The Journal of Geometric Analysis (2024) 34:44
https://doi.org/10.1007/s12220-023-01490-9

# Growth of Subsolutions of $\Delta_p u = V|u|^{p-2}u$ and of a General Class of Quasilinear Equations

Luis J. Alías[1] · Giulio Colombo[2] · Marco Rigoli[3]

Received: 18 July 2023 / Accepted: 13 November 2023


## Abstract

In this paper we prove some integral estimates on the minimal growth of the positive part $u_+$ of subsolutions of quasilinear equations

$$\operatorname{div} A(x, u, \nabla u) = V|u|^{p-2}u$$

on complete Riemannian manifolds $M$, in the non-trivial case $u_+ \not\equiv 0$. Here $A$ satisfies the structural assumption $|A(x, u, \nabla u)|^{p/(p-1)} \le k\langle A(x, u, \nabla u), \nabla u\rangle$ for some constant $k > 0$ and for $p > 1$ the same exponent appearing on the RHS of the equation, and $V$ is a continuous positive function, possibly decaying at a controlled rate at infinity. We underline that the equation may be degenerate and that our arguments do not require any geometric assumption on $M$ beyond completeness of the metric. From these results we also deduce a Liouville-type theorem for sufficiently slowly growing solutions.

**Keywords** Quasilinear equation · $p$-Laplacian · Growth estimates · Liouville theorem

**Mathematics Subject Classification** Primary: 35J62 · 35J70 · 35J92 · 58J05; Secondary: 35B53

✉ Luis J. Alías
ljalias@um.es

Giulio Colombo
giulio.colombo@unina.it

Marco Rigoli
marco.rigoli@unimi.it

[1] Departamento de Matemáticas, Universidad de Murcia, Campus de Espinardo, 30100 Espinardo, Murcia, Spain

[2] Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli “Federico II”, Via Vicinale Cupa Cintia 26, 80126 Naples, Italy

[3] Dipartimento di Matematica “F. Enriques”, Università degli Studi di Milano, Via Saldini 50, 20133 Milan, Italy

Published online: 16 December 2023

## 1 Introduction

In the recent paper [4] (Lemma 8), the following theorem was established. Let $M$ be a complete Riemannian manifold (without boundary), $\lambda > 0$ a constant and $u \in C^2(M)$. If the superlevel set $\Omega_+ := \{x \in M : u(x) > 0\}$ is not empty and $u$ satisfies

$$\Delta u \geq \lambda u \qquad \text{on } \Omega_+ \tag{1}$$

then for any fixed point $x_0 \in M$ we have

$$\liminf_{R\to+\infty} \frac{1}{R} \log \int_{B_R(x_0)} u_+^2 > 0 \tag{2}$$

where $u_+ := \max\{u, 0\}$ is the positive part of $u$ and $B_R(x_0)$ is the geodesic ball of radius $R$ centered at $x_0$. Indeed, inspection of the proof also shows that there exists a constant $C(\lambda)$, not depending on $M$ or $u$, such that

$$\liminf_{R\to+\infty} \frac{1}{R} \log \int_{B_R(x_0)} u_+^2 \geq C(\lambda) > 0 \tag{3}$$

and that the optimal value for $C(\lambda)$ is not smaller than $\frac{\log 2}{4}\sqrt{\lambda}$. This can be regarded as a sort of "gap" theorem for subsolutions of $\Delta u = \lambda u$: if $u \in C^2(M)$ satisfies

$$\Delta u \geq \lambda u \qquad \text{on } M$$

then either $u \leq 0$ or the positive part of $u$ has to be sufficiently large in an integral sense (that is, its $L^2$ norm on $B_R(x_0)$ must grow at least exponentially with respect to $R$). In fact, the result from [4] is more general and also covers the case of weighted Laplacians and locally Lipschitz weak solutions of (1).

In this paper we generalize the above theorem by considering differential inequalities for a wider class of (possibly degenerate) quasilinear elliptic operators in divergence form, including the $p$-Laplace operator

$$\Delta_p u := \mathrm{div}(|\nabla u|^{p-2}\nabla u), \qquad 1 < p < +\infty,$$

and also replacing the constant $\lambda$ by a positive continuous function $V$ possibly decaying at infinity at a controlled rate, namely, not faster than a negative power $r(x)^{-\mu}$, $\mu > 0$, of the distance $r(x) = \mathrm{dist}(x, o)$ from some fixed point $o \in M$. More precisely, for a given pair of parameters $\lambda > 0$ and $\mu \geq 0$ we shall assume that

$$\begin{array}{lll} V \geq \lambda & & \text{if } \mu = 0 \\ \liminf_{x\to\infty} [\mathrm{dist}(x, o)^{\mu} V(x)] \geq \lambda & \text{for some } o \in M & \text{if } \mu > 0. \end{array} \tag{$V_{\lambda,\mu}$}$$

These conditions are clearly satisfied, for instance, if

$$V(x) \geq \frac{\lambda}{1 + \mathrm{dist}(x, o)^{\mu}} \qquad \text{on } M\,.$$

Also, in case $\mu > 0$ the triangle inequality implies that the validity of ($V_{\lambda,\mu}$) does not depend on the choice of the reference base point $o \in M$.

To give an example of our main result, we state it in the model case of the $p$-Laplace operator. To do so, we have to precise some terminology. For a function $u \in W^{1,p}_{\mathrm{loc}}(M)$, we denote by $\Omega_+ := \{x \in M : u(x) > 0\}$ its positivity set and for a given measurable function $V \geq 0$ we say that $u$ satisfies

$$\Delta_p u \geq V u^{p-1} \qquad \text{weakly on } \Omega_+ \tag{4}$$

if

$$-\int_M \langle |\nabla u|^{p-2} \nabla u, \nabla \varphi \rangle \geq \int_M V u^{p-1} \varphi \qquad \forall \varphi \in D^+(\Omega_+)$$

where

$$D^+(\Omega_+) := \{\varphi \in W^{1,p}_c(M) :\ \varphi \geq 0 \text{ on } M,\\ \varphi = 0 \text{ and } \nabla\varphi = 0 \text{ a.e. on } M \setminus \Omega_+\}\,.$$

(Note that if $|\Omega_+| > 0$ then the space $D^+(\Omega_+)$ of test functions is non-trivial because it contains at least elements of the form $\varphi = u_+ \psi$, with $0 \leq \psi \in C^\infty_c(M)$, so (4) is a meaningful condition.) In particular, (4) is always satisfied if

$$\Delta_p u \geq V |u|^{p-2} u \qquad \text{weakly on } M \tag{5}$$

or even if

$$\Delta_p u_+ \geq V u_+^{p-1} \qquad \text{weakly on } M \tag{6}$$

since $\nabla u_+ = \mathbf{1}_{\Omega_+} \nabla u$ almost everywhere on $M$. Note that (6) is a weaker condition than (5), as follows from work of Le, [7].

**Theorem 1** *Let $M$ be a complete Riemannian manifold, $p \in (1, +\infty)$, $\mu \in [0, p]$, $\lambda > 0$, and $V : M \to (0, +\infty)$ a continuous function satisfying ($V_{\lambda,\mu}$).*

*Let $u \in W^{1,p}_{\mathrm{loc}}(M)$. If $\Omega_+ := \{x \in M : u(x) > 0\}$ is of positive measure and*

$$\Delta_p u \geq V u^{p-1} \qquad \textit{weakly on } \Omega_+$$

*then for any* $x_0 \in M$ *and* $q \in (p-1, +\infty)$ *we have*

$$\liminf_{R\to+\infty} \frac{1}{R^{1-\frac{\mu}{p}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{C_0}{1-\frac{\mu}{p}} > 0 \qquad \text{if } \mu \in [0, p) \tag{7}$$

$$\liminf_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_1 > p \qquad \text{if } \mu = p \tag{8}$$

*where* $C_0$ *and* $C_1$ *are explicitely given by*

$$C_0 = \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \lambda^{1/p}, \qquad C_1^{1/p}(C_1 - p)^{1/p'} = C_0 \tag{9}$$

*where* $p' = \frac{p}{p-1}$ *is the exponent conjugate to* $p$*. Moreover, in case* $\mu = p$ *we have*

$$\lim_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_0 + p \tag{10}$$

*whenever the limit on the LHS exists.*

***Remark 2*** Note that the value $C_1 > p$ determined by (9) satisfies $C_1 < C_0 + p$, hence (10) gives a stronger estimate than (8) when its LHS is well defined.

The constants appearing in (7) and (10) are sharp, that is, for each combination of values of $p$, $\mu$, $\lambda$ and $q$ it is possible to find $M$ and $u$ for which the equality in (7) or (10) is attained. This is shown by explicit examples described at the end of Sect. 3. We don't know whether the value of $C_1 > p$ in (9) is sharp or not for the validity of (8). It seems worth to underline that the case $p = 2$, $q = 2$, $\mu = 0$ in the above theorem implies that the optimal value for $C(\lambda)$ in (3) is $C(\lambda) = 2\sqrt{\lambda}$.

**Theorem 3** *Let* $M$ *be a complete Riemannian manifold,* $\mu \in [0, 2]$*,* $\lambda > 0$ *and* $V : M \to (0, +\infty)$ *a continuous function satisfying* ($V_{\lambda,\mu}$)*.*

*Let* $u \in W^{1,2}_{loc}(M)$*. If* $\Omega_+ := \{x \in M : u(x) > 0\}$ *is of positive measure and*

$$\Delta u \geq Vu \qquad \text{weakly on } \Omega_+$$

*then for any* $x_0 \in M$ *and* $q \in (1, +\infty)$ *we have*

$$\liminf_{R\to+\infty} \frac{1}{R^{1-\frac{\mu}{2}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{2\sqrt{q-1}\sqrt{\lambda}}{1-\frac{\mu}{2}} \qquad \text{if } \mu \in [0, 2)$$

$$\liminf_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq 1 + \sqrt{1 + 4(q-1)\lambda} > 2 \qquad \text{if } \mu = 2$$

*and in case* $\mu = 2$

$$\lim_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq 2(1 + \sqrt{q-1}\sqrt{\lambda})$$

*provided the limit exists.*

In full generality, in our main theorem we deal with differential inequalities involving quasilinear differential operators $L$ formally defined by

$$Lu := \operatorname{div}(A(x, u, \nabla u)) \tag{11}$$

where $A : \mathbb{R} \times TM \to TM$ is a continuous function (or, more generally, a Carathéodory-type function as specified in Sect. 2) satisfying

$$\langle A(x, s, \xi), \xi \rangle \geq 0 \quad \text{and} \quad |A(x, s, \xi)| \leq k \langle A(x, s, \xi), \xi \rangle^{\frac{p}{p-1}} \tag{12}$$

for all $x \in M$, $s \in \mathbb{R}$, $\xi \in T_x M$ with some constant $k > 0$. If these conditions are satisfied, we say that the differential operator $L$ defined by (11) is weakly $p$-coercive with coercivity constant $k$. The $p$-Laplace operator falls in this class since it can be expressed as in (11) for the choice $A(x, s, \xi) = |\xi|^{p-2}\xi$, which fulfills (12) with $k = 1$. In analogy with what we did above, we say that a function $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfies

$$Lu \geq V u^{p-1} \qquad \text{weakly on } \Omega_+ := \{u > 0\}$$

if

$$-\int_M \langle A(x, u, \nabla u), \nabla \varphi \rangle \geq \int_M V u^{p-1} \varphi \qquad \forall \varphi \in D^+(\Omega_+)\,.$$

**Theorem 4** *Let $M$ be a complete Riemannian manifold, $p \in (1, +\infty)$, $\mu \in [0, p]$ and $\lambda > 0$. Let $L$ be a weakly $p$-coercive operator as in (11) with coercivity constant $k > 0$ and $V : M \to (0, +\infty)$ a continuous function satisfying ($V_{\lambda,\mu}$).*

*Let $u \in W^{1,p}_{\mathrm{loc}}(M)$. If $\Omega_+ := \{x \in M : u(x) > 0\}$ is of positive measure and*

$$Lu \geq V u^{p-1} \qquad \textit{weakly on } \Omega_+$$

*then for any $x_0 \in M$ and $q \in (p-1, +\infty)$ we have*

$$\liminf_{R \to +\infty} \frac{1}{R^{1-\frac{\mu}{p}}} \log \int_{B_R(x_0)} u_+^q \geq \frac{C_0}{1-\frac{\mu}{p}} \qquad \textit{if } \mu \in [0, p) \tag{13}$$

$$\liminf_{R \to +\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_1 \qquad \textit{if } \mu = p \tag{14}$$

*where $C_0 > 0$ and $C_1 > p$ are determined by*

$$C_0 = \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \frac{\lambda^{1/p}}{k}\,, \qquad C_1^{1/p}(C_1 - p)^{1/p'} = C_0$$

*with* $p' = \frac{p}{p-1}$. *Moreover, in case* $\mu = p$ *we have*

$$\lim_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R(x_0)} u_+^q \geq C_0 + p \tag{15}$$

*whenever the limit exists.*

We point out that the RHS's of (14) and (15) both converge to $p$ from above as $\lambda \to 0^+$. Hence, if $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfies

$$Lu \geq Vu^{p-1} \qquad \text{weakly on } \Omega_+ = \{u > 0\}$$

with $|\Omega_+| \neq 0$ and $V$ a continuous positive function decaying to 0 faster than $r(x)^{-p}$ as $x \to \infty$, then on arbitrary manifolds we couldn't expect the possible validity of an estimate stronger than

$$\liminf_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R} u_+^q \geq p\,.$$

In fact, we are able to prove a weaker growth estimate (with lim inf replaced by lim sup) holds more generally for any $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfying

$$Lu \geq f \qquad \text{weakly on } \Omega_+ \tag{16}$$

for some measurable function $f : M \to [0, +\infty]$ such that $f > 0$ on a set $E \subseteq \Omega_+$ of positive measure. Of course, by (16) we mean that

$$-\int_M \langle A(x, u, \nabla u), \nabla\varphi\rangle \geq \int_M f\varphi \qquad \forall\, \varphi \in D^+(\Omega_+)\,. \tag{17}$$

Note that if (16) holds with $f$ as above then there exists $\varphi \in D^+(\Omega_+)$ for which the LHS of (17) is strictly positive (this follows by considering a test function of the form $\varphi = u_+\psi$ for some $0 \leq \psi \in C^\infty_c(M)$ strictly positive on a portion of $E$ of positive measure), and then it must also be $A(x, u, \nabla u) \neq 0$ on a subset $E_0 \subseteq \Omega_+$ of positive measure.

