceq g_n$ for all $n \in \mathbb{N}$. If there exists $m$ such that $f_n \sim g_n$, for all $n > m$, then by (1) $f_n = g_n$ which means that $f = g$ and hence $f \sim g$. Therefore, assume that there exist two infinite sub-sequences $\{f_{n_i}\}_{i\in\mathbb{N}}$ and $\{g_{n_i}\}_{i\in\mathbb{N}}$ such that $f_{n_i} \prec g_{n_i}$, for all $i \in \mathbb{N}$. As a consequence of condition (2), we have that that $f_{n_i}(a) \leq g_{n_i}(a)$ and $f_{n_i}(b) \leq g_{n_i}(b)$. Moreover, since $\{f_{n_i}\}_{i\in\mathbb{N}} \longrightarrow f$ and $\{g_{n_i}\}_{i\in\mathbb{N}} \longrightarrow g$, it also holds $f(a) \leq g(a)$ and $f(b) \leq g(b)$. Then, if $f(a) < g(a)$ or $f(b) < g(b)$, by condition (2), $f \prec g$. Conversely, if $f(a) = g(a)$ and $f(b) = g(b)$, then also $f(c) = g(c)$ and hence $f \sim g$.

It remains to prove that $\preceq$ does not satisfy Axiom 7. Let $f$, $g$ and $h$ be defined as follows:

$$\begin{aligned} g &= \frac{1}{2}[a] + \frac{1}{2}[b] \\ f &= [c] \\ h &= \frac{1}{8}[a] + \frac{1}{4}[b] + \frac{5}{8}[c] \end{aligned}$$

By condition 2, $f \prec h \prec g$ and by condition 1 $\langle \alpha, f, g\rangle \sim h$ iff $h = \langle \alpha, f, g\rangle$. However, $a$ and $b$ have the same probability in any convex combination of $f$ and $g$, whereas they have different probabilities in $h$. This means that there does not exist an $\alpha$ such that $h = \langle \alpha, f, g\rangle$. □

**Lemma 8** *There exists a preference relation that satisfies PDT$_2$ and does not satisfy Axioms 3 and 8.*

*Proof* Let $\mathcal{A} = \{a, b\}$ and $\preceq$ be the preference relation defined as follows, for all $f, g \in \Delta(\mathcal{A})$,

$$f \preceq g \text{ iff } \frac{1}{g(b)} \leq \frac{1}{f(b)} \text{ or } f = g = [a].$$

Notice that, since $\frac{1}{0}$ is undefined, $[a]$ is comparable only with itself. Moreover, in case $f \neq [a]$ and $g \neq [a]$, we have that $g(b) - f(b) \geq 0$ (resp. $g(b) - f(b) > 0$) iff $f \preceq g$ (resp. $f \prec g$).

First, we show that all the axioms of PDT$_2$ are satisfied. Reflexivity and transitivity hold by construction. Assume that $f_1 \preceq g_1$ and $f_2 \preceq g_2$. For some $0 < \alpha < 1$, let $h = \langle \alpha, f_1, f_2\rangle$ and $k = \langle \alpha, g_1, g_2\rangle$. Then, $h(b) - k(b) = \alpha(g_1(b) - f_1(b)) + (1 - \alpha)(g_2(b) - f_2(b))$. Now, consider the following four cases: Case (1) $f_1 = [a]$ and $f_2 = [a]$. In this case also $g_1 = g_2 = [a]$, therefore trivially $h \preceq k$. Case (2) $f_1 = [a]$ and $f_2 \neq [a]$. Then, $g_1 = [a]$ and $g_2 \neq [a]$ which means that both $h$ and $k$ are not $[a]$. Moreover, since $h(b) - k(b) = (1 - \alpha)(g_2(b) - f_2(b))$ and $g_2(b) - f_2(b)$ is a positive number by hypothesis, it holds that $h \preceq k$. Case (3) $f_1 \neq [a]$ and $f_2 = [a]$. Analogously to the previous case, it follows that $h(b) - k(b) = \alpha(g_1(b) - f_1(b)) \geq 0$, $h \preceq k$. Case (4) $f_1 \neq [a]$ and $f_2 \neq [a]$. This means that also $g_1 \neq [a]$ and $g_2 \neq [a]$ and hence both $g_1(b) - f_1(b)$ and $g_2(b) - f_2(b)$ are positive by construction. Then, $h(b) - k(b) = \alpha(g_1(b) - f_1(b)) + (1 - \alpha)(g_2(b) - f_2(b)) \geq 0$, which means that $h \preceq k$. Since in all four cases $h \preceq k$, Axiom 5 holds. Analogously, in can be shown that Axiom 6 is satisfied as well.

Finally, let the antecedent of Axiom 7 hold, $h \prec f \prec g$. Since $[a]$ is comparable with itself only, we have that $h \neq [a]$ and consequently $0 < h(b) < f(b) < g(b) \leq 1$. Then, by setting $\alpha = \frac{f(b)-h(b)}{g(b)-h(b)}$, it is easy to verify that $\alpha$ belongs to the interval (0, 1) and $\alpha h(b) + (1 - \alpha)g(b) = f(b)$. Consequently, $f \sim \langle \alpha, g, h\rangle$.

It remains to prove that Axioms 3 and 8 are not satisfied. With respect to Axiom 3, let $f = [a]$, $g = [b]$, $h = \frac{1}{2}[a] + \frac{1}{2}[b]$ and $\alpha = \frac{1}{2}$. We have that $\langle \alpha, f, h\rangle = \frac{3}{4}[a] + \frac{1}{4}[b]$ and $\langle \alpha, g, h\rangle = \frac{1}{4}[a] + \frac{3}{4}[b]$, consequently $\langle \alpha, f, h\rangle \preceq \langle \alpha, g, h\rangle$. However, by construction $f$ and $g$ are incomparable, hence $f \not\preceq g$.

With respect to Axiom 8, let $f_n = \langle 1 - \frac{1}{2n}, [a], [b]\rangle$ and $g_n = \langle \frac{1}{2} - \frac{1}{2n}, [a], [b]\rangle$, with $n \in \mathbb{N}$. Since $f_n(b) = \frac{1}{2n}$ and $g_n(b) = \frac{1}{2} + \frac{1}{2n}$, we have that $f_n \prec g_n$ for all $n \in \mathbb{N}$. Moreover, $\{f_n\}_{n\in\mathbb{N}} \longrightarrow [a]$ and $\{g_n\}_{n\in\mathbb{N}} \longrightarrow \frac{1}{2}[a] + \frac{1}{2}[b]$. Consequently, since $[a]$ and $\frac{1}{2}[a] + \frac{1}{2}[b]$ are by construction incomparable, $\preceq$ does not satisfy Axiom 8. □

**Lemma 9** *There exists a preference relation that satisfies PDT$_3$ and does not satisfy Axiom 8.*

*Proof* Let $\mathcal{A} = \{a, b, c\}$ and $\preceq$ be the preference relation defined as follows:

$$f \sim g \text{ iff } f = g \tag{3}$$

$$f \prec g \text{ iff } f(c) = g(c) \text{ and } f(b) < g(b) < 1 \tag{4}$$

First, we show that $\preceq$ satisfies $\text{PDT}_3$. Axioms 1 and 2 are trivial. With respect to Axiom 3, given $0 < \alpha < 1$ and a lottery $h \in \Delta(\mathcal{A})$, let $k_1 = \langle \alpha, f, h\rangle$ and $k_2 = \langle \alpha, g, h\rangle$. The first condition above implies that:

$$f \sim g \Leftrightarrow f = g \Leftrightarrow k_1 = k_2 \Leftrightarrow k_1 \sim k_2 .$$

Moreover, by (4), we also have that $f \prec g$ iff $f(c) = g(c)$ and $f(b) < g(b) < 1$. Then, $f(c) = g(c)$ implies $k_1(c) = k_2(c)$ and $f(b) < g(b)$ implies that $k_1(b) < k_2(b)$. Moreover, since $0 < \alpha < 1$, $k_2(b)$ is in the open interval with endpoints $g(b)$ and $h(b)$. Therefore, since $g(b) < 1$, we also have that $k_2(b) < 1$ and hence $k_1 \prec k_2$.

With respect to Axiom 7, assume that $h \prec f \prec g$. By construction, $h(c)$, $f(c)$, and $g(c)$ are the same value $v$. This means that the projection of $h$, $f$ and $g$ on the alternatives $a$ and $b$ lie on the same straight line given by the equation $x + y = 1 - v$. Consequently, it is straightforward to see that

$$\alpha = \frac{g(b) - f(b)}{g(b) - h(b)}$$

makes $f = \langle \alpha, h, g\rangle$. Moreover, since $h(b) < f(b) < g(b)$, $\alpha$ is in the interval $(0, 1)$. Consequently, Axiom 7 is satisfied by $\preceq$.

Finally, consider sequences of lotteries $f_n = [a]$ and $g_n = \frac{1}{n}[a] + (1 - \frac{1}{n})[b]$. By construction, for all $n \in \mathbb{N}$ $f_n \preceq g_n$. Moreover, we have that $\{f_n\}_n \longrightarrow [a]$ and $\{g_n\}_n \longrightarrow [b]$. Since $[a]$ and $[b]$ are incomparable, $\preceq$ does not satisfy Axiom 8. $\square$

Finally, Theorem 3 summarizes the previous lemmas and provides a complete characterization of $\text{PDT}_0$–$\text{PDT}_6$ in terms of the entailment relation (see Fig. 3). In particular, by replacing totality with reflexivity, $\text{PDT}_0$–$\text{PDT}_6$ actually characterize different sets of preference relations and lie in a hierarchy of strict entailments.

**Theorem 3** *The following entailment relations hold: $PDT_0 < PDT_1$, $PDT_0 < PDT_2$, $PDT_1 < PDT_3$, $PDT_2 < PDT_3$, $PDT_1 < PDT_{4/5}$, $PDT_3 < PDT_6$, and $PDT_{4/5} < PDT_6$. Moreover, no other entailment holds.*

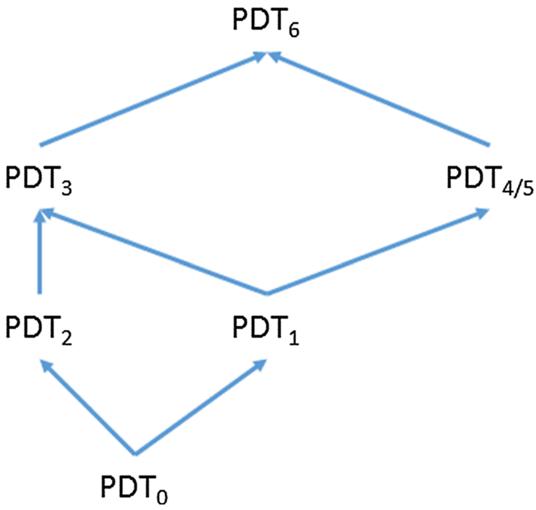
*Proof* By Lemma 1, it follows that $\text{PDT}_0 \leq \text{PDT}_1$, $\text{PDT}_2 \leq \text{PDT}_3$. That $\text{PDT}_0 \leq \text{PDT}_2$ and $\text{PDT}_1 \leq \text{PDT}_3$ derives from Lemma 3. Moreover, Lemma 5 implies that $\text{PDT}_1 \leq \text{PDT}_{4/5}$.

Now, we show that the previous entailments are strict. By $\text{PDT}_0 \leq \text{PDT}_1 \leq \text{PDT}_{4/5}$ and Lemma 7, there exists a preference relation that satisfies $\text{PDT}_1$ and $\text{PDT}_0$ and does not satisfy Axiom 7. Consequently, $\text{PDT}_0 < \text{PDT}_2$ and $\text{PDT}_1 < \text{PDT}_3$. Similarly, $\text{PDT}_0 \leq \text{PDT}_2$ and Lemma 8 implies $\text{PDT}_0 < \text{PDT}_1$ and $\text{PDT}_2 < \text{PDT}_3$, whereas $\text{PDT}_1 \leq \text{PDT}_{4/5}$ and Lemma 9 implies $\text{PDT}_1 < \text{PDT}_{4/5}$.

By Lemma 7 and Lemma 9, we have that $\text{PDT}_3 \not\leq \text{PDT}_{4/5}$ and vice versa $\text{PDT}_{4/5} \not\leq \text{PDT}_3$. Moreover, since $\text{PDT}_1 \leq \text{PDT}_{4/5}$, Lemma 7 also implies that $\text{PDT}_2 \not\leq \text{PDT}_1$. Similarly, Lemma 8 implies that $\text{PDT}_1 \not\leq \text{PDT}_2$.

Finally, we clearly have that $\text{PDT}_3 \leq \text{PDT}_6$ and $\text{PDT}_{4/5} \leq \text{PDT}_6$. However, since $\text{PDT}_3 \not\leq \text{PDT}_{4/5}$ and $\text{PDT}_{4/5} \not\leq \text{PDT}_3$ hold, $\text{PDT}_6$ cannot be equivalent to one of them and hence we have that $\text{PDT}_3 < \text{PDT}_6$ and $\text{PDT}_{4/5} < \text{PDT}_6$. $\square$

**Fig. 3** The hierarchical structure of $PDT_0$–$PDT_6$




Theorem 3 provides some useful information on how properties "propagate" from one theory to another. For example, if a property holds in $PDT_0$, then it holds in all the other theories (dually, if something does not hold in $PDT_3$, then it does not hold in $PDT_0$, $PDT_1$, and $PDT_2$).

In the next sections we continue our formal analysis by showing first some relevant properties that all the theories share and then we shed light on the what distinguish one theory from another in terms of comparability strength and topological properties.

## 5 Shared properties of partial preferences

In this section, we show some properties that hold in $PDT_0$–$PDT_6$ and hence cannot be used to advocate for any of them. Clearly, due to Theorem 3, it will suffice to show that such properties hold in $PDT_0$. Thus, in the rest of this section we implicitly assume that $\preceq$ satisfies $PDT_0$.

The following theorem shows that $PDT_0$ preserves monotonicity under convex combinations.

**Theorem 4** *If* $f \prec g$ *and* $0 \leq \beta < \alpha \leq 1$*, then* $\langle \alpha, f, g \rangle \prec \langle \beta, f, g \rangle$.

*Proof* Let $h = \langle \alpha, f, g \rangle$, by Axiom 6 and $f \prec g$, we have that $h \prec \langle \alpha, g, g \rangle = g$. Now, let $\gamma = \frac{\beta}{\alpha}$, since $\beta < \alpha$, $\gamma$ is in the semi-open interval $[0, 1)$. Moreover, it is straightforward to see that $\langle \beta, f, g \rangle = \langle \gamma, h, g \rangle$. Then, by $h \prec g$ and Axiom 6, we have that $h = \langle \gamma, h, h \rangle \prec \langle \gamma, h, g \rangle$. Hence, $\langle \alpha, f, g \rangle \prec \langle \beta, f, g \rangle$. □

*Example 2* Consider the preference relation in Fig. 1 and the following lotteries:

$$f_1 = \frac{1}{3}[16; 250; 0.2] + \frac{2}{3}[16; 250; 0.6]$$
$$f_2 = \frac{1}{2}[16; 250; 0.2] + \frac{1}{2}[16; 250; 0.6]$$

Since $[16; 250; 0.6] \prec [16; 250; 0.2]$, by Theorem 4, $f_1 \prec f_2$. □

Another property that can be proved in $\mathrm{PDT}_0$ is dominance which extends Axiom 5 to generic convex combinations.

