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Mathematical derivation and analysis of a mixture model of tumor growth

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ABSTRACT

We derive, through the periodic homogenization theory in thin heterogeneous domains, a $2D$ model consisting of Hele-Shaw equation coupled with the convective Cahn-Hilliard equation with non-constant mobility. The upscaled set of equations, which models in particular tumor growth, is then analyzed and we prove some regularity results. We heavily rely on the two-scale convergence concept in thin heterogeneous media associated to some Sobolev inequalities such as the Gagliardo-Nirenberg and Agmon inequalities to achieve our goal.

1. Introduction and the main result

Mathematical modelling of biological processes is widely used to enhance quantitative understanding of biomedical phenomena. This is particularly relevant in the area of cancer biology, taking into account the huge rate of prevalence of cancer in the world. As seen in [1], there are several mathematical models of tumor growth ranging from simple models attempting to simulate the tumor growth volume, to very sophisticated ones combining many biologically important molecular processes. These models are built and developed from the following three standpoints: microscopic or discrete ones, macroscopic or continuous ones, and hybrid or micro-macroscopic ones.

In this work we focus on the hybrid model. Indeed, due to the complexity of the biology, there is a need to link microscopic models to macroscopic ones in order to get efficient models. This imposes the use of multiscale models instead of just macroscopic ones. We start from a microscopic model belonging to the class of diffuse interface models that are used to describe the behavior of multi-phase fluids. To be more precise, our micro-model reads as follows.

Let Ω be a bounded open Lipschitz domain in $\mathbb{R}^2$ and let $h_1, h_2 \in W^{1,\infty}(\mathbb{R}^2) \cap C_{\text{per}}(Y)$ with $Y = (0,1)^2$, where $C_{\text{per}}(Y)$ stands for the space of Y-periodic continuous functions in $\mathbb{R}^2$. For $\varepsilon > 0$, we define the thin heterogeneous domain Ω_ε in $\mathbb{R}^3$ by

$$\Omega_\varepsilon = \left\{(\overline{x}, x_3) \in \mathbb{R}^3 \, : \, \overline{x} \in \Omega \text{ and } \varepsilon h_1^\varepsilon(\overline{x}) < x_3 < \varepsilon h_2^\varepsilon(\overline{x})\right\},$$

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where $h_i^\varepsilon(\overline{x}) = h_i(\overline{x}/\varepsilon)$ for $\overline{x} \in \Omega$, $i = 1,2$, and $\max_{\overline{Y}} h_1 < \min_{\overline{Y}} h_2$ and $0 \in [\min_{\overline{Y}} h_1, \max_{\overline{Y}} h_2]$. In the thin layer Ω_ε, the flow of two-phase immiscible fluids at the micro-scale is described by the following Stokes-Cahn-Hilliard system with non-constant mobility:

$$\begin{cases} -\alpha\varepsilon^2\Delta\mathbf{u}_\varepsilon + \nabla p_\varepsilon - \mu_\varepsilon\nabla\varphi_\varepsilon = \mathbf{g} \text{ in } Q_\varepsilon = (0,\infty)\times\Omega_\varepsilon, \\ \operatorname{div}\mathbf{u}_\varepsilon = 0 \text{ in } Q_\varepsilon, \\ \dfrac{\partial\varphi_\varepsilon}{\partial t} + \mathbf{u}_\varepsilon\cdot\nabla\varphi_\varepsilon - \operatorname{div}(m^\varepsilon(\cdot,\varphi_\varepsilon)\nabla\mu_\varepsilon) = 0 \text{ in } Q_\varepsilon, \\ \mu_\varepsilon = -\beta\Delta\varphi_\varepsilon + \lambda F'(\varphi_\varepsilon) \text{ in } Q_\varepsilon, \\ \dfrac{\partial\mu_\varepsilon}{\partial\nu} = 0,\ \dfrac{\partial\varphi_\varepsilon}{\partial\nu} = 0 \text{ and } \mathbf{u}_\varepsilon = 0 \text{ on } (0,\infty)\times\partial\Omega_\varepsilon, \\ \varphi_\varepsilon(0,x) = \varphi_0^\varepsilon(x) \text{ in } \Omega_\varepsilon, \end{cases} \tag{1.1}$$

where α, β and λ are positive fixed parameters, and ν is a unit outward normal to $\partial\Omega_\varepsilon$. Here, $\mathbf{u}_\varepsilon$, p_ε, φ_ε and μ_ε are respectively the unknown velocity, pressure, the order parameter and the chemical potential; m^ε is the oscillating mobility, $\mathbf{g}$ stands for an external force density acting on the fluid mixture and F is the configuration potential accounting for the presence of two phases. The order parameter φ_ε is the difference of the fluid relative concentrations and usually takes values between -1 and 1. However in the case of the assumption (1.3) below on the function F, φ_ε may take any value in $\mathbb{R}$. In (1.1), ∇ (resp. div and Δ) denotes the usual gradient (resp. divergence and Laplacian) operator in Ω_ε. The function $\mathbf{g}$ has the form

$$\mathbf{g}(t,x) = (\mathbf{g}_1(t,\overline{x}),0) \text{ for a.e. } (t, x = (\overline{x},x_3)) \in (0,\infty)\times\Omega\times(h_1^-, h_2^+), \tag{1.2}$$

