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Effective conductivity in steady well-type flows through porous formations

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Abstract

A steady flow generated by a well of given strength takes place in a two-dimensional heterogeneous porous formation where the conductivity K is modeled as a random space function (RSF). As a consequence, the flow-variables become RSF s, and we wish to compute the effective conductivity (EC) by means of the self-consistent approximation. Toward this aim, the porous formation is sought as a collection of circular, non overlapping, inclusions with different (and statistically independent) conductivities. We compute the EC by adapting a procedure which was originally developed for mean uniform flows. Overall, the EC is found to be position-dependent, and therefore it can not be regarded as a medium's property (unless one is dealing with large distances from the well). Then, it is shown how results of the present study can be used for practical applications.

Keywords Porous media · Sink-flow · Heterogeneity · Stochastic modelling · Self-consistent approximation · Effective conductivity

1 Introduction

The classical equations of groundwater flow assume that the properties of the aquifer and of the flowing fluid(s) are regarded as constant over the entire domain (Bear 2013). However, in natural porous media these assumptions are not supported by measurements (see, e.g. Comegna et al. 2010). In particular, it is a rule (rather than an exception) that samples taken at different locations exhibit significantly different conductivity-values, even if samples are homogeneous at the laboratory scale (see, e.g. Dagan 1989; Rubin 2003; Severino et al. 2010). In addition, owing to several logistic and economic limitations, hydraulic properties can be measured only at a limited number of positions, and inferring parameters at points where measurements are not available entails random errors (see, e.g. Fallico et al. 2016). As matter of fact, these errors and uncertainties render the formation's properties RSF s, and concurrently the flow (and transport) variables become random fields (Severino et al. 2005).

One of the central problems in the environmental fluid mechanics is to determine the equations for the average flow variables. In particular, here we aim at deriving the effective conductivity $\mathbf{K}_{\text{eff}}$, relating the mean flux $\langle \mathbf{q} \rangle$ to the mean head gradient $\nabla \langle h \rangle$, i.e. $\langle \mathbf{q}(\boldsymbol{x}) \rangle = -\mathbf{K}_{\text{eff}} \nabla \langle h(\boldsymbol{x}) \rangle$ (effective Darcy law). Determining the EC has a long tradition starting from Beran (1968), and subsequently forwarded to several contexts ranging from electricity, wave scattering and the theory of elasticity (for a wide review, see Milton 2002; Torquato 2013, and references therein). In the mechanics of fluids in porous media, the same problem is traced back to Dagan (1989) in the case of linear flows, and more recently it has been also tackled in the context of nonlinear flows (Tartakovsky et al. 2003; Severino et al. 2003; Severino and Santini 2005; Severino and Coppola 2012).

In the present study, the EC in well-type flows is computed by means of the SCA. The physical model underlying the SCA regards the porous formation as a collection of many (homogeneous) inclusions set at random in space, and the fluctuation of the head field induced by each

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inclusion is computed by assuming that it is surrounded by a homogeneous matrix of unknown conductivity. Hence, the EC is computed by requiring that: “it is equal to the conductivity of the medium as a whole” (Sahimi 2003). In spite of their approximate nature, the simplicity of the results makes them particularly useful in comparison with those attained by numerical simulations, and (more important) they lead to simple estimates of statistical flow properties with no limitation imposed on the magnitude of the variance σ_Y^2 of the log-conductivity $Y \equiv \ln K$.

The simplest case, largely documented both in the literature of composites (Milton 2002; Torquato 2013) and in the literature of porous media (Dagan 1979, 1981; Renard and De Marsily 1997; Fiori et al. 2003), is that of steady flows which are uniform in the average (see, e.g. Zarlenga et al. 2018). However, in the presence of spatially distributed sources, the assumption of mean uniform flow is not anymore applicable. Hence, the natural question is whether one can derive also for source/sink flows a self-consistent approximation of the EC. This problem has received little attention in the literature, its importance for the applications, notwithstanding.

With the exception for a few exact closed formulae, all results concerning the computation of the EC were obtained under various assumptions about the fluctuations of the log-conductivity field Y, and they are quoted in a few papers (e.g. Indelman 1996, 2001; Tartakovsky et al. 2003; Severino 2011b), and monographs (e.g. Dagan 1989; Rubin 2003). In his pioneering studies, Shvidler (1964) has developed a perturbation procedure valid for $\sigma_Y^2 \ll 1$, and he has obtained two asymptotics for the EC valid close and far from the well. These results were subsequently refined by Severino (2011b) who investigated the transitional behavior of the EC from the near to the far field. The main conclusion, in line with the fundamental results of Indelman and Abramovich (1994) and Indelman (2001), was that a local (i.e. depending only upon the statistics of the conductivity) EC can not be defined.