**Theorem 5** *Let* $M$ *be a complete Riemannian manifold,* $p \in (1, +\infty)$*,* $L$ *a weakly* $p$*-coercive operator as in* (11) *and* $u \in W^{1,p}_{\mathrm{loc}}(M)$ *such that* $\Omega_+ := \{x \in M : u(x) > 0\}$ *has positive measure. If* $u$ *satisfies*

$$Lu \geq 0 \qquad \textit{weakly on } \Omega_+$$

*and further*

$$A(x, u, \nabla u) \neq 0 \qquad \textit{on a set } E_0 \subseteq \Omega_+ \textit{ of positive measure} \tag{18}$$

*then for any* $q \in (p-1, +\infty)$

$$\limsup_{R\to+\infty} \frac{1}{R^p} \int_{B_R} u_+^q = +\infty\,. \tag{19}$$

*In particular,*

$$\limsup_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R} u_+^q \geq p\,.$$

As said, (18) holds if $u$ satisfies (16) for some measurable $f : M \to [0, +\infty]$ with $f$ not a.e. vanishing on $\Omega_+$. Alternatively, (18) is satisfied also when $u$ is not constant on $M$ and positive somewhere (so that $|\Omega_+| > 0$) and $A$ obeys the following mild non-degeneracy condition:

$$A(x, s, \xi) = 0 \qquad \text{only if} \qquad \xi = 0\,. \tag{20}$$

Theorem 5 is a consequence of the next Theorem 6, proved in the last part of the paper where we extend some arguments from [8] to general weakly $p$-coercive operators $L$ of the form (11).

**Theorem 6** *Let $M$ be a complete, non-compact Riemannian manifold, $p \in (1, +\infty)$, $L$ a weakly $p$-coercive operator as in (11) and $u \in W^{1,p}_{\mathrm{loc}}(M)$. If $\{u > 0\}$ has positive measure, $u$ satisfies*

$$Lu \geq 0 \qquad \textit{weakly on } \{u > 0\} \tag{21}$$

*and for some $q > p-1$ it holds*

$$\lim_{R\to+\infty} \int_r^R \left( \int_{\partial B_s} u_+^q \right)^{-\frac{1}{p-1}} \mathrm{d}s = +\infty \qquad \forall\, r > 0\,, \tag{22}$$

*then $A(x, u, \nabla u) = 0$ almost everywhere on $\{u > 0\}$. In particular, if the structural condition (20) holds, then $u$ is constant on $M$.*

We remark that condition (22) amounts to saying that the function $\varphi : (0, +\infty) \to [0, +\infty]$ given by

$$\varphi(s) = \left( \int_{\partial B_s} u_+^q \right)^{-\frac{1}{p-1}} \qquad \forall\, s > 0$$

is not in $L^1((r, +\infty))$ for any $r > 0$. In fact, as proved in Lemma 19 below, in the assumptions of Theorem 6 there exists $r_0 \geq 0$ such that $\varphi$ is finite a.e. on $(r_0, +\infty)$ and $\varphi \in L^1((r, R))$ for any $r_0 < r < R < +\infty$, so that (22) is satisfied if and only if $\varphi$ is not integrable in a neighborhood of $+\infty$. Note that in general $\varphi$ may be integrable at $+\infty$ and still satisfy $\varphi = +\infty$ on $(0, r_0)$ for some $r_0 > 0$. For instance, for fixed $n \in \mathbb{N}$ and $p > n$, the function

$$u(x) := |x|^{\frac{p-n}{p-1}} - 1 \qquad \text{on } \mathbb{R}^n$$

satisfies $\Delta_p u = 0$ on $\Omega_+ = \mathbb{R}^n \backslash \overline{B_1}$, and for any $q > p - 1$

$$\varphi(s) = \begin{cases} +\infty & \text{for } 0 < s \le 1 \\ \left[ C s^{n-1} \left( s^{q\frac{p-n}{p-1}} - 1 \right) \right]^{-\frac{1}{p-1}} & \text{for } s > 1 \end{cases}$$

(with $C = |\partial B_1|$) is integrable at $+\infty$: indeed,

$$\varphi(s) \sim C^{-\frac{1}{p-1}} s^{-\frac{(n-1)(p-1)+q(p-n)}{(p-1)^2}} \qquad \text{as } s \to +\infty$$

and (under the assumption $p > n$) we have $-\frac{(n-1)(p-1)+q(p-n)}{(p-1)^2} < -1$ if and only if $q > p - 1$. This shows that the clause "$\forall\, r > 0$" in (22) cannot in general be replaced by "for some $r > 0$".

Note that (22) is a condition about the growth of the integral of $u_+^q$ on geodesic spheres $\partial B_s$. This can be related to the growth of the integral of $u_+^q$ on balls $B_s$. More precisely, (22) is implied (see Proposition 1.3 in [8]) by the stronger condition

$$\lim_{R \to +\infty} \int_r^R \left( \frac{s}{\int_{B_s} u_+^q} \right)^{\frac{1}{p-1}} \mathrm{d}s = +\infty \qquad \forall\, r > 0$$

which in turn is satisfied, for instance, when

$$\int_{B_R} u_+^q = O(R^p) \qquad \text{as } R \to +\infty\,.$$

Since this last condition is exactly the negation of condition (19) above, Theorem 5 follows at once from Theorem 6.

As hinted at the beginning of this Introduction, our main Theorem 4 can be also interpreted as a "gap" theorem for functions $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfying

$$Lu \ge V|u|^{p-2}u \qquad \text{on } M\,.$$

Namely, if $u$ satisfies the above differential inequality, then either $u \le 0$ a.e. on $M$ or the positive part of $u$ must grow sufficiently fast. As an easy consequence we have the following Liouville-type result (for its proof it is enough to apply Theorem 4 to both $u$ and $-u$). For the sake of simplicity, we only state it in case $V$ is a positive constant, but the interested reader can immediately generalize it to the case where $V$ is a function satisfying ($V_{\lambda,\mu}$) for some $\lambda > 0$ and $\mu \in [0, p]$.

**Theorem 7** *Let $M$ be a complete Riemannian manifold, $p \in (1, +\infty)$, $\lambda > 0$ and $L$ a weakly $p$-coercive operator as in (11) with coercivity constant $k > 0$. Let $u \in W^{1,p}_{\text{loc}}(M)$ satisfy*

$$Lu = \lambda |u|^{p-2} u \quad \text{on } M\,.$$

*If for some $x_0 \in M$ and $q \in (p-1, +\infty)$*

$$\int_{B_R(x_0)} |u|^q \leq e^{CR} \quad \text{for all sufficiently large } R$$

*for some constant $C < \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \frac{\lambda^{1/p}}{k}$, then $u \equiv 0$.*

We conclude this introduction with a few comments on some technical points. First, in all the results stated above, except for Theorem 6, $M$ is not explicitely assumed to be non-compact. Indeed, if $M$ is compact (without boundary) and $u$ satisfies

$$Lu \geq f \geq 0 \quad \text{on } \Omega_+$$

for some measurable $f$, then necessarily $f = 0$ and $A(x, u, \nabla u) = 0$ a.e. on $\Omega_+$ (see Lemma 8 in Sect. 2). Hence, in the assumptions of Theorems 1, 3, 4 and 5, $M$ is necessarily non-compact. Secondly, in all our results we do not make additional regularity assumptions on the subsolutions beside their belonging to the appropriate Sobolev class $W^{1,p}_{\text{loc}}(M)$. Since we do not know if Sobolev subsolutions of possibly degenerate equations of the form

$$\operatorname{div} A(x, u, \nabla u) = V|u|^{p-2}u$$

are always locally essentially upper bounded (that is, if they necessarily satisfy $u_+ \in L^\infty_{\text{loc}}(M)$), in some of our arguments we have to follow more winding roads using approximation procedures.

The paper is organized as follows. In Sect. 2 we collect the notation and all relevant definitions. In Sect. 3 we prove the main Theorem 4 and we provide examples showing sharpness of the constants in the statements. Section 4 is devoted to the proof of Theorem 6, from which Theorem 5 can be easily deduced (see Corollary 22 and Remark 23).

Comparison results and the case $p = 1$ will appear in a forthcoming paper.

We recently learned that on arXiv has just appeared a paper by Bisterzo, Farina and Pigola [2] which is somehow related to our work, at least where $L$ is the Laplace–Beltrami operator. However, even in the above overlapping case, the two papers are different in setting, scope and sharpness of the results.

## 2 Definitions and Notation

Throughout this paper, $M$ will always be a connected Riemannian manifold withouth boundary. We denote by $TM$ its tangent bundle and by $\langle\ ,\ \rangle$ its Riemannian metric. For any $p \in (1, +\infty)$ we also denote by $W^{1,p}_{\mathrm{loc}}(M)$ the space of Sobolev functions $u$ whose restrictions to any relatively compact set $\Omega \subseteq M$ belong to $W^{1,p}(\Omega)$. This is equivalent to requiring that $u \circ \psi^{-1} \in W^{1,p}_{\mathrm{loc}}(\psi(U))$ for any local chart $\psi : U \subseteq M \to \mathbb{R}^m$, where $m = \dim M$. We also denote by $W^{1,p}_c(M)$ the subspace of $W^{1,p}_{\mathrm{loc}}(M)$ consisting of functions with compact support.

We consider quasilinear differential operators $L$ in divergence form weakly defined on functions $u \in W^{1,p}_{\mathrm{loc}}(M)$ by

$$Lu(x) = \operatorname{div} A(x, u, \nabla u)\,. \tag{23}$$

Here $A : \mathbb{R} \times TM \to TM$ is a function such that

$$A(x, s, \xi) \in T_x M \qquad \forall x \in M,\ s \in \mathbb{R},\ \xi \in T_x M$$

and whose local representation $\tilde{A} : \psi(U) \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$ in any chart $\psi : U \subseteq M \to \mathbb{R}^m$ satisfies the Carathéodory conditions

- $\tilde{A}(y, \cdot, \cdot)$ is continuous for a.e. $y \in \psi(U)$
- $\tilde{A}(\cdot, s, v)$ is measurable for every $(s, v) \in \mathbb{R} \times \mathbb{R}^m$.

(The representation $\tilde{A}$ is defined by

$$\tilde{A}(\psi(x), s, v) := A\left(x, s, \sum_{i=1}^{m} v^i \left.\frac{\partial}{\partial y^i}\right|_x\right) \forall x \in U,\ s \in \mathbb{R},\ v = (v^1, \dots, v^m) \in \mathbb{R}^m$$

where $y^1, \dots, y^m$ are the coordinates induced by $\psi$.) In particular, these conditions on $\tilde{A}$ are satisfied whenever $A$ is a continuous function of its arguments. Following terminology from [5, Definition 2.1], we say that $A$ and the corresponding operator $L$ given by (23) are *weakly-$p$-coercive* for some $p \in (1, +\infty)$ if

$$\langle A(x, s, \xi), \xi \rangle \geq 0 \qquad \forall x \in M,\ s \in \mathbb{R},\ \xi \in T_x M \tag{24}$$

$$|A(x, s, \xi)| \leq k \langle A(x, s, \xi), \xi \rangle^{\frac{p-1}{p}} \qquad \forall x \in M,\ s \in \mathbb{R},\ \xi \in T_x M \tag{25}$$

for some constant $k > 0$ that we will call the *coercivity constant* of $A$. Note that the above conditions imply that

$$|A(x, s, \xi)| \leq k^p |\xi|^{p-1} \qquad \forall x \in M,\ s \in \mathbb{R},\ \xi \in T_x M\,. \tag{26}$$

Indeed, this is clearly true when $A(x, s, \xi) = 0$; otherwise, by Cauchy–Schwarz inequality and (25) we have $|A(x, s, \xi)|^p \leq k^p |A(x, s, \xi)|^{p-1} |\xi|^{p-1}$, and then (26) follows dividing both sides by $|A(x,s,\xi)|^{p-1}$. In particular, we have

$$A(x,s,0)=0 \qquad \forall\, x\in M,\ s\in\mathbb{R}. \tag{27}$$

On the other hand, in general we do not assume non-degeneracy of $A$, that is, we do not assume that $A(x,s,\xi)\neq 0$ when $\xi\neq 0$.

Let $A$ be a weakly $p$-coercive function for some $p\in(1,+\infty)$. For any given $u\in W^{1,p}_{\mathrm{loc}}(M)$ and any $s_0\in\mathbb{R}$ we set

$$\Omega_{s_0}:=\{x\in M : u(x)>s_0\}$$

and for any non-negative measurable $f: M\to[0,+\infty]$ we say that $u$ satisfies

$$Lu\geq f \qquad \text{(weakly) on } \Omega_{s_0} \tag{28}$$

if

$$-\int_M \langle A(x,u,\nabla u),\nabla\varphi\rangle \geq \int_M f\varphi \qquad \forall\,\varphi\in D^+(\Omega_{s_0}) \tag{29}$$

where

$$\begin{aligned} D^+(\Omega_{s_0}):=\{\varphi\in W^{1,p}_c(M):\ &\varphi\geq 0 \text{ on } M,\\ &\varphi=0 \text{ and } \nabla\varphi=0 \text{ a.e. on } M\setminus\Omega_{s_0}\}. \end{aligned}$$

We remark that our assumptions on $A$ and $u$ imply that $|A(x,u,\nabla u)|\in L^{p'}_{\mathrm{loc}}(M)$, with $p'=\frac{p}{p-1}$ the exponent conjugate to $p$, and that $\langle A(x,u,\nabla u),\nabla\varphi\rangle$ is measurable for each $\varphi\in D^+(\Omega_{s_0})$ (see for instance [9, Lemma 2.4]). Hence, the LHS of (29) is well defined and finite for each $\varphi\in D^+(\Omega_{s_0})$.

The next lemma justifies our focus on complete, non-compact manifolds in the introduction and in the following sections.

**Lemma 8** *Let $M$ be a compact manifold without boundary, $p\in(1,+\infty)$ and $L$ a weakly $p$-coercive operator as in (23). If $u\in W^{1,p}(M)$ satisfies*

$$Lu\geq f\geq 0 \qquad \text{on } \Omega_{s_0}:=\{u>s_0\}$$

*for some measurable $f: M\to\mathbb{R}$ and some $s_0\in\mathbb{R}$, then*

$$f=0 \quad \text{and} \quad A(x,u,\nabla u)=0 \qquad \text{a.e. on } \Omega_{s_0}. \tag{30}$$

***Proof*** Considering the test function $\varphi=(u-s_0)_+\in D^+(\Omega_{s_0})$ we have

$$\int_{\Omega_{s_0}}\langle A(x,u,\nabla u),\nabla u\rangle \leq -\int_{\Omega_{s_0}}(u-s_0)_+ f\leq 0$$

and by the weak coercivity condition (25) we obtain

$$\int_{\Omega_{s_0}} |A(x,u,\nabla u)|^{\frac{p}{p-1}} \leq 0\,.$$

By non-negativity of $f$ and of $|\cdot|$, this immediately yields (30). □

Lastly, we precise the following terminology. For an open interval $I \subseteq \mathbb{R}$ we say that a function $F : I \to \mathbb{R}$ is piecewise $C^1$ if $F$ is continuous on $I$ and there exists a discrete (possibly empty) set $E \subseteq I$ such that

(*i*) $F'$ exists and is continuous on $I \setminus E$
(*ii*) $\forall\, a \in E \quad \lim_{x\to a^-} F'(x) \quad$ and $\quad \lim_{x\to a^-} F'(x) \quad$ exist and are finite.