where $\mathbf{g}_1 \in L^2(Q)^2$ ($Q := (0,\infty)\times\Omega$) and $h_1^- = \min_{\overline{Y}} h_1$, $h_2^+ = \max_{\overline{Y}} h_2$. The function $F \in C^3(\mathbb{R})$ satisfies

$$\liminf_{|r|\to\infty} F''(r) > 0 \text{ and } |F'''(r)| \le c_F(1+|r|)\ \forall r\in\mathbb{R}, \tag{1.3}$$

where c_F is a positive constant. The mobility $m^\varepsilon(\cdot,\varphi_\varepsilon)(t,x) = m(x/\varepsilon,\varphi_\varepsilon(t,x))$ is such that the function $m : \mathbb{R}^3\times\mathbb{R}\to\mathbb{R}$, lies in $\mathrm{BUC}(\mathbb{R}^3; C^{0,1}_{\mathrm{loc}}(\mathbb{R}))$ (where BUC stands for bounded uniformly continuous) and there exist two constants $m_1, m_2 > 0$ such that

$$m_1 \le m(y,r) \le m_2 \text{ for a.e. } y\in\mathbb{R}^3 \text{ and for all } r\in\mathbb{R}. \tag{1.4}$$

Finally the initial condition $\varphi_0^\varepsilon \in H^1(\Omega_\varepsilon)$ satisfies the hypothesis

$$\left\|\varphi_0^\varepsilon\right\|_{H^1(\Omega_\varepsilon)} \le C\varepsilon^{\frac{1}{2}} \text{ and } \int_{\Omega_\varepsilon} F(\varphi_0^\varepsilon)\,\mathrm{d}x \le C\varepsilon, \tag{1.5}$$

where $C > 0$ is a constant independent of ε, and we assume without loss of generality that

$$\varepsilon^{-\frac{1}{2}}\left\|\varphi_0^\varepsilon - \varphi^0\right\|_{L^2(\Omega_\varepsilon)} \to 0 \tag{1.6}$$

when $\varepsilon\to 0$, where $\varphi^0 \in H^1(\Omega)$.

It follows from (1.3) that

$$\begin{aligned} &|F''(r)| \le C(1+|r|^2),\ |F'(r)| \le C(1+|r|^3), \\ &|F''(r) - F''(s)| \le C(1+|r|+|s|)|r-s| \text{ and} \\ &|F'(r) - F'(s)| \le C(1+|r|^2+|s|^2)|r-s| \quad \forall r,s\in\mathbb{R}, \end{aligned} \tag{1.7}$$

for a positive constant C depending on F.

Throughout the work, the notation $(1.1)_i$ stands for the ith equation of system (1.1); the same holds true for any other equation.

The micro-model (1.1) consists of a convective Cahn-Hilliard equation with non-constant oscillating mobility coupled with the stationary Stokes equation through the surface tension term $\mu_\varepsilon\nabla\varphi_\varepsilon$. Our goal is to study the limiting behaviour as $\varepsilon\to 0$, of the sequence of solutions of (1.1).

There are a few work dealing with the homogenization theory in thin heterogeneous domains; see e.g. [5–12,19–21,26,28,29], to cite some of them. Our model problem is stated in a highly heterogeneous thin domain with oscillating lateral boundaries and whose heterogeneities are uniformly distributed inside. Therefore the two-scale convergence method for thin periodic structures introduced in [26] for flat parallel lateral boundaries is extended here to the case of oscillating boundaries, and will be our essential tool for the limit passage in the homogenization process. Hence, using the above concept of convergence together with some Sobolev-type inequalities, we obtain the following homogenized model (1.9) which is a mixture model of tumor growth. Problems in domains with oscillating boundaries were also studied using the unfolding operator, see e.g. [2,3].

Theorem 1.1. *For each $\varepsilon > 0$, let $(\mathbf{u}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon,p_\varepsilon)$ be a solution of (1.1).*

(A) ***Derivation of the homogenized system.*** *Given any ordinary sequence E of generic term $\varepsilon > 0$, and any $T\in(0,\infty]$, there exists a subsequence E' of E such that the sequence $(\mathbf{u}_\varepsilon,\mu_\varepsilon,p_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges (as $\varepsilon\to 0$) in $L^2(Q_\varepsilon^T)^3\times L^2(Q_\varepsilon^T)\times L^2(Q_\varepsilon^T)$ towards $(\mathbf{u}_0,\mu_0,p_0)$ and the subsequence $(\varphi_\varepsilon)_{\varepsilon\in E'}$ strongly two-scale converges in $L^2(Q_\varepsilon^T)$ towards φ_0 where $\varphi_0\in L^\infty([0,\infty);H^1(\Omega))$,*

$\mathbf{u}_0 \in L^2(Q; H^1_{per}(Y; H^1_0(I))^3)$, $\mu_0 \in L^2_{\text{uloc}}([0,\infty); H^1(\Omega))$ *and* $p_0 \in L^2_{\text{uloc}}([0,\infty); H^1(\Omega) \cap L^2_0(\Omega))$. *Setting*

$$M_\varepsilon \phi(t,\overline{x}) = \fint_{\varepsilon h_1\left(\frac{\overline{x}}{\varepsilon}\right)}^{\varepsilon h_2\left(\frac{\overline{x}}{\varepsilon}\right)} \phi(t,\overline{x},\zeta)\,\mathrm{d}\zeta \text{ for } (t,\overline{x}) \in Q,$$

and

$$\mathbf{u}(t,\overline{x}) = \fint_Z \mathbf{u}_0(t,\overline{x},y)\,\mathrm{d}y \equiv (\overline{\mathbf{u}}(t,\overline{x}), u_3(t,\overline{x})),$$