The plan of the paper is as follows: we present the problem at stake; we focus on the derivation of the governing equation for the EC; we discuss the general properties, and in particular we highlight the potential applications of our results; we end up with concluding remarks.

2 Problem statement

A steady incompressible flow takes place into a two-dimensional unbounded porous medium, and it is generated by a “point-like” (representing a well at regional scale) of strength $\mathcal{Q} \equiv Q/(2\pi)$, being Q the discharge for unit length of well. The constitutive model and continuity equation write as

$$\mathbf{q}(\boldsymbol{x}) = -\exp\left[Y(\boldsymbol{x})\right]\nabla h(\boldsymbol{x}), \qquad \nabla\cdot\mathbf{q}(\boldsymbol{x}) = -\mathcal{Q}\,\delta(\boldsymbol{x}), \qquad Y(\boldsymbol{x}) \equiv \ln K(\boldsymbol{x}), \tag{1}$$

where $\mathbf{q} \equiv (q_1, q_2)^\top$ is the (*Darcy*) flux, h is the hydraulic head (energy per unit weight), and $\boldsymbol{x} \equiv (x_1, x_2) \in \mathbb{R}^2$ is the position. This potential flow provides solution to steady flow to wells in order to identify the aquifer’s parameters (Gómez et al. 2009; Fallico et al. 2018). Moreover, techniques of superposition (such as an injecting well superimposed on a uniform flow field) enable one to tackle quality problems in which only advective solute flux is considered (a comprehensive review can be found in Jankovic et al. 2017, and references therein).

We assume that the porous formation is of random and stationary Gaussian Y with mean $\langle Y \rangle$ and variance σ_Y^2. By following the stand point of the SCA (a comprehensive exposition can be found in Kanaun and Levin 2007), the porous formation is sought as a collection of a large number of randomly arranged, homogeneous, non overlapping inclusions of different conductivity-values. Then, by invoking the ergodic argument one can replace the above formation with the ensemble average, and therefore the simultaneous interaction among the numerous inclusions can be approached by focusing upon a single one implanted into a medium homogenized by a background (unknown at the moment) conductivity. A major simplification, which facilitates the derivation of a simple (closed form) expression for the flow-field, is to represent inclusions by regular, well defined, shapes. Nevertheless, the use of such a shape does not constitute a severe limitation since it leads to results which match very well numerical simulations (Jankovic et al. 2003). For this reason, we deal hereafter with a porous medium which is sought as a collection of circular inclusions. In addition, due to the isotropy of the porous formation, the EC is a scalar in this case.

We consider a single inclusion Ω_0 (with conductivity K_0) whose position and size are $\boldsymbol{r}_0$ and $\mathcal{R}_0$, respectively (Fig. 1). For such a "single realization", we consider the flux $\mathbf{q}$, and concurrently the fluctuation

$$\mathbf{q}'(\boldsymbol{x}) = \mathbf{q}(\boldsymbol{x}) + K_\infty \nabla h(\boldsymbol{x}), \tag{2}$$

where K_∞ (unknown for the moment) is the conductivity of the background $\Omega_\infty \equiv \mathbb{R}^2/\Omega_0$ (Fig. 1). The decomposition (2) is now averaged over all possible positions, sizes, and conductivities to come up with

$$\langle \mathbf{q}(\boldsymbol{x}) \rangle = -K_\infty \nabla \langle h(\boldsymbol{x}) \rangle + \langle \mathbf{q}'(\boldsymbol{x}) \rangle, \tag{3}$$

which suggests into a straightforward manner that the self-consistent requirement is such that

$$\langle \mathbf{q}'(\boldsymbol{x}) \rangle \equiv 0, \qquad \mathrm{K}_{eff} \equiv K_\infty. \tag{4}$$

Hence, it is clear that the crux of the matter consists into solving a sink flow as disturbed by a single inclusion.

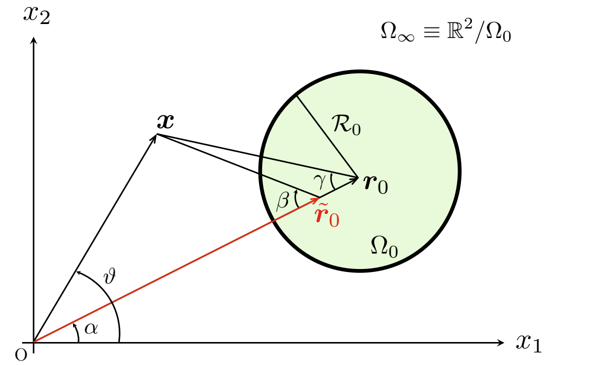

Fig. 1 A circular inclusion Ω_0 of radius $\mathcal{R}_0$, and center implanted at $\boldsymbol{r}_0 \equiv r_0(\cos\alpha, \sin\alpha)$. The position of any point belonging to the flow domain $\mathbb{R}^2 \equiv \Omega_\infty \cup \Omega_0$ is represented by the vector $\boldsymbol{x} \equiv x(\cos\vartheta, \sin\vartheta)$, whereas $\tilde{\boldsymbol{r}}_0$ is a vector parallel to $\boldsymbol{r}_0$