If $u \in W^{1,p}_{\mathrm{loc}}(M)$ with $u(M) \subseteq I$ and $F'$ is bounded on $I \setminus E$, then by Stampacchia's lemma the function $v = F(u)$ is also in $W^{1,p}_{\mathrm{loc}}(M)$ and

$$\nabla v = \begin{cases} F'(u)\nabla u & \text{a.e. on } M \setminus u^{-1}(E) \\ 0 & \text{a.e. on } u^{-1}(E)\,, \end{cases}$$

see for instance Theorem 7.8 in [6]. (Here and in the following statements, "a.e." always referes to the $m$-dimensional Riemannian volume measure of $M$.) Since $\nabla u = 0$ a.e. on each level set of $u$, we can further write

$$\nabla v = F'(u)\nabla u \qquad \text{a.e. on } M\,.$$

## 3 Proof of the Main Theorem

The aim of this section is to prove the main Theorem 13 below, which is slightly more general than Theorem 4 from the Introduction. To do so, we have to collect some preliminary lemmas about functions $u$ satisfying $Lu \geq 0$ on some superlevel set $\Omega_{s_0} := \{x \in M : u(x) > s_0\}$, $s_0 \in \mathbb{R}$. Note that for the validity of the following lemmas it is not necessary to assume that $|\Omega_{s_0}| > 0$, that is, $s_0$ may be a priori larger than or equal to $\operatorname{ess\,sup}_M u$ (in which case it is clearly true that $Lu \geq 0$ on $\Omega_{s_0}$ in the sense of (29), and the thesis of each lemma holds trivially).

**Lemma 9** *Let $M$ be a Riemannian manifold, $p > 1$ and $L$ a weakly $p$-coercive operator as in (23) with coercivity constant $k > 0$. Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \geq f \geq 0 \qquad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{31}$$

*for some $s_0 \in \mathbb{R}$ and some measurable $f : M \to \mathbb{R}$. Let $F$ be a non-negative, non-decreasing, piecewise $C^1$ function on $(0,+\infty)$. Then for every $0 \leq \eta \in C^\infty_c(M)$*

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| \geq k^{-p'}\int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta F(w) f\,, \tag{32}$$

*where* $w := (u - s_0)_+$, $A_u := A(x, u, \nabla u)$ *and* $p' = \frac{p}{p-1}$.

***Proof*** Let $0 \le \eta \in C_c^\infty(M)$ be given and let

$$w := (u - s_0)_+ \in W^{1,p}_{\mathrm{loc}}(M)\,, \qquad A_u := A(x, u, \nabla u)$$

as in the statement. Let $\lambda \in C^\infty(\mathbb{R})$ be such that

$$\lambda(s) = 0 \quad \text{if } s \le 1\,, \qquad \lambda(s) = 1 \quad \text{if } s \ge 2\,, \qquad \lambda' \ge 0 \quad \text{on } \mathbb{R} \tag{33}$$

and for any $\varepsilon > 0$ define $\lambda_\varepsilon \in C^\infty(\mathbb{R})$ by

$$\lambda_\varepsilon(s) := \lambda(s/\varepsilon)\,. \tag{34}$$

Clearly we have

$$0 \le \lambda_\varepsilon \le \mathbf{1}_{(0,+\infty)} \quad \forall\, \varepsilon > 0 \qquad \text{and} \qquad \lambda_\varepsilon \nearrow \mathbf{1}_{(0,+\infty)} \quad \text{as } \varepsilon \to 0^+\,, \tag{35}$$

where $\mathbf{1}$ denotes the indicator function and $\nearrow$ denotes monotone convergence from below. Let $h > 0$ be fixed and for any $\varepsilon \in (0, h/2)$ let $F_{\varepsilon,h} : \mathbb{R} \to [0, +\infty)$ be given by

$$F_{\varepsilon,h}(s) = \begin{cases} 0 & \text{if } s < 0 \\ \lambda_\varepsilon(s) F(s) & \text{if } 0 \le s < h \\ F(h) & \text{if } s \ge h\,. \end{cases}$$

By our choice of $\lambda$ and our assumptions on $F$, the function $F_{\varepsilon,h}$ is non-negative, non-decreasing, piecewise $C^1$ on $\mathbb{R}$ (with an additional corner point at $s = h$) and globally Lipschitz, so $F_{\varepsilon,h}(w) \in W^{1,p}_{\mathrm{loc}}(M)$ with

$$\nabla F_{\varepsilon,h}(w) = F'_{\varepsilon,h}(w)\nabla u \qquad \text{a.e. on } M\,.$$

In particular we have

$$F'_{\varepsilon,h}(s) = \begin{cases} \lambda'_\varepsilon(s)F(s) + \lambda_\varepsilon(s)F'(s) \ge \lambda_\varepsilon(s)F'(s) & \text{if } \varepsilon < s < h \\ 0 & \text{if } s \le \varepsilon \text{ or } s > h\,. \end{cases}$$

Set

$$\varphi = \varphi_{\varepsilon,h} := \eta F_{\varepsilon,h}(w)\,.$$

We have $0 \le \varphi \in W^{1,p}_c(M)$ and by the choice of $\lambda_\varepsilon$ we also have that $\varphi$ vanish outside $\{w > 0\} \equiv \Omega_{s_0}$. So $\varphi$ is an admissible test function for (29) and we have

$$-\int_M \langle A_u, \nabla\varphi\rangle \ge \int_M f\varphi\,. \tag{36}$$

By direct computation and using that $\eta F(w)\lambda'_\varepsilon(w)\langle A_u, \nabla u\rangle \geq 0$ by our assumptions on $\lambda_\varepsilon$, $F$, $\eta$ and $A$, together with weak $p$-coercivity (25) of $A$ and Cauchy–Schwarz inequality we have

$$\begin{aligned}\langle A_u, \nabla\varphi\rangle &= F_{\varepsilon,h}(w)\langle A_u, \nabla\eta\rangle + \eta F'_{\varepsilon,h}(w)\langle A_u, \nabla u\rangle \\ &\geq F_{\varepsilon,h}(w)\langle A_u, \nabla\eta\rangle + \eta F'(w)\langle A_u, \nabla u\rangle\lambda_\varepsilon(w)\mathbf{1}_{\{\varepsilon<w<h\}} \\ &\geq -F_{\varepsilon,h}(w)|A_u||\nabla\eta| + k^{-p'}\eta F'(w)|A_u|^{p'}\lambda_\varepsilon(w)\mathbf{1}_{\{\varepsilon<w<h\}}\,.\end{aligned}$$

We substitute into (36) and rearrange terms to get

$$\int_{\Omega_{s_0}} F_{\varepsilon,h}(w)|A_u||\nabla\eta| \geq k^{-p'}\int_{\{\varepsilon<w<h\}} \eta\lambda_\varepsilon(w)F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta F_{\varepsilon,h}(w)f\,.$$

Using non-negativity of $F$, $F'$, $f$, $\eta$, monotonicity of $F$ and (35), by the monotone convergence theorem we get

$$\lim_{\substack{\varepsilon\to 0^+\\ h\to+\infty}} \int_{\Omega_{s_0}} F_{\varepsilon,h}(w)|A_u||\nabla\eta| = \int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta|$$

$$\lim_{\substack{\varepsilon\to 0^+\\ h\to+\infty}} \int_{\{\varepsilon<w<h\}} \eta\lambda_\varepsilon(w)F'(w)|A_u|^{p'} = \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'}$$

$$\lim_{\substack{\varepsilon\to 0^+\\ h\to+\infty}} \int_{\Omega_{s_0}} \eta F_{\varepsilon,h}(w)f = \int_{\Omega_{s_0}} \eta F(w)f$$

and then we obtain (85). □

We underline that the LHS of (32) can be further estimated from above via Young's inequality in two different ways, both useful in what will follow.

**(1)** Suppose that $F' > 0$ on $(0, +\infty)$. By Hölder's and Young's inequalities with conjugate exponents $p$ and $p'$, for any $\sigma > 0$ we get

$$\begin{aligned}\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| &\leq \left(\int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta|\right)^{1/p}\left(\int_{\Omega_{s_0}} F'(w)|A_u|^{p'}|\nabla\eta|\right)^{1/p'} \\ &\leq \frac{\sigma^p}{p}\int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta| + \frac{\sigma^{-p'}}{p'}\int_{\Omega_{s_0}} F'(w)|A_u|^{p'}|\nabla\eta|\,.\end{aligned} \tag{37}$$

**(2)** If $0 \leq \psi \in C_c^\infty(M)$, then applying (32) with $\eta := \psi^p \in C_c^\infty(M)$ we get

$$p\int_{\Omega_{s_0}} \psi^{p-1}F(w)|A_u||\nabla\psi| \geq k^{-p'}\int_{\Omega_{s_0}} \psi^p F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \psi^p F(w)f \tag{38}$$

and by Young's inequality we have, again for any $\sigma > 0$,

$$p \int_{\Omega_{s_0}} \psi^{p-1} F(w)|A_u||\nabla\psi| \le \frac{p^p\sigma^p}{p} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla\psi|^p + \frac{\sigma^{-p'}}{p'} \int_{\Omega_{s_0}} \psi^p F'(w)|A_u|^{p'} . \tag{39}$$

By suitably choosing $\sigma$ in (39) and rearranging terms we deduce the following

**Lemma 10** *In the assumptions of Lemma 9, if*

$$F'(w)|A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M) \tag{40}$$

*then for any $\varepsilon > 0$ and for any $0 \le \eta \in C^\infty_c(M)$ we have*

$$\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla\eta|^p \ge (1-\varepsilon)k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F(w) f . \tag{41}$$

*In particular, (40) holds under one of the following assumptions:*

(a) $F(s) = O(s)$ *as* $s \to +\infty$
(b) $u_+ \in L^r_{\mathrm{loc}}(M)$ *and* $F(s) = O(s^{r/p})$ *as* $s \to +\infty$*, for some* $r > p$
(c) $u_+ \in L^\infty_{\mathrm{loc}}(M)$.

***Proof*** If $\varepsilon > 0$ is given then for $\sigma = (\varepsilon p')^{-1/p'} k$ we have

$$\frac{\sigma^{-p'}}{p'} = \varepsilon k^{-p'} , \qquad \frac{p^p\sigma^p}{p} = \frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}}$$

and then from (39) we get

$$\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}} \int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}} |\nabla\eta|^p + \varepsilon k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w)|A_u|^{p'} \ge k^{-p'} \int_{\Omega_{s_0}} \eta^p F'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F(w) f . \tag{42}$$

In the assumption (40) we can rearrange terms to obtain (41). In view of (32) and since $f \ge 0$ on $\Omega_{s_0}$, condition (40) is automatically satisfied if $F(w)|A_u|\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$. In particular this is always the case if $F(w)\mathbf{1}_{\Omega_{s_0}} \in L^p_{\mathrm{loc}}(M)$, because then $F(w)|A_u|\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$ by Hölder inequality (recall that $u \in W^{1,p}_{\mathrm{loc}}(M)$, so $|A_u| \le k^p|\nabla u|^{p-1} \in L^{p'}_{\mathrm{loc}}(M)$), and condition $F(w)\mathbf{1}_{\Omega_{s_0}} \in L^p_{\mathrm{loc}}(M)$ is in turn satisfied in either one of the cases (a), (b) or (c). □

A case that will be relevant for our subsequent discussion is where $u_+ \in L^q_{\mathrm{loc}}(M)$ and $F(s) = s^{q-p+1}$ for some $q \in (p-1, +\infty)$. In this setting the desired inequality takes the form

$$\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}\gamma^{p-1}}\int_{\Omega_{s_0}} w^q|\nabla\eta|^p \geq (1-\varepsilon)k^{-p'}\int_{\Omega_{s_0}}\eta^p w^{q-p}|A_u|^{p'} + \int_{\Omega_{s_0}}\eta^p w^p f$$

where $\gamma := q - p + 1 \in (0, +\infty)$. Note that for $p - 1 < q \leq p$ we have $0 < \gamma \leq 1$, hence $F(s) = s^{q-p+1} = s^\gamma = O(s)$ and this scenario is covered by alternative (a) in Lemma 10, while for $q > p$ (and without assuming $u_+ \in L^\infty_{\mathrm{loc}}(M)$) we cannot refer to (b) or (c).

**Lemma 11** *Let $M$ be a Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23) with coercivity constant $k > 0$. Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \geq f \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{43}$$

*for some $s_0 \in \mathbb{R}$ and some measurable $f : M \to \mathbb{R}$. Let $w := (u - s_0)_+$ and $A_u := A(x, u, \nabla u)$. Then for any $q \in (p-1, +\infty)$ and for every $0 \leq \eta \in C^\infty_c(M)$*

$$\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}\min\{1,\gamma^{p-1}\}}\int_{\Omega_{s_0}} w^q|\nabla\eta|^p \geq (1-\varepsilon)\gamma k^{-p'}\int_{\Omega_{s_0}}\eta^p w^{q-p}|A_u|^{p'} + \int_{\Omega_{s_0}}\eta^p w^{q-p+1} f \tag{44}$$

*where $\gamma := q - p + 1$. If $u_+ \in L^q_{\mathrm{loc}}(M)$, this can be strengthened to*

$$\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}\gamma^{p-1}}\int_{\Omega_{s_0}} w^q|\nabla\eta|^p \geq (1-\varepsilon)\gamma k^{-p'}\int_{\Omega_{s_0}}\eta^p w^{q-p}|A_u|^{p'} + \int_{\Omega_{s_0}}\eta^p w^{q-p+1} f\,. \tag{45}$$

*In particular, if $u_+ \in L^\infty_{\mathrm{loc}}(M)$ then this holds for any $q \in (p-1, +\infty)$.*

***Proof*** Let $0 \leq \eta \in C^\infty_c(M)$, $q \in (p-1, +\infty)$ be given and set $F(s) = s^\gamma$ for $s > 0$, where $\gamma := q - p + 1$ as in the statement of the Lemma.

If $p - 1 < q \leq p$ then $0 < \gamma \leq 1$ and by Lemma 10 we have the validity of (45) for any $\varepsilon \in (0, 1]$. (Note that in this case (44) and (45) coincide.)