(Z is the cell of periodicity, see (3.1)), one has, as $E' \ni \varepsilon \to 0$,

$$\begin{aligned} &M_\varepsilon \mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(Q)^3\text{-weak},\ \ M_\varepsilon \varphi_\varepsilon \to \varphi_0 \text{ in } L^2(Q)\text{-strong}, \\ &M_\varepsilon \mu_\varepsilon \to \mu_0 \text{ in } L^2(Q)\text{-weak and } M_\varepsilon p_\varepsilon \to p_0 \text{ in } L^2(Q)\text{-weak}, \end{aligned} \tag{1.8}$$

and the quadruplet $(\overline{\mathbf{u}}, \varphi_0, \mu_0, p_0)$ is solution of the effective 2D problem

$$\begin{cases} \overline{\mathbf{u}} = A(\mathbf{g}_1 + \mu_0 \nabla_{\overline{x}}\varphi_0 - \nabla_{\overline{x}} p_0) \text{ in } Q, \\ \operatorname{div}_{\overline{x}} \overline{\mathbf{u}} = 0 \text{ in } Q \text{ and } \overline{\mathbf{u}} \cdot \boldsymbol{n} = 0 \text{ on } (0,\infty) \times \partial\Omega, \\ \dfrac{\partial \varphi_0}{\partial t} + \overline{\mathbf{u}} \cdot \nabla_{\overline{x}}\varphi_0 - \operatorname{div}_{\overline{x}}(\widehat{m}(\varphi_0)\nabla_{\overline{x}}\mu_0) = 0 \text{ in } Q, \\ \mu_0 = -\beta \Delta_{\overline{x}}\varphi_0 + \lambda F'(\varphi_0) \text{ in } Q, \\ \dfrac{\partial \varphi_0}{\partial \boldsymbol{n}} = \dfrac{\partial \mu_0}{\partial \boldsymbol{n}} = 0 \text{ on } (0,\infty) \times \partial\Omega, \\ \varphi_0(0) = \varphi^0 \text{ in } \Omega, \end{cases} \tag{1.9}$$

where $A = (a_{ij})_{1\le i,j\le 2}$ is a symmetric positive definite 2×2 matrix defined by its entries $a_{ij} = \int_Z \omega^i(y) e_j\,\mathrm{d}y$. Here $\omega^j = (\omega^j_i)_{1\le i\le 3}$ ($j = 1,2$) is the unique solution in $H^1_{0,\#}(Z)^3$ of the auxiliary Stokes system

$$\begin{cases} -\alpha \Delta_y \omega^j + \nabla_y \pi^j = e_j \text{ in } Z, \quad \operatorname{div}_y \omega^j = 0 \text{ in } Z, \\ \int_Z \omega^j_3(\overline{y},\zeta)\,\mathrm{d}\overline{y}\,\mathrm{d}\zeta = 0, \end{cases}$$

e_j being the jth vector of the canonical basis in $\mathbb{R}^3$. The mobility coefficient $\widehat{m}(\varphi_0)$ is defined by

$$\widehat{m}(\varphi_0)(t,\overline{x}) = \int_Z m(y, \varphi_0(t,\overline{x}))(I_2 + \nabla_{\overline{y}}\varpi(t,\overline{x},y))\,\mathrm{d}y,\ \ (t,\overline{x}) \in Q^T,$$

where I_2 is the 2×2 identity matrix and ϖ is the unique solution of the cell problem

$$\begin{cases} \text{Find } \pi_{\xi,r} \equiv \pi_{\xi,r}(t,\overline{x},\cdot) \in H^1_\#(Z)/\mathbb{R} \text{ such that} \\ -\operatorname{div}_y(m(\cdot,r)(\xi + \nabla_y \pi_{\xi,r}(t,\overline{x},\cdot)) = 0 \text{ in } Z \end{cases}$$

corresponding to $r = \varphi_0(t,\overline{x})$ and $\xi = \nabla_{\overline{x}}\mu_0(t,\overline{x})$. Any limit point of the sequence $(\mathbf{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon, p_\varepsilon)_{\varepsilon\in E}$ in the sense of (1.8) solves (1.9).

(B) ***Regularity results****. It holds that $\mu \in L^4_{\text{uloc}}([0,\infty); L^2(\Omega))$. Assume further that Ω is of class C^3. Then $\varphi_0 \in \mathrm{BC}([0,\infty); H^1_N(\Omega)) \cap L^2_{\text{uloc}}([0,\infty); H^3(\Omega)) \cap L^4_{\text{uloc}}([0,\infty); H^2(\Omega))$. Moreover, for any $r \ge 2$, $\varphi \in L^2_{\text{uloc}}([0,\infty); W^{2,r}(\Omega))$.*

The Eq. $(1.9)_1$ is a Hele-Shaw type equation, and the coupled system (1.9) is therefore a Hele-Shaw-Cahn-Hilliard system (HSCH). It has many applications in two-phase flow in porous media and Hele-Shaw cell, but also widely used to model tumor growth [24,38].