2.1 Sink-flow in the presence of an inclusion of conductivity different from the matrix

We consider a steady flow toward a point-like sink located at the origin. A permeable circular inclusion Ω_0 (of conductivity K_0 and radius $\mathcal{R}_0$) is implanted at $\boldsymbol{r}_0$, being $r_0 > \mathcal{R}_0$ (Fig. 1). As such, the origin belongs to $\Omega_\infty \equiv \mathbb{R}^2/\Omega_0$ for any choice of the pair $(r_0, \mathcal{R}_0)$, whereas the flow domain consists of two sub-domains, i.e. $\mathbb{R}^2 \equiv \Omega_\infty \cup \Omega_0$, with Ω_0 separating the portion of $\mathbb{R}^2$ laying within the inclusion from the external domain Ω_∞ (of conductivity $K_\infty \neq K_0$). The analytical solution of this flow was derived by Wheatcraft and Winterberg (1985). In the "Appendix" we generalize such a solution by accounting for any anomaly α of the vector $\boldsymbol{r}_0$ (correcting also for an error and a few typos appearing in the paper of Wheatcraft and Winterberg 1985). Thus, the hydraulic head h and the stream function ψ are:

$$\psi(\boldsymbol{x}) = \cancel{Q} \begin{cases} \vartheta - \dfrac{1-\kappa}{1+\kappa}(\beta - \gamma) & \boldsymbol{x} \in \Omega_\infty \\[2ex] \dfrac{2\,\kappa}{1+\kappa}\vartheta + \dfrac{1-\kappa}{1+\kappa}\alpha & \boldsymbol{x} \in \Omega_0 \end{cases} \tag{6}$$

where $\kappa \equiv K_0/K_\infty$ is the *contrast ratio*, and $\tilde{\boldsymbol{r}}_0 \equiv \left[1 - (\mathcal{R}_0/r_0)^2\right]\boldsymbol{r}_0$ is a vector parallel to $\boldsymbol{r}_0$. For $K_\infty \equiv K_0$ (i.e. $\kappa \equiv 1$) one has $h \sim \ln x$ and $\psi \sim \vartheta$, in agreement with the well known result valid for homogeneous media. The same conclusion is drawn at large distances from the inclusion (i.e. $x \gg r_0$), since in the far field the disturbance due to Ω_0 becomes immaterial. The scaled flow-net $(h_s, \psi_s) \equiv (h\,K_\infty, \psi)/\cancel{Q}$, as computed by Eqs. (5)–(6), is depicted in the Fig. 2. For low values of κ the stream lines $\psi_s \equiv$ const tend to circumvent the inclusion (and concurrently the curves $h_s \equiv$ const get denser inside the

$$h(\boldsymbol{x}) = \frac{\cancel{Q}}{K_\infty} \begin{cases} \ln x + \dfrac{1-\kappa}{1+\kappa} \ln\left(\left| \dfrac{\boldsymbol{x} - \tilde{\boldsymbol{r}}_0}{\boldsymbol{x} - \boldsymbol{r}_0} \right| r_0 \right) - \dfrac{2}{1+\kappa} \ln(r_0 - \mathcal{R}_0) & \boldsymbol{x} \in \Omega_\infty \\[2ex] \dfrac{2}{1+\kappa} \ln\left(\dfrac{x}{r_0 - \mathcal{R}_0} \right) & \boldsymbol{x} \in \Omega_0 \end{cases} \tag{5}$$