If $q > p$ then we proceed by approximating $F$ from below with globally Lipschitz functions. For any $h > 0$ let $F_h : (0, +\infty) \to (0, +\infty)$ be defined by

$$F_h(s) = \begin{cases} s^\gamma & \text{if } 0 < s \leq h \\ h^{\gamma-1}s & \text{if } s > h\,. \end{cases}$$

Then $F_h$ is piecewise smooth with a corner point at $s = h$ and satisfies $F_h(s) = O(s)$ as $s \to +\infty$, so by Lemma 10 we have

$$\begin{aligned} &\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}}\int_{\Omega_{s_0}} \frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}|\nabla\eta|^p \\ &\quad\geq (1-\varepsilon)k^{-p'}\int_{\Omega_{s_0}}\eta^p F_h'(w)|A_u|^{p'} + \int_{\Omega_{s_0}}\eta^p F_h(w) f\,. \end{aligned}$$

By direct computation we have

$$F_h'(w)|A_u|^{p'} = \gamma w^{q-p}|A_u|^{p'}\mathbf{1}_{\{0<w\leq h\}} + h^{q-p}|A_u|^{p'}\mathbf{1}_{\{w>h\}} \qquad \text{a.e. on } \Omega_{s_0}$$
$$\frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}} \leq \frac{w^q}{\gamma^{p-1}}\mathbf{1}_{\{0<w\leq h\}} + h^{q-p}w^p\mathbf{1}_{\{w>h\}} \leq w^q \qquad \text{on } \Omega_{s_0}$$

We substitute the second estimate into the previous inequality to obtain

$$\begin{aligned}\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}}&\int_{\Omega_{s_0}} w^q|\nabla\eta|^p \\ &\geq (1-\varepsilon)k^{-p'}\int_{\Omega_{s_0}} \eta^p F_h'(w)|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p F_h(w) f\end{aligned}$$

and then letting $h \to +\infty$ we get, by the monotone convergence theorem,

$$\begin{aligned}\frac{k^p(p-1)^{p-1}}{\varepsilon^{p-1}}&\int_{\Omega_{s_0}} w^q|\nabla\eta|^p \\ &\geq (1-\varepsilon)\gamma k^{-p'}\int_{\Omega_{s_0}} \eta^p w^{q-p}|A_u|^{p'} + \int_{\Omega_{s_0}} \eta^p w^{q-p+1} f\end{aligned}$$

proving (44).

If additionally $u_+ \in L^q_{\mathrm{loc}}(M)$, then for any given $0 \leq \eta \in C_c^\infty(M)$

$$\int_{\Omega_{s_0}} \frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta|^p \equiv \frac{1}{\gamma^{p-1}}\int_{\Omega_{s_0}} w^q|\nabla\eta|^p < +\infty$$

and from (44) applied for any $\varepsilon \in (0, 1)$ we deduce (since $f \geq 0$) that also

$$\int_{\Omega_{s_0}} \eta^p F'(w)|A_u|^{p'} \equiv \gamma\int_{\Omega_{s_0}} \eta^p w^{q-p}|A_u|^{p'} < +\infty\,.$$

Since this holds for any $0 \leq \eta \in C_c^\infty(M)$ we have that $F'(w)|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$, that is, the hypothesis (40) in Lemma 10 is satisfied, and then (45) directly follows from that lemma. □

We briefly comment on the condition $u_+ \in L^\infty_{\mathrm{loc}}(M)$. If the function $A$ satisfies the additional coercivity condition

$$|A(x, s, \xi)| \geq k_2|\xi|^{p-1} \qquad \forall\, x \in M,\ s \in \mathbb{R},\ \xi \in T_xM \tag{46}$$

for some constant $k_2 > 0$ (note that this is the case for the $p$-Laplacian $L = \Delta_p$) then subsolutions of $Lu = 0$ on $M$ are locally essentially bounded above, that is, condition $u_+ \in L^\infty_{\mathrm{loc}}(M)$ is automatically satisfied for any $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfying

$$Lu \geq 0 \qquad \text{weakly on } M\,. \tag{47}$$

More generally, $u_+ \in L^\infty_{\mathrm{loc}}(M)$ holds for functions $u \in W^{1,p}_{\mathrm{loc}}(M)$ such that, for some $s_0 \in \mathbb{R}$, the truncation $w := (u - s_0)_+$ satisfies $Lw \geq 0$ weakly on $M$.

**Proposition 12** *Let $M$ be a Riemannian manifold, $p > 1$ and $L$ as in (23) a weakly $p$-coercive operator for which (46) holds. Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$L(u - s_0)_+ \geq 0 \qquad \text{weakly on } M \tag{48}$$

*for some $s_0 \in \mathbb{R}$. Then $u_+ \in L^\infty_{\mathrm{loc}}(M)$.*

***Sketch of proof*** For $p > \dim M$ the thesis holds because $W^{1,p}_{\mathrm{loc}}(M) \subseteq C(M)$ by (local) Sobolev embeddings, while for $1 < p \leq \dim M$ the statement can be proved by Moser iteration technique, using the Caccioppoli-type inequality

$$\frac{2^p(p-1)^{p-1}k^{pp'}}{\gamma \min\{1, \gamma^{p-1}\}} \int_M |\nabla \eta|^p (u - s_0)_+^q \geq k_2^{p'} \int_M \eta^p (u - s_0)_+^{q-p} |\nabla u|^p$$

obtained by (44) (with the choices $\varepsilon = 1/2$ and $f = 0$) and (46), together with the fact that every point $x \in M$ has a relatively compact neighborhood $U \subseteq M$ on which a Sobolev inequality holds. In fact, the Moser technique can be used to prove that $(u - s_0)_+ \in L^\infty_{\mathrm{loc}}(M)$, from which $u_+ \in L^\infty_{\mathrm{loc}}(M)$ immediately follows. □

Since the argument above is of local nature, clearly it also applies in case (46) is satisfied with $k_2 : M \to (0, +\infty)$ a continuous function possibly decaying to zero at infinity. However, in our analysis we are not assuming coercivity conditions of the form (46), and in fact we don't know whether a function $u \in W^{1,p}_{\mathrm{loc}}(M)$ such that $Lu \geq 0$ on some superlevel set $\{u > s_0\}$, with $L$ only satisfying assumptions (24)–(25) from Sect. 2, is necessarily locally upper bounded.

We are now ready to state and prove the main theorem of this section.

**Theorem 13** *Let $M$ be a complete Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23) with coercivity constant $k > 0$. Let $\lambda > 0$, $\mu \in [0, p]$ and $V : M \to (0, +\infty)$ be a continuous function satisfying*

$$\begin{aligned} &V \geq \lambda && \text{if } \mu = 0 \\ &\liminf_{x \to \infty} [\mathrm{dist}(x, o)^\mu V(x)] \geq \lambda \quad \text{for some } o \in M && \text{if } \mu \in (0, p]\,. \end{aligned} \tag{49}$$

*Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy, for some $0 \leq s_0 < \operatorname{ess\,sup}_M u$,*

$$Lu \geq V u^{p-1} \qquad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}\,.$$

*Then for any $x_0 \in M$ and $q \in (p - 1, +\infty)$ we have*

$$\liminf_{R \to +\infty} \frac{1 - \frac{\mu}{p}}{R^{1 - \frac{\mu}{p}}} \log \int_{B_R} (u - s_0)_+^q \geq C_0 > 0 \qquad \text{if } \mu \in [0, p) \tag{50}$$

$$\liminf_{R \to +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q \geq C_1 > p \qquad \text{if } \mu = p \tag{51}$$

*where $C_0$ and $C_1$ are determined by*

$$C_0 := \frac{p(q-p+1)^{1/p'}\lambda^{1/p}}{(p-1)^{1/p'}k}\,, \qquad C_1^{1/p}(C_1-p)^{1/p'} = C_0\,.$$

*Moreover, in case $\mu = p$ we have*

$$\lim_{R\to+\infty}\frac{1}{\log R}\log\int_{B_R}(u-s_0)_+^q \geq C_0 + p \tag{52}$$

*whenever the limit on the LHS exists.*

***Remark 14*** Note that $C_0 + p > C_1 > C_0$ always.

***Proof*** Let us set $w := (u-s_0)_+$ and $A_u := A(x,u,\nabla u)$. Let $x_0 \in M$ and $q \in (p-1,+\infty)$ be given. For the sake of brevity, for any $R > 0$ we shall write $B_R$ to denote the geodesic ball $B_R(x_0)$. Without loss of generality we can assume $w^q \in L^1_{\mathrm{loc}}(M)$, since otherwise $\int_{B_R} w^q = +\infty$ for each sufficiently large $R > 0$ and the conclusion is trivial. Note that under this assumption we also have $w^{q-p}|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$, as a consequence of (45) in Lemma 11. Let $G, H : (0,+\infty) \to [0,+\infty)$ be defined by

$$G(t) := \int_{B_t} w^q\,, \qquad H(t) := \int_{\Omega_{s_0}\cap B_t} w^{q-p}|A_u|^{p'}\,. \tag{53}$$

By the previous observation, the functions $G$ and $H$ are well defined, non-decreasing and absolutely continuous on any compact interval contained in $(0,+\infty)$. In particular, they are differentiable a.e. on $(0,+\infty)$.

Since $s_0 \geq 0$, we have $u^{p-1} \geq w^{p-1}$ on $\Omega_{s_0}$. Then by applying Lemma 9 with the choices $F(s) = s^{q-p+1}$ and $f = Vw^{p-1}$ we have

$$\int_M w^{q-p+1}|A_u||\nabla\eta| \geq \gamma k^{-p'}\int_{\Omega_{s_0}}\eta w^{q-p}|A_u|^{p'} + \int_M V\eta w^q \tag{54}$$

for any $0 \leq \eta \in C_c^\infty(M)$, where $\gamma := q-p+1 > 0$, and applying Young's inequality as in (37) we have, for any $\sigma > 0$,

$$\int_M w^{q-p+1}|A_u||\nabla\eta| \leq \frac{\sigma^p}{p}\int_M w^q|\nabla\eta| + \frac{\sigma^{-p'}}{p'}\int_{\Omega_{s_0}} w^{q-p}|A_u|^{p'}|\nabla\eta|\,. \tag{55}$$

Let $\varepsilon \in (0,\lambda)$ be given. By condition (49) and continuity and (strict) positivity of $V$, there exists $R_0 = R_0(x_0,\varepsilon) > 0$ large enough so that

$$V(x) \geq \frac{\lambda-\varepsilon}{\mathrm{dist}(x,x_0)^\mu} \qquad \text{for all } x \in M\setminus B_{R_0} \tag{56}$$

and

$$\inf_{B_R} V \geq \frac{\lambda-\varepsilon}{R^\mu} \qquad \forall R > R_0\,. \tag{57}$$

Indeed, for $\mu = 0$ this is clearly true since $V \geq \lambda$ everywhere on $M$ by assumption (49). In case $\mu > 0$, note that it is possible to first find $r_0 > 0$ such that

$$V(x) \geq \frac{\lambda - \varepsilon}{\mathrm{dist}(x, x_0)^\mu} \qquad \text{for all } x \in M \setminus B_{r_0} \tag{58}$$

since from (49) and the triangle inequality we have

$$\liminf_{x\to\infty} [\mathrm{dist}(x, x_0)^\mu V(x)] \geq \lambda \,,$$

and then for any $R > r_0$ we get

$$\inf_{B_R} V \geq \min\left\{\inf_{B_{r_0}} V \,, \frac{\lambda - \varepsilon}{R^\mu}\right\} . \tag{59}$$

From the assumption that $V$ is continuous and strictly positive on $M$ we have $\inf_{B_{r_0}} V > 0$, so we can find $R_0 \geq r_0$ such that $\inf_{B_{r_0}} V \geq (\lambda - \varepsilon)/R_0^\mu$. Then for any $R > R_0$ the RHS in (59) is just $(\lambda - \varepsilon)/R^\mu$, and so (56)–(57) hold for such $R_0$.

Let $t > R_0$ be a value for which $G'(t)$ and $H'(t)$ both exist. For any $0 < \delta < t$ choose $\eta_\delta \in C_c^\infty(M)$ satisfying

$$\begin{array}{rlll} (i) & \eta_\delta \equiv 1 & \text{on } B_{t-\delta} \,, \\ (ii) & \eta_\delta \equiv 0 & \text{on } M \setminus B_t \,, \\ (iii) & 0 \leq \eta_\delta \leq 1 & \text{on } B_t \setminus B_{t-\delta} \\ (iv) & |\nabla \eta_\delta| \leq \dfrac{1}{\delta} + 1 & \text{on } M \,. \end{array}$$

Since $|\nabla \eta_\delta| \leq (1 + \delta^{-1})\mathbf{1}_{B_R \setminus B_{R-\delta}}$ we have

$$\int_M w^q |\nabla \eta_\delta| \leq \left(\frac{1}{\delta} + 1\right) \int_{B_t \setminus B_{t-\delta}} w^q = (1 + \delta) \frac{G(t) - G(t - \delta)}{\delta}$$

and letting $\delta \searrow 0$ we get

$$\limsup_{\delta\to 0^+} \int_M w^q |\nabla \eta_\delta| \leq G'(t) \,.$$

Similarly, we have

$$\limsup_{\delta\to 0^+} \int_{\Omega_{s_0}} w^{q-p} |A_u|^{p'} |\nabla \eta_\delta| \leq H'(t) \,.$$

On the other hand, since $\eta_\delta = 0$ on $M \setminus B_t$ and $\eta_\delta \to \mathbf{1}_{B_t}$ pointwise as $\delta \to 0$, by the dominated convergence theorem and also using (57) we get

$$\lim_{\delta\to 0^+} \int_{\Omega_{s_0}} \eta_\delta w^{q-p} |A_u|^{p'} = \int_{\Omega_{s_0}\cap B_t} w^{q-p} |A_u|^{p'} = H(t)$$
$$\lim_{\delta\to 0^+} \int_M V \eta_\delta w^q = \int_{B_t} V w^q\,.$$

Thus, in view of (54)–(55) we have, for any $\sigma > 0$,

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \int_{B_t} V w^q + \gamma k^{-p'} H(t) \tag{60}$$

and using (57) to further estimate

$$\int_{B_t} V w^q \geq \frac{\lambda - \varepsilon}{t^\mu} \int_{B_t} w^q = \frac{\lambda - \varepsilon}{t^\mu} G(t)$$

we obtain

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \frac{\lambda - \varepsilon}{t^\mu} G(t) + \gamma k^{-p'} H(t)\,.$$

We apply the above reasoning to each value $t > R_0$ for which $G$ and $H$ are simultaneously differentiable to deduce that for any $\sigma : (0, +\infty) \to (0, +\infty)$

$$\frac{[\sigma(t)]^p}{p} G'(t) + \frac{[\sigma(t)]^{-p'}}{p'} H'(t) \geq \frac{\lambda - \varepsilon}{t^\mu} G(t) + \gamma k^{-p'} H(t) \qquad \text{for a.e. } t > R_0$$

that is, multiplying everything by $p[\sigma(t)]^{-p}$ and recalling that $p + p' = pp'$,

$$G'(t) + \frac{p-1}{[\sigma(t)]^{pp'}} H'(t) \geq \frac{p(\lambda - \varepsilon)}{[\sigma(t)]^p t^\mu} \left( G(t) + \frac{\gamma}{(\lambda - \varepsilon) k^{p'}} t^\mu H(t) \right) \tag{61}$$

for a.e. $t > R_0$. We now consider separately the cases $\mu \in [0, p)$ and $\mu = p$.
**Case** $\mu \in [0, p)$. Assume that $\mu \in [0, p)$. Choosing

$$c_1 = c_{1,\varepsilon} = (p-1)^{\frac{1}{pp'}} (\lambda - \varepsilon)^{\frac{1}{pp'}} \gamma^{-\frac{1}{pp'}} k^{1/p}$$
$$c_2 = c_{2,\varepsilon} = \frac{(p-1)}{c_1^{pp'}} \equiv \gamma (\lambda - \varepsilon)^{-1} k^{-p'}$$
$$c_3 = c_{3,\varepsilon} = \frac{p(\lambda - \varepsilon)}{c_1^p} \equiv \frac{p \gamma^{1/p'} (\lambda - \varepsilon)^{1/p}}{(p-1)^{1/p'} k}$$
$$\sigma(t) = c_1 t^{-\frac{\mu}{pp'}}$$

we get

$$G'(t) + c_2 t^\mu H'(t) \geq c_3 t^{-\frac{\mu}{p}} \big(G(t) + c_2 t^\mu H(t)\big) \qquad \text{for a.e. } t > R_0 \,.$$