In fact, the parameters that appear in the homogenized HSCH system (1.9) can be related to measurable quantities in tumor growth and biological tissue mechanics. The effective viscosity tensor $A = (a_{ij})$ represents the mechanical resistance of the heterogeneous tissue to deformation and flow: it has an anisotropic structure that reflects the directional dependence of stiffness that has been induced, for example, by collagen fibers alignment or by spatial heterogeneities in the extracellular matrix (ECM) [17,27]. The mobility coefficient $\widehat{m}(\varphi_0)$, which in the model determines the magnitude of diffusive fluxes caused by the chemical potential, governs the rate of diffusive fluxes driven by the chemical potential: it corresponds biologically to the motility and rearrangement capacity of tumor cells within the surrounding matrix. It follows that regions with higher effective mobility can therefore be interpreted as zones of enhanced cell migration, usually located at the invasive tumor front, while the lower mobility characterizes necrotic or fibrotic regions [23,38].

The order parameter φ_0 serves as a phase variable that distinguishes tumor and host tissues. In general, $\varphi_0 \approx 1$ represents the tumor-dominated regions, $\varphi_0 \approx -1$ corresponds to healthy tissue, while an intermediate value φ_0 between these two extrema describes the diffuse transition layer at the tumor interface. The chemical potential $\mu_0 = -\beta\Delta_x\varphi_0 + \lambda F'(\varphi_0)$ describes the local thermodynamic force that determines tumor-tissue segregation. The gradient term represents the interfacial tension between the two phases and penalizes abrupt transitions: therefore, it models the mechanical cost of interface formation. The mass term $F'(\varphi_0)$ represents the local preference of one phase over the other and can be interpreted in terms of the energetics of cell-cell adhesion and proliferation [14].

Finally, the Hele–Shaw relation

$$u = A\big(\mathbf{g}_1 + \mu_0 \nabla_x \varphi_0 - \nabla_x p_0\big)$$

describes the macroscopic movement of the tumor tissue mixture, driven by gradients of mechanical pressure and interfacial stresses. Here p_0 denotes the mechanical pressure due to the proliferation of cells and compression of tissues, while $\mathbf{g}_1$ may represent external forces or nutrient-induced stresses [30]. So, these parameters of the upscaled model encode key biological mechanisms-tissue stiffness, cell motility, adhesion, and interfacial tension-that link the mechanical formulation of the continuum with the experimentally observable properties of tumor growth.

One of our main aim in this work will be to make a qualitative analysis of (1.9) in order to prove some regularity results.

The HSCH system is mainly known in the case when the velocity has a diagonal form, that is, when $\overline{\mathbf{u}}$ expresses as

$$\overline{\mathbf{u}} = \mathbf{g}_1 + \mu_0 \nabla_{\overline{x}} \varphi_0 - \nabla_{\overline{x}} p_0 \text{ in } Q.$$

In that case, few studies have been made in the literature as far as its analysis is concerned. Indeed, in [37], the system (1.9) with $A = Id$ has been studied numerically. It has also been studied analytically in [16] where existence and uniqueness of weak solutions in two/three dimensional bounded domains were proved, and in [35,40] where the well-posedness and longtime behaviour of strong solutions in two or three dimensional torus were considered. In [24], a systematic analysis of the (1.9) (with $A = Id$) was considered in a $2D$ rectangle or in a $3D$ parallelepiped. In our model, though the velocity does not act on a diagonal form as known in the literature, however the analysis of our homogenized model will follows the same way of reasoning as in [24], due to the ellipticity property of the matrix A. Also, the mobility coefficient $\widehat{m}(\varphi_0)$ is not a scalar-valued function of φ_0, but rather a matrix-valued function.

Let us point out a few differences between the model problem considered in the current work and the one in [10]. In (1.1) the convective Cahn-Hilliard system is local like in the work [10]. However, the mobility coefficient here is not constant and depend on the phase variable. This renders more delicate the proof of the existence of solutions to (1.1) as seen in Section 2. Also, the domain Ω_ε has oscillating lateral boundaries while in [10], the lateral boundaries are flat. This adds an additional difficulty in controlling the pressure term as seen in Proposition 2.3 where a suitable decomposition of the pressure is needed to overcome that difficulty. Last but not least, the ε-problem is stated here in the infinite time interval $[0,\infty)$, requiring us to seek solutions to the problem in an unusual space; for instance, the chemical potential lies in $L^2_{\text{uloc}}([0,\infty); H^1(\Omega_\varepsilon))$; see Section 2 below.

This paper consists of four additional sections. In Section 2, we state and prove the existence result and we derive uniform estimates for the sequence of solutions of (1.1). In Section 3 we define the concept of two-scale convergence in thin periodic domains with oscillating boundaries, and we gather some essential compactness results useful for the homogenization process. Section 4 deals with the limit passage in (1.1) and the derivation of the homogenized model. We next analyze the upscaled model in Section 5. We close Section 5 with the proof of the main result of the paper.