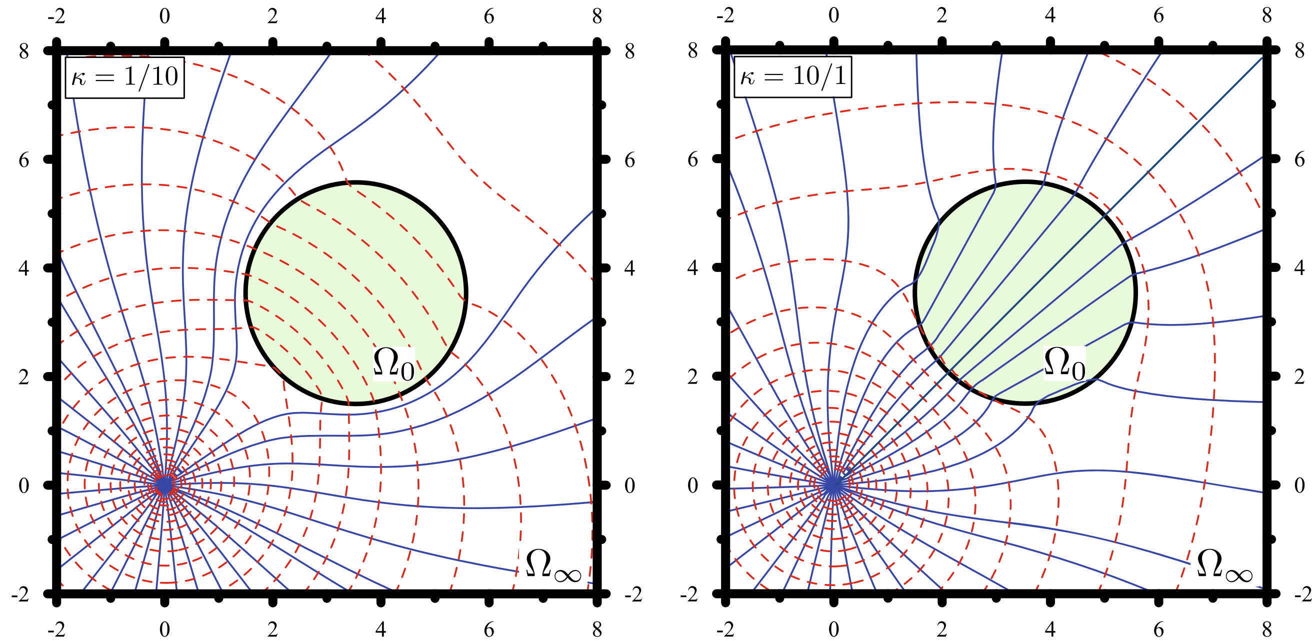

Fig. 2 Contour-plot of the scaled hydraulic head h_s (red dashed lines), and stream function ψ_s (blue continuous lines) for two largely different values of the contrast ratio κ. The center of the inclusion Ω_0 is at $\boldsymbol{r}_0 \equiv 5(\cos \pi/4, \sin \pi/4)$

inclusion). The behavior of the iso-heads and stream lines is completely reversed for large κ-values.

3 Discussion

In the fluid mechanics of porous media, the implementation of the SCA was applied by Dagan (1989) to mean uniform flows. In the present study, we have adapted the SCA to a sink-type flow leading to the condition (4). Hence, the resulting effective (constitutive) flow model, together with the mass conservation law, i.e. $\nabla \cdot \langle \mathbf{q}(\boldsymbol{x}) \rangle = -\mathcal{Q}\, \delta(\boldsymbol{x})$, can be used to solve several problems of practical concern (see, e.g. Severino 2011a; Severino et al. 2011, 2012, and references therein).

In order to derive an equation for K_{eff}, we consider the mass conservation of the ensemble average of the fluctuation which, according to the first of (4), can be written as

$$\left\langle \int_{\Omega_R} \mathrm{d}\boldsymbol{x}\, \nabla \cdot \mathbf{q}'(\boldsymbol{x}) \right\rangle = \mathbf{0}, \tag{7}$$

being Ω_R a large circle of radius $R \gg r_0$ surrounding the inclusion Ω_0, whereas the fluctuation $\mathbf{q}'$ is computed from (5) as

$$\mathbf{q}' \equiv \mathbf{q} + \mathcal{Q}\, \nabla \ln x = -\mathcal{Q}\, \frac{1-\kappa}{1+\kappa} \nabla \begin{cases} \ln \left| \dfrac{\boldsymbol{x} - \tilde{\boldsymbol{r}}_0}{\boldsymbol{x} - \boldsymbol{r}_0} \right| & \boldsymbol{x} \in \Omega_\infty \\ -\ln |\boldsymbol{x}| & \boldsymbol{x} \in \Omega_0. \end{cases} \tag{8}$$

The integral appearing into (7) is computed by applying Green's theorem and accounting for the fact that in the limit $R \to \infty$ the exterior term drops out, i.e.