**Theorem 5** *Let $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$ and $g = \beta_1 g_1 + \cdots + \beta_m g_m$. If $f_i \preceq g_j$, for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$, then $f \preceq g$.*

*Proof* We first show that for all $f_i$, $i \in \{1, \ldots, n\}$, $f_i \preceq g$. The proof is by induction on $m$ where the base case $m = 1$ is trivial. Induction step $m > 1$. Let $g_{-1} = \beta'_2 g_2 + \cdots + \beta'_m g_m$ where $\beta'_j = \frac{\beta_j}{1-\beta_1}$, it is easy to see that $g = \langle \beta_1, g_1, g_{-1} \rangle$. Thus, we know that $f_i \preceq g_1$ and, by induction hypothesis, $f_1 \preceq g_{-1}$, then by Axiom 5, $f_i \preceq g$.

The rest of the proof is by induction on $n$, where the case $n = 1$ has just been proved. For the induction step, we decompose now $f$ in $f_1$ and $f_{-1} = \alpha'_2 f_2 + \cdots + \alpha'_n f_n$ where $\alpha'_i = \frac{\alpha_i}{1-\alpha_1}$. Since $f_1 \preceq g$ and by induction $f_{-1} \preceq g$, we finally have that $f \preceq g$. □

In particular, Theorem 5 ensures that if each alternative coming from a distribution $f$ is worse than all the alternatives from $g$, then $f \preceq g$. By applying Axiom 6, it is easy to see that $f \prec g$ if, in the hypothesis of Theorem 5, for at least one pair $f_i$ is strictly preferred to $g_j$.

*Example 3* Consider the preference relation in Fig. 2 and the following lotteries:

$$f = \frac{1}{2}[8; 500; 0.4] + \frac{1}{2}[8; 500; 0.6]$$
$$g = \frac{1}{4}[16; 250; 0.2] + \frac{3}{4}[16; 250; 0.4]$$

Since both [16; 250; 0.4] and [16; 250; 0.2] are strictly preferred to [8; 500; 0.4] and [8; 500; 0.6], it holds that $f \prec g$. □

Partial preferences also preserve a weaker notion of dominance which occurs when $g$ can be obtained from $f$ by shifting positive amounts of probability in favour of strictly preferred alternatives. Given two distributions $f, g \in \Delta(\mathcal{A})$, we write $f \rightarrow g$ in case there exist $\epsilon > 0$ and two alternatives $a_1$ and $a_2$ such that $[a_1] \prec [a_2]$, $g(a_1) = f(a_1) - \epsilon$, $g(a_2) = f(a_2) + \epsilon$, and for all $a \in \mathcal{A} \setminus \{a_1, a_2\}$, $g(a) = f(a)$. Informally, $f \rightarrow g$ means that $g$ is obtained from $f$ by shifting an amount $\epsilon$ of probability from an alternative $a_1$ to a strictly preferred alternative $a_2$. Then, by $f \twoheadrightarrow g$ we denote the transitive closure of $\rightarrow$.

**Lemma 10** *Given $f, g \in \Delta(\mathcal{A})$, if $f \twoheadrightarrow g$, then $f \prec g$.*

*Proof* It suffices to show that $f \rightarrow g$ implies $f \prec g$. Assume that for some $a_1$ and $a_2$, where $[a_1] \prec [a_2]$, there exists $\epsilon > 0$ such that $f(a_1) = g(a_1) + \epsilon$ and $f(a_2) = g(a_2) - \epsilon$. Let $\gamma = g(a_1) + g(a_2) = f(a_1) + f(a_2)$, $\alpha = \frac{f(a_2)}{\gamma}$ and $\beta = \frac{g(a_2)}{\gamma}$. Then, $g$ can be written as $\langle \gamma, g', h \rangle$ and $f$ as $\langle \gamma, f', h \rangle$, where $h$ is a probability distribution such that $h(a_1) = h(a_2) = 0$, $g' = \langle \beta, [a_2], [a_1] \rangle$ and $f' = \langle \alpha, [a_2], [a_1] \rangle$. Since $[a_1] \prec [a_2]$ and $\alpha < \beta$, Theorem 4 implies that $f' \prec g'$. Then, by Axiom 6, $f \prec g$. □

*Example 4* Consider the preference relation in Fig. 2 and the following lotteries:

$$f = \frac{1}{4}[8; 500; 0.2] + \frac{1}{2}[8; 500; 0.4] + \frac{1}{4}[8; 500; 0.6]$$
$$g = \frac{1}{3}[16; 250; 0.2] + \frac{2}{3}[16; 250; 0.4]$$

Note that [16; 250; 0.4] and [8; 500; 0.2] are incomparable, hence we cannot apply Theorem 5. However, it holds that:

$$
\begin{aligned}
&\frac{1}{4}[8;500;0.2]+\frac{1}{2}[8;500;0.4]+\frac{1}{4}[8;500;0.6]\\
&\quad\to\frac{1}{4}[16;250;0.2]+\frac{1}{2}[8;500;0.4]+\frac{1}{4}[8;500;0.6]\\
&\quad\to\frac{1}{4}[16;250;0.2]+\frac{1}{2}[8;500;0.4]+\frac{1}{4}[16;250;0.4]\\
&\quad\to\frac{1}{3}[16;250;0.2]+\frac{5}{12}[8;500;0.4]+\frac{1}{4}[16;250;0.4]\\
&\quad\to\frac{1}{3}[16;250;0.2]+\frac{2}{3}[16;250;0.4]\,.
\end{aligned}
$$

Consequently, $f \twoheadrightarrow g$ and hence, by Theorem 6, $f \prec g$. □

The previous example shows that, as such, Lemma 10 would be with little practical use without a procedure to verify whether $f \twoheadrightarrow g$. Consider the labelled bipartite graph $\mathsf{G} = \langle V_1, V_2, E, h\rangle$ such that $h = f - g$, $V_1 = \{a \in \mathcal{A} \mid h(a) > 0\}$, $V_2 = \{b \in \mathcal{A} \mid h(b) < 0\}$, and $E$ is the subset of $V_1 \times V_2$ such that $(a, b) \in E$ iff $[a] \prec [b]$. In particular, with $a\uparrow$ we mean set of edges starting from $a \in V_1$, vice versa $b\downarrow$ is the set of edges reaching $b \in V_2$. Incidentally, note that $\sum_{a\in V_1} h(a) - \sum_{b\in V_2} h(b) = 0$.

A function $\mathsf{mc} : E \to [0, 1]$ is a *matching* if and only if for all $a \in V_1$ and $b \in V_2$,

$$
\begin{aligned}
h(a) &= \sum_{e\in a\uparrow} \mathsf{mc}(e)\\
h(b) &= -\sum_{e\in b\downarrow} \mathsf{mc}(e)\,.
\end{aligned}
$$

Intuitively, a matching function describes how the values of $h$ can "flow" from $V_1$ to $V_2$ through the edges in $E$. As the following theorem shows, looking for a matching function is equivalent to checking whether $f \twoheadrightarrow g$.

**Theorem 6** *Given two lotteries $f, g \in \Delta(\mathcal{A})$, let $\mathsf{G} = \langle V_1, V_2, E, h\rangle$ be the relative labelled bipartite graph. Then, $f \twoheadrightarrow g$ iff $\mathsf{G}$ admits a matching function $\mathsf{mc}$.*

*Proof* [$\Leftarrow$] Let $(a_1, b_1, \epsilon_1), \dots, (a_n, b_n, \epsilon_n)$ be the sequence of all triples such that $(a_i, b_i) \in E$ and $\epsilon_i = \mathsf{mc}((a_i, b_i)) > 0$, $i \in \{1, \dots, n\}$. Consider the sequence $f_0, \dots, f_n$ such that $f_0 = f$ and each $f_{i+1}$, $i \in \{0, \dots, n-1\}$, is obtained from $f_i$ by shifting the positive amount of probability $\epsilon_i$ from $a_i$ to $b_i$. Since $(a_i, b_i) \in E$, then by construction $[a_i] \prec [b_i]$. Therefore, we have that $f = f_0 \twoheadrightarrow f_n$. Moreover, by definition of $\mathsf{mc}$, for all $i \in \{1, \dots, n\}$, $f_n(a_i) = f(a_i) - h(a_i) = g(a_i)$ and $f_n(b_i) = f(b_i) - h(b_i) = g(a_i)$. Consequently, $f_n = g$.

[$\Rightarrow$] Assume that $f \twoheadrightarrow g$, this means that for some $f_1, \dots, f_n$, we have that $f = f_0 \to \cdots \to f_n = g$. As before, such a chain can be represented by a sequence of triples $(a_1, b_1, \epsilon_1), \dots, (a_n, b_n, \epsilon_n)$. It is easy to see that any permutation does not change the final result (i.e. $g$), therefore, we can assume w.l.o.g. that the first components of the sequence preserve the preference $\preceq$—that is, if $a_i \prec a_j$ then $i < j$. Moreover, for any $(a_i, b_i, \epsilon_i)$ and $(a_j, b_j, \epsilon_j)$ such that $a_j = b_i$, if $\epsilon_i \leq \epsilon_j$, then we can replace the two triples with $(a_i, b_j, \epsilon_i)$ and $(a_j, b_j, \epsilon_j - \epsilon_i)$. Conversely, $\epsilon_i > \epsilon_j$, we replace them with $(a_i, b_i, \epsilon_i - \epsilon_j)$ and $(a_i, b_j, \epsilon_j)$. Accordingly, we can assume that the set $\mathcal{A}_1 = \{a \in \mathcal{A} \mid \exists 1 \leq i \leq n\, a = a_i\}$

and $\mathcal{A}_2 = \{b \in \mathcal{A} \mid \exists 1 \leq j \leq n\ b = b_j\}$ are disjoint. Finally, if two triples $(a_i, b_i, \epsilon_i)$ and $(a_j, b_j, \epsilon_j)$ have the same source and the same target, $a_i = a_j$ and $b_i = b_j$, we replace them with a single triple $(a_i, b_i, \epsilon_i + \epsilon_j)$. By construction, for all $a \in \mathcal{A}_1$ (resp. $a \in \mathcal{A}_2$), $h(a) = f(a) - g(a) > 0$ (resp. $h(b) = f(b) - g(b) < 0$). Then, since all the other alternatives remain unchanged, in the corresponding labelled bipartite graph $\mathsf{G} = \langle V_1, V_2, E, h\rangle$ we have that $V_1 = \mathcal{A}_1$, $V_2 = \mathcal{A}_2$. Moreover, for each triple $(a_i, b_i, \epsilon_i)$ $a_i \prec b_i$, which means that $(a_i, b_i) \in E$, for all $i \in \{1, \ldots, n\}$. Then, by setting $\mathsf{mc}((a_i, b_i)) = \epsilon_i$ and $\mathsf{mc}((a, b)) = 0$, for all the other edges $(a, b)$ in $E$, it is straightforward to verify that $\mathsf{mc}$ is a matching function. □

Theorem 6 reduces the problem of establishing whether $f \twoheadrightarrow g$ to a bipartite matching problem. In turn, the latter can be reduced to a maximum-flow problem by adding a source node $s$ and a target node $t$ where $s$ is connected to all the nodes of $V_1$ and all the nodes in $V_2$ are connected to $t$. Then, the capacity of an edge of type $(s, a)$ is equal to $h(a)$, the capacity of an edge $(b, t)$ is $|h(b)|$, and the capacity of each edge in $E$ is $\infty$. Then, it is immediate to see that the maximum flow passing from the source is equal to $\sum_{a \in V_1} h(a)$ iff there exists a matching function.

*Example 5* Let $\mathcal{A} = \{a_1, \ldots, a_6\}$ and a preference relation over alternatives which is depicted in Fig. 4. Then, consider the lotteries $f$ and $g$ given by the following table:

| | $a_1$ | $a_2$ | $a_3$ | $a_4$ | $a_5$ | $a_6$ |
|---|---|---|---|---|---|---|
| $f$ | 0.3 | 0.3 | 0.1 | 0.2 | 0 | 0.1 |
| $g$ | 0 | 0.1 | 0.2 | 0.1 | 0.2 | 0.4 |
| $h$ | 0.3 | 0.2 | −0.1 | 0.1 | −0.2 | −0.3 |

where the last row is the function $h = f - g$. Figure 5 shows the relative maximum-flow problem, each edge is labelled with a pair of values $v/c$ where $c$ is the capacity of the edge and $v$ is the intensity of the maximum flow. Since the flow is equal to $0.6 = h(a_1)+h(a_2)+h(a_4)$, then $f \twoheadrightarrow g$, and hence $f \preceq g$. In particular, $g$ can be obtained from $f$ by shifting $\epsilon_1 = 0.1$ from $a_1$ to $a_3$, $\epsilon_2 = 0.2$ from $a_1$ to $a_6$, $\epsilon_3 = 0.2$ from $a_2$ to $a_5$, and $\epsilon_4 = 0.1$ from $a_4$ to $a_6$.

Finally, we show that the relation $\twoheadrightarrow$ is related to the notion of first-order stochastic dominance (FSD) [37]. FSD assumes that the restriction of a preference relation $\preceq$ on degenerate lotteries is a linear order—i.e. a total and antisymmetric preorder. The cumulative distribution $F$ of a lottery $f$ is defined as:

$$F(a) = \sum_{[b] \preceq [a]} f(b)\,. \tag{5}$$

Then, we say that $g$ first-order stochastically dominates $f$ if and only if for all $a \in \mathcal{A}$, $G(a) \leq F(a)$, with strict inequality at some $a$. The following theorem shows that, if the restriction of $\preceq$ on degenerate lotteries is a linear order, then the relation $\twoheadrightarrow$ and FSD coincide.

**Theorem 7** *Assume that the restriction of a preference relation $\preceq$ on degenerate lotteries is a linear order. Then, $f \twoheadrightarrow g$ iff $g$ first-order stochastically dominates $f$.*

*Proof* First of all, since the restriction of $\preceq$ on degenerate lotteries is a linear order, then the set of degenerate lotteries $[\mathcal{A}]$ can be ordered as a sequence $[a_1] \prec \cdots \prec [a_N]$.

[⇒] Let $f = f_0 \rightarrow f_1 \rightarrow \cdots \rightarrow f_m = g$. As in Theorem 6, we represent this chain as a sequence of triples $(a_{i_1}, a_{j_1}, \epsilon_1), \ldots, (a_{i_m}, a_{j_m}, \epsilon_m)$ where w.l.o.g. we assume that

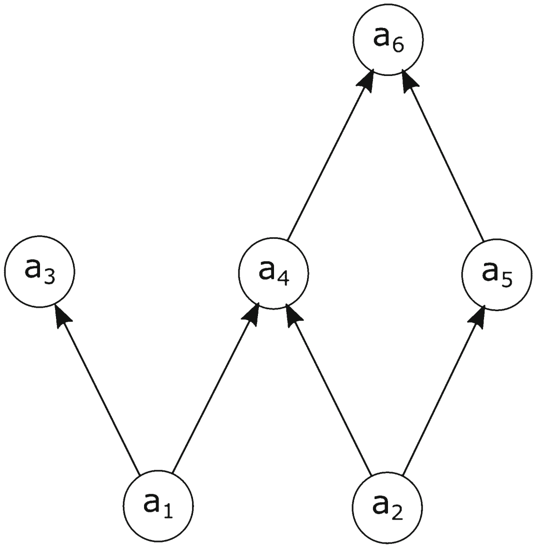



**Fig. 4** The preference relation in Example 5

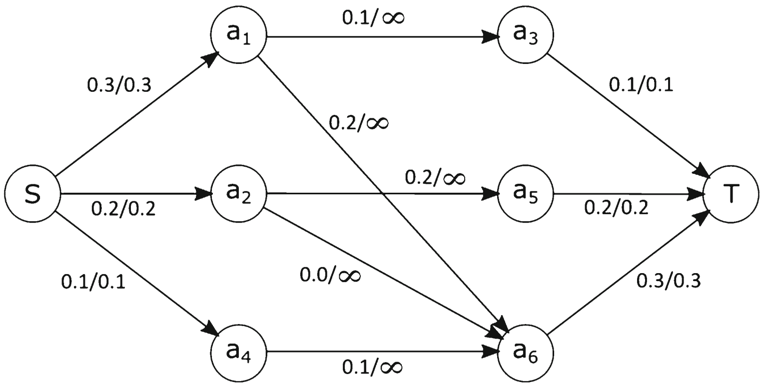



**Fig. 5** The maximum-flow problem in Example 5

$i_1 \leq i_2 \leq \cdots \leq i_m$. Since by definition $[a_{i_h}] \prec [a_{j_h}]$, with $1 \leq h \leq m$, it is easy to verify that for all $1 \leq i \leq n$, $G(a_i) = F_m(a_i) \leq F_{m-1}(a_i) \leq \cdots \leq F_0(a_i) = F(a_i)$. Moreover, by construction $G(a_{i_1}) \leq F(a_{i_1}) - \epsilon_1$. Consequently, $g$ first-order stochastically dominates $f$.