Let $\Phi : (0, +\infty) \to [0, +\infty)$ be defined by

$$\Phi(t) = G(t) + c_2 t^\mu H(t) \,.$$

The function $\Phi$ is absolutely continuous on each compact subset of $(0, +\infty)$ with

$$\Phi'(t) = G'(t) + c_2 t^\mu H'(t) + \mu c_2 t^{\mu-1} H(t) \qquad \text{for a.e. } t \in (0, +\infty) \,. \tag{62}$$

Then, in view of the previous inequality and since $\mu c_2 t^{\mu-1} H(t) \geq 0$, we get

$$\Phi'(t) \geq c_{3,\varepsilon} t^{-\frac{\mu}{p}} \Phi(t) \qquad \text{for a.e. } t > R_0 \,. \tag{63}$$

We have $|\Omega_{s_0}| > 0$ because $s_0 < \operatorname{ess\,sup}_M u$, so there exists $R_1 > R_0$ such that $G(R_1) > 0$. Let $R > R_1$ be given. By monotonicity of $G$ and since $c_2 t^\mu H(t) \geq 0$, we have $\Phi(t) \geq G(t) \geq G(R_1) > 0$ for all $t \in [R_1, R]$. Since $[G(R_1), +\infty) \ni s \mapsto \log s$ is Lipschitz, the function $\log \Phi$ is absolutely continuous on $[R_1, R]$ with

$$(\log \Phi)'(t) = \frac{\Phi'(t)}{\Phi(t)} \qquad \text{for a.e. } t \in [R_1, R] \,.$$

Thus, integrating (63) and using that $\Phi(R_1) \geq G(R_1) > 0$ we get

$$\log \Phi(R) \geq \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}} R^{1-\frac{\mu}{p}} + \log G(R_1) - \frac{c_{3,\varepsilon}}{1 - \frac{\mu}{p}} R_1^{1-\frac{\mu}{p}} \qquad \forall\, R > R_1 \,. \tag{64}$$

Note that dividing both sides by $R^{1-\frac{\mu}{p}}$, letting $R \to +\infty$ and then $\varepsilon \to 0^+$ we would obtain

$$\liminf_{R \to +\infty} \frac{1 - \frac{\mu}{p}}{R^{1-\frac{\mu}{p}}} \log \Phi(R) \geq \lim_{\varepsilon \to 0^+} c_{3,\varepsilon} = \frac{p(q - p + 1)^{1/p'} \lambda^{1/p}}{(p - 1)^{1/p'} k} \,,$$

which is (formally) weaker than (50) since $\Phi(R) \geq G(R)$. To show that the same inequality holds with $\log G(R)$ in place of $\Phi(R)$, we proceed as follows. Let $R > R_1$ and $h > 0$ be given. By inequality (45) in Lemma 11 applied with the choice $\varepsilon = \frac{1}{2}$ and with a cut-off function $0 \leq \eta \in C_c^\infty(M)$ satisfying

$$\begin{array}{lll} (i) & \eta \equiv 1 & \text{on } B_R \,, \\ (ii) & \eta \equiv 0 & \text{on } M \setminus B_{R+h} \,, \\ (iii) & 0 \leq \eta \leq 1 & \text{on } B_{R+h} \setminus B_R \\ (iv) & |\nabla \eta| \leq \dfrac{2}{h} & \text{on } M \end{array}$$

we get

$$\frac{k^{pp'}(p-1)^{p-1}4^p}{\gamma\min\{1,\gamma^{p-1}\}}G(R+h)\geq h^p H(R) \tag{65}$$

and thus, choosing $h = R^{\mu/p}$,

$$\begin{aligned}\Phi(R) &= G(R) + c_2 R^\mu H(R)\\ &\leq G(R) + \frac{c_2 k^{pp'}(p-1)^{p-1}4^p}{\gamma\min\{1,\gamma^{p-1}\}}G(R+R^{\mu/p})\\ &\leq \left(1+\frac{c_2 k^{pp'}(p-1)^{p-1}4^p}{\gamma\min\{1,\gamma^{p-1}\}}\right)G(R+R^{\mu/p}) =: C_2 G(R+R^{\mu/p})\end{aligned}$$

where in the last inequality we used monotonicity of $G$. Then, from (64) we get

$$\log G(R+R^{\mu/p}) \geq \frac{c_{3,\varepsilon}}{1-\frac{\mu}{p}}R^{1-\frac{\mu}{p}} + \log\frac{G(R_1)}{C_2} - \frac{c_{3,\varepsilon}}{1-\frac{\mu}{p}}R_1^{1-\frac{\mu}{p}} \qquad \forall\, R > R_1\,. \tag{66}$$

Dividing both sides by $(R+R^{\mu/p})^{1-\frac{\mu}{p}}$ and then letting $R\to+\infty$ we get

$$\liminf_{R\to+\infty}\frac{\log G(R+R^{\mu/p})}{(R+R^{\mu/p})^{1-\frac{\mu}{p}}} \geq \lim_{R\to+\infty}\frac{c_{3,\varepsilon}}{1-\frac{\mu}{p}}\left(\frac{R}{R+R^{\mu/p}}\right)^{1-\frac{\mu}{p}} = \frac{c_{3,\varepsilon}}{1-\frac{\mu}{p}}$$

that is,

$$\lim_{R\to+\infty}\frac{1-\frac{\mu}{p}}{R^{1-\frac{\mu}{p}}}\log G(R) \geq c_{3,\varepsilon}$$

and letting $\varepsilon\to 0^+$ we obtain (50).
**Case** $\mu = p$. Assume now that $\mu = p$. We first prove (51), and then (52) in the assumption that its LHS is well defined.
**Proof of** (51). Choosing

$$\sigma(t) = c_4 t^{-1/p'}$$

for a suitable constant $c_4 = c_{4,\varepsilon}$ to be suitably selected later, from (61) we get

$$G'(t) + \frac{p-1}{c_4^{pp'}}t^p H'(t) \geq \frac{p(\lambda-\varepsilon)}{c_4^p t}\left(G(t) + \frac{\gamma}{(\lambda-\varepsilon)k^{p'}}t^p H(t)\right) \tag{67}$$

for a.e. $t > R_0$. In analogy with the previous case, we aim at using this to deduce an inequality of the form

$$\Phi'(t) \geq c_5 t^{-1}\Phi(t) \qquad \text{for a.e. } t > R_0 \tag{68}$$

with

$$\Phi(t) = G(t) + c_6 t^p H(t) \tag{69}$$

for suitable constants $c_5 = c_{5,\varepsilon}$ and $c_6 = c_{6,\varepsilon}$. Computing $\Phi'$ and rearranging terms we see that the desired inequality takes the form

$$\begin{aligned} G'(t) + c_6 t^p H'(t) &\geq c_5 t^{-1}(G(t) + c_6 t^p H(t)) - p c_6 t^{p-1} H(t) \\ &= c_5 t^{-1}\left(G(t) + c_6\left(1 - \frac{p}{c_5}\right) t^p H(t)\right) \end{aligned} \tag{70}$$

so we want to choose $c_4$, $c_5$ and $c_6$ matching the following relations:

$$\frac{p-1}{c_4^{pp'}} = c_6\,, \qquad \frac{p(\lambda-\varepsilon)}{c_4^p} = c_5\,, \qquad c_6\left(1 - \frac{p}{c_5}\right) = \frac{\gamma}{(\lambda-\varepsilon)k^{p'}}\,.$$

Expressing everything in terms of $c_5$ this amounts to

$$\begin{gathered} c_4 = \frac{p^{1/p}(\lambda-\varepsilon)^{1/p}}{c_5^{1/p}}\,, \qquad c_6 = \frac{(p-1)c_5^{p'}}{p^{p'}(\lambda-\varepsilon)^{p'}}\,, \\ \frac{\gamma}{(\lambda-\varepsilon)k^{p'}} = \frac{c_6}{c_5}(c_5 - p) = \frac{(p-1)c_5^{p'-1}(c_5-p)}{p^{p'}(\lambda-\varepsilon)^{p'}}\,. \end{gathered} \tag{71}$$

that is, raising everything to the power $1/p'$ in the last relation, we choose $c_5 = c_{5,\varepsilon}$ as the unique value in $(p, +\infty)$ satisfying

$$c_5^{1/p}(c_5 - p)^{1/p'} = \frac{p\gamma^{1/p'}(\lambda-\varepsilon)^{1/p}}{(p-1)^{1/p'}k} \; (= c_{3,\varepsilon}\,)$$

and then we let $c_4$ and $c_6$ be defined accordingly by (71). Summarizing, for there choices of $c_4$, $c_5$ and $c_6$ we have that (67) and (70) coincide, and each of them is equivalent to (68) for $\Phi$ defined as in (69). Then choosing $R_1 > R_0$ such that $G(R_1) > 0$ and reasoning as in the previous case we see that

$$\log \Phi(R) \geq c_{5,\varepsilon} \log R + \log G(R_1) - c_{5,\varepsilon} \log R_1 \qquad \forall\, R > R_1$$

and then by applying (65) with $h = R$ we obtain

$$\log G(2R) \geq c_{5,\varepsilon} \log R + \log G(R_1) - c_{5,\varepsilon} \log R_1 - \log C_2 \qquad \forall\, R > R_1\,.$$

Dividing both sides by $\log(2R)$ and using that $\log(2R) \sim \log R$ as $R \to +\infty$ we get (after relabeling)

$$\liminf_{R\to+\infty} \frac{\log G(R)}{\log R} \geq c_{5,\varepsilon}$$

and then letting $\varepsilon \to 0$ we get (51).
**Proof of** (52). Assume that

$$\ell := \lim_{R\to+\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q = \lim_{R\to+\infty} \frac{\log G(R)}{\log R}$$

exists. From (51) we already know that $\ell \geq C_1 > p$. If $\ell = +\infty$ then (52) is trivially satisfied, so let us assume that $\ell < +\infty$. Let $\varepsilon > 0$ be as above and small enough so that $\ell - \varepsilon > p$. Then there exists $R_2 > R_0$ such that

$$R^p < R^{\ell-\varepsilon} < G(R) < R^{\ell+\varepsilon} \qquad \forall\, R > R_2\,. \tag{72}$$

We recall, from the discussion preceding the treatment of case $\mu < p$, that for each $t > R_2$ such that $G'(t)$ and $H'(t)$ exist we have (60), that is,

$$\frac{\sigma^p}{p} G'(t) + \frac{\sigma^{-p'}}{p'} H'(t) \geq \int_{B_t} V w^q + \gamma k^{-p'} H(t)$$

for any $\sigma > 0$. Using the co-area formula twice together with (56) we get

$$\begin{aligned} \int_{B_t} V w^q \geq \int_{B_t \setminus B_{R_2}} V w^q &= \int_{R_2}^t \left( \int_{\partial B_s} V w^q \,\mathrm{d}\mathcal{H}^{m-1} \right) \mathrm{d}s \\ &\geq \int_{R_2}^t \frac{\lambda - \varepsilon}{s^p} \left( \int_{\partial B_s} w^q \,\mathrm{d}\mathcal{H}^{m-1} \right) \mathrm{d}s \\ &= \int_{R_2}^t \frac{\lambda - \varepsilon}{s^p} G'(s) \,\mathrm{d}s \end{aligned}$$

where $m = \dim M$ and $\mathcal{H}$ is the Hausdoff measure induced by the Riemannian structure. Substituting into the above inequality and multiplying both sides by $p\sigma^{-p}t^{-p}$ we get

$$\frac{G'(t)}{t^p} + \frac{p-1}{\sigma^{pp'}} \frac{H'(t)}{t^p} \geq \frac{(\lambda - \varepsilon)p}{\sigma^p t^p} \left[ \int_{R_2}^t \frac{G'(s)}{s^p} \,\mathrm{d}s + \frac{\gamma}{(\lambda - \varepsilon)k^{-p'}} H(t) \right]$$

and then choosing

$$\begin{aligned} c_1 &= c_{1,\varepsilon} = (p-1)^{\frac{1}{pp'}} (\lambda - \varepsilon)^{\frac{1}{pp'}} \gamma^{-\frac{1}{pp'}} k^{1/p} \\ c_2 &= c_{2,\varepsilon} = \frac{p-1}{c_1^{pp'}} \equiv \gamma (\lambda - \varepsilon)^{-1} k^{-p'} \\ c_3 &= c_{3,\varepsilon} = \frac{p(\lambda - \varepsilon)}{c_1^p} \equiv \frac{p\gamma^{1/p'} (\lambda - \varepsilon)^{1/p}}{(p-1)^{1/p'} k} \\ \sigma &= \sigma(t) = c_1 t^{-1/p'} \end{aligned}$$

this yields

$$\frac{G'(t)}{t^p} + c_2 H(t) \geq \frac{c_3}{t}\left[\int_{R_2}^{t} \frac{G'(s)}{s^p}\mathrm{d}s + c_2 H(t)\right] \qquad \text{for a.e. } t > R_2\,. \tag{73}$$

Let $\Psi : (R_2, +\infty) \to [0, +\infty)$ be defined by

$$\Psi(t) = \int_{R_2}^{t} \frac{G'(s)}{s^p}\,\mathrm{d}s + c_2 H(t)\,.$$

The function $\Psi$ is absolutely continuous on each compact interval contained in $(R_2, +\infty)$ and inequality (73) can be restated as

$$\Psi'(t) \geq \frac{c_{3,\varepsilon}}{t}\Psi(t) \qquad \text{for a.e. } t > R_2\,. \tag{74}$$

Reasoning as in the previous cases, since $\Psi \not\equiv 0$ we reach the conclusion

$$\liminf_{R\to+\infty} \frac{\log \Psi(R)}{\log R} \geq c_{3,\varepsilon}\,. \tag{75}$$

We now use this to deduce (52). Let $R > R_2$ be given. Applying (65) with $h = R$, integrating by parts and then using (72) twice we get

$$\begin{aligned}
\Psi(R) &\leq \int_{R_2}^{R} \frac{G'(s)}{s^p}\,\mathrm{d}s + C_2\frac{G(2R)}{R^p}\\
&= \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + p\int_{R_2}^{R}\frac{G(s)}{s^{p+1}}\,\mathrm{d}s + C_2\frac{G(2R)}{R^p}\\
&\leq \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + p\int_{R_2}^{R} s^{\ell+\varepsilon-p-1}\,\mathrm{d}s + C_2\frac{G(2R)}{R^p}\\
&= \frac{G(R)}{R^p} - \frac{G(R_2)}{R_2^p} + \frac{pR^{\ell+\varepsilon-p}}{\ell+\varepsilon-p} - \frac{pR_2^{\ell+\varepsilon-p}}{\ell+\varepsilon-p} + C_2\frac{G(2R)}{R^p}\\
&\leq \frac{G(R)}{R^p} + \frac{pR^{2\varepsilon}}{\ell+\varepsilon-p}\frac{G(R)}{R^p} + C_2\frac{G(2R)}{R^p} - \frac{G(R_2)}{R_2^p} - \frac{pR_2^{\ell+\varepsilon-p}}{\ell+\varepsilon-p}\,.
\end{aligned}$$