Unless otherwise specified, the vector spaces throughout are assumed to be real vector spaces, and the scalar functions are assumed to take real values. We shall always assume that the numerical space $\mathbb{R}^m$ (integer $m \geq 1$) and its open sets are each provided with the Lebesgue measure denoted by $\mathrm{d}x = \mathrm{d}x_1 \dots \mathrm{d}x_m$. Finally we will adopt the following notation in the remaining part of the work. If $A = (a_{ij})_{1\leq i,j\leq m}$ and $B = (b_{ij})_{1\leq i,j\leq m}$, we denote $A \cdot B := \sum_{i,j=1}^m a_{ij} b_{ij}$; we use the same notation for the scalar product in $\mathbb{R}^m$, namely, if $\mathbf{u} = (u_i)_{1\leq i\leq m}$ and $\mathbf{v} = (v_i)_{1\leq i\leq m}$, then $\mathbf{u}\cdot\mathbf{v} = \sum_{i=1}^m u_i v_i$. Finally, throughout the work, the letter C will denote a positive constant that may vary from line to line.

2. Existence result and uniform estimates

2.1. Existence result

In order to define the notion of weak solutions we will deal with in this work, we first introduce the functional setup. Let X be a Banach space. The notation $\langle \cdot,\cdot\rangle$ will stand for the duality pairings between X and its topological dual X' while $\mathbb{X}$ will denote the space $X\times X\times X$ endowed with the product structure. If in particular X is a real Hilbert space with inner product $(\cdot,\cdot)_X$, then we denote by $\|\cdot\|_X$ the induced norm. Especially, by $\mathbb{H}_\varepsilon$ and $\mathbb{V}_\varepsilon$ we denote the Hilbert spaces defined as the closure in $\mathbb{L}^2(\Omega_\varepsilon) = L^2(\Omega_\varepsilon)^3$ (resp. $\mathbb{H}^1_0(\Omega^\varepsilon) = H^1_0(\Omega_\varepsilon)^3$) of the space $\{\mathbf{u}\in \mathbb{C}_0^\infty(\Omega_\varepsilon) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_\varepsilon\}$ where $\mathbb{C}_0^\infty(\Omega_\varepsilon) = C_0^\infty(\Omega_\varepsilon)^3$. Then $\mathbb{V}_\varepsilon = \{\mathbf{u}\in\mathbb{H}^1_0(\Omega_\varepsilon) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_\varepsilon\}$ and $\mathbb{H}_\varepsilon = \{\mathbf{u}\in\mathbb{L}^2(\Omega_\varepsilon) : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega_\varepsilon \text{ and } \mathbf{u}\cdot\nu = 0 \text{ on } \partial\Omega_\varepsilon\}$ where ν is the outward unit normal to $\partial\Omega_\varepsilon$. The space $\mathbb{H}_\varepsilon$ is endowed with the scalar product denoted by $(\cdot,\cdot)$ whose associated norm is denoted by $\|\cdot\|_{\mathbb{H}_\varepsilon}$. The space $\mathbb{V}_\varepsilon$ is equipped with the scalar product

$$(\mathbf{u},\mathbf{v}) = (\nabla\mathbf{u},\nabla\mathbf{v}) \quad (\mathbf{u},\mathbf{v}\in\mathbb{V}_\varepsilon)$$

whose associated norm is the norm of the gradient and is denoted by $\|\cdot\|_{\mathbb{V}_\varepsilon}$. Owing to the Poincaré inequality, the norm in $\mathbb{V}_\varepsilon$ is equivalent to the $\mathbb{H}^1(\Omega_\varepsilon)$-norm. We also define the space $L^2_0(\Omega_\varepsilon) = \{v\in L^2(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} v\,\mathrm{d}x = 0\}$. We denote by $\mathbb{V}$ (resp. $\mathbb{H}$) the space defined as $\mathbb{V}_\varepsilon$ (resp. $\mathbb{H}_\varepsilon$) when replacing Ω_ε by Ω. For the sake of simplicity, we shall often use the notation $\|\cdot\|_{H^s}$ to denote the norm in $H^s(D)$ for s an integer and D any open subset of $\mathbb{R}^m$ (integer $m\geq 1$). Finally, for a Banach space X and any real number $p\geq 1$, we denote by $L^p_{\mathrm{uloc}}([0,\infty);X)$ the Banach space of those functions u in $L^p_{\mathrm{loc}}([0,\infty);X)$ satisfying

$$\|u\|_{L^p_{\mathrm{uloc}}([0,\infty);X)} := \sup_{t\geq 0}\left(\int_t^{t+1}\|u(\tau)\|_X^p\,\mathrm{d}\tau\right)^{1/p} < \infty.$$

It is worth to note that $L^p_{\mathrm{uloc}}([0,\infty);X)$ is the Wiener amalgam space $(L^p,l^\infty)([0,\infty);X)$ (see [4,18,36]) for the definition of scalar-valued function spaces $(L^p,l^q)([0,\infty))$ for $1\leq p,q\leq\infty$). If X is a reflexive Banach space and $1<p<\infty$, then $(L^p,l^\infty)([0,\infty);X)$ is the topological dual space of $(L^{p'},l^1)([0,\infty);X')$, where $p'=p/(p-1)$.

This being so, the concept of weak solution we will deal with in this work, is defined as follows.