[$\Leftarrow$] Let $a_i$ be the least preferred alternative such that $G(a_i) < F(a_i)$. Since $G(a_j) = F(a_j)$ for all $j < i$, then $f(a_j) = g(a_j)$ and $f(a_i) = g(a_i) + \epsilon_i$, where $\epsilon_i = F(a_i) - G(a_i)$. Then, let $I = \{a_{i_1}, \ldots, a_{i_m}\}$ be the first set of alternatives greater than $a_i$ such that $g$ can absorb $\epsilon_i$. Formally, $I$ satisfies the following conditions: (1) $i < i_1 < \cdots < i_m$; (2) $f(a_{i_h}) < g(a_{i_h})$, with $1 \leq h \leq m$; (3) for all $i < j < i_m$ if $a_j \notin I$, then $g(a_j) \leq f(a_j)$; (4) it holds that

$$\delta_i = \sum_{h=1}^{m-1} \left( g(a_{i_h}) - f(a_{i_h}) \right) < \epsilon_i$$

and

$$\delta_i + \left( g(a_{i_m}) - f(a_{i_m}) \right) \geq \epsilon_i \, .$$

The existence of $I$ is ensured by the fact that $F(a_N) = G(a_N) = 1$ and hence

$$\sum_{h=i+1}^{n} g(a_h) = \sum_{h=i+1}^{n} f(a_h) + \epsilon_i \,.$$

Consider the function $f_1$ where $I$ absorbs $\epsilon_i$ from $a_i$. Formally, we have that $f_1(a_j) = f(a_j)$, for all $j \notin \{i, i_1, \ldots, i_m\}$; $f_1(a_i) = f(a_i) - \epsilon_i = g(a_i)$; $f_1(a_{i_h}) = g(a_{i_h})$, with $1 \leq h \leq m-1$; and $f_1(a_{i_m}) = f(a_{i_m}) + \epsilon_i - \delta_i$. Since for all $1 \leq h \leq m$ $[a_i] \prec [a_{i_h}]$, we have that $f \twoheadrightarrow f_1$.

Now, we prove that for all $1 \leq k \leq N$, $G(a_k) \leq F_1(a_k)$. Clearly, for all $1 \leq k \leq i$, $f_1(a_k) = g(a_k)$ and hence $F_1(a_k) = G(a_k)$. By construction, for all $i < k < i_m$ if $k \in I$, then $f_1(a_k) = g(a_k)$, otherwise $f_1(a_k) = f(a_k)$. Then, by condition (3), it holds $g(a_k) \leq f_1(a_k)$ and hence $G(a_k) \leq F_1(a_k)$. For $i_m$, we have that

$$F_1(a_{i_m}) - F(a_{i_m}) = f_1(a_i) - f(a_i) + \sum_{h=1}^{m-1} \big(f_1(a_{i_h}) - f(a_{i_h})\big) + f_1(a_{i_m}) - f(a_{i_m})\,.$$

By construction, we have that $f_1(a_i) = f(a_i) - \epsilon_i$; $f_1(a_{i_h}) = g(a_{i_h})$, with $1 \leq h \leq m-1$; and $f_1(a_{i_m}) = f(a_{i_m}) + \epsilon_i - \delta_i$. Consequently, $F_1(a_{i_m}) - F(a_{i_m}) = f(a_i) - \epsilon_i - f(a_i) + \delta_i + f(a_{i_m}) + \epsilon_i - \delta_i - f(a_{i_m}) = 0$. Then, since by assumption $G(a_{i_m}) \leq F(a_{i_m})$, we also have $G(a_{i_m}) \leq F_1(a_{i_m})$. Finally, for $k > i_m$, $f_1(a_k) = f(a_k)$ and hence $G(a_k) \leq F_1(a_k)$.

Now, we distinguish two cases. In the first case, for all $1 \leq k \leq N$, $G(a_k) = F_1(a_k)$. This means that $f_1 = g$ and hence we have the thesis $f \twoheadrightarrow g$. In the second case, for all $1 \leq k \leq N$, $G(a_k) \leq F_1(a_k)$ and there exists a $1 \leq h \leq N$ such that $G(a_h) < F_1(a_h)$. Clearly, this means that $g$ first-order stochastically dominates $f_1$. Therefore, we can consider the least preferred alternative $a_{i'}$ such that $G(a_{i'}) < F_1(a_{i'})$. By construction $i < i'$ and, since for all $j < i'$ $G(a_j) = F_1(a_j)$, we have that $f_1(a_j) = g(a_j)$. Then, by reproducing the same procedure as before, we obtain a lottery $f_2$ such that $f_1 \twoheadrightarrow f_2$ and for all $1 \leq k \leq n$, $G(a_k) \leq F_2(a_k)$. If $f_2 \neq g$, then $g$ first-order stochastically dominates $f_2$ too. So we can iterate the procedure again obtaining a sequence $i < i' < i'' < \cdots$. Since $\mathcal{A}$ is finite, the iteration eventually ends with a lottery $f_n = g$. Consequently, $f \twoheadrightarrow g$. □

## 6 Comparability propagation in partial preferences

In the previous section we have shown that $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ form a hierarchy where, for instance, $\mathrm{PDT}_2$ is "stronger" than $\mathrm{PDT}_0$, in the sense that the set of preferences satisfying $\mathrm{PDT}_2$ is strictly contained in the set satisfying $\mathrm{PDT}_0$. A natural question is: how much stronger is $\mathrm{PDT}_2$ than $\mathrm{PDT}_0$? In this section we provide some insights on this question. First, we need to choose a criterion to compare the six theories. Intuitively, since $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ are all implied by the classical theory, the idea is to use the latter as a "yardstick". In the classical theory, two convex combinations $\langle \alpha, f_1, f_2 \rangle$ and $\langle \beta, g_1, g_2 \rangle$ are, by totality, always comparable. Conversely, comparability does not hold a priori in $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ and generally depends on which preferences subsist between their components $f_1$, $f_2$, $g_1$, and $g_2$ (see Axiom 5 for instance).

Therefore, we study when two convex combinations $\langle \alpha, f_1, f_2 \rangle$ and $\langle \beta, g_1, g_2 \rangle$ are forced to be comparable provided some hypotheses on the comparability of the relative components—clearly, the weaker such hypotheses are the closer a theory is to the classical one. In particular, we consider the following three hypotheses:

I. for all $i, j \in \{1, 2\}$ $f_i \bowtie f_j$, $g_i \bowtie g_j$, and $f_i \bowtie g_j$;
II. for all $i, j \in \{1, 2\}$ $f_i \bowtie g_j$;
III. for all $i \in \{1, 2\}$ $f_i \bowtie g_i$.

where $f \bowtie g$ means that $f$ and $g$ are comparable (i.e. either $f \preceq g$ or $g \preceq f$). Clearly, these hypotheses are not independent, hypothesis I implies hypothesis II which in turn implies III.

Intuitively, hypothesis I reproduces the same preconditions that hold in the classical theory, namely all the components $f_1$, $f_2$, $g_1$, and $g_2$ are mutually comparable. With respect to hypothesis II, a mixture $\langle \alpha, f_1, f_2\rangle$ can be seen as a random choice which picks $f_1$ with probability $\alpha$ and $f_2$ with probability $1 - \alpha$. Thus, to compare $f = \langle \alpha, f_1, f_2\rangle$ with another mixture $g = \langle \beta, g_1, g_2\rangle$, we can imagine to run these two lotteries in parallel obtaining four possible draws: $(f_1, g_1)$, $(f_1, g_2)$, $(f_2, g_1)$, and $(f_2, g_2)$ with probabilities $\alpha\beta$, $\alpha(1 - \beta)$, $(1-\alpha)\beta$ and $(1-\alpha)(1-\beta)$, respectively. Now, assuming that all these draws are comparable, then $f$ and $g$ are comparable as well? Finally, hypothesis III further relaxes hypothesis II by assuming that only the first and the last draws are comparable. Note that in case of homogeneous mixtures Axiom 5 is a special case of hypothesis III where the preferences between $f_1, g_1$ and $f_2, g_2$ are concordant.

In what follows, we will distinguish the general case of *heterogeneous* mixtures ($\alpha$ and $\beta$ are whatever) and the more specific case of *homogeneous* mixtures ($\alpha = \beta$). Moreover, we shortly say that $PDT_x$, with $x = 0, 1, 2, 3, 4/5, 6$, and hypothesis $y$, $y = I, II, III$, imply the comparability of homogeneous (resp. heterogeneous) mixtures meaning that for all preference relations $\preceq$ satisfying $PDT_x$, if $f_1, f_2, g_1, g_2 \in \Delta(\mathcal{A})$ satisfy hypothesis $y$, then for all $0 < \alpha < 1$ $\langle \alpha, f_1, f_2\rangle \bowtie \langle \alpha, g_1, g_2\rangle$ (resp. for all $0 < \alpha, \beta < 1$ $\langle \alpha, f_1, f_2\rangle \bowtie \langle \beta, g_1, g_2\rangle$).

To start, the following lemma tells us that, whenever $f$ and $g$ are comparable, all their convex combinations are comparable as well.

**Lemma 11** *Let $\preceq$ be a preference relation satisfying $PDT_0$. Given two lotteries $f, g \in \Delta(\mathcal{A})$, if $f \preceq g$, then $\langle \alpha, f, g\rangle \bowtie \langle \beta, f, g\rangle$, for all* $0 \leq \alpha, \beta \leq 1$.

*Proof* In case $\alpha = \beta$, comparability immediately derives from reflexivity, therefore we assume that $\alpha \neq \beta$. Let $f \sim g$. By Axiom 5, it is easy to see that $f \sim \langle \alpha, f, g\rangle$ and $f \sim \langle \beta, f, g\rangle$, hence $\langle \alpha, f, g\rangle \sim \langle \beta, f, g\rangle$. Finally, assume that $f \prec g$. Then, by Theorem 4, if $\alpha < \beta$, $\langle \beta, f, g\rangle \prec \langle \alpha, f, g\rangle$. Conversely, $\beta < \alpha$, $\langle \alpha, f, g\rangle \prec \langle \beta, f, g\rangle$. □

The following two theorems provide positive results, in particular Theorem 8 concerns the general case of heterogeneous mixtures and shows that comparability is already ensured if we take $\mathrm{PDT}_2$ and hypothesis II.

**Theorem 8** *$PDT_2$ and hypothesis II imply the comparability of heterogeneous mixtures.*

*Proof* We prove the thesis case by case. By hypothesis II, each pair $(f_1, g_1)$, $(f_1, g_2)$, $(f_2, g_1)$, and $(f_2, g_2)$ is comparable. Then, for each pair $(f_i, g_j)$, $i, j \in \{1, 2\}$, we assign 0 if $f_i \preceq g_j$ or 1 if $g_j \preceq f_i$ (in case of equivalence you can indifferently assign 0 or 1). Doing so, we come up with 16 different cases that can be enumerate by the corresponding assignments (e.g. case 5, 0101, corresponds to $f_1 \preceq g_1$, $f_1 \succeq g_2$, $f_2 \preceq g_1$, and $f_2 \succeq g_2$).

Case 0 ($f_1 \preceq g_1$, $f_1 \preceq g_2$, $f_2 \preceq g_1$, and $f_2 \preceq g_2$). Let $f = \langle \alpha, f_1, f_2\rangle$ and $g = \langle \beta, g_1, g_2\rangle$, since $f_1 \preceq g_1$ and $f_2 \preceq g_1$, by Axiom 5 it holds that $\langle \alpha, f_1, f_2\rangle \preceq \langle \alpha, g_1, g_1\rangle$, that is $f \preceq g_1$. Similarly, from $f_1 \preceq g_2$ and $f_2 \preceq g_2$, we have that $f \preceq g_2$. Then, by applying Axiom 5 again, we obtain $\langle \beta, f, f\rangle \preceq \langle \beta, g_1, g_2\rangle$, that is $f \preceq g$.

Case 1 ($f_1 \preceq g_1$, $f_1 \preceq g_2$, $f_2 \preceq g_1$, and $f_2 \succeq g_2$). In this case lotteries are totally ordered, $f_1 \preceq g_2 \preceq f_2 \preceq g_1$. By Axiom 7, there exists some $0 \leq \delta \leq 1$ such that

$f_2 \sim \langle \delta, f_1, g_1 \rangle$ and some $0 \leq \lambda \leq 1$ such that $g_2 \sim \langle \lambda, f_1, g_1 \rangle$. Thus, we have that $f = \langle \alpha, f_1, f_2 \rangle \sim \alpha f_1 + (1-\alpha)\delta f_1 + (1-\alpha)(1-\delta)g_1$, that is $f \sim \langle \gamma_1, f_1, g_1 \rangle$, where $\gamma_1 = \alpha + (1-\alpha)\delta$. Similarly, $g = \langle \beta, g_1, g_2 \rangle \sim \langle \gamma_2, f_1, g_1 \rangle$, where $\gamma_2 = (1-\beta)\lambda$. Subsequently, by Lemma 11 $f$ and $g$ are comparable.

Case 2 ($f_1 \preceq g_1$, $f_1 \preceq g_2$, $f_2 \succeq g_1$, and $f_2 \preceq g_2$). As in the previous case, the lotteries are totally ordered, $f_1 \preceq g_1 \preceq f_2 \preceq g_2$. Similarly, for some $0 \leq \delta, \lambda \leq 1$, we have that $f_2 \sim \langle \delta, f_1, g_2 \rangle$ and $g_1 \sim \langle \lambda, f_1, g_2 \rangle$. Consequently, $f \sim \langle \gamma_1, f_1, g_2 \rangle$ and $g \sim \langle \gamma_2, f_1, g_2 \rangle$, where $\gamma_1 = \alpha + (1-\alpha)\delta$ and $\gamma_2 = \beta\lambda$. Again, Lemma 11 ensures that $f$ and $g$ are comparable.