Since $G$ is non-decreasing, we have $G(R) \leq G(2R)$ and then

$$\Psi(R) \leq \left(\frac{p}{\ell+\varepsilon-p} + (1+C_2)R^{-2\varepsilon}\right) R^{-p+2\varepsilon}G(2R) + O(1)$$

as $R \to +\infty$. By (72) we see that $R^{-p+2\varepsilon}G(2R) > 2^p R^{2\varepsilon} \to +\infty$, so

$$\log\left[\left(\frac{p}{\ell+\varepsilon-p} + (1+C_2)R^{-2\varepsilon}\right) R^{-p+2\varepsilon}G(2R) + O(1)\right] \sim \log(R^{-p+2\varepsilon}G(2R))$$

as $R \to +\infty$, and then

$$\liminf_{R\to+\infty} \frac{\log \Psi(R)}{\log R} \le \liminf_{R\to+\infty} \frac{\log(R^{-p+2\varepsilon}G(2R))}{\log R}$$
$$= -p + 2\varepsilon + \liminf_{R\to+\infty} \frac{\log(G(2R))}{\log R}$$

and then, using that $\log R \sim \log(2R)$, after relabeling we get

$$\liminf_{R\to+\infty} \frac{\log \Psi(R)}{\log R} \le -p + 2\varepsilon + \lim_{R\to+\infty} \frac{\log G(R)}{\log R} .$$

Substituting this into (75) yields

$$\lim_{R\to+\infty} \frac{\log G(R)}{R} \ge c_{3,\varepsilon} + p - 2\varepsilon$$

and then letting $\varepsilon \to 0^+$ we finally obtain (52). □

***Remark 15*** As a byproduct of the previous proof (namely, inequality (66) above), we showed that if $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfies

$$Lu \ge Vu^{p-1} \qquad \text{on } \Omega_{s_0} = \{u > s_0\}$$

with $V : M \to (0, +\infty)$ continuous and matching (49) for some $\lambda > 0$ and $\mu \in [0, p]$, then for each $\varepsilon \in (0, \lambda)$ and $R_0 > 0$ large enough (so that (56)–(57) are satisfied) and for each $R_1 > R_0$ such that

$$I_1 := \int_{B_{R_1}} (u - s_0)^q_+ > 0$$

we have

$$\log \int_{B_{R+R^{\mu/p}}} (u - s_0)^q_+ \ge C_{0,\varepsilon} \int_{R_1}^{R} t^{-\mu/p}\, \mathrm{d}t + \log \frac{I_1}{C_2} \qquad \forall R > R_1 \tag{76}$$

where

$$C_{0,\varepsilon} = \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} \frac{(\lambda-\varepsilon)^{1/p}}{k}, \qquad C_{2,\varepsilon} = 1 + \frac{k^p(p-1)^p 4^p}{(\lambda-\varepsilon)\min\{1, \gamma^{p-1}\}}$$

do not depend on $u$. Inequality (76) only involves the integrals of $w = (u - s_0)^q_+$ on geodesic balls, so it would still hold for functions $u \in L^q_{\mathrm{loc}}(M)$ that can be approximated pointwise and in $L^q$ norm on balls $B$ of arbitrary large radii by Sobolev functions $\tilde{u} \in W^{1,p}_{\mathrm{loc}}(B)$ satisfying

$$L\tilde{u} \ge V|\tilde{u}|^{p-2}\tilde{u} \qquad \text{on } B .$$

For instance, when $L = \Delta$ is the Laplace–Beltrami operator and $V \equiv 1$, a non-trivial result concerning local smooth monotone approximation of distributional $L^1_{\mathrm{loc}}$ subsolutions of $\Delta u = u$ (namely, Theorem D in [3]) allows to extend the estimate

$$\liminf_{R\to+\infty} \frac{1}{R} \int_{B_R} u_+^q \geq 2\sqrt{q-1}$$

to distributional and not everywhere negative $L^1_{\mathrm{loc}}$ subsolutions of $\Delta u = u$.

The following examples are aimed at showing the sharpness of the constant appearing in (50) and (52). Let $M$ be a model surface, that is, a complete Riemannian manifold diffeomorphic to $\mathbb{R}^2$ and radially symmetric around some point $o \in M$ so that in global polar coordinates $(r, \theta)$ centered at $o$ the metric takes the form

$$\langle\, , \,\rangle = \mathrm{d}r^2 + g(r)^2 \mathrm{d}\theta^2$$

for a smooth $g : (0, +\infty) \to (0, +\infty)$ satisfying $g'(0^+) = 1$ and $g^{(2k)}(0^+) = 0$ for each $k \in \{0\} \cup \mathbb{N}$. Let $v : [0, +\infty) \to \mathbb{R}$ be smooth and such that

$$v^{(k)}(0) = 0 \quad \forall k \in \mathbb{N} \qquad \text{and} \qquad v'(t) > 0 \quad \forall t > 0\,.$$

Then $u := v \circ r \in C^\infty(M)$, $|\nabla u| \neq 0$ on $M \setminus \{o\}$ and for any $p > 1$ we have

$$\Delta_p u = \left[ (p-1)(v')^{p-2} v'' + \frac{g'}{g}(v')^{p-1} \right] \circ r \qquad \text{on } M \setminus \{o\}\,. \tag{77}$$

**Case** $\mu \in [0, p)$. Let $p > 1$ and $\mu \in [0, p)$ be given. Consider $a, c \in \mathbb{R}$ satisfying

$$c > 0\,, \qquad (p-1)c + a > 0 \tag{78}$$

and set

$$\beta := 1 - \frac{\mu}{p} \in (0, 1]\,.$$

Choose $g$ and $v$ satisfying the above requirements and such that

$$g(t) = \begin{cases} t & \text{for } 0 < t \leq 1/2 \\ \exp(a t^\beta) & \text{for } t \geq 1 \end{cases}$$

and

$$v(t) = \exp(c t^\beta) \qquad \text{for } t \geq 1\,.$$

By (77) we have

$$\Delta_p u = V u^{p-1} \qquad \text{on } \Omega := M \setminus \overline{B_1} \tag{79}$$

where

$$V = \left( (p-1) \left( 1 + \frac{\beta - 1}{c\beta r^\beta} \right) c + a \right) \frac{\beta^p c^{p-1}}{r^\mu} . \tag{80}$$

Let $s_0 > e^c$. Since $v$ is non-decreasing, the set $\Omega_{s_0} := \{u > s_0\}$ coincides with $M \setminus \overline{B_{t_0}}$, where $t_0 = [(\log s_0)/c]^{1/\beta} > 1$, so in particular $\Omega_{s_0} \subseteq \Omega$. Also, for any $q > p - 1$ we have

$$\int_{B_R} (u - s_0)_+^q = \int_{t_0}^R g(s)(v(s) - s_0)^q \, \mathrm{d}s \sim \int_{t_0}^R \exp((a + qc)s^\beta) \, \mathrm{d}s \qquad \text{as } R \to +\infty$$

where the asymptotic equivalence between the integrals holds because

$$g(s)(v(s) - s_0)^q \sim \exp((a + qc)s^\beta) \to +\infty \qquad \text{as } s \to +\infty .$$

(Recall that $a + qc > a + (p-1)c > 0$ due to our assumptions on $a$ and $c$.) Integrating by parts yields

$$\begin{aligned}
\int_{t_0}^R \exp((a + qc)s^\beta) \, \mathrm{d}s &= \int_{t_0}^R \frac{\frac{\mathrm{d}}{\mathrm{d}s} \exp((a + qc)s^\beta)}{(a + qc)\beta s^{\beta-1}} \, \mathrm{d}s \\
&= \frac{1}{(a + qc)\beta} \left( \frac{\exp((a + qc)R^\beta)}{R^{\beta-1}} - \frac{\exp((a + qc)t_0^\beta)}{t_0^{\beta-1}} \right) \\
&\quad - \frac{1 - \beta}{(a + qc)\beta} \int_{t_0}^R s^{-\beta} \exp((a + qc)s^\beta) \, \mathrm{d}s \\
&\geq \frac{1}{(a + qc)\beta} \left( \frac{\exp((a + qc)R^\beta)}{R^{\beta-1}} - \frac{\exp((a + qc)t_0^\beta)}{t_0^{\beta-1}} \right) \\
&\quad - \frac{(1 - \beta)t_0^{-\beta}}{(a + qc)\beta} \int_{t_0}^R \exp((a + qc)s^\beta) \, \mathrm{d}s
\end{aligned}$$

hence, rearranging terms and using that $\beta \in (0, 1]$, we get

$$\frac{\exp((a + qc)R^\beta)}{a_1 R^{\beta-1}} + O(1) \geq \int_{t_0}^R \exp((a + qc)s^\beta) \, \mathrm{d}s \geq \frac{\exp((a + qc)R^\beta)}{a_2 \beta R^{\beta-1}} + O(1)$$

for $R \to +\infty$, with

$$a_1 = (a + qc)\beta , \qquad a_2 = (a + qc)\beta + (1 - \beta)t_0^{-\beta} .$$

Passing to logarithms, we obtain

$$\log \int_{B_R} (u - s_0)_+^q \sim \log \int_{t_0}^R \exp((a + qc)s^\beta) \, \mathrm{d}s \sim (a + qc)R^\beta$$

as $R \to +\infty$, that is, multiplying both sides by $\beta R^{-\beta}$ and recalling that $\beta = 1 - \frac{\mu}{p}$,

$$\lim_{R\to+\infty} \frac{1-\frac{\mu}{p}}{R^{1-\frac{\mu}{p}}} \log \int_{B_R} (u - s_0)_+^q = (a + qc)\beta\,.$$

On the other hand, from (80) we clearly have

$$\lim_{x\to\infty} r(x)^\mu V(x) = \lambda \qquad \text{with} \qquad \lambda = \beta^p c^{p-1}((p-1)c + a)\,. \tag{81}$$

Since the $p$-Laplacian is weakly $p$-coercive with coercivity constant $k = 1$, to prove that estimate (50) is sharp it is enough to show that for any $p$ and $q > p - 1$ there exist $a$ and $c$ satisfying (78) and such that

$$a + qc = \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} c^{1/p'}((p-1)c + a)^{1/p}\,. \tag{82}$$

This can be done by picking any $a$ and $c > 0$ such that

$$(p-1)a = (q - p(p-1))c$$

since this would yield

$$a + qc = p((p-1)c + a) = \frac{p(q-p+1)}{p-1} c > 0$$

and then

$$\begin{aligned} a + qc = (a+qc)^{1/p'}(a+qc)^{1/p} &= \left(\frac{p(q-p+1)}{p-1}c\right)^{1/p'} (p((p-1)c + a))^{1/p} \\ &= \frac{p(q-p+1)^{1/p'}}{(p-1)^{1/p'}} c^{1/p'}((p-1)c + a)^{1/p} \end{aligned}$$

as desired. For instance, a feasible choice for $a$ and $c$ would be the following:

$$\begin{cases} a = -1 \text{ and } c = \dfrac{p-1}{p(p-1)-q} & \text{if } p - 1 < q < p(p-1) \\ a = 0 \text{ and } c = 1 & \text{if } q = p(p-1) \\ a = 1 \text{ and } c = \dfrac{p-1}{q-p(p-1)} & \text{if } q > p(p-1)\,. \end{cases} \tag{83}$$

**Case** $\mu = p$. Let $p > 1$ be given, consider $a, c \in \mathbb{R}$ satisfying (78) and choose $g$ and $v$ satisfying the general requirements and such that

$$g(t) = \begin{cases} t & \text{for } 0 < t \le 1/2 \\ t^{a+p-1} & \text{for } t \ge 1 \end{cases}$$

and

$$v(t) = t^c \qquad \text{for } t \geq 1 .$$

By (77) we have

$$\Delta_p u = V u^{p-1} \qquad \text{on } \Omega = M \setminus \overline{B_1}$$

with

$$V = \frac{c^{p-1}((p-1)c + a)}{r^p} .$$

Let $s_0 > 1$ be given. Then $\Omega_{s_0} := \{u > s_0\}$ is contained in $\{u > 1\} = M \setminus \overline{B_1}$ and for any $q > p - 1$ we have

$$\log \int_{B_R} (u - s_0)_+^q \sim \log \int_{s_0}^R s^{a+p-1+qc} \, \mathrm{d}s \sim (a + p + qc) \log R \qquad \text{as } R \to +\infty$$

that is,

$$\lim_{R \to +\infty} \frac{1}{\log R} \log \int_{B_R} (u - s_0)_+^q = (a + qc) + p$$

and then again to prove sharpness of (52) we need to show that for any $p > 1$ and $q > p - 1$ we can choose $a$ and $c$ satisfying (78) and

$$a + qc = \frac{p(q - p + 1)^{1/p'} c^{1/p'} ((p-1)c + a)^{1/p}}{(p-1)^{1/p'}} ,$$

but this is precisely what we did in the previous case.

## 4 The Case $Lu \geq 0$

In this section we are concerned with lower bounds on the growth of functions $u$ satisfying the differential inequality $Lu \geq 0$ on a non-empty superlevel set. The main result of this section is Theorem 20 below, corresponding to Theorem 6 from the Introduction. The starting point in this case is again Lemma 9. For ease of the reader we point out that in this case it takes the following form.

**Lemma 16** *Let $M$ be a Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23). Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \geq 0 \qquad on \ \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{84}$$

*for some* $s_0 \in \mathbb{R}$*. Then for any* $0 \le \eta \in C_c^\infty(M)$ *and for any non-negative, non-decreasing, piecewise* $C^1$ *function on* $(0, +\infty)$ *we have*

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| \ge \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \tag{85}$$

*where* $w := (u - s_0)_+$ *and* $A_u := A(x, u, \nabla u)$*.*

The main tool to prove Theorem 20 is the next proposition.