with $\vartheta_0 \equiv \mathcal{R}_0/r_0 \leq 1$. Hence, from Eq. (7) one finally ends up with

$$
\int_{\Omega_R} \mathrm{d}\boldsymbol{x}\, \nabla \cdot \mathbf{q}'(\boldsymbol{x}) \overset{R\to\infty}{=} \mathcal{Q}\, \frac{1-\kappa}{1+\kappa} \int_{\partial\Omega_0} \mathrm{d}s\, \boldsymbol{e}_0 \cdot \nabla \left(\ln \left| \frac{\boldsymbol{x}+\boldsymbol{r}_0-\tilde{\boldsymbol{r}}_0}{\boldsymbol{x}} \right| + \ln |\boldsymbol{x}+\boldsymbol{r}_0| \right) = \\
\int_0^{2\pi} \mathrm{d}\alpha \cos\alpha \left(\frac{1+\vartheta_0 \cos\alpha}{1+\vartheta_0^2+2\vartheta_0\cos\alpha} + \frac{1+\vartheta_0^{-1}\cos\alpha}{1+\vartheta_0^{-2}+2\vartheta_0^{-1}\cos\alpha} \right) = -2\pi \mathcal{Q}\, \frac{1-\kappa}{1+\kappa}\, \vartheta_0, \tag{9}
$$

$$
\left\langle \frac{\mathrm{K}_{eff}-K_0}{\mathrm{K}_{eff}+K_0}\, \vartheta_0 \right\rangle = 0, \tag{10}
$$

where the ensemble average is sought over all possible values of the RSF s K_0 and ϑ_0. The nonlinear Eq. (10), representing the main achievement of the present study, enables one to determine $\mathrm{K}_{\mathrm{eff}}$ once the joint probability distribution $f(K_0, \vartheta_0)$ is selected. The most important feature which is detected from (10) is the dependence of $\mathrm{K}_{\mathrm{eff}}$ upon the (relative) distance ϑ_0, that prevents *de facto* considering $\mathrm{K}_{\mathrm{eff}}$ as a local medium's property. Such a result was also obtained by Indelman and Abramovich (1994) and subsequently by Severino (2011b) via a perturbation approach which regards the variance σ_Y^2 as a small parameter (first-order approximation, FOA). Here, the same conclusion is extended to any formation (irrespective of the magnitude of σ_Y^2).

At this stage, it is worth recapitulating the approximations underlying Eq. (10): (1) the matrix surrounding each block can be replaced by a homogeneous background of conductivity K_∞. This approximation is bound to be quite accurate if one deals with a multiphase material, and (more important) if interactions between inclusions are not accounted for (at least into a direct manner). (2) Blocks are circular, which is again a quite accurate approximation for an isotropic formation. (3) The averaging domain is large compared to that of the inclusion.

Although the assumptions (1)–(3) are clearly an approximation of the model, they nevertheless do not limit the accuracy of the final result, as it was assessed by Jankovic et al. (2003) by means of very accurate numerical simulations, and by Hashin and Shtrikman (1962) in a completely different context. For illustration purposes, we consider $\boldsymbol{r}_0$ as a constant so that Eq. (10) writes as:

$$
\int_0^\infty \int_0^{r_0} \mathrm{d}K_0\, \mathrm{d}\mathcal{R}_0 f(K_0, \mathcal{R}_0) \frac{\mathrm{K}_{eff}-K_0}{\mathrm{K}_{eff}+K_0}\, \mathcal{R}_0 = 0, \tag{11}
$$

which is easily solved for given bivariate probability distribution function $f \equiv f(K_0, \mathcal{R}_0)$. In particular, it allows to investigate the flow behavior in both the near and the far field. More precisely, close to the sink Eq. (11) reads as:

$$
\int_0^\infty \mathrm{d}K_0 f(K_0, r_0) \frac{\mathrm{K}_{\mathrm{eff}}-K_0}{\mathrm{K}_{\mathrm{eff}}+K_0} \simeq 0 \qquad (\text{small } \; r_0), \tag{12}
$$

which clearly shows the non locality of the EC. At the other extreme of large r_0, one has:

$$
\int_0^\infty \int_0^\infty \mathrm{d}K_0\, \mathrm{d}\mathcal{R}_0 f(K_0, \mathcal{R}_0) \frac{\mathrm{K}_{\mathrm{eff}}-K_0}{\mathrm{K}_{\mathrm{eff}}+K_0}\, \mathcal{R}_0 = 0. \tag{13}
$$

In this case the EC is not anymore a function of the position, and therefore one can claim that in the far field $\mathrm{K}_{\mathrm{eff}}$ is a medium's property, in agreement with the results valid for weakly heterogeneous formations (Severino et al. 2008; Severino 2011b). In the sequel, we focus on the case of lack of correlation between K_0 and ϑ_0, which translates (10) into

$$
\left\langle \frac{\mathrm{K}_{\mathrm{eff}}-K_0}{\mathrm{K}_{\mathrm{eff}}+K_0} \right\rangle \equiv \int_0^\infty \mathrm{d}K_0 f(K_0) \frac{\mathrm{K}_{\mathrm{eff}}-K_0}{\mathrm{K}_{\mathrm{eff}}+K_0} = 0. \tag{14}
$$

The nonlinear Eq. (14) coincides with that valid for mean uniform flows (Dagan 1989). Nevertheless, it is known (-see, e.g. Indelman 2001; Severino et al. 2008) that away from the sink (far field) the flow behaves like a mean uniform one, and therefore we conclude that the neglect of correlation between K_0 and ϑ_0 is a good working hypothesis when one is interested in the far field behavior. Moreover, it has been recently demonstrated by Di Dato et al. (2017) that the above approximation works quite well even in the vicinity of the sink.