Definition 2.1. Let $\varphi_0^\varepsilon\in H^1(\Omega_\varepsilon)$ with $F(\varphi_0^\varepsilon)\in L^1(\Omega_\varepsilon)$. The triplet $(\mathbf{u}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon)$ is a weak solution to (1.1) if:

- It holds that
  (i) $\mathbf{u}_\varepsilon\in L^2([0,\infty);\mathbb{V}_\varepsilon)$,
  (ii) $\varphi_\varepsilon\in L^\infty([0,\infty);H^1(\Omega_\varepsilon))$ with $\partial\varphi_\varepsilon/\partial t\in L^2(0,\infty;H^1(\Omega_\varepsilon)')$,
  (iii) $\mu_\varepsilon\in L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega_\varepsilon))$;
- The variational formulation

$$\alpha\varepsilon^2\int_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon\cdot\nabla\psi\,\mathrm{d}x+\int_{\Omega_\varepsilon}(\psi\cdot\nabla\mu_\varepsilon)\varphi_\varepsilon\,\mathrm{d}x=\int_{\Omega_\varepsilon}\mathbf{g}\psi\,\mathrm{d}x, \tag{2.1}$$

$$\left\langle\frac{\partial\varphi_\varepsilon}{\partial t},\phi\right\rangle_{H^1(\Omega_\varepsilon)',H^1(\Omega_\varepsilon)}-\int_{\Omega_\varepsilon}(\mathbf{u}_\varepsilon\cdot\nabla\phi)\varphi_\varepsilon\,\mathrm{d}x+\int_{\Omega_\varepsilon}m^\varepsilon(\cdot,\varphi_\varepsilon)\nabla\mu_\varepsilon\cdot\nabla\phi\,\mathrm{d}x=0, \tag{2.2}$$

$$\int_{\Omega_\varepsilon}\mu_\varepsilon\chi\,\mathrm{d}x=\beta\int_{\Omega_\varepsilon}\nabla\varphi_\varepsilon\cdot\nabla\chi\,\mathrm{d}x+\lambda\int_{\Omega_\varepsilon}F'(\varphi_\varepsilon)\chi\,\mathrm{d}x, \tag{2.3}$$

  holds a.e. in $[0,\infty)$ for all $\phi,\chi\in H^1(\Omega_\varepsilon)$ and $\psi\in\mathbb{V}_\varepsilon$,
- $\varphi_\varepsilon(0)=\varphi_0^\varepsilon$ a.e. in Ω_ε.
  Furthermore to each weak solution $(\mathbf{u}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon)$ is associated a pressure $p_\varepsilon\in L^2_{\mathrm{uloc}}([0,\infty);L^2_0(\Omega_\varepsilon))$ that satisfies $(1.1)_1$ in the distributional sense.

The main aim of this section is to prove the following result.

Theorem 2.1. *For each fixed $\varepsilon>0$, let $\varphi_0^\varepsilon\in H^1(\Omega_\varepsilon)$ with $F(\varphi_0^\varepsilon)\in L^1(\Omega_\varepsilon)$. Then under assumptions (1.2), (1.3) and (1.4), there exists at least a weak solution $(\mathbf{u}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon)$ to (1.1) in the sense of Definition 2.1. Furthermore, to $(\mathbf{u}_\varepsilon,\varphi_\varepsilon,\mu_\varepsilon)$ is attached a unique $p_\varepsilon\in L^2_{\mathrm{uloc}}([0,\infty);L^2_0(\Omega_\varepsilon))$ such that $(1.1)_1$ is satisfied in the distributional sense.*

The proof of Theorem 2.1 is based on a semi-Galerkin discretization of the convective Cahn-Hilliard subsystem.

2.2. Galerkin approximation

Proceeding as in [15], we aim at constructing approximate solutions by using a Galerkin approximation with respect to φ_ε and μ_ε, and solve for $\mathbf{u}_\varepsilon$ and p_ε in the corresponding whole function spaces. We consider the Neumann-Laplace operator $-\Delta$ on $L^2(\Omega_\varepsilon)$ with domain $H^2_N(\Omega_\varepsilon)=\{\phi\in H^2(\Omega_\varepsilon):\frac{\partial\phi}{\partial\nu}=0\text{ on }\partial\Omega_\varepsilon\}$. By the spectral theory, there is a sequence of positive eigenvalues $(\lambda_i)_{i\geq1}$ associated with $-\Delta$ such that $\lambda_1=1$, $\lambda_i\leq\lambda_{i+1}$ and $\lambda_i\to\infty$ as $i\to\infty$. The sequence of the corresponding eigenfunctions $(w_i)_{i\geq1}\subset H^2_N(\Omega_\varepsilon)$ such that $-\Delta w_i=\lambda_i w_i$ forms an orthonormal basis in $L^2(\Omega_\varepsilon)$ as well as in $H^1(\Omega_\varepsilon)$. By elliptic regularity, we get that $(w_i)_{i\geq1}$ is also an orthonormal basis in $H^2_N(\Omega_\varepsilon)$. For each fixed integer $k\geq1$, we define the finite dimensional subspace

$$\mathfrak{W}_k=\mathrm{span}(w_1,\ldots,w_k),$$

and denote by $\mathcal{P}_k$ the orthogonal projection on $\mathfrak{W}_k$ with respect to the inner product in $L^2(\Omega_\varepsilon)$. Set $\varphi_{0,k}^\varepsilon=\mathcal{P}_k\varphi_0^\varepsilon$. We look for an approximate solution $(\varphi_{\varepsilon,k},\mu_{\varepsilon,k})$ of the form

$$\varphi_{\varepsilon,k}(t,x)=\sum_{i=1}^k a_i^k(t)w_i(x),\quad \mu_{\varepsilon,k}(t,x)=\sum_{i=1}^k b_i^k(t)w_i(x)$$