Case 3 ($f_1 \preceq g_1$, $f_1 \preceq g_2$, $f_2 \succeq g_1$, and $f_2 \succeq g_2$). We have that $f_1 \preceq g_1 \preceq f_2$ and $f_1 \preceq g_1 \preceq f_2$, where $g_1$ and $g_2$ can be possibly incomparable. Thus, by applying the continuity Axiom 7, both $g_1$ and $g_2$ are equivalent to a convex combination of $f_1$ and $f_2$, say $g_1 \sim \langle \delta, f_1, f_2 \rangle$ and $g_2 \sim \langle \lambda, f_1, f_2 \rangle$. Consequently, it holds that $g \sim \langle \gamma, f_1, f_2 \rangle$, where $\gamma = \beta\delta + (1-\beta)\lambda$. Since $f_1 \preceq f_2$, by Lemma 11 $f$ and $g$ are comparable.

Case 4 ($f_1 \preceq g_1$, $f_1 \succeq g_2$, $f_2 \preceq g_1$, and $f_2 \preceq g_2$). Also in this case the lotteries are totally ordered $f_2 \preceq g_2 \preceq f_1 \preceq g_1$, similarly as the cases 1 and 2, it results that both $f$ and $g$ can be expressed as a combination of $f_2$ and $g_2$, therefore they are comparable.

Case 5 ($f_1 \preceq g_1$, $f_1 \succeq g_2$, $f_2 \preceq g_1$, and $f_2 \succeq g_2$). This case is somehow symmetrical to case 3. In particular, it holds that $g_2 \preceq f_1 \preceq g_1$ and $g_2 \preceq f_2 \preceq g_1$. This means that $f_1$ and $f_2$ can be respectively replaced with a mixture of $g_1$ and $g_2$ and hence $f$ itself is equivalent to a mixture $g_1$ and $g_2$. Consequently, $f$ and $g$ are comparable.

Case 6 ($f_1 \preceq g_1$, $f_1 \succeq g_2$, $f_2 \succeq g_1$, and $f_2 \preceq g_2$). It is easy to see that, by construction, $f_1 \sim f_2 \sim g_1 \sim g_2$. Consequently, $f \sim g$.

Case 7 ($f_1 \preceq g_1$, $f_1 \succeq g_2$, $f_2 \succeq g_1$, and $f_2 \succeq g_2$). The proof is analogous to the cases 1,2, and 4. In particular, since $g_2 \preceq f_1 \preceq g_1 \preceq f_2$, both $f$ and $g$ can be represented as a mixture of $f_2$ and $g_2$.

For the other cases, each of them can be obtained from a previous one by inverting the preference relation (e.g. case 8 is complementary to case 7), therefore the proofs are analogous. □

The following theorem shows that, by relaxing in Theorem 8 hypothesis II in favour of hypothesis III, the comparability of homogeneous mixtures is still guaranteed.

**Theorem 9** *$PDT_2$ and hypothesis III imply comparability of homogeneous mixtures.*

*Proof* Let $f = \langle \alpha, f_1, f_2 \rangle$, $g = \langle \alpha, g_1, g_2 \rangle$. By hypothesis III, we have that $f_1$ is comparable to $g_1$ and $f_2$ is comparable to $g_2$. Clearly, in case $f_1 \preceq g_1$ and $f_2 \preceq g_2$ (resp. $f_1 \succeq g_1$ and $f_2 \succeq g_2$), then we directly have by Axiom 5 that $f \preceq g$ (resp. $f \succeq g$). Therefore, assume that $f_1 \preceq g_1$ and $f_2 \succeq g_2$; by Axiom 5 it holds that

$$\langle \alpha, f_1, g_2 \rangle \preceq f \preceq \langle \alpha, g_1, f_2 \rangle$$
$$\langle \alpha, f_1, g_2 \rangle \preceq g \preceq \langle \alpha, g_1, f_2 \rangle \,.$$

Then, Axiom 7 ensures that for some $0 \leq \gamma, \delta \leq 1$, $f \sim \langle \gamma, h, k \rangle$ and $g \sim \langle \delta, h, k \rangle$, where $h = \langle \alpha, f_1, g_2 \rangle$ (resp. $k = \langle \alpha, g_1, f_2 \rangle$). Consequently, $h \sim k$ or $\gamma = \delta$ imply $f \sim g$, whereas $h \prec k$ and $\delta < \gamma$ (resp. $\gamma < \delta$) imply $f \prec g$ (resp. $g \prec f$). So, $f$ and $g$ are comparable.

The last case, $f_1 \succeq g_1$ and $f_2 \preceq g_2$, can be proved analogously. □

The following two theorems show negative results. Theorem 10 tells us that $\mathrm{PDT}_{4/5}$ does not ensure comparability of mixtures, even if we consider the more restrictive hypothesis I and the special case of homogeneous mixtures.

**Theorem 10** *$PDT_{4/5}$ and hypothesis I do not imply comparability of homogeneous mixtures.*

*Proof* Consider the preference relation $\preceq$ defined in Lemma 7 and the following lotteries:

$$\begin{aligned} f_1 &= \frac{1}{4}[a] + \frac{1}{8}[b] + \frac{5}{8}[c] \\ f_2 &= \frac{1}{4}[a] + \frac{1}{2}[b] + \frac{1}{4}[c] \\ g_1 &= \frac{3}{16}[a] + \frac{1}{16}[b] + \frac{3}{4}[c] \\ g_2 &= \frac{1}{2}[a] + \frac{1}{2}[b] \end{aligned}$$

Since $g_1 \prec f_1 \prec f_2 \prec g_2$, hypothesis I is satisfied. Then, consider $f = \langle \frac{1}{2}, f_1, f_2 \rangle$ and $g = \langle \frac{1}{2}, g_1, g_2 \rangle$. It is straightforward to see that:

$$\begin{aligned} f &= \frac{1}{4}[a] + \frac{5}{16}[b] + \frac{7}{16}[c] \\ g &= \frac{11}{32}[a] + \frac{9}{32}[b] + \frac{3}{8}[c] \end{aligned}$$

Then, since $\frac{11}{32} > \frac{1}{4}$ and $\frac{9}{32} < \frac{5}{16}$, $f$ and $g$ are incomparable. □

Looking back on Theorem 9, if we strengthen $\mathrm{PDT}_2$ with $\mathrm{PDT}_6$, we could perhaps ensure the comparability of heterogeneous mixtures too. The following theorem shows that this is not the case, even in the more restrictive case where the preferences between components are concordant.

**Theorem 11** *$\mathrm{PDT}_6$ and hypothesis III do not imply comparability of heterogeneous mixtures. The thesis holds even if $f_1 \preceq g_1$ and $f_2 \preceq g_2$.*

*Proof* Consider a slight modification of the preference relation in Lemma 9 where $\mathcal{A} = \{a, b, c\}$ and $\preceq$ is such that:

$$f \preceq g \quad \text{iff} \quad f(c) = g(c) \text{ and } f(b) \leq g(b) \tag{6}$$

First, since $f(a) + f(b) + f(c) = g(a) + g(b) + g(c) = 1$, then $f \sim g$ iff $f = g$. The proof that $\preceq$ satisfies $\mathrm{PDT}_3$ is essentially the same as in Lemma 9. To prove that $\preceq$ satisfies $\mathrm{PDT}_6$ we only need to show that it satisfies Axiom 8. Consider two convergent sequences $\{f_n\}_{n\in\mathbb{N}} \longrightarrow f$ and $\{g_n\}_{n\in\mathbb{N}} \longrightarrow g$ such that $f_n \preceq g_n$ for all $n \in \mathbb{N}$. By construction, we have that $f_n(c) = g_n(c)$ and $f_n(b) \leq g_n(b)$ which implies that $f(c) = g(c)$ and $f(b) \leq g(b)$.

Let $f_1$, $f_2$, $g_1$, and $g_2$ be the following lotteries:

$$\begin{aligned} f_1 &= \frac{3}{8}[a] + \frac{1}{8}[b] + \frac{1}{2}[c] \\ f_2 &= \frac{1}{2}[a] + \frac{1}{4}[b] + \frac{1}{4}[c] \\ g_1 &= \frac{1}{8}[a] + \frac{3}{8}[b] + \frac{1}{2}[c] \\ g_2 &= \frac{1}{4}[a] + \frac{1}{2}[b] + \frac{1}{4}[c] \end{aligned}$$

**Table 3** Comparability propagation over homogeneous and heterogeneous mixtures

| | Homogeneous mixtures | | | Heterogeneous mixtures | | |
|---|---|---|---|---|---|---|
| | Hyp. I | Hyp. II | Hyp. III | Hyp. I | Hyp. II | Hyp. III |
| $PDT_0$ | × | × | × | × | × | × |
| $PDT_1$ | × | × | × | × | × | × |
| $PDT_{4/5}$ | × | × | × | × | × | × |
| $PDT_2$ | ✓ | ✓ | ✓ | ✓ | ✓ | × |
| $PDT_3$ | ✓ | ✓ | ✓ | ✓ | ✓ | × |
| $PDT_6$ | ✓ | ✓ | ✓ | ✓ | ✓ | × |

By construction, we have that $f_1 \prec g_1$ and $f_2 \prec g_2$. Finally, let $f = \langle \frac{1}{2}, f_1, f_2 \rangle$ and $g = \langle \frac{1}{3}, g_1, g_2 \rangle$; it is straightforward to see that:

$$\begin{aligned} f &= \frac{7}{16}[a] + \frac{3}{16}[b] + \frac{3}{8}[c] \\ g &= \frac{5}{24}[a] + \frac{11}{24}[b] + \frac{1}{3}[c] \end{aligned}$$

Consequently, since $f(c) \neq g(c)$, $f$ and $g$ are incomparable. □

From Theorem 3, the entailments among hypotheses I–III and the fact that homogeneous mixtures are just a special case of the heterogeneous ones, it is easy to see that Theorems 8–11 provide a complete picture of the comparability of mixtures w.r.t. $PDT_0$–$PDT_6$ and hypotheses I–III; such a picture is summarized in Table 3.

First, even when the same conditions of classical Decision Theory hold (hypothesis I) $PDT_0$, $PDT_1$, and $PDT_{4/5}$ do not guarantee comparability. Conversely, in $PDT_2$, $PDT_3$, and $PDT_6$ heterogeneous mixtures are comparable even if we relax hypothesis I with hypothesis II. Moreover, in case of homogeneous mixtures, we can further relax hypothesis II by requiring the comparability of the homologous components $f_i, g_i, i \in \{1, 2\}$ only (hypothesis III).

By comparing $PDT_0$ and $PDT_2$ (resp. $PDT_1$ and $PDT_3$), it is evident that the notion of continuity provided by Axiom 7 has a notable effect on the comparability of mixtures. Conversely, since $PDT_1$ and $PDT_{4/5}$ show the same behaviour, replacing Axiom 4 with 8 seems to be ineffective. Furthermore, the considered hypotheses are not able to distinguish between $PDT_0$ and $PDT_1$ (resp. $PDT_2$ and $PDT_3$). However, we know from Theorem 3 that these theories are not equivalent; so which property distinguishes between $PDT_0$ and $PDT_1$ (resp. $PDT_2$ and $PDT_3$)? A first relevant difference between Axioms 5–6 and Axiom 3 lies in the bi-directional nature of the latter which allows one to infer that convex combinations of two incomparable lotteries are incomparable too.

**Theorem 12** *Let $f$ and $g$ be two incomparable lotteries, $f \not\bowtie g$. If $\preceq$ satisfies $PDT_1$, then for all $0 < \alpha \neq \beta < 1$, $\langle \alpha, f, g \rangle \not\bowtie \langle \beta, f, g \rangle$.*

*Proof* First, $f \not\bowtie g$ implies that $\langle \lambda, f, h \rangle \not\bowtie \langle \lambda, g, h \rangle$, for all $h \in \Delta(\mathcal{A})$ and $\lambda$ in the semi-open interval $(0, 1]$—the case $\lambda = 1$ is trivial, $\langle \lambda, f, h \rangle = f$ and $\langle \lambda, g, h \rangle = g$, the case $0 < \lambda < 1$ is a direct consequence of Axiom 3.

Then, assume w.l.o.g. that $\alpha > \beta$ and let $\delta$ and $\gamma$ be $\alpha - \beta$ and $\frac{\beta}{1-\alpha+\beta}$, respectively. Since $0 < \beta < \alpha < 1$, it is immediate to see that $\gamma$ and $\delta$ belong to the open interval $(0, 1)$. Moreover, it holds that $\langle \alpha, f, g \rangle = \langle \delta, f, h \rangle$ and $\langle \beta, f, g \rangle = \langle \delta, g, h \rangle$, where $h = \langle \gamma, f, g \rangle$. Consequently, $\langle \alpha, f, g \rangle \not\bowtie \langle \beta, f, g \rangle$. □

Note that the preference relation defined in Lemma 8 can be easily used to show that Theorem 12 does not hold in $\mathrm{PDT}_0$ and $\mathrm{PDT}_2$—it suffices to consider $f = [a]$, $g = [b]$, $\alpha = \frac{1}{2}$, and $\beta = \frac{1}{3}$. Moreover, in Theorem 12 incomparability is propagated to distinct mixtures of the same lotteries $f$ and $g$. So, it is a natural question whether incomparability is also propagated when convex combinations come, as in the hypotheses I–III, from different lotteries. The following theorem shows that the incomparability of homogeneous mixtures is not preserved even if we consider the more restricted case where $\mathrm{PDT}_6$ is the underlying theory and all the components are pairwise incomparable:

IV. $f_1 \not\bowtie f_2$, $g_1 \not\bowtie g_2$, $f_1 \not\bowtie g_1$, and $f_2 \not\bowtie g_2$.

**Theorem 13** *$PDT_6$ and hypothesis IV do not imply the incomparability of homogeneous mixtures.*

*Proof* Let $\preceq$ be defined as in Theorem 11 and $f_1$, $f_2$, $g_1$, and $g_2$ be the following lotteries:

$$\begin{aligned} f_1 &= \frac{7}{40}[a] + \frac{1}{8}[b] + \frac{7}{10}[c] \\ f_2 &= \frac{23}{40}[a] + \frac{1}{8}[b] + \frac{3}{10}[c] \\ g_1 &= \frac{1}{8}[a] + \frac{1}{4}[b] + \frac{5}{8}[c] \\ g_2 &= \frac{3}{8}[a] + \frac{1}{4}[b] + \frac{3}{8}[c] \end{aligned}$$

Since $f_1(c) \neq f_2(c) \neq g_1(c) \neq g_2(c)$, the four lotteries are pairwise incomparable. Then, let $f = \langle \frac{1}{2}, f_1, f_2 \rangle$ and $g = \langle \frac{1}{2}, g_1, g_2 \rangle$. It is straightforward to see that

$$\begin{aligned} f &= \frac{3}{8}[a] + \frac{1}{8}[b] + \frac{1}{2}[c] \\ g &= \frac{1}{4}[a] + \frac{1}{4}[b] + \frac{1}{2}[c]. \end{aligned}$$

Consequently, by construction, $f \prec g$. □

Summarizing, the continuity notion provided by Axiom 7 significantly enforces the comparability of homogeneous and heterogeneous mixtures. Conversely, Axioms 5–6 in $\mathrm{PDT}_2$ do not propagate incomparability whereas Axiom 3 provides a restricted propagation between convex combinations of the same lotteries.

## 7 Topological properties

In this section we extend the results in Sect. 6 to generic convex combinations. This will also emphasize the differences between $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ under a topological perspective. Consider the sets $\mathcal{C}_1 = \{(f, g) \in \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \mid f \preceq g\}$, $\mathcal{C}_2 = \{(f, g) \in \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \mid f \succ g\}$ and the set of comparable lotteries $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 = \{(f, g) \in \Delta(\mathcal{A}) \times \Delta(\mathcal{A}) \mid f \bowtie g\}$. Clearly, for classical preferences, $\mathcal{C}$ is just the entire product $\Delta(\mathcal{A}) \times \Delta(\mathcal{A})$. However, in case $\preceq$ is partial, how do $\mathrm{PDT}_0 - \mathrm{PDT}_6$ change the "shape" of $\mathcal{C}$?