**Proposition 17** *Let* $M$ *be a complete, non-compact Riemannian manifold,* $p \in (1, +\infty)$ *and* $L$ *a weakly* $p$*-coercive operator as in* (23)*. Let* $u \in W^{1,p}_{\mathrm{loc}}(M)$ *satisfy*

$$Lu \ge 0 \qquad on\ \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{86}$$

*for some* $s_0 \in \mathbb{R}$*.*

***(a)*** *For any* $q > p - 1$ *and for any* $x_0 \in M$ *and* $0 < r < R$

$$\int_{B_r(x_0)\cap\Omega_{s_0}} w^{q-p}|A_u|^{p'} \le \frac{(p-1)^{p-1}}{\min\{1, \gamma^p\}} \left( \int_r^R \left( \int_{\partial B_s(x_0)} w^q \right)^{1/(1-p)} \mathrm{d}s \right)^{1-p} \tag{87}$$

*where* $w := (u - s_0)_+$*,* $A_u := A(x, u, \nabla u)$ *and* $\gamma := q - p + 1$*.*

***(b)*** *If* $u_+ \in L^\infty_{\mathrm{loc}}(M)$ *and* $F$ *is a non-negative, piecewise* $C^1$ *function on* $(0, +\infty)$ *such that* $F' > 0$ *everywhere on* $(0, +\infty)$*, then*

$$\int_{B_r(x_0)\cap\Omega_{s_0}} F'(w)|A_u|^{p'} \le (p-1)^{p-1} \left( \int_r^R \left( \int_{\Omega_{s_0}\cap\partial B_s(x_0)} \frac{[F(w)]^p}{[F'(w)]^{p-1}} \right)^{1/(1-p)} \mathrm{d}s \right)^{1-p} \tag{88}$$

*for every* $x_0 \in M$ *and* $0 < r < R$*, with* $w$ *and* $A_u$ *as above.*

***Remark 18*** We remark that the exponents $1-p$ and $1/(1-p)$ appearing on the RHS's of (87) and (88) are negative. With the agreement that $0^a = +\infty$ and $(+\infty)^a = 0$ for any $a \in (-\infty, 0)$, the inequalities make sense also in case one or more of the integrals on the RHS's are either vanishing or diverging.

***Proof*** Let $w$ and $A_u$ be as in the statement. We first prove (b), since the proof of (a) relies on the same idea coupled with suitable approximation arguments.

**Proof of (b).** Suppose that $u_+ \in L^\infty_{\mathrm{loc}}(M)$ and let $F$ be as in the statement. The function $F$ satisfies all the requirements in Lemma 16 and therefore

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta| \ge \int_{\Omega_{s_0}} \eta F'(w)|A_u|^{p'} \tag{89}$$

for any $0 \le \eta \in C_c^\infty(M)$. Note that both integrals are finite since $F(w), F'(w) \in L^\infty(\Omega_{s_0})$ and $|A_u|\mathbf{1}_{\Omega_{s_0}} \in L^{p'}_{\mathrm{loc}}(M)$. Applying Hölder inequality with conjugate exponents $p$ and $p'$ as in (37) we further obtain

$$\left(\int_{\Omega_{s_0}} F'(w)|A_u|^{p'}|\nabla\eta|\right)^{1/p'}\left(\int_{\Omega_{s_0}}\frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta|\right)^{1/p}\geq\int_{\Omega_{s_0}}\eta F'(w)|A_u|^{p'} \tag{90}$$

where the middle integral is again finite since $[F(w)]^p/[F'(w)]^{p-1}\in L^\infty(\Omega_{s_0})$. Let $x_0\in M$ be fixed and let us write $B_s$ for the geodesic ball $B_s(x_0)$, for any $s>0$. Let $G,H:(0,+\infty)\to[0,+\infty)$ be defined by

$$G(s):=\int_{\Omega_{s_0}\cap B_s} F'(w)|A_u|^{p'}, \qquad H(s):=\int_{\Omega_{s_0}\cap B_s}\frac{[F(w)]^p}{[F'(w)]^{p-1}}. \tag{91}$$

Since $F'(w)|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}}\in L^1_{\text{loc}}(M)$ and $[F(w)]^p/[F'(w)]^{p-1}\mathbf{1}_{\Omega_{s_0}}\in L^\infty(M)\subseteq L^1_{\text{loc}}(M)$, the functions $G$ and $H$ are well defined, non-decreasing and absolutely continuous on any compact interval contained in $(0,+\infty)$. In particular, they are differentiable a.e. on $(0,+\infty)$. Let $s>0$ be a value for which $G'(s)$ and $H'(s)$ both exist. For any $\varepsilon>0$ choose $\eta_\varepsilon\in C^\infty_c(M)$ satisfying

$$\begin{array}{lll}(i) & \eta_\varepsilon\equiv 1 & \text{on } B_s\,,\\ (ii) & \eta_\varepsilon\equiv 0 & \text{on } M\setminus B_{s+\varepsilon}\,,\\ (iii) & 0\leq\eta_\varepsilon\leq 1 & \text{on } B_{s+\varepsilon}\setminus B_s\\ (iv) & |\nabla\eta_\varepsilon|\leq\dfrac{1}{\varepsilon}+1 & \text{on } M\,.\end{array}$$

Then

$$\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta_\varepsilon|\leq\left(\frac{1}{\varepsilon}+1\right)\int_{\Omega_{s_0}\cap B_{s+\varepsilon}\setminus B_s} F(w)|A_u|\leq(1+\varepsilon)\frac{G(s+\varepsilon)-G(s)}{\varepsilon}$$

and passing to limits as $\varepsilon\to 0^+$ we get

$$\limsup_{\varepsilon\to 0^+}\int_{\Omega_{s_0}} F(w)|A_u||\nabla\eta_\varepsilon|\leq G'(s)\in[0,+\infty)\,.$$

Similarly, we obtain

$$\limsup_{\varepsilon\to 0^+}\int_{\Omega_{s_0}}\frac{[F(w)]^p}{[F'(w)]^{p-1}}|\nabla\eta_\varepsilon|\leq H'(s)$$

and by dominated convergence theorem we also have

$$\lim_{\varepsilon\to 0^+}\int_{\Omega_{s_0}}\eta_\varepsilon F'(w)|A_u|^{p'}=G(s)\,.$$

Then by (90) we deduce

$$[H'(s)]^{p'/p} G'(s) \geq [G(s)]^{p'} \qquad \text{for a.e. } s > 0\,. \tag{92}$$

Moreover, by the co-area formula we have

$$H'(s) = \int_{\Omega_{s_0} \cap \partial B_s} \frac{[F(w)]^p}{[F'(w)]^{p-1}} =: \varphi(s) \qquad \text{for a.e. } s > 0\,. \tag{93}$$

Let $0 < r < R$ be given. If $G(r) = 0$ then (88) is trivially satisfied. If $G(r) > 0$ then by monotonicity of $G$ we have that $G(s) \geq G(r)$ for all $s \in [r, R]$. Since $G'(s)$ is finite for a.e. $s \in [r, R]$, from (92) and (93) we infer that $\varphi(s) > 0$ for a.e. $s \in [r, R]$ and then

$$\frac{G'(s)}{[G(s)]^{p'}} \geq [\varphi(s)]^{-p'/p} \qquad \text{for a.e. } s \in [r, R]\,. \tag{94}$$

Since $G(s) \geq G(r) > 0$ for all $s \in [r, R]$ and $[G(r), +\infty) \ni t \mapsto t^{1/(1-p)}$ is Lipschitz, the function $G^{1/(1-p)} \equiv G^{1-p'}$ is absolutely continuous on $[r, R]$ with

$$\frac{\mathrm{d}}{\mathrm{d}s}[G(s)]^{1-p'} = \frac{1}{1-p} \frac{G'(s)}{[G(s)]^{p'}} \qquad \text{for a.e. } s \in [r, R]\,.$$

Thus, integrating (94) we get (noting that $p'/p = 1/(p-1)$)

$$(p-1)\left[G(r)^{-1/(p-1)} - G(R)^{-1/(p-1)}\right] = \int_r^R \frac{G'(s)}{G(s)^{p'}}\,\mathrm{d}s \geq \int_r^R [\varphi(s)]^{1/(1-p)}\,\mathrm{d}s\,.$$

Discarding the term containing $G(R)$ and raising everything to $1 - p$ we get

$$G(r) \leq (p-1)^{p-1} \left( \int_r^R [\varphi(s)]^{1/(1-p)}\,\mathrm{d}s \right)^{1-p}$$

that is, (88).

**Proof of (a).** We observe that the argument developed above can be applied straightforwardly, without the assumption $u_+ \in L^\infty_{\mathrm{loc}}(M)$, as long as we consider a piecewise $C^1$ function $F : (0, +\infty) \to (0, +\infty)$ with $F' > 0$ such that

$$F'(w)|A_u|^{p'} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)\,, \qquad \frac{[F(w)]^p}{[F'(w)]^{p-1}} \mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)\,. \tag{95}$$

Indeed, if the conditions in (95) are satisfied then all the integrals appearing in (89) and (90) are finite and the functions $G$ and $H$ defined as in (91) are again finite-valued, non-decreasing and absolutely continuous on every compact interval contained in $(0, +\infty)$.

**Case** $q \geq p$. Set $\gamma := q - p + 1 \geq 1$. For any $h > 0$ define $F_h$ by

$$F_h(s) := \begin{cases} \dfrac{s^\gamma}{\gamma} & \text{if } 0 < s < h \\ \dfrac{h^\gamma}{\gamma} + (s-h)h^{\gamma-1} & \text{if } s \geq h\,. \end{cases} \tag{96}$$

Note that $F_h$ is positive and $C^1$ on $(0, +\infty)$ with

$$F_h'(s) = \begin{cases} s^{\gamma-1} & \text{if } 0 < s < h \\ h^{\gamma-1} & \text{if } s \geq h\,. \end{cases} \tag{97}$$

We have $F_h' > 0$ everywhere on $(0, +\infty)$ and $F_h'(w) \in L^\infty(\Omega_{s_0})$, therefore also $F_h'(w)|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$, due to (26) and $u \in W^{1,p}_{\mathrm{loc}}(M)$. Moreover,

$$\begin{aligned} \frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} &= \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + h^{\gamma-1}\left(w - \frac{\gamma-1}{\gamma}h\right)^p \mathbf{1}_{\{w\geq h\}} \\ &\leq \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + h^{\gamma-1}w^p\mathbf{1}_{\{w\geq h\}} \end{aligned} \tag{98}$$

so in particular

$$\frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} \leq h^{\gamma-1}w^p \in L^1_{\mathrm{loc}}(M)$$

since $\gamma \geq 1$ and $u \in W^{1,p}_{\mathrm{loc}}(M)$. Hence, conditions (95) are satisfied for $F = F_h$ and we can repeat the argument in the proof of **(a)** up to obtaining

$$[\varphi_h(s)]^{p'/p}G_h'(s) \geq [G_h(s)]^{p'} \qquad \text{for a.e. } s > 0$$

with

$$G_h(s) = \int_{\Omega_{s_0}\cap B_s} F_h'(w)|A_u|^{p'}, \qquad \varphi_h(s) = \int_{\Omega_{s_0}\cap\partial B_s} \frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}\,.$$

From (98) and recalling that $\gamma = q - p + 1$ we also have

$$\frac{[F_h(w)]^p}{[f_h(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} \leq w^q \qquad \text{on } M$$

hence

$$\varphi_h(s) \leq \varphi(s) := \int_{\partial B_s} w^q \qquad \forall\, s > 0\,.$$

Reasoning again as in the proof of **(a)** we deduce that either $G_h(r) = 0$ or

$$\begin{cases} G_h(s) \geq G_h(r) > 0 & \forall s \in [r, R] \\ \dfrac{G_h'(s)}{[G_h(s)]^{p'}} \geq [\varphi_h(s)]^{1/(1-p)} \geq [\varphi(s)]^{1/(1-p)} & \text{for a.e. } s \in [r, R]\,. \end{cases}$$

In any case we get

$$\int_{B_r} \min\{w, h\}^{\gamma-1} |A_u|^{p'} = G_h(r) \leq (p-1)^{p-1} \left( \int_r^R [\varphi(s)]^{1/(1-p)} \mathrm{d}s \right)^{1-p}$$

and the conclusion follows by the monotone convergence theorem letting $h \to +\infty$.

**Case** $p - 1 < q < p$. Set $\gamma := q - p + 1$ as in the previous case and note that now $\gamma \in (0, 1)$. For any $h > 0$ let $F_h$ be defined as in (96). We note that $F_h$ is positive and $C^1$ on $(0, +\infty)$ in this case too, with $F_h' > 0$ everywhere on $(0, +\infty)$. Then from Lemma 16 we get

$$\int_{\Omega_{s_0}} F_h(w)|A_u||\nabla\eta| \geq \int_{\Omega_{s_0}} \eta F_h'(w)|A_u|^{p'} \qquad \forall\, 0 \leq \eta \in C_c^\infty(M)\,. \tag{99}$$

From the expression (96) we see that $F_h(w) \leq C_{h,\gamma}(1 + w)$, hence $F_h(w)|A_u|\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)$ by Hölder inequality. By (99) this also yields

$$F_h'(w)|A_u|^{p'}\mathbf{1}_{\Omega_{s_0}} \in L^1_{\mathrm{loc}}(M)\,.$$

On the other hand, we have

$$\begin{aligned} \frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} &= \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + h^{\gamma-1}\left(w - h + \frac{h}{\gamma}\right)^p \mathbf{1}_{\{w \geq h\}} \\ &\leq \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + h^{\gamma-1}\left(\frac{w-h}{\gamma} + \frac{h}{\gamma}\right)^p \mathbf{1}_{\{w \geq h\}} \\ &= \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + \frac{h^{\gamma-1}w^p}{\gamma^p}\mathbf{1}_{\{w \geq h\}} \end{aligned} \tag{100}$$

where the inequality in the middle holds because $w - h < (w - h)/\gamma$ on $\{w > h\}$, since $0 < \gamma < 1$ in this case. From this estimate we get

$$\frac{[F_h(w)^p]}{[F_h'(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} \leq \max\left\{\frac{h^q}{\gamma^p}, \frac{h^{\gamma-1}}{\gamma^p}w^p\right\} \in L^1_{\mathrm{loc}}(M)\,.$$

Hence, both conditions in (95) are satisfied. Setting again

$$G_h(s) = \int_{\Omega_{s_0} \cap B_s} F_h'(w)|A_u|^{p'}\,, \qquad \varphi_h(s) = \int_{\Omega_{s_0} \cap \partial B_s} \frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}$$

we can repeat once more the general argument to get that either $G_h(r) = 0$ or

$$\begin{cases} G_h(s) \geq G_h(r) > 0 & \forall s \in [r, R] \\ \dfrac{G_h'(s)}{[G_h(s)]^{p'}} \geq [\varphi_h(s)]^{1/(1-p)} & \text{for a.e. } s \in [r, R] \end{cases}$$

and in any case we get

$$\int_{B_r} F_h'(w)|A_u|^{p'} = G_h(r) \leq (p-1)^{p-1} \left( \int_r^R [\varphi_h(s)]^{1/(1-p)} \mathrm{d}s \right)^{1-p} . \tag{101}$$

We now let $h \to +\infty$ in both sides of (101). By Fatou's lemma we have

$$\liminf_{h \to +\infty} \int_{\Omega_{s_0} \cap B_r} F_h'(w)|A_u|^{p'} \geq \int_{\Omega_{s_0} \cap B_r} w^{\gamma-1}|A_u|^{p'} \equiv \int_{\Omega_{s_0} \cap B_r} w^{q-p}|A_u|^{p'} . \tag{102}$$