The normalized $\mathrm{K}_{\mathrm{eff}}/K_G$ (being $K_G \equiv \exp\langle Y \rangle$ the geometric mean), as computed by (14), is depicted in the -Fig. 3. This latter has been produced by adopting the lognormal model for the univariate probability density function $f(K_0)$ (see the data-survey in Rubin 2003). To compare with the results known for weakly heterogeneous media, we have also depicted $\mathrm{K}_{\mathrm{eff}}/K_G = \exp\left(-\sigma_Y^2/2\right) + 3\sinh\left(\sigma_Y^2/2\right)$ which is valid for small σ_Y^2 (Severino 2011b). The striking (and quite evident) result is that the FOA provides a lower bound for $\mathrm{K}_{\mathrm{eff}}$ due to the neglect of the higher order effects in the effective medium behavior. In particular, the SCA is found to be in close agreement with the FOA till to $\sigma_Y^2 \lesssim 0.2$. The insert displays the dependence

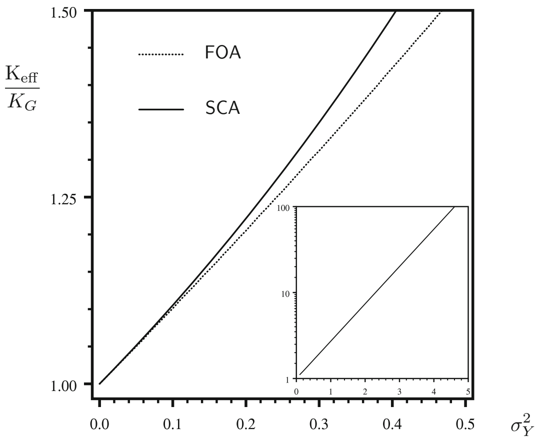

Fig. 3 The dimensionless effective conductivity K_{eff}/K_G according to: (i) the SCA (continuous line), and (ii) the FOA (dashed line) as a function of the log-conductivity variance σ_Y^2. The insert shows the dependence of K_{eff}/K_G upon σ_Y^2 beyond the validity of the FOA

of the EC upon σ_Y^2 in the regime where the FOA does not apply.

4 Concluding remarks

The behavior of flow variables in porous media depends to a great extent upon the formation's heterogeneity. One of the central problem pertains to the computation of the EC, that has been intensively studied for flows uniform in the mean (Dagan et al. 2013). However, there are numerous applications (such as well-type flows) for which the above flow conditions do not apply. Despite the importance for the applications, and with the exception of a series of papers all dealing with weakly heterogeneous media (see, e.g. Severino et al. 2008; Severino 2011b, and references therein), to our knowledge there are no analytical studies aiming at deriving the EC for a formation of any degree of heterogeneity.

The focus of this study was the computation of the EC for a steady sink-type flow taking place in a heterogeneous medium whose local conductivity K is a stationary, and isotropic RSF. To this aim, we have adapted to the problem at stake the SCA, which was originally employed for mean uniform flows. Thus, we have first revisited the flow as disturbed by a single circular inclusion, and subsequently we have derived an approximate equation which enables one to compute the EC for any σ_Y^2. The main result, in agreement with previous studies dealing with weakly heterogeneous media (see, e.g. Indelman and Abramovich 1994), is that K_{eff} is not a local property. Nevertheless, it is shown that the far field limit, authorizing to regard the EC as a local property, is a robust tool to tackle problems of practical concern.

The approximate procedure developed in the present study can be extended to three dimensional flows, as well. This is part of ongoing research projects. To conclude, this paper is a preliminary investigation toward the computation of the EC in non uniform sink/source flows. As such, it is hoped that it will stimulate further developments especially with respect to the complex problem of transport in highly heterogeneous porous formations. At the occurrence, it is worth noting that, although a 2D flow solution seems to be limiting, it can be applied to three dimensional formations after employing the adapting methodology of Severino et al. (2011).