First, we consider the usual component-wise extension of sum and product, $(f_1, g_1) + (f_2, g_2) = (f_1 + g_1, f_2 + g_2)$ and $\alpha(f, g) = (\alpha f, \alpha g)$. It is immediate to see that a convex combination $\alpha_1(f_1, g_1) + \cdots + \alpha_n(f_n, g_n)$, with $\alpha_1 + \cdots + \alpha_n = 1$, is the pair $(f, g) \in \Delta(\mathcal{A}) \times \Delta(\mathcal{A})$ such that $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$ and $g = \alpha_1 g_1 + \cdots + \alpha_n g_n$.

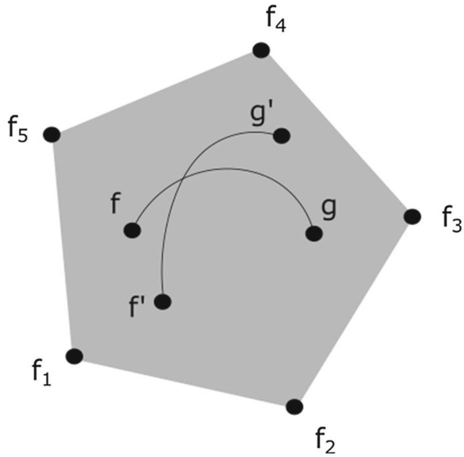



**Fig. 6** The convex hull of the pairwise comparable lotteries $F = \{f_1, \ldots, f_5\}$

Axioms 5 and 6 state that both $\mathcal{C}_1$ and $\mathcal{C}_2$ are convex sets in all the theories $\mathrm{PDT}_0$–$\mathrm{PDT}_6$. With respect to their union $\mathcal{C}$, Theorem 10 states $\mathcal{C}$ is not convex set in $\mathrm{PDT}_{4/5}$ (and hence also in $\mathrm{PDT}_0$ and $\mathrm{PDT}_1$).

Conversely, Theorem 9 states that if $\preceq$ satisfies $\mathrm{PDT}_2$, then $\mathcal{C}$ is convex. Then, a general result in convex theory (see Theorem 2.2 of [35]) is that $\mathcal{C}$ is closed under convex combinations.

**Theorem 14** *Let $\preceq$ satisfy PDT$_2$ and $\mathcal{F} = \{(f_1, g_1), \ldots, (f_n, g_n)\} \subseteq \Delta(\mathcal{A}) \times \Delta(\mathcal{A})$. If $\mathcal{F} \subseteq \mathcal{C}$, then $\mathcal{H}_c(\mathcal{F}) \subseteq \mathcal{C}$.*

Theorem 14 generalizes Theorem 9 from convex combinations of two lotteries to generic combinations of several lotteries. Similarly, the next theorem can be seen as an analogous generalization of Theorem 8, if two sets of lotteries are reciprocally comparable, then all the relative mixtures are comparable too.

**Theorem 15** *Let $\preceq$ satisfy PDT$_2$. Given two finite subsets $F$ and $G$ of $\Delta(\mathcal{A})$, if $F \times G \subseteq \mathcal{C}$, then $\mathcal{H}_c(F) \times \mathcal{H}_c(G) \subseteq \mathcal{C}$.*

*Proof* Let $F = \{f_1, \ldots, f_n\}$ and $G = \{g_1, \ldots, g_m\}$, by hypothesis $f_i \bowtie g_j$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$, we have to show that all convex combinations $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$ and $g = \beta_1 g_1 + \cdots + \beta_m g_m$ are comparable, $f \bowtie g$.

First of all, each $f_i \in F$ can be clearly represented as $f_i = \beta_1 f_i + \cdots + \beta_m f_i$. Then, by Theorem 14, $f_i \bowtie g$, for all $1 \leq i \leq n$.

The rest of the proof is by induction on $n$, where the base case $n = 1$ has just been proved. For the induction step, as in Theorem 14, $f = \langle \alpha_1, f_1, f_{-1} \rangle$, where $f_{-1} \in \mathcal{H}_c(f_2, \ldots, f_n)$. We already know that $f_1 \bowtie g$ and, by induction hypothesis, it also holds that $f_{-1} \bowtie g$. Then, by representing $g$ as $\langle \alpha_1, g, g \rangle$, Theorem 9 implies that $f \bowtie g$. □

As special case of Theorem 15, given $F = \{f_1, \ldots, f_n\}$, if all lotteries in $F$ are pairwise comparable then any two convex combinations of $\mathcal{H}_c(F)$ are in $\mathrm{PDT}_2$ comparable too (e.g. $f$ and $g$ in Fig. 6). Conversely, Theorem 12 can be easily generalized so that if the lotteries in $F$ are pairwise incomparable then any two distinct convex combinations of $\mathcal{H}_c(F)$ are incomparable in $\mathrm{PDT}_1$. In this sense, $\mathrm{PDT}_1$ and $\mathrm{PDT}_2$ show a complementary behaviour.

Another important property of $PDT_1$ concerns the utility representation problem. In [3] Aumann shows that $PDT_1$ satisfies a restricted form of the classical expected utility theorem where only the left-to-right implication holds.

**Theorem 16** (Aumann [3]) *Let $\preceq$ satisfy $PDT_1$. There exists a utility function $u$ such that for all $f, g \in \Delta(\mathcal{A})$ $f \prec g$ (resp. $f \sim g$) implies $f \cdot u < g \cdot u$ (resp. $f \cdot u = g \cdot u$).*

$PDT_3$ inherits Theorems 14 and 16 from $PDT_2$ and $PDT_1$, respectively. Therefore, comparable lotteries form a convex set $\mathcal{C}$ which is divided in the two sets $\mathcal{C}_1$ and $\mathcal{C}_2$ by an hyperplane orthogonal to a vector $(-u, u)$. Due to Lemma 9, $\mathcal{C}$ is not necessarily a closed set in $PDT_3$, whereas Axiom 8 in $PDT_{4/5}$ implies that $\mathcal{C}$ is closed.

Dubra et al. also show that in $PDT_{4/5}$ we can gain back the expected utility theorem in its bidirectional form provided that a partial preference $\preceq$ is represented by a set of utilities.

**Theorem 17** (Dubra et al. [15,43]) *Let $\preceq$ satisfy $PDT_{4/5}$. There exists a non-empty set of utilities, $U \subseteq \mathbb{R}^N$, such that $f \preceq g$ iff $f \cdot u \leq g \cdot u$, for all $u \in U$.*

Theorem 17 can be further refined if we consider $PDT_6$. In particular, the following theorem shows that if a preference relation satisfies $PDT_6$, then the set of utilities $U$ in Theorem 17 can be finite and characterizes a half subspace of $\mathbb{R}^N$.

**Theorem 18** *Let $\preceq$ satisfy $PDT_6$. There exist $m$ utility functions $u_1, \ldots, u_m$, with $m \leq N$, and a utility function $u$ such that $f \preceq g$ iff $f \cdot u \leq g \cdot u$ and $f \cdot u_i = g \cdot u_i$, for all $1 \leq i \leq m$.*

*Proof* From Theorem 1.6 in [42], there exists a linearly closed convex cone $D$ such that for all $f, g \in \Delta(\mathcal{A})$, $f \preceq g$ iff $g - f \in D$.[5] Clearly, by reflexivity, the null vector is in $D$. By $D_n$ we denote the smallest subspace of $\mathbb{R}^N$ containing $D$, where $n \leq N$ is the dimension of $D_n$. Then, let $B$ be an orthogonal basis for $D_n$ and $B \cup \{u_1, \ldots, u_m\}$ be an orthogonal basis for $\mathbb{R}^N$ extending $B$, with $m = N - n$. It is well known that a vector $v$ is in $D_n$ iff $v \cdot u_i = 0$, for all $1 \leq i \leq m$. Then, for each pair of lotteries $f$ and $g$ it holds that

$$g - f \in D_n \text{ iff } (g - f) \cdot u_i = 0 \text{ iff } f \cdot u_i = g \cdot u_i\,, \tag{7}$$

for all $1 \leq i \leq m$. If $D = D_n$, then by setting $u$ equal to the null vector we have the thesis. Note that in this case, since $f - g = -(g - f)$ and $D$ is a subspace of $\mathbb{R}^N$, we have that

$$f \preceq g \Leftrightarrow g - f \in D \Leftrightarrow f - g \in D \Leftrightarrow g \preceq f\,.$$

Consequently, each pair $f$ and $g$ of lotteries are either equivalent ($g - f \in D$) or incomparable ($g - f \notin D$).

Assume that $D \subset D_n$. If $D_n$ is a line (i.e. $n = 1$), then $D$ is a half-line contained in $D_n$. Let $u$ be any vector in $D$, we have that $g - f \in D$ iff $g - f \in D_n$ and $(g - f) \cdot u \geq 0$ (i.e. $f \cdot u \leq g \cdot u$). Consequently, by Theorem 1.6 in [42] and (7) we have the thesis. Then, assume that $n \geq 2$, by convexity and Corollary 11.5.2 in [35], $D$ is contained in a closed half-space $H$. We will show that $D = H$.

For the sake of simplicity, let's fix a certain lottery $f$. By Theorem 6.2 in [35] there exist $n$ lotteries $g_1, \ldots, g_n$ such that the vectors $v_1 = (g_1 - f), \ldots, v_n = (g_n - f)$ belong to the relative interior of $D$ and are linearly independent. Moreover, by Corollary 6.5.2 in [35], $v_1, \ldots, v_n$ are also in the interior of $H$. This means that $-v_1 = (f - g_1), \ldots, -v_n = (f - g_n)$

[5] By "linearly closed" the authors means that the intersection of $D$ with any line is a closed set. This in particular implies that $D$ is closed.

are not in $H$ and hence they are not in $D$. Then, by applying again Theorem 1.6 in [42], we have that $f \prec g_j$, for all $1 \leq j \leq n$.

Consider the vectors $f - v_j$, with $1 \leq j \leq n$, we have that $f - v_j = f + (f - g_j)$ and hence

$$\sum_{a\in\mathcal{A}} \left(f - v_j\right)(a) = \sum_{a\in\mathcal{A}} \left(2f(a) - g_j(a)\right) = 2\sum_{a\in\mathcal{A}} f(a) - \sum_{a\in\mathcal{A}} g_j(a) = 1\,.$$

Therefore, $f - v_j$ is a lottery $h_j$ and, since $f - h_j = g_j - f$, we have that $h_j \prec f$.

Then, consider the pairs $h_j$ and $g_{j+1}$ with $1 \leq j \leq n-1$. Since $h_j \prec f \prec g_{j+1}$, by applying Axiom 7, there exists some $0 < \alpha_j < 1$ such that $f \sim \langle \alpha_j, h_j, g_{j+1}\rangle$. Since $h_j = f - v_j$ and $g_{j+1} = f + v_{j+1}$, by construction we have that $\langle \alpha_j, h_j, g_{j+1}\rangle = f + w_j$, where $w_j = \langle \alpha_j, -v_j, v_{j+1}\rangle$. Consequently, we have that $f \sim f + w_j$, which means that $w_j \in D$, for all $1 \leq j \leq n-1$. We will show that these vectors are linearly independent. By contradiction, assume that $w_j$ can be expressed as a linear combination of the other vectors:

$$w_j = c_1 w_1 + \cdots + c_{j-1} w_{j-1} + c_{j+1} w_{j+1} + \cdots + c_{n-1} w_{n-1}$$

where $c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n$ are real numbers. By expanding the previous equation, we have that

$$\begin{aligned} w_j = & -c_1\alpha_1 v_1 + c_1(1-\alpha_1)v_2 + \cdots - c_{j-1}\alpha_{j-1} v_{j-1} + c_{j-1}(1-\alpha_{j-1})v_j \\ & -c_{j+1}\alpha_{j+1} v_{j+1} + c_{j+1}(1-\alpha_{j+1})v_{j+2} + \cdots \\ & -c_{n-1}\alpha_{n-1} v_{n-1} + c_{n-1}(1-\alpha_{n-1})v_n\,. \end{aligned}$$

Since $w_j = -\alpha_j v_j + (1-\alpha_j)v_{j+1}$, $c_{j-1}$ and $c_{j+1}$ are different from zero whereas all the other constants are equal to zero. Then, we have

$$\begin{aligned} -\alpha_j v_j + (1-\alpha_j)v_{j+1} = & -c_{j-1}\alpha_{j-1} v_{j-1} + c_{j-1}(1-\alpha_{j-1})v_j \\ & -c_{j+1}\alpha_{j+1} v_{j+1} + c_{j+1}(1-\alpha_{j+1})v_{j+2}\,. \end{aligned}$$

where the only possible solution is $c_{j-1} = -\alpha_j$, $c_{j+1} = -(1-\alpha_j)$, $\alpha_{j-1} = 0$, and $\alpha_{j+1} = 1$. However, this is against the hypothesis $0 < \alpha_{j-1}, \alpha_{j+1} < 1$.

Therefore, $w_1, \ldots, w_{n-1}$ are linearly independent and hence they form an hyperplane $P$ of dimension $n-1$. Clearly, since $f \sim f + w_j$, $P$ is the border of $D$ — recall that for each $g - f$ in the relative interior of $D$ it holds that $f \prec g$. Consequently, $D$ is the half-space $H$.

Finally, let $u$ be a vector of $D$ that is orthogonal to $P$. As before, we have that $g - f \in D$ iff $g - f \in D_n$ and $(g - f)\cdot u \geq 0$. By applying Theorem 1.6 of [42] and (7) we have the thesis. □

*Example 6* Consider the preference relation defined in Theorem 11, in this case we only need two utility functions $u_1$ and $u$. The utility $u_1$ is any function such that $u_1(a) = u_1(b) = 0$ and $u(c) = \lambda$, where $\lambda$ is a strictly positive real number. In particular, by setting $\lambda = 1$, $u_1$ is the lottery $[c]$. Similarly, $u$ is such that $u(a) = u(c) = 0$ and $u(b) = \gamma$, with $\gamma > 0$; as before, we can set $u = [b]$. Therefore, we have that $f \preceq g$ iff $f \cdot [c] = g \cdot [c]$ and $f \cdot [b] \leq g \cdot [b]$—that is, $f(c) = g(c)$ and $f(b) \leq g(b)$. □

In Theorem 18 if $m = 0$, then $\preceq$ is classical preference (i.e. $\preceq$ is a total relation).

Finally, the following theorem shows that in $\mathrm{PDT}_6$, given a lottery $f$, the set of lotteries that are comparable to $f$ is totally ordered.

**Theorem 19** *Let $\preceq$ satisfy $PDT_6$. If $f \bowtie g_1$ and $f \bowtie g_2$, then $g_1 \bowtie g_2$.*

*Proof* Assume that $f \bowtie g_1$ and $f \bowtie g_2$, clearly if $g_1 \preceq f \preceq g_2$ or $g_2 \preceq f \preceq g_1$, then by transitivity $g_1 \bowtie g_2$. Therefore, assume that $f \preceq g_1$ and $f \preceq g_2$ (the case $g_1 \preceq f$ and $g_2 \preceq f$ is symmetrical). This means that both $v_1 = g_1 - f$ and $v_2 = g_2 - f$ are in $D$ (see Theorem 18). Since $D$ is a half-subspace of $\mathbb{R}^N$, then $v_2 - v_1$ is in $D$ too. However, $v_2 - v_1 = g_2 - g_1$ and hence $g_1$ and $g_2$ are comparable.