satisfying

$$\varphi_{\varepsilon,k},\mu_{\varepsilon,k}\in C^1([0,\infty);\mathfrak{W}_k), \tag{2.4}$$

$$\int_{\Omega_\varepsilon}\frac{\partial\varphi_{\varepsilon,k}}{\partial t}v\,\mathrm{d}x=-\int_{\Omega_\varepsilon}m^\varepsilon(.,\varphi_{\varepsilon,k})\nabla\mu_{\varepsilon,k}\cdot\nabla v\,\mathrm{d}x-\int_{\Omega_\varepsilon}(\mathbf{u}_{\varepsilon,k}\nabla\varphi_{\varepsilon,k})v\,\mathrm{d}x, \tag{2.5}$$

$$\int_{\Omega_\varepsilon}\mu_{\varepsilon,k}v\,\mathrm{d}x=\beta\int_{\Omega_\varepsilon}\nabla\varphi_{\varepsilon,k}\cdot\nabla v\,\mathrm{d}x+\lambda\int_{\Omega_\varepsilon}F'(\varphi_{\varepsilon,k})v\,\mathrm{d}x, \tag{2.6}$$

for all $v \in \mathfrak{W}_k$, and with the initial condition $\varphi_{\varepsilon,k}(0) = \varphi^{\varepsilon}_{0,k}$. Furthermore, we define the velocity $\mathbf{u}_{\varepsilon,k}$ and the pressure $p_{\varepsilon,k}$ as the solution of

$$\begin{cases} -\alpha\varepsilon^2 \Delta \mathbf{u}_{\varepsilon,k} + \nabla p_{\varepsilon,k} - \mu_{\varepsilon,k} \nabla \varphi_{\varepsilon,k} = \mathbf{g} \text{ in } Q_\varepsilon, \\ \operatorname{div} \mathbf{u}_{\varepsilon,k} = 0 \text{ in } Q_\varepsilon, \quad \mathbf{u}_{\varepsilon,k} = 0 \text{ on } (0,\infty) \times \partial\Omega_\varepsilon. \end{cases} \tag{2.7}$$

We start by observing that, given any couple $(\varphi_{\varepsilon,k}, \mu_{\varepsilon,k})$ of functions satisfying (2.4), (2.5) and (2.6), there exists a unique pair $(\mathbf{u}_{\varepsilon,k}, p_{\varepsilon,k}) \in L^2([0,\infty); \mathbb{H}^2(\Omega_\varepsilon) \cap \mathbb{V}_\varepsilon) \times L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon))$ solution of (2.7).

Now, taking $v = w_j$ $(1 \le j \le k)$ in (2.5)–(2.6) and defining as in [15] the matrices $S^k_m = ((S^k_m)_{ji})_{1\le i,j\le k}$, $S = (S_{ij})_{1\le i,j\le k}$, $C^k = ((C^k)_{ji})_{1\le i,j\le k}$, and the vector-function $F^k = (F^k_j)^T_{1\le j\le k}$, where

$$(S^k_m)_{ji} = \int_{\Omega_\varepsilon} m^\varepsilon(.,\varphi_{\varepsilon,k}) \nabla w_i \cdot \nabla w_j \,\mathrm{d}x, \quad S_{ij} = \int_{\Omega_\varepsilon} \nabla w_i \cdot \nabla w_j \,\mathrm{d}x,$$

$$(C^k)_{ji} = \int_{\Omega_\varepsilon} (\nabla w_i \cdot \mathbf{u}_{\varepsilon,k}) w_j \,\mathrm{d}x, \quad F^k_j = \int_{\Omega_\varepsilon} F'(\varphi_{\varepsilon,k}) w_j \,\mathrm{d}x,$$

we obtain a system of ordinary differential equations equivalent to (2.4)–(2.6) and amounting to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}^k = -S^k_m \mathbf{b}^k - C^k \mathbf{a}^k, \tag{2.8}$$

$$\mathbf{b}^k = \beta S \mathbf{a}^k + \lambda F^k, \tag{2.9}$$

where $\mathbf{a}^k = (a^k_1, \dots, a^k_k)^T$ and $\mathbf{b}^k = (b^k_1, \dots, b^k_k)^T$. The system is completed with the initial condition

$$a^k_i(0) = \int_{\Omega_\varepsilon} \varphi^0_{\varepsilon,k} w_i \,\mathrm{d}x, \quad 1 \le i \le k. \tag{2.10}$$

It is easy to see that

$$\left\| \sum_{i=1}^k a^k_i(0) w_i \right\|_{H^1(\Omega_\varepsilon)} \le \left\| \varphi^\varepsilon_0 \right\|_{H^1(\Omega_\varepsilon)}.$$