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Appendix: sink-type flow disturbed by a single circular inclusion of conductivity different from the background

For the purposes of the present study, we generalize here the approximate analytical solution of Wheatcraft and Winterberg (1985) by allowing for any position $\boldsymbol{r}_0$ of the circular inclusion (Fig. 1). The starting point is the well-known solution (Milne-Thomson 1968, ch.8.61) of a flow past a circular obstacle of radius $\mathcal{R}_0$ located at $z_0 = r_0 \exp(i\alpha)$ in the complex plane $\mathbb{C} = \{z : \mathrm{Re}z \equiv x_1, \mathrm{Im}z \equiv x_2\}$. With the sink at the origin, the complex potential $w^\star \equiv w^\star(z)$ is determined by the *circle-theorem* (Milne-Thomson 1940) as

$$w^\star(z) = \mathcal{Q}\ln z + \mathcal{Q}\ln\left(z - z_0'\right) - \mathcal{Q}\ln(z - z_0) + C^\star \tag{15}$$

being $z_0' = [1 - (\mathcal{R}_0/r_0)^2]z_0 \equiv |\mathrm{OA}|(\cos\alpha, \sin\alpha)$ (Fig. 1). The arbitrary constant $C^\star$ is generally a complex number. Thus, according to (15) the flow generated by a sink at $z \equiv 0$, and disturbed by a circular obstacle implanted at $z = z_0$, can be sought as the superposition of flows determined by: i) a real sink at $z \equiv 0$, ii) and a system of fictitious sink/source of equal strength at $z = z_0'$ and $z = z_0$, respectively. Since $\mathcal{R}_0 < r_0$, the system of fictitious sink/-source lies inside the obstacle, and it acts such that no fluid particle enters/leaves it in order to honor the boundary condition of impermeable inclusion.

For a permeable inclusion (with conductivity K_0), the flow domain consists of two sub-domains, i.e. $\mathbb{R}^2 \equiv \Omega_\infty \cup \Omega_0$, where $\Omega_0 = \{z \in \mathbb{C} : |z - z_0| \leqslant \mathcal{R}_0\}$ separates the portion of $\mathbb{R}^2$ inside the inclusion from the external domain $\Omega_\infty = \mathbb{R}^2/\Omega_0$ (of conductivity $K_\infty \neq K_0$). and ii) to preserve energy and mass. To extract the real and the imaginary part of the complex potentials w_∞ and w_0, it is convenient to represent the (polar) coordinates of each source and sink as follows

$$z = |z| \exp(i\vartheta), \quad z - z_0' = |z - z_0'| \exp(-i\beta), \\ z - z_0 = |z - z_0| \exp(-i\gamma). \tag{19}$$

This leads to the general expression for the hydraulic head and the stream function

$$h_\infty = K_\infty^{-1} \left[\mathcal{Q} \ln|z| + Q_\infty \ln \left| \frac{z - z_0'}{z - z_0} \right| + \mathrm{Re}\,(C_\infty) \right], \quad h_0 = K_0^{-1} \left[Q_0 \ln|z| + \mathrm{Re}\,(C_0) \right] \tag{20}$$

$$\psi_\infty = \mathcal{Q}\,\vartheta - Q_\infty\,(\beta - \gamma) + \mathrm{Im}\,(C_\infty)\,, \qquad \psi_0 = Q_0\,\vartheta + \mathrm{Im}\,(C_0)\,. \tag{21}$$

In this case, the flow inside Ω_∞ is still generated by a sink at $z \equiv 0$ of strength equal to $\mathcal{Q}$, whereas the system of fictitious sink/source system is now characterized by a strength $Q_\infty \neq \mathcal{Q}$ (unknown at the moment), i.e.

Hence, by requiring that $h_\infty \equiv h_0$ and $\psi_\infty \equiv \psi_0$ along the boundary $\partial\Omega_0$ one has

$$(\kappa\,\mathcal{Q} - Q_0) \ln \overline{\mathrm{PO}} + \kappa\, Q_\infty \ln \left(\frac{\overline{\mathrm{PA}}}{\overline{\mathrm{PC}}} \right) = \mathrm{Re}\,(C_0 - \kappa\, C_\infty) \qquad (\kappa \equiv K_0/K_\infty) \tag{22}$$

$$w_\infty(z) = \mathcal{Q} \ln z + Q_\infty \ln\,(z - z_0') - Q_\infty \ln(z - z_0) + C_\infty. \tag{16}$$

Inside the inclusion the flow is driven by the sink at the origin with unknown strength $Q_0 \neq \mathcal{Q}$, i.e.

$$w_0(z) = Q_0 \ln z + C_0. \tag{17}$$

The values of Q_∞ and Q_0 are determined by requiring the continuity of: i) the hydraulic head h (computed by dividing the real part of the complex potential by the medium conductivity), and ii) the stream function ψ:

$$h(\boldsymbol{x}) = \begin{cases} h_\infty \equiv K_\infty^{-1}\,\mathrm{Re}(w_\infty) & \boldsymbol{x} \in \Omega_\infty \\ h_0 \equiv K_0^{-1}\,\mathrm{Re}(w_0) & \boldsymbol{x} \in \Omega_0, \end{cases} \qquad \psi(\boldsymbol{x}) = \begin{cases} \psi_\infty \equiv \mathrm{Im}(w_\infty) & \boldsymbol{x} \in \Omega_\infty \\ \psi_0 \equiv \mathrm{Im}(w_0) & \boldsymbol{x} \in \Omega_0, \end{cases} \tag{18}$$

along the boundary $\partial\Omega_0$ of the inclusion. In other words, the system of sink/source inside the inclusion acts now such that: i) to allow the flow passing through the inclusion,

$$(\mathcal{Q} - Q_0)\vartheta - Q_\infty(\beta - \gamma) = \mathrm{Im}\,(C_0 - C_\infty)\,. \tag{23}$$