## 8 Integrating with preference elicitation methodologies

A preference can be completely specified as in Lemmas 7 or 8, however in several concrete applications it is reasonable to assume that the user specifies a preference over alternatives only, charging the software agent to compare generic lotteries. Moreover, as in Examples 1 and 5, preference elicitation may not cover all possible pairs of alternatives and this clearly raises the question of how to interpret the unspecified pairs. In [32,33] the authors distinguish between incomparability, which they interpret as the inability to make a decision, and incompleteness which represents the fact that the software agent does not know in which relation two alternatives are. As in [33], we abstract away from a specific user/agent interface and assume that the preference elicitation process ends up with a set $\mathsf{S}$ of user statements indicating equivalence $a \equiv b$, strict preference $a < b$, and incomparability $a \not\lesseqgtr b$.[6]

A preorder $\preceq \subseteq \Delta(\mathcal{A}) \times \Delta(\mathcal{A})$ satisfies $\mathsf{S}$ iff (i) for each statement $a \equiv b$ in $\mathsf{S}$, $[a] \sim [b]$; (ii) for each statement $a < b$ in $\mathsf{S}$, $[a] \prec [b]$; (iii) for each statement $a \not\lesseqgtr b$ in $\mathsf{S}$, $[a] \not\bowtie [b]$. With $\mathcal{P}_\mathsf{S}$ we denote the set of preorders that satisfy $\mathsf{S}$.

Consider Example 5, where $\mathsf{S}$ consists of the set of $<$-statements represented by the edges in Fig. 4. Since, no statement concerns the pair $a_3$ and $a_6$, we have that $\mathcal{P}_\mathsf{S}$ contains some preorders $\preceq_1, \ldots, \preceq_4$ such that $[a_3] \sim_1 [a_6]$, $[a_3] \prec_2 [a_6]$, $[a_6] \prec_3 [a_3]$, and $[a_6] \not\bowtie_4 [a_3]$. This means that $a_3$ and $a_6$ can be in any relation in $\mathcal{P}_\mathsf{S}$ and hence underspecification is interpreted as incompleteness.

Clearly, $\mathcal{P}_\mathsf{S}$ is defined only on the base of degenerate lotteries and hence it includes a large class of relations that are meaningless for us in the sense that they do not satisfy any of $\mathrm{PDT}_0$–$\mathrm{PDT}_6$. For this reason we denote with $\mathcal{P}_\mathsf{S}[PDT_x]$, where $x = 0, 1, 2, 3, 4/5, 6$, the set of preferences of $\mathcal{P}_\mathsf{S}$ satisfying $PDT_x$. In other words, $\mathcal{P}_\mathsf{S}[PDT_x]$ represents the set of preferences that are consistent with the input statements $\mathsf{S}$ assuming that the software agent adopts $PDT_x$ and no assumption is made on the unspecified pairs—that is, incompleteness is used as default. Then, the software agent can derive that two generic lotteries $f$ and $g$ are in certain relation if this holds in the whole of $\mathcal{P}_\mathsf{S}[PDT_x]$. In particular, we say analogously to the previous sections that $\mathcal{P}_\mathsf{S}[PDT_x]$ implies $f \star g$ (where $\star$ can be any positive or negated relation $\preceq, \prec, \sim, \bowtie$) just in the case all preferences in $\mathcal{P}_\mathsf{S}[PDT_x]$ satisfy $f \star g$.

As said above, underspecification is interpreted in $\mathcal{P}_\mathsf{S}$ as incompleteness, another possibility is to assume that each unspecified pair of alternatives is, by default, incomparable. Let $\mathcal{I}_\mathsf{S} \subseteq \mathcal{P}_\mathsf{S}$ be the set of preferences $\preceq$ such that for all $a, b \in \mathcal{A}$, $[a] \preceq [b]$ iff for all $\preceq' \in \mathcal{P}_\mathsf{S}$, $[a] \preceq' [b]$. In the previous example, since $[a_3] \prec_2 [a_6]$, $[a_6] \prec_3 [a_3]$, then for all $\preceq \in \mathcal{I}_\mathsf{S}$ we have that $[a_3] \not\preceq [a_6]$ and $[a_6] \not\preceq [a_3]$, and this means that $\mathcal{I}_\mathsf{S}$ forces $[a_3] \not\bowtie [a_6]$. The set $\mathcal{I}_\mathsf{S}[PDT_x]$ is defined as before.

The formal results provided so far show that the theories $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ behave differently w.r.t. $\mathcal{P}_\mathsf{S}$ and $\mathcal{I}_\mathsf{S}$. For instance, in Example 5, it is easy to see that $[a_4]$ and $[a_5]$ can be in any relation in $\mathcal{P}_\mathsf{S}[\mathrm{PDT}_1]$. In particular, they can be incomparable. On the contrary, if

[6] We refer to [1] for a concrete application domain where a user/agent interface provides a set of statements $\mathsf{S}$ over service-level agreements.

$\mathrm{PDT}_2$ is adopted, both $[a_4]$ and $[a_5]$ are equivalent to a mixture of $[a_2]$ and $[a_6]$ and hence, by Lemma 11, they are comparable. This means that, for each preference $\preceq \in \mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$, either $[a_4] \preceq [a_5]$ or $[a_5] \preceq [a_4]$, consequently $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$ implies $[a_4] \bowtie [a_5]$. Similarly, if the assertions $a < b$ in $\mathsf{S}$ form a complete lattice, as in Fig. 1, then $\mathrm{PDT}_2$ forces all the alternatives to be pairwise comparable. Subsequently, by Theorem 15, for all $f, g \in \Delta(\mathcal{A})$, $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$ implies $f \bowtie g$. In other words, the agent assumes that the user has a classical preference $\preceq$ even if it does not know which one—$\preceq$ can be any preference in $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$.

It is worth noting that $\mathrm{PDT}_2$, $\mathrm{PDT}_3$, and $\mathrm{PDT}_6$ may also be inconsistent with the user's statements. For instance, if in Example 5 the user adds a statement $a_4 \nleq a_5$, then for each preference in $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$ both $[a_4] \bowtie [a_5]$ and $[a_4] \not\bowtie [a_5]$ should hold, which means that $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_2]$ is empty. The same considerations lead to the fact that $\mathrm{PDT}_2$, $\mathrm{PDT}_3$, and $\mathrm{PDT}_6$ cannot in general be adopted when incomparability is used as default. In particular, $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_2]$ turns out to be empty whenever the assertions of type $a < b$ contain "diamonds" as in Examples 1 and 5. Conversely, the following theorem shows that $\mathrm{PDT}_0$ and $\mathrm{PDT}_1$ do not clash with the user's statements.[7]

**Theorem 20** *Let $\mathsf{S}$ be a set of statements. If $\mathcal{P}_{\mathsf{S}}$ is not empty, then $\mathcal{I}_{\mathsf{S}}[PDT_0]$ and $\mathcal{I}_{\mathsf{S}}[PDT_1]$ are also not empty.*

*Proof* Let $\mathsf{S}_{\equiv}$ be the set of pairs $(a, b)$ such that $a \equiv b \in \mathsf{S}$ and similarly $\mathsf{S}_{<}$ the set of pairs $(a, b)$ where $a < b \in \mathsf{S}$. Let $\equiv_c \subseteq \mathcal{A} \times \mathcal{A}$ be the minimal equivalence relation that contains the identity and $\mathsf{S}_{\equiv}$. Analogously, let $<_c$ be the minimal strict partial order (i.e. transitive and asymmetric relation) that contains $\mathsf{S}_{<}$ and conforms with $\equiv_c$, that is, if $a <_c b$ and $b \equiv_c d$ (resp. $a \equiv_c d$), then $a <_c d$ (resp. $d <_c b$).

Then, let $\preceq_c$ be the preference relation satisfying the following two constraints:

- $f \sim_c g$ iff $f([\![a]\!]_c) = g([\![a]\!]_c)$ for all $a \in \mathcal{A}$;
- $f \prec_c g$ iff $f \twoheadrightarrow_c g$ ,

where $[\![a]\!]_c = \{b \in \mathcal{A} \mid a \equiv_c b\}$, $f([\![a]\!]_c) = \sum_{b \in [\![a]\!]_c} f(b)$, and $\twoheadrightarrow_c$ is defined as the operator $\twoheadrightarrow$ in Sect. 5 where the probability amounts $\epsilon$ are shifted according to the strict partial order $<_c$. It is straightforward to see that if $\mathcal{P}_{\mathsf{S}}$ is not empty, then by construction $\preceq_c$ is in $\mathcal{I}_{\mathsf{S}}$. We will also show that $\preceq_c$ satisfies $\mathrm{PDT}_1$.

Reflexivity and transitivity are straightforward. W.r.t. Axiom 3, let $k_1 = \langle\alpha, f, h\rangle$ and $k_2 = \langle\alpha, g, h\rangle$, with $0 < \alpha < 1$. We have that $f \sim_c g$ iff $f([\![a]\!]_c) = g([\![a]\!]_c)$ iff $\alpha f([\![a]\!]_c) + (1-\alpha)h([\![a]\!]_c) = \alpha g([\![a]\!]_c) + (1-\alpha)h([\![a]\!]_c)$, that is, $k_1([\![a]\!]_c) = k_2([\![a]\!]_c)$, for all $a \in \mathcal{A}$. Moreover, by construction, $f \prec_c g$ iff $f \twoheadrightarrow_c g$. By definition, there exists a chain $f = f_0 \rightarrow_c \cdots \rightarrow_c f_n = g$, such that each $f_{i+1}$ is obtained from $f_i$ by shifting $\epsilon_i$ from an alternative $a_i$ to a better alternative $a_{i+1}$. This is equivalent to say that $\alpha f_{i+1} + (1-\alpha)h$ is obtained from $\alpha f_i + (1-\alpha)h$ by shifting $\alpha\epsilon_i$ from $a_i$ to $a_{i+1}$. Consequently, we have $f \prec_c g$ iff $f \twoheadrightarrow_c g$ iff $k_1 \twoheadrightarrow_c k_2$ iff $k_1 \prec_c k_2$.

It remains to show that Axiom 4 holds. Assume that $h \prec \langle\alpha, f, g\rangle$ , for all $0 < \alpha < 1$ and $f \prec_c h$. By construction, $f \prec_c h$ implies $f \twoheadrightarrow_c h$. Since $h$ is obtained from $f$ by shifting probabilities "upwards" w.r.t. $<_c$, there exists an alternative $a$ such that $h(a) = f(a) + \epsilon$, for some $\epsilon > 0$, whereas $h(b) = f(b)$ for all $b$ such that $a \prec_c b$. Then, by continuity of convex combinations, there exists a $\gamma$ sufficiently close to 1 such that $h(a) > \langle\gamma, f, g\rangle(a) + \frac{\epsilon}{2}$, and

$$\sum_{b \,:\, a \prec_c b} (\langle\gamma, f, g\rangle(b) - h(b)) < \frac{\epsilon}{2}\,.$$

[7] It is an open question whether Theorem 20 holds for $\mathrm{PDT}_{4/5}$.

This means that $h \not\twoheadrightarrow_c \langle\gamma, f, g\rangle$ and hence $h \not\prec_c \langle\gamma, f, g\rangle$, a contradiction.

Summing up, $\preceq_c$ is in $\mathcal{I}_{\mathsf{S}}$ and satisfies $\mathrm{PDT}_1$, hence $\preceq_c \in \mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$. Moreover, since $\mathrm{PDT}_0 < \mathrm{PDT}_1$, we have that $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1] \subseteq \mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_0]$ and hence $\preceq_c$ is also in $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_0]$. □

The proof of Theorem 20 provides a canonical preference $\preceq_c$ which allows one to decide whether $f \preceq g$ holds in $\mathcal{P}_{\mathsf{S}}[PDT_x]$, with $x = 0, 1$.

**Theorem 21** *For all lotteries $f, g \in \Delta(\mathcal{A})$, $f \preceq_c g$ iff $\mathcal{P}_{\mathsf{S}}[PDT_x]$ implies $f \preceq g$, with $x = 0, 1$.*

*Proof* As shown in the proof of Theorem 20, the right-to-left direction is direct consequence of the fact that $\preceq_c \in \mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1] \subseteq \mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_0]$. For the left-to-right direction, since $\preceq_c \in \mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_0]$, we have that for all $a, b \in \mathcal{A}$ and $\preceq \in \mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$, (i) $[a] \sim_c [b]$ iff $[a] \sim [b]$ and (ii) $[a] \prec_c [b]$ iff $[a] \prec [b]$.

Assume that $f \prec_c g$, by construction we have that $f \twoheadrightarrow_c g$. Then, let $\preceq$ be a preference of $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$, by (ii) and $f \twoheadrightarrow_c g$ we also have that $f \twoheadrightarrow g$ and hence, by Lemma 10, $f \prec g$.

Then, assume that $f \sim_c g$ and let $[\![a_1]\!]_c, \ldots, [\![a_n]\!]_c$ be the set of equivalent classes of $\preceq_c$. By (i), for all $1 \leq i \leq n$ and $\preceq \in \mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$, $[\![a_i]\!] = [\![a_i]\!]_c$ and, since $f([\![a_i]\!]_c) = g([\![a_i]\!]_c)$, we also have that $f([\![a_i]\!]) = g([\![a_i]\!])$, for all $1 \leq i \leq n$. Now, let $f_i$ (resp. $g_i$) be the lottery such that $f_i(b)$ is equal to $\frac{f(b)}{f([\![a_i]\!])}$ (resp. $\frac{g(b)}{g([\![a_i]\!])}$), if $b \in [\![a_i]\!]$, 0 otherwise. By Theorem 5, it holds that $f_i \sim g_i$, for all $1 \leq i \leq n$. Moreover, $f$ and $g$ can be decomposed as $f = \alpha_1 f_1 + \cdots + \alpha_n f_n$ and $g = \alpha_1 g_1 + \cdots + \alpha_n g_n$, where $\alpha_i = f([\![a_i]\!]) = g([\![a_i]\!])$. Then, it is straightforward to see that $f \sim g$. Subsequently, we have that if $f \preceq_c g$, then $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$ implies $f \preceq g$. Finally, since $\mathcal{P}_{\mathsf{S}}[PDT_1]$ is contained in $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$, the same holds for $\mathcal{P}_{\mathsf{S}}[PDT_1]$. □

First, it is worth noting that checking whether $f \sim_c g$ or $f \prec_c g$ can be done in polynomial time,[8] hence Theorem 21 provides correct and complete procedure to decide whether $f \preceq g$ holds in all preferences of $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$ or $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_1]$.

Secondly, since $\preceq_c \in \mathcal{I}_{\mathsf{S}}[PDT_x]$ and $\mathcal{I}_{\mathsf{S}}[PDT_x] \subseteq \mathcal{P}_{\mathsf{S}}[PDT_x]$, with $x = 0, 1$, it is immediate to see that Theorem 21 holds even if we replace $\mathcal{P}_{\mathsf{S}}[PDT_x]$ with $\mathcal{I}_{\mathsf{S}}[PDT_x]$. In particular, this means that in respect of inferring $f \star g$, with $\star =\sim, \prec, \bowtie$, it makes no difference if we interpret underspecification as incompleteness or incomparability and which theory we adopt. Conversely, these settings generally affect the negated relations and in particular incomparability.