Concerning the RHS of (101), we aim at showing that

$$\lim_{h \to +\infty} \int_r^R [\varphi_h(s)]^{1/(1-p)} \mathrm{d}s = \int_r^R [\varphi(s)]^{1/(1-p)} \mathrm{d}s \tag{103}$$

with

$$\varphi(s) := \frac{1}{\gamma^p} \int_{\partial B_s} w^q .$$

From (100) and recalling that $\gamma + p - 1 = q$ we have

$$0 \leq \varphi_h(s) - \frac{1}{\gamma^p} \int_{\partial B_s \cap \{w<h\}} w^q \leq \frac{h^{\gamma-1}}{\gamma^p} \int_{\partial B_s \cap \{w \geq h\}} w^p . \tag{104}$$

Since $w \in W^{1,p}_{\mathrm{loc}}(M)$, for a.e. $s \in [r, R]$ we have $w \in L^p(\partial B_s)$ by the co-area formula. Then, using the monotone convergence theorem on the first integral in (104) together with the fact that $h^{\gamma-1} \to 0$ as $h \to +\infty$ (due to $\gamma < 1$) we get

$$\lim_{h \to +\infty} \varphi_h(s) = \frac{1}{\gamma^p} \int_{\partial B_s} w^q = \varphi(s) \qquad \text{for a.e. } s \in [r, R] . \tag{105}$$

If $\varphi^{1/(1-p)} \notin L^1([r, R])$, then by (105) and Fatou's lemma we have

$$\liminf_{h \to +\infty} \int_r^R \varphi_h^{1/(1-p)} \geq \int_r^R \varphi^{1/(1-p)} = +\infty$$

so (103) holds with both sides equalling $+\infty$. Suppose, instead, that $\varphi^{1/(1-p)} \in L^1([r, R])$. From the first line in (100) we also deduce the reversed estimate

$$\begin{aligned}\frac{[F_h(w)]^p}{[F_h'(w)]^{p-1}}\mathbf{1}_{\Omega_{s_0}} &\geq \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + h^{\gamma-1}w^p\mathbf{1}_{\{w\geq h\}} \\ &\geq \frac{w^{\gamma+p-1}}{\gamma^p}\mathbf{1}_{\{0<w<h\}} + w^{\gamma-1}w^p\mathbf{1}_{\{w\geq h\}} \\ &\geq w^{\gamma+p-1} = w^q\end{aligned}$$

where in the second inequality we exploited again the fact that $0 < \gamma < 1$. Then, for every $h > 0$ we also have $\varphi_h \geq \gamma^p \varphi$ and therefore

$$\varphi_h^{1/(1-p)} \leq \gamma^{-p/(p-1)}\varphi^{1/(1-p)} \qquad \text{on } [r, R]\,.$$

Hence, if $\varphi^{1/(1-p)} \in L^1([r, R])$ then (103) follows by the dominated convergence theorem. In any case, from the continuity of $[0, +\infty] \ni t \mapsto t^{1-p} \in [0, +\infty]$ with the agreement that $0^{1-p} = +\infty$ and $(+\infty)^{1-p} = 0$ we get

$$\lim_{h\to+\infty}\left(\int_r^R [\varphi_h(s)]^{1/(1-p)}\mathrm{d}s\right)^{1-p} = \left(\int_r^R [\varphi(s)]^{1/(1-p)}\mathrm{d}s\right)^{1-p}. \tag{106}$$

By (101), (102) and (106) we obtain the desired conclusion. □

From Proposition 17 we easily deduce the following lemma.

**Lemma 19** *Let $M$ be a complete, non-compact Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23). Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \geq 0 \qquad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\} \tag{107}$$

*for some $s_0 \in \mathbb{R}$ and also suppose that*

$$A(x, u, \nabla u) \neq 0 \qquad \text{on a set } E_0 \subseteq \Omega_{s_0} \text{ of positive measure.} \tag{108}$$

*Then there exists $r_0 \geq 0$ such that for any $q > p - 1$*

$$\int_r^R \left(\int_{\partial B_s}(u - s_0)_+^q\right)^{1/(1-p)} \mathrm{d}s < +\infty \qquad \forall\, r_0 < r < R < +\infty\,. \tag{109}$$

*In particular,*

$$\mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_r) > 0 \qquad \text{for a.e. } r > r_0 \tag{110}$$

*where $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure. Moreover, if $u_+ \in L^\infty_{\mathrm{loc}}(M)$ then also*

$$0 < \int_r^R \left(\mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s)\right)^{1/(1-p)} \mathrm{d}s < +\infty \qquad \forall\, r_0 < r < R < +\infty\,. \tag{111}$$

***Proof*** Choose $r_0 \geq 0$ such that $|B_r \cap E_0| > 0$ for every $r > r_0$, where $E_0$ is as in (108). Then, for every $r > r_0$

$$\int_{B_r \cap \Omega_{s_0}} w^{q-p} |A_u|^{p'} \geq \int_{B_r \cap E_0} w^{q-p} |A_u|^{p'} > 0$$

and then applying Proposition 17.(a) we see that the RHS of (87) must be strictly positive for any $R > r$, that is (since $1 - p < 0$),

$$\int_r^R \left( \int_{\partial B_s} w^q \right)^{1/(1-p)} \mathrm{d}s < +\infty \qquad \forall\, R > r\,.$$

In particular, $\left( \int_{\partial B_s} w^q \right)^{1/(1-p)}$ must be finite for a.e. $s > r$, hence for a.e. $s > r_0$ by arbitrariness of $r > r_0$, and therefore it must be $\int_{\partial B_s} w^q > 0$ for a.e. $s > r_0$, yielding (110). If $u_+ \in L^\infty_{\mathrm{loc}}(M)$, to prove (111) we start from the two-sided estimate

$$\mathcal{H}^{m-1}(\partial B_s) \geq \mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s) \geq \frac{1}{(1 + \operatorname{ess\,sup}_{B_R} w)^p} \int_{\Omega_{s_0} \cap \partial B_s} (1 + w)^p\,,$$

holding for each $s > r_0$, from which we deduce

$$\begin{aligned} \left( \mathcal{H}^{m-1}(\partial B_s) \right)^{1/(1-p)} &\leq \left( \mathcal{H}^{m-1}(\Omega_{s_0} \cap \partial B_s) \right)^{1/(1-p)} \\ &\leq (1 + \operatorname{ess\,sup}_{B_R} w)^{p/(p-1)} \left( \int_{\Omega_{s_0} \cap \partial B_s} (1 + w)^p \right)^{1/(1-p)} . \end{aligned}$$

The function $v(r) := \mathcal{H}^{m-1}(\partial B_r)$ satisfies

$$v(r) > 0 \quad \text{for } r > 0 \qquad \text{and} \qquad v, \frac{1}{v} \in L^\infty_{\mathrm{loc}}((0, +\infty)) \tag{112}$$

see Proposition 1.6 in [1], so we have

$$\int_r^R \left( \mathcal{H}^{m-1}(\partial B_s) \right)^{1/(1-p)} \mathrm{d}s > 0 \qquad \forall\, 0 < r < R$$

and by Proposition 17.(b) applied with the choice $f \equiv 1$ and $F(s) = 1 + s$ we get

$$\int_r^R \left( \int_{\partial B_s} (1 + w)^p \right)^{1/(1-p)} \mathrm{d}s < +\infty \qquad \forall\, r_0 < r < R\,.$$

Putting together all inequalities above we obtain (111). □

We are now ready for the proof of the main result of this section.

**Theorem 20** *Let $M$ be a complete, non-compact Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23). Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \geq 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}$$

*for some $s_0 \in \mathbb{R}$ and suppose that for some $x_0 \in M$ and $q > p - 1$ it holds*

$$\lim_{R \to +\infty} \int_r^R \left( \int_{\partial B_s(x_0)} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} \mathrm{d}s = +\infty \qquad \forall r > 0 . \tag{113}$$

*Then $A(x, u, \nabla u) = 0$ a.e. on $\Omega_{s_0}$. Thus, if $A$ satisfies the structural condition*

$$A(x, s, \xi) = 0 \quad \text{if and only if} \quad \xi = 0 . \tag{114}$$

*then either $u \equiv c$ a.e. on $M$ for some constant $c > s_0$, or $u \leq s_0$ a.e. on $M$.*

***Remark 21*** Condition (113) can be stated, more briefly, as

$$\left( \int_{\partial B_s} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} \notin L^1(+\infty)$$

with this notation meaning that the function $\varphi : (0, +\infty) \to [0, +\infty]$ given by

$$\varphi(s) = \left( \int_{\partial B_s} (u - s_0)_+^q \right)^{-\frac{1}{p-1}} \qquad \forall s > 0$$

is not in $L^1((r, +\infty))$ for any $r > 0$. The previous Lemma 19 implies that this is a meaningful condition, since in general only two cases are possible:

(i) $\varphi = +\infty$ a.e. on $(0, +\infty)$, and then $\Omega_{s_0}$ has zero measure while condition (113) is obviously satisfied, or
(ii) there exists $r_0 \geq 0$ such that $\varphi < +\infty$ a.e. on $(r_0, +\infty)$ and $\varphi \in L^1((r, R))$ for any $r_0 < r < R < +\infty$, so that (113) is satisfied if and only if $\varphi$ is not integrable in a neighborhood of $+\infty$.

Concerning case (ii), note that in general $\varphi$ may be integrable at $+\infty$ and still satisfy $\varphi = +\infty$ on $(0, r_0)$ for some $r_0 > 0$ (for instance, on $\mathbb{R}^n$ this may happen if $u$ satisfies $u \leq s_0$ on $B_{r_0}$ and $u(x) \geq |x|^a$ as $x \to \infty$ for some $a > (p - n)/q$), so the clause "$\forall r > 0$" in (113) cannot in general be replaced by "for some $r > 0$".

***Proof of Theorem 20*** Suppose, by contradiction, that $A_u := A(x, u, \nabla u)$ is non-zero on a set $E_0 \subseteq \Omega_{s_0}$ of positive measure. Then reasoning as in the proof of Lemma 19 we see that there exists $r > 0$ such that

$$\int_{\Omega_{s_0} \cap B_r} w^{q-p} |A_u|^{p'} > 0$$

and by Proposition 17 this implies that

$$\int_r^R \left( \int_{\partial B_s} (u - s_0)_+^q \right)^{1/(1-p)} ds \le \left( \frac{\min\{1, \gamma^p\}}{(p-1)^{p-1}} \int_{\Omega_{s_0} \cap B_r} w^{q-p} |A_u|^{p'} \right)^{1/(1-p)}$$

for all $R > r$, with $\gamma = q - p + 1$. Since the RHS of this inequality is finite, letting $R \to +\infty$ in the LHS we reach the desired contradiction. So, we conclude that $A(x, u, \nabla u) = 0$ a.e. on $\Omega_{s_0}$.

If $A$ satisfies the non-degeneracy condition (114) then we further deduce that $\nabla u = 0$ a.e. on $\Omega_{s_0}$, and since the function $w := (u - s_0)_+ \in W^{1,p}_{\mathrm{loc}}(M)$ has weak gradient $\nabla w = \mathbf{1}_{\Omega_{s_0}} \nabla u$ this yields $\nabla w \equiv 0$ a.e. on $M$. By connectedness of $M$ this implies that $w = a$ a.e. on $M$ for some constant $a \ge 0$. If $a > 0$ then $u = c := s_0 + a$ a.e. on $M$ (and $\Omega_{s_0}$ is of full measure), while if $a = 0$ then $u \le s_0$ a.e. on $M$ (and $\Omega_{s_0}$ has zero measure). □

As a consequence of Theorem 20 we have the following Liouville-type theorem.

**Corollary 22** *Let $M$ be a complete, non-compact Riemannian manifold, $p \in (1, +\infty)$ and $L$ a weakly $p$-coercive operator as in (23). Let $u \in W^{1,p}_{\mathrm{loc}}(M)$ satisfy*

$$Lu \ge 0 \quad \text{on } \Omega_{s_0} := \{x \in M : u(x) > s_0\}$$

*for some $s_0 \in \mathbb{R}$ and suppose that for some $x_0 \in M$ and $q > p - 1$ it holds*

$$\lim_{R \to +\infty} \int_r^R \left( \frac{s}{\int_{B_s} (u - s_0)_+^q} \right)^{\frac{1}{p-1}} ds = +\infty \qquad \forall r > 0 . \tag{115}$$

*Then $A(x, u, \nabla u) = 0$, and if $A$ satisfies the structural condition (114) then either $u \equiv c$ a.e. on $M$ for some $c > s_0$ or $u \le s_0$ a.e. on $M$.*

***Proof*** The corollary is a direct consequence of Theorem 20 since (115) implies (113). For the details, see the proof of Proposition 1.3 in [8] (the parameter $\delta$ there corresponds to $p - 1$ in our setting). □

***Remark 23*** Note, in particular, that (115) holds if

$$\int_{B_R} (u - s_0)_+^q = O(R^p) \qquad \text{as } R \to +\infty \tag{116}$$

or even if, for some $n \in \mathbb{N}$

$$\int_{B_R} (u - s_0)_+^q = O(R^p g_n^{p-1}(R)) \qquad \text{as } R \to +\infty . \tag{117}$$

where

$$g_n(t) = (\log t)(\log\log t) \cdots \underbrace{(\log\log \cdots \log t)}_{n \text{ iterations}} \qquad \text{for } t >> 1 .$$

**Acknowledgements** This research is part of the grant PID2021-124157NB-I00, funded by MCIN / AEI / 10.13039 / 501100011033/ "ERDF A way of making Europe", Spain, and is also supported by Comunidad Autónoma de la Región de Murcia, Spain, within the framework of the Regional Programme in Promotion of the Scientific and Technical Research (Action Plan 2022), by Fundación Séneca, Regional Agency for Science and Technology, REF. 21899/PI/22. The second named author wants to express gratitute for the hospitality provided by the Department of Mathematics of the University of Murcia, where part of this work was done.

**Funding** Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

**Data Availability** Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.



## References

1. Bianchini, B., Mari, L., Rigoli, M.: On some aspects of oscillation theory and geometry. Mem. AMS **225**, 1056 (2013)
2. Bisterzo, A., Farina, A., Pigola, S.: $L^p_{\mathrm{loc}}$ positivity preservation and Liouville-type theorems, available at arXiv:2304.00745
3. Bisterzo, A., Marini, L.: The $L^\infty$-positivity preserving property and stochastic completeness. Potential Anal. (2022). https://doi.org/10.1007/s11118-022-10041-w
4. Colombo, G., Mari, L., Rigoli, M.: Einstein-type structures, Besse's conjecture, and a uniqueness result for a $\varphi$-CPE metric in its conformal class. J. Geom. Anal. **32**(11), Paper No. 267 (2022)
5. D'Ambrosio, L., Mitidieri, E.: Quasilinear elliptic equations with critical potentials. Adv. Nonlinear Anal. **6**(2), 147–164 (2017)
6. Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Reprint of the 1998 edition. Classics in Mathematics. Springer, Berlin (2001)
7. Le, V.K.: On some equivalent properties of sub- and supersolutions in second order quasilinear elliptic equations. Hiroshima Math. J. **28**(2), 373–380 (1998)
8. Rigoli, M., Setti, A.G.: Liouville type theorems for $\varphi$-subharmonic functions. Rev. Mat. Iberoamericana **17**(3), 471–520 (2001)
9. Rindler, F.: Calculus of Variations. Universitext. Springer, Cham (2018)