Since, according to the circle-theorem, A and O are inverse points with respect to the circle Ω_0 (ch. 5.15 in Milne-Thomson 1968), it results $\overline{\mathrm{CA}} \cdot \overline{\mathrm{CO}} = \overline{\mathrm{CP}}^2$ (see Fig. 4), and concurrently triangles PCA and PCO are similar (by virtue of the second criterium of similarity). This implies that:

$$\vartheta - \alpha = \beta - \gamma. \tag{24}$$

The application of the law of sines to the sides: i) PC and PA of the triangle PCA, and ii) PO and CO of the triangle PCO leads to:

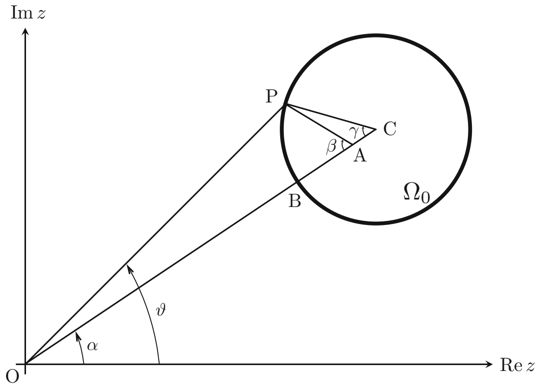

Fig. 4 Geometrical sketch leading to Eqs. (24)–(25). The real sink is at the origin O, whereas the fictitious sink and source (both laying inside the inclusion Ω_0) are at the points A and C, respectively

$$\frac{\overline{\mathrm{PA}}}{\overline{\mathrm{PC}}} = \frac{\overline{\mathrm{PO}}}{\overline{\mathrm{CO}}}. \tag{25}$$

The relationship (25) is known as "the inversion theorem". Insertion of (24)–(25) into (22)–(23), and choosing the arbitrary constants $\mathrm{Re}(C_0 - \kappa\, C_\infty)$ and $\mathrm{Im}(C_0 - C_\infty)$ such that:

$$\mathrm{Re}(C_0 - \kappa\, C_\infty) = -\kappa\, Q_\infty \ln \overline{\mathrm{CO}}, \qquad \mathrm{Im}(C_0 - C_\infty) = \alpha\, Q_\infty, \tag{26}$$

gives

$$Q_\infty = \frac{1-\kappa}{1+\kappa}\, \mathcal{Q}, \qquad Q_0 = \frac{2\kappa}{1+\kappa}\, \mathcal{Q} \tag{27}$$

(in agreement with Wheatcraft and Winterberg 1985). The constant C_∞ is determined by specifying w_∞ at an arbitrary point belonging to $\Omega_\infty \equiv \mathbb{R}^2/\Omega_0$. In particular, the algebra is simplified by requiring that $w_\infty \equiv i\alpha\mathcal{Q}$ at $\mathrm{B} \equiv (\mathrm{r}_0 - \mathcal{R}_0)(\cos\alpha, \sin\alpha)$ (see Fig. 1). From (19) one has the following values:

$$|z| \equiv r_0\left(1 - \frac{\mathcal{R}_0}{r_0}\right), \quad |z - z_0'| \equiv \mathcal{R}_0\left(1 - \frac{\mathcal{R}_0}{r_0}\right), \tag{28}$$
$$|z - z_0| \equiv \mathcal{R}_0, \quad \vartheta \equiv \alpha, \quad \beta \equiv \gamma = 0,$$

which lead to:

$$C_\infty = -Q_\infty\left[\frac{2}{1-\kappa}\ln(r_0 - \mathcal{R}_0) - \ln r_0\right]. \tag{29}$$

The left constant C_0 is straightforwardly determined by the second of (26) and (29), i.e.

$$C_0 = -Q_\infty\left[\frac{2\kappa}{1-\kappa}\ln(r_0 - \mathcal{R}_0) - i\alpha\right]. \tag{30}$$

Summarizing, the complex potentials (16)–(17) are determined uniquely by means of (27), and (29)–(30). The quantity $\mathrm{Re}\,(w)/K_\infty$ and (21) provide eqs (5)–(6).

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