Clearly, one can derive more incomparability in $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_0]$ with respect to $\mathcal{P}_{\mathsf{S}}[\mathrm{PDT}_0]$. However, since in $\mathrm{PDT}_0$ no axiom specifies how incomparability propagates from lotteries to mixtures thereof, this is limited to degenerate lotteries only. On the contrary, we know that in $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ the incomparability of degenerate lotteries propagates to their convex combinations according to Theorem 12. Note that inferring incomparability is not worthless, consider for instance the problem of finding a lottery in a set $F = \{f_1, \ldots f_n\}$ that is optimal. First of all, we have to formalize what we mean by an optimal lottery. A natural definition could be that $f_i$, $1 \leq i \leq n$, is optimal w.r.t. to a preference $\preceq$ iff for all $1 \leq j \leq n$, $f_j \preceq f_i$. However, since $\preceq$ is partial, the existence of a lottery which dominates all the others will be rarely satisfied in practice. For this reason we use, as in [3], a weaker notion where $f_i$ is

8 Instantiating the relations $=_c$ and $<_c$ is not more complex than calculating the transitive closure of a relation, hence it can be done in $O(|\mathcal{A}|^3)$. Then, on the one hand, checking $f \sim_c g$ means to verify that, for all equivalent classes $[a]\!]$ induced by $=_c$, $f([\![a]\!]) = g([\![a]\!])$, which can be done in $O(|\mathcal{A}|)$. On the other hand, checking $f \prec_c g$ means to verify that $f \twoheadrightarrow_c g$, by Theorem 6 this is essentially equivalent to a bipartite-matching problem that can be solved again in $O(|\mathcal{A}|^3)$ [47].

optimal iff no other lottery strictly dominates $f_i$, that is, for all $1 \leq j \leq n$, $f_i \not\prec f_j$. Note that $f_i \not\prec f_j$ means that either $f_j \preceq f_i$ or $f_j \not\bowtie f_i$, consequently, weak optimality allows the existence of several local maxima that are incomparable with each other.

*Example 7* Consider $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ where $\mathsf{S}$ is the set of strict preferences depicted in Fig. 4. Moreover, let $f_1 = [a_1]$, $f_2 = [a_3]$, and $f_3 = \frac{1}{3}[a_6] + \frac{2}{3}[a_5]$. Clearly, $f_3$ dominates $f_1$ and hence $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ implies $f_1 \prec f_3$. Moreover, since $a_3$, $a_4$ and $a_5$ are pairwise incomparable, two distinct mixtures of them are in $\mathrm{PDT}_1$ incomparable as well. Then, (i) $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ implies $[a_3] \not\bowtie \frac{1}{3}[a_4] + \frac{2}{3}[a_5]$. Moreover, since $\frac{1}{3}[a_4] + \frac{2}{3}[a_5] \twoheadrightarrow \frac{1}{3}[a_6] + \frac{2}{3}[a_5]$, we have that (ii) $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ implies $\frac{1}{3}[a_4] + \frac{2}{3}[a_5] \prec \frac{1}{3}[a_6] + \frac{2}{3}[a_5]$. Finally, from (i) and (ii), $\mathcal{I}_{\mathsf{S}}[\mathrm{PDT}_1]$ implies $\frac{1}{3}[a_6] + \frac{2}{3}[a_5] \not\prec [a_3]$. Consequently, $f_3$ is weakly optimal. □

## 9 Related work

The present paper extends [39] in several ways. First of all, $\mathrm{PDT}_{4/5}$ and $\mathrm{PDT}_6$ were not considered in [39] and $\mathrm{PDT}_0$ was redundant as it included monotonicity under convex combinations as an axiom. Conversely, we have shown in Theorem 4 that such a property can be derived from the other axioms. Secondly, we have reduced the problem of checking whether $f \twoheadrightarrow g$ to a bipartite matching problem (see Theorem 6) and we have shown that if a preference relation is a linear order on the degenerate lotteries, then the relation $\twoheadrightarrow$ is equivalent to the notion of first-order stochastic dominance. Third, here we extend the comparison of $\mathrm{PDT}_0$–$\mathrm{PDT}_6$ by analysing the degree of comparability with respect the classical Decision Theory and topological properties (Sects. 6 and 7). In particular, $\mathrm{PDT}_6$ allows to gain back a form of the expected utility theorem that is similar to the classical one (see Theorem 18). Finally, besides comparing $\mathrm{PDT}_0$–$\mathrm{PDT}_6$, we have also considered how these theories can be integrated with preference elicitation processes which typically lead to underspecified preferences over alternatives (Sect. 8).

### 9.1 Partial preferences in multi-agent systems

Partial preferences have been used in different contexts of Multi-agent Systems. First, in order to improve usability several preference elicitation methodologies, e.g. CP nets, do not force preferences to be total. In CP nets [9,23] alternatives are represented as instantiations of a finite set $V$ of variables. Each variable $x \in V$ depends on a set of parent variables $W$ and is conditionally preferentially independent from the remaining variables $Z = V \setminus (W \cup \{x\})$. Roughly speaking, each instantiation of $W$ induces, when all the other variables in $Z$ are held fixed, a conditional preference over the possible values of $x$. It is easy to see that a CP net does not ensure the totality of the resulting preference which in general is just a preorder over alternatives. Moreover, in the original framework the user has to specify a preference over the $x$'s values for each possible instantiation of the parent variables $W$. Thus, the corresponding data structure, called a conditional preference table, may easily grow exponentially. General CP nets relax this constraint by allowing partial specifications of the conditional preference tables. In recent years, CP nets have been extended in different ways. Briefly, in [10] it is possible to state that a variable $x$ is more important than $y$ in the sense that, all other variables being equal, the preference on $x$ overrides the preference on $y$ (see also Example 1). In [49] stronger statements are introduced which express that a value $v_x$ of the variable $x$ is preferred to a another value $v'_x$ irrespective of the values of other variables. It is worth noting that in all these formalisms the user cannot explicitly assert that two alternatives are incomparable.

This means that all the theories $PDT_0$–$PDT_6$ we have considered are compatible with CP nets in case incompleteness is adopted as default. In particular, since the preorder resulting from a CP net is a lattice, $PDT_2$ ensures the comparability of each pair of lotteries (see Sect. 8).

Partial preferences has been also used in the context of preference aggregation. In [32,33] agents' preferences may include both incomparable and unspecified pairs of alternatives, where the latter can be completed with any relation (equivalence, incomparability, or strict preference) which preserves consistency. Consequently, a social welfare function may return different optimal alternatives for each possible way the agents' preferences can be completed. An alternative is then a possible winner (resp. necessary winner) if it is optimal in at least one completion (resp. in all possible completions). The complexity of checking possible or necessary winners is studied in different settings. Roughly, since preferences can be completed in polynomial time and the number of possible completions may increase exponentially, checking possible and necessary winners are in the general case NP-complete and coNP-complete, respectively. However, under the mild assumptions that the social welfare function satisfies the properties of monotonicity and independence from irrelevant alternatives (such as the Pareto social welfare function), the previous decision problems are both polynomial. In [50] a similar preference aggregation problem is studied where we do not have incomparable alternatives but only underspecified preferences. Moreover, in contrast to [32,33], the authors focus on a bundle of well-known social welfare functions (e.g. Veto, Copeland, Bucklin) instead of classes of functions satisfying some specific properties. The complexity of possible and necessary winner problems are studied. As in [32,33], some cases (e.g. Veto) are tractable and other social welfare functions, such as Copeland, raise the complexity to NP (possible winner) and coNP (necessary winner). However, there are also some functions, such as Bucklin and Maxmin, where the possibility winner problem is NP-complete and the necessarily winner problem is in P. Clearly, in these works preferences are limited to alternatives only whereas our work focuses on lotteries of alternatives. However, we have considered a similar scenario where a preference elicitation process involves alternatives only. In particular, we have seen that $PDT_0$ and $PDT_1$ are more appropriate in the context described in [32,33] where incomparability can be explicitly asserted by the user. Conversely, if incomparability cannot be asserted and the user statements form a lattice then $PDT_2$, $PDT_3$, and $PDT_6$ express, as in [50], the set of all total preferences that are consistent with the user's statements.

Recently, partial preferences had been also used in mechanisms without money like contract-based procurement auctions [1,6–8,28,34]. In particular, preferences are modelled as preorders over a set of possible contracts where the unspecified pairs of contracts are assumed to be incomparable by default. In [8] the authors focus on deterministic auctions adopting a priority relation over bidders to break possible ties. In this contest, it has been proved that no auction can be designed which preserves the properties of relative optimality, no positive transfer, and truthfulness at the same time. Roughly speaking, relative optimality is a revenue guarantee for the auctioneer assuring that the mechanism properly uses the competition among the bidders. The no positive transfer property simply means that a mechanism does not allocate a contract to a bidder if this contract is not included in her offer. Finally, truthfulness means that the best strategy for a bidder, no matter what the other bidders do, is always to bid exactly the set of all affordable contracts. Then, a mechanism is proposed that satisfies relative optimality, no positive transfer and admits overbidding as the only profitable deviation from playing truthful. In particular, from a bidder's point of view, overbidding can be profitable only in case of tie-breaking, otherwise it may yield an equivalent outcome or even cause a loss. Interestingly, for the auctioneer's perspective, overbidding is never worse than playing truthfully. In [1] non-deterministic auctions are considered which, by breaking ties randomly, generate lotteries over contract allocations. However, instead of defining a theory which extends preferences to lotteries, this work proposes *ad hoc* rules which have no clear relation with Classical Decision Theory and are rather a notion of dominance among strategies. On the contrary, the theories presented in our work directly derive from (different formulations of) Classical Decision Theory. In particular, since incomparability is used as a default, and during the mechanism's process agents look for optimal lotteries, the best candidate to support these non-deterministic auctions is $\mathrm{PDT}_1$.

### 9.2 Partial preferences in decision theory

Since the pioneering work [3], a theoretical investigation on partial preferences has received an increasing attention in the last decades. The majority of the works focused on the representation problem, that is, how and at which extent partial preferences (which are ordinal in their nature) can be quantitatively represented by different forms of utility functions.

In Aumann [3] considers $\mathrm{PDT}_1$ as axiomatic system. The main result of the paper is Theorem 16 which can be seen as a weaker form of the classical expected utility theorem where only the left-to-right direction holds. Note that, according to Theorem 16, a lottery $g$ which maximizes the scalar product with $u$ (that is, for all $f \in \Delta(\mathcal{A})$ $f \cdot u \leq g \cdot u$) is also weakly optimal w.r.t. $\preceq$. Consequently, the utility function $u$ can be used in the optimality problem in order to find at least one weakly optimal lottery.

In Peleg [31] considers the representation problem of partial preferences in a more abstract setting. Given a topological space $\mathcal{X}$ and a strict partial order $\prec$ over it (i.e. an irreflexive and transitive relation) the author investigate sufficient conditions for the existence of a continuous utility function $u : \mathcal{X} \to \mathbb{R}$ such that for all $x, y \in \mathcal{X}$, $x \prec y \Rightarrow u(x) < u(y)$. The paper provides a first answer to this issue, a continuous ordering-preserving utility is guaranteed to exist if $\prec$ is continuous, separable, and spacious, where (i) continuous means that $L(x) = \{y \in \mathcal{X} \mid y \prec x\}$ is open, for all $x \in \mathcal{X}$; (ii) $\prec$ is separable iff there exists a countable subset $\mathcal{S} \subseteq \mathcal{X}$ such that for each pair $x \prec y$ in $\mathcal{X}$ there exists a $z \in \mathcal{S}$ with $x \prec z \prec y$; (iii) $\prec$ is spacious whenever $x \prec y$ implies that the closure of $L(x)$ is strictly contained in $L(y)$. In [11] point (iii) is replaced with the condition that also the set $G(x) = \{y \in \mathcal{X} \mid x \prec y\}$ is open. In [45] another sufficient condition is given for strict partial orders, if $\prec$ is continuous and $\mathcal{X}$ is second countable (i.e. it contains a countable basis of open sets) then $\prec$ is represented by an upper semi-continuous function $u$.

Fishburn focuses on the problem of linearizing a partial order (i.e. an antisymmetric preorder) and shows that this problem can be more insidious than expected [19]. In particular, given a partial order $\preceq$ on a finite set $X$, the linear extension majority graph is a directed graph $\langle X, <_m \rangle$ in which $x <_m y$ if and only if $S(y, x) < S(x, y)$, where $S(x, y)$ is number of total orders $\preceq'$ that extend $\preceq$ and such that $x \prec' y$. Apparently, $<_m$ can be seen as a reasonable method to linearize $\preceq$, however the author shows an example with 304 elements where $<_m$ has a cycle and hence it is not a strict order.

In Walley [48] focuses on probabilistic reasoning in case of uncertainty and imperfect information. Imperfect information is modelled by a set $\Omega$ of possible states of affairs. A gamble $X$ is a bounded real-valued function on $\Omega$ representing an uncertain rewards in probability currency. That is, if $\omega \in \Omega$ is the "true" state and you accept the gamble $X$, then you will receive a number of lottery tickets proportional to $X(\omega)$ that increases the probability to win a given prize from a initial value $\rho$ to $\rho + \alpha X(\omega)$. In particular, Walley provides an axiomatization of preferences that avoid sure loss. Roughly, a gamble $Y$ is preferred to $X$,

$X \leqq Y$, whenever you are disposed to accept $Y + \delta$ instead of $X$, for all $\delta > 0$.[9] Moreover, $\leqq$ avoids a sure loss if $\sup_{\omega \in \Omega} \sum_{i=1}^{n} Y_i(\omega) - X_i(\omega) \geq 0$ whenever $n \geq 1$ and $X_i \leqq Y_i$. The axiomatization essentially defines a partial order which is monotone (if $X(\omega) \leq Y(\omega)$, for all $\omega \in \Omega$, then $X \leqq Y$), homogeneous (if $X \leqq Y$ and $\lambda > 0$, then $\lambda X \leqq \lambda Y$), continuous (if $X \leqq Y + \delta$, for all $\delta > 0$, then $X \leqq Y$), and satisfies the cancellation principle ($X \leqq Y$ iff $0 \leqq Y - X$). Finally, avoiding sure loss is represented by the axiom $0 \nleqq -1$.

In [24] the authors propose a variant of [3] where the continuity axiom is slightly weaker than Axiom 8. The resulting axiomatization is then very close to $\mathrm{PDT}_{4/5}$ and it is easy to see that it verifies all the corresponding properties in Sects. 4–6. With respect to [3] where the set of alternatives are assumed to be finite, here the support $\{a \in \mathcal{A} \mid f(a) > 0\}$ of a lottery $f$ can also be countable. Then, the main result refines the one of [3] as follows: if the lotteries are assumed to have a finite support, then each preference relation $\preceq$ satisfying the proposed axiomatization is represented by a real-valued function $u$ such that $f \prec g$ (resp. $f \sim g$) $\Rightarrow$ $f \cdot u < g \cdot u$ (resp. $f \cdot u = g \cdot u$). If, conversely, lotteries admit countable supports, then $\preceq$ is *weakly* represented by $u$, which means that $f \preceq g \Rightarrow f \cdot u \leq g \cdot u$ (hence, it could be the case that $f \prec g$ and $f \cdot u = g \cdot u$). A further result is that each compliant preference relation $\preceq$ can be extended with a preference relation $\preceq'$ over $\mathcal{A} \cup \{\bot, \top\}$ such that (i) the restriction of $\preceq'$ to $\Delta(\mathcal{A})$ is $\preceq$, (ii) $[\bot] \preceq' f \preceq' [\top]$, for all $f \in \Delta(\mathcal{A})$, and (iii) also $\preceq'$ is compliant with the proposed axiomatization. This result, which can be easily extended to $\mathrm{PDT}_{4/5}$, emphasizes another difference between $\mathrm{PDT}_{4/5}$ and $\mathrm{PDT}_3$. In fact, a partial preference $\preceq$ of $\mathrm{PDT}_3$ cannot be extended as above to another $\mathrm{PDT}_3$ preference $\preceq'$ because conditions (ii) forces $\preceq'$ to be a total order. Also Fishburn in [20] examines the left-to-right representation problem for a strict partial preference $\prec$. Interestingly, $\mathcal{A}$ is possibly infinite and $\prec$ is acyclic but not necessarily transitive. Moreover, he also assumes that if $f_1 \prec g_1$ and $f_2 \prec g_2$ then (i) for all $0 < \alpha < 1$ $\langle \alpha, f_1, f_2 \rangle \prec \langle \alpha, g_1, g_2 \rangle$ (this is essentially Axiom 6), and (ii) for some $0 < \beta < 1$ $\langle \beta, g_1, f_2 \rangle \prec \langle \beta, g_2, f_1 \rangle$. Under these hypotheses, $f \prec g \Rightarrow f \cdot u < g \cdot u$.

Clearly, since $\mathbb{R}$ is totally ordered, partial preferences $\preceq$ cannot be isomorphically represented by real-valued utilities $u$ such that $x \preceq y \Leftrightarrow u(x) \leq u(y)$; consequently, the previous works focused on order-preserving utilities that preserve only the left-to-right direction. However, this means to arbitrarily "linearize" the preference $\preceq$ and, subsequently, to loose any information about the original indecisiveness [41]. Clearly, the fact that we cannot faithfully represent $\preceq$ by means of a real valued function does not prevent from finding isomorphic representations of other types. For instance, some orders admit an isomorphic representation in terms of functions over real intervals [18,21]. More specifically, if a partial order over a generic set $X$ has a countable quotient set $X \backslash \sim$ and for all $x, y, w, z \in X$

$$x \prec y \text{ and } w \prec z \Rightarrow w \prec y \text{ or } x \prec z\,, \tag{8}$$

then there exists a function $I : X \to \mathbb{I}$ such that $x \prec y \Leftrightarrow I(x) < I(y)$, where $\mathbb{I}$ is the set of closed intervals of $\mathbb{R}$ and $I(x) < I(y)$ means that minimum of $I(y)$ is greater than the maximum of $I(x)$. In [29] the author considers the problem of finding finite-dimension utility vectors $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ such that $x \preceq y \Leftrightarrow \mathbf{u}(x) \leq \mathbf{u}(y)$, where $\mathbf{u}(x) \leq \mathbf{u}(y)$ is the standard Pareto condition $u_i(x) \leq u_i(y)$ for all $1 \leq i \leq n$. The first result is a necessary condition: there exists a finite-dimension utility vector such that $x \preceq y \Leftrightarrow \mathbf{u}(x) \leq \mathbf{u}(y)$ only if $\preceq$ can be obtained as the intersection of finitely many total orders. Conversely, a sufficient condition in order to be isomorphically represented by a finite-dimension utility vector is that $\preceq$ is near-complete and upper separable, where near-complete means that all anti-chains have finite cardinality and upper separability refines the notion of separability used in [31]

9 Here, $\delta$ is interpreted as a constant function and hence $(Y + \delta)(\omega) = Y(\omega) + \delta$.

by adding the condition that for each pair $x \not\bowtie y$ there exists $z \in S$ such that $z \prec x$ and $z \not\bowtie y$.

According to Theorem 12, preferences over lotteries are in general not near-complete. Therefore, isomorphic representations may require to check the Pareto condition over an infinite set of utilities. In particular, in [15] the authors show that if $\preceq$ satisfies $\mathrm{PDT}_{4/5}$, then there exists a closed and convex set of utilities $U$ such that $f \preceq g$ iff for all $u \in U$, $f \cdot u \leq g \cdot u$ (see Theorem 17). This result is specialized in [43] in the context multiperson utilities. In particular, it is assumed that each agent $i$ has a total preference represented by a utility $u_i$ and the preference of a group of agents $S = \{1, \ldots, n\}$, $\preceq_S$, is aggregated through a Pareto condition. The resulting preference $\preceq_S$ satisfies $\mathrm{PDT}_{4/5}$ (whereas it is not hard to see that $\preceq_S$ does not satisfy $\mathrm{PDT}_2$) and the corresponding set of utilities $U$ is contained in the convex hull generated by the utilities $u_1, \ldots, u_n$. This has important consequences, in particular the partial preference of a group of agents can be determined as the composition of bilateral agreements (preferences of pairs of agents).

In Dubra [14] considers an axiomatization that is slightly weaker than $\mathrm{PDT}_{4/5}$. In particular, he shows that if we add the archimedean property

$$\text{if } f \prec g, \text{ then there exists some } 1 < \alpha < 1 \text{ such that } f \prec \langle \alpha, h, g \rangle\,,$$

the resulting theory is a classical one, i.e. $\preceq$ is total. The archimedean property says that even if $h$ is incomparable to $f$ and $g$, if $g$ is strictly preferred to $f$ and $\alpha$ is sufficiently small, then the mixture $\langle \alpha, h, g \rangle$ is strictly preferred to $f$. Intuitively, the smaller is $\alpha$ the less $g$ is preferred to $f$. In this sense, the archimedean property reintroduces a notion of measurability and hence it is not completely surprising that we gain back a preference relation that can be represented by a utility function. Now, consider the preference relation in Theorem 11 and assume that $f \prec g$, $h \not\bowtie f$, and $h \not\bowtie g$; by construction, $f(c) = g(c)$ and $h(c) \neq f(c)$. Then, for all $0 < \alpha < 1$, it holds that $\langle \alpha, h, g \rangle(c) \neq f(c)$ and subsequently $\langle \alpha, h, g \rangle \not\bowtie f$. This means that $\mathrm{PDT}_6$, and hence all the theories we have considered, does not satisfy the archimedean property.

In Ok and Dubra [30] investigate a procedural model of decision making which relies on Axiom 3 only. Starting from a "primitive" preorder $R$ over lotteries which is directly provided by an individual, the authors study the properties of the smallest extension $\preceq^R$ of $R$ that satisfies Axiom 3 (this essentially means that underspecification is interpreted as incomparability). As first result, if $R$ is a finite, the same representation theorem in [15] holds, namely there exists a closed convex set of utilities $U$ such that $f \preceq^R g$ iff the expected utility of $g$ is greater than the expected utility of $f$, for each utility in $U$. Note that this property holds for any set of alternatives which is a compact metric space (e.g. the interval [0, 1] of real numbers). However, if the set of alternatives is finite, as in our settings, then the authors show that $U$ is finite too. This result has some analogies with the representation theorem for $\mathrm{PDT}_6$, namely Theorem 18. The main difference is that in [30] $U$ represents a generic polyhedral cone whereas $\mathrm{PDT}_6$ forces it to be a half-subspace of $\mathbb{R}^N$. Then, the authors focuses on a specific $R$ over degenerate lotteries given by the standard linear order on [0, 1]. In this settings, they show that first-order stochastic dominance provides the smallest extension of $R$ which satisfies both Axiom 3 and the notion of continuity provided by Axiom 8.

As shown in the introduction, a well-established tradition in Decision Theory models preferences in terms of which observable choices are made [38]. This perspective has been formally studied in the classical settings where rational choice functions are introduced as first-class citizens and then it is shown that they correspond to the maximization of a total preorder [2]. In particular, given a generic set of items $\Omega$, a choice function $c : 2^\Omega \to 2^\Omega$ is essentially a selection rule that for each set of offered items $S \subseteq \Omega$ returns a subset

$c(S) \subseteq S$. Then, a choice function is "rational" if it satisfies the *weak axiom of revealed preference* (warp), for all sets of items $T \subseteq S$, (i) if $x \in c(S)$, then $x \in c(T)$ and (ii) if $x, y \in c(T)$ and $x \in c(S)$, then $y \in c(S)$. Informally, condition (i) tells us that if $x$ is not inferior to any other item in $S$, then it cannot be dominated in a smaller set $T$. Condition (ii), instead, is related to the notion of completeness of a preference relation: since $x$ and $y$ are both selected in $T$, $x$ is not inferior to $y$ and vice versa $y$ is not inferior to $x$. Then, this necessarily means $x$ and $y$ are equivalent and hence if $x \in c(S)$, then $y \in c(S)$. It can be shown that a choice function satisfies warp if and only it corresponds to the maximization of a total preorder $\preceq$, i.e. $c(S) = \{x \in S \mid \forall y \in S \ \ y \preceq x\}$, for all $S \subseteq \Omega$. In [16] the authors extends this choice-theoretic approach to a specific class of incomplete preorders called regular preferences. A preorder $\preceq$ is regular in case for each pair $x \not\bowtie y$ there exists a $z$ such that either $x \not\bowtie z$ and $y$ and $z$ are strictly ordered by $\prec$, or $y \not\bowtie z$ and $x$ and $z$ are strictly ordered by $\prec$. Such a property is naturally satisfied when $\preceq$ derives from the maximization of multiple criteria. Then, in order to represent regular preferences, warp is replaced with the weaker condition: for all $S \subseteq \Omega$ and $y \in S$, if for every $x \in c(S)$ there exists $T \subseteq \Omega$ with $y \in c(T)$ and $x \in T$, then $y \in c(S)$. It is proved that the previous condition satisfies only condition (i) of warp and represents the maximization of a regular preference. Finally, the authors consider the case where $\Omega$ is a probability space and provide some further axioms for $c$ in order to be represented as in [15] by the maximization of multiple expected utility.

In [17] generic partial preferences over topological spaces are considered but, differently from [31], the authors focus on isomorphic representations through sets of utilities [15,29]. More specifically, a preorder $\preceq$ over a topological space $\mathcal{X}$ has a continuous multi-utility representation if there exists a set $\mathcal{U}$ of continuous functions $u : \mathcal{X} \to \mathbb{R}$ such that $x \preceq y \Leftrightarrow u(x) \leq u(y)$ for all $u \in \mathcal{U}$. The main result is that a preorder has a continuous multi-utility representation iff it is *semi-normal* which roughly means that for each pair $x$ and $y$ such that $x \not\preceq y$ there exists a dense collection of open sets $\mathbb{L}$ such that for all $\mathcal{L} \in \mathbb{L}$, $x \in \mathcal{L}$ and $y \in \mathcal{X} \setminus \mathcal{L}$. The authors also identify several sufficient conditions, for instance every continuous preorder on a nonempty closed subset of a Euclidean space admits a continuous multi-utility representation.

Clearly, since quantitative utilities provide an effective method to compare or find optimal lotteries, the representation problem has a central role in Decision Theory. Nevertheless, in the context of MAS, quantitative approaches are not always feasible and, as we have seen before, several elicitation methodologies return ordinal preferences over degenerate lotteries. This in particular means that we do not have in our context a single preference relation but an entire set of all possible completions of the user's statements (see Sect. 8). Consequently, using a utility function (or a set of utilities as in [15]) to represent all these preferences requires to preliminary provide numerical ranks to alternatives. However, there exists infinite different rankings that are consistent with the user's statements and each of them may arbitrarily affect the expected utility and hence the comparison of two lotteries.[10]

Finally, in [26] the authors consider incomparability between different alternatives as well as other deviations from the classical theory. For instance, preference relations can be modelled by semi-orders. In a semi-order strict preference $\prec$ is, as usual, an asymmetric and transitive relation whereas indifference $\sim$ is symmetric and reflexive but not necessarily transitive. Moreover, it also holds that if $x \prec y$, $y \sim z$, and $z \prec w$, then $x \prec w$. Semi-orders are justified in case the alternatives differ from each other for a quantity that increase

[10] Note that this can happen even if the user's statements are a total order of $\mathcal{A}$. For example, consider the user's statements $a < b < c < d$. Two possible utilities are: $u_1(a) = 1$, $u_1(b) = 5$, $u_1(c) = 6$, $u_1(d) = 10$, and $u_2(a) = 1$, $u_2(b) = 4$, $u_2(c) = 5$, $u_2(d) = 10$. Then, given $f = \frac{1}{2}[a] + \frac{1}{2}[d]$ and $g = \frac{1}{2}[b] + \frac{1}{2}[c]$, it is immediate to see that $f \cdot u_1 < g \cdot u_1$ whereas $f \cdot u_2 > g \cdot u_2$.

gradually. For example, you could be indifferent between 1000 dollars and 1001 dollars, 1001 dollars and 1002 dollars, and so on. However, a chain of indifferent pairs of alternatives may eventually accumulate a valuable amount of money. Semi-orders can be represented by a utility function $u$ and an indifference threshold $q$ as follows: $x \sim y$ iff $|u(x) - u(y)| \leq q$ and $x \prec y$ iff $u(y) > u(x) + q$.

## 10 Conclusion

To make software agents act on behalf of real users, effective methodologies are required to elicit preferences [4,5,9,12,13,36,49]. However, in several concrete scenarios a user may feel uncomfortable providing a complete preference relation due to vacillating desires, lack of knowledge or simply because this process would take too long. Consequently, in several contexts of Multiagent Systems preferences are assumed to be partial.

Nevertheless, this is in contrast with the classical assumption of totality which means that, strictly speaking, there is a mismatch between Multiagent System (or at least part of it) and classical Decision Theory. Actually, such a mismatch does not represent a notable problem as far as preferences deal with certain outcomes only—it simply suffices to assume that preference relations are preorders instead of weak orders—however in the presence of uncertain environments or randomized protocols specific solutions have been defined ad hoc to model preferences over lotteries [1,7]. Unfortunately, these solutions strongly depend on the specific context and do not have a solid foundation as well as a clear connection with Decision Theory. For this reason, in this work we challenged the customary decision-theoretic assumption of totality. This is not a straightforward task, since different formalizations of the classical theory, once totality is weakened in favour of reflexivity, are no longer equivalent.

Clearly, this gives room to debate which axiomatization is the most appropriate in general (if any) or with respect to a specific application domain. In this work we do not advocate any specific foundational or cognitive consideration; conversely we help fuel the debate by providing a detailed comparison of the different theories and their formal properties. In particular, we consider six distinct axiomatizations and show that they lie in a hierarchy of strict entailments. We also enrich our analysis by showing first what is shared by all these theories, specifically monotonicity under convex combinations and some dominance notions, and then what discriminates them. In particular, the six theories combine three distinct continuity axioms and two axioms that propagate the preference relation over mixtures. We show that, the different notions of continuity have an impact specifically on the comparability of lotteries, whereas the considered propagation rules affects incomparability. Moreover, the theory $\mathrm{PDT}_6$ satisfies a suitable form of the expected utility theorem where comparability is characterized by a finite set of utility functions.

Finally, we combine these theories with preference elicitation methodologies. Here, depending on which kind of statements the user can assert and how incomplete information is interpreted (see also [33]), some theories are more appropriate than others. For instance, in case the user can explicitly state that two different alternatives are incomparable, or incomplete information is by default interpreted as incomparability, the theories that enforce more comparability, namely $\mathrm{PDT}_2$, $\mathrm{PDT}_3$, and $\mathrm{PDT}_6$, can easily be inconsistent with the user statements. Conversely, $\mathrm{PDT}_3$ and $\mathrm{PDT}_6$ can be used in those cases where the user conforms with total, classical preferences but for efficiency reasons the elicitation processes does not completely inject such a preference into the software agent. On the other hand, when incomparability is explicitly stated or assumed by default, $PDT_1$ and $PDT_{4/5}$ behave better than $PDT_0$ with respect to the (weak) optimality problem.

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