Therefore, relying on the local Lipschitz continuity of $m(y,\cdot)$ and F' together with the stability of (2.7) under perturbations, we see that the right-hand side of (2.8)–(2.9) is a locally Lipschitz function of $\mathbf{a}^k$. Thus, there exists $T_k \in (0,\infty]$ such that (2.8)–(2.10) possesses a unique solution $(\mathbf{a}^k, \mathbf{b}^k) \in [C^1([0,T_k); \mathbb{R}^k)]^2$. This shows that (2.5), (2.6) admits a unique solution $(\varphi_{\varepsilon,k}, \mu_{\varepsilon,k}) \in [C^1([0,T_k); \mathfrak{W}_k)]^2$. Since $\mathfrak{W}_k \subset H^2_N(\Omega_\varepsilon) \hookrightarrow L^\infty(\Omega_\varepsilon)$, we easily see that $\mu_{\varepsilon,k} \nabla \varphi_{\varepsilon,k} \in L^2([0,T_k); L^2(\Omega_\varepsilon)^3)$, so that $\mathcal{P}_k \mathbf{g} - \mu_{\varepsilon,k} \nabla \varphi_{\varepsilon,k} \in L^2([0,T_k); L^2(\Omega_\varepsilon)^3)$. This being so, with $(\varphi_{\varepsilon,k}, \mu_{\varepsilon,k})$ obtained as above, there exists a unique $(\mathbf{u}_{\varepsilon,k}, p_{\varepsilon,k}) \in L^2(0,T_k; \mathbb{H}^2(\Omega_\varepsilon) \cap \mathbb{V}_\varepsilon) \times L^2(0,T_k; H^1(\Omega_\varepsilon) \cap L^2_0(\Omega_\varepsilon))$ solving (2.7) in $(0,T_k) \times \Omega_\varepsilon$. We infer the existence of a unique quadruple $(\mathbf{u}_{\varepsilon,k}, \varphi_{\varepsilon,k}, \mu_{\varepsilon,k}, p_{\varepsilon,k})$ solving (2.5), (2.6) and (2.7) with boundary and initial conditions.

In order to conclude with the proof, we need to find suitable estimates in order to pass to the limit and hence to show that $T_k = \infty$.

2.3. A priori estimates

At this level, we shall establish a priori estimates that are uniform in both k and ε as they will be useful for the existence of the solution to (1.1) as well as for the homogenization process. Toward that end, we shall need the following preliminary result which stems from Poincaré's and Sobolev's inequalities; see e.g. [25, Lemmas 8, 10 and Remark 5] for the proof in the case of flat lateral boundaries.

Lemma 2.1. *One has*

$$\|u\|_{L^2(\Omega_\varepsilon)} \le C\varepsilon \|\nabla u\|_{L^2(\Omega_\varepsilon)^3} \tag{2.11}$$

and

$$\|u\|_{L^4(\Omega_\varepsilon)} \le C\varepsilon^{\frac{1}{2}} \|\nabla u\|_{L^2(\Omega_\varepsilon)^3} \tag{2.12}$$

for any $u \in H^1_0(\Omega_\varepsilon)$, where $C > 0$ is independent of ε.

In the estimates that follow below, C is a positive constant independent of k and ε, and may change from one line to another.

First, we take the scalar product in $\mathbb{H}_\varepsilon$ of (2.7) with $\mathbf{u}_{\varepsilon,k}$ and use the boundary condition $\mathbf{u}_{\varepsilon,k} = 0$ on $\partial\Omega_\varepsilon$ to obtain

$$-\int_{\Omega_\varepsilon} \mu_{\varepsilon,k} (\nabla \varphi_{\varepsilon,k} \cdot \mathbf{u}_{\varepsilon,k}) \,\mathrm{d}x + \alpha\varepsilon^2 \int_{\Omega_\varepsilon} \left|\nabla \mathbf{u}_{\varepsilon,k}\right|^2 \mathrm{d}x = \int_{\Omega_\varepsilon} \mathbf{g} \cdot \mathbf{u}_{\varepsilon,k} \,\mathrm{d}x. \tag{2.13}$$

Next, choosing $v = b^k_j w_j$ in (2.5), $v = \frac{\mathrm{d}a^k_j}{\mathrm{d}t} w_j$ in (2.6) and then summing each of the resulting identities over $j = 1, \dots, k$, and finally adding the obtained equalities, we obtain

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\beta}{2}\int_{\Omega_\varepsilon} \left|\nabla \varphi_{\varepsilon,k}\right|^2 \mathrm{d}x + \lambda \int_{\Omega_\varepsilon} F(\varphi_{\varepsilon,k}) \,\mathrm{d}x\right] + \int_{\Omega_\varepsilon} m^\varepsilon(\cdot,\varphi_{\varepsilon,k}) \nabla \mu_{\varepsilon,k} \cdot \nabla \mu_{\varepsilon,k} \,\mathrm{d}x \\ + \int_{\Omega_\varepsilon} \mu_{\varepsilon,k} \nabla \varphi_{\varepsilon,k} \cdot \mathbf{u}_{\varepsilon,k} \,\mathrm{d}x = 0. \end{cases} \tag{2.14}$$

Let us notice the fact that in getting (2.14) we have used the equation $\operatorname{div}\mathbf{u}_{\varepsilon,k}=0$ and $\frac{\partial\varphi_{\varepsilon,k}}{\partial\nu}=0$. Now summing up (2.13) and (2.14) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\beta}{2}\|\nabla\varphi_{\varepsilon,k}(t)\|_{L^2}^2+\lambda\int_{\Omega_\varepsilon}F(\varphi_{\varepsilon,k}(t))\,\mathrm{d}x\right]+\alpha\varepsilon^2\|\nabla\mathbf{u}_{\varepsilon,k}(t)\|_{L^2}^2+\left\|\sqrt{m^\varepsilon(\cdot,\varphi_{\varepsilon,k}(t))}\nabla\mu_{\varepsilon,k}(t)\right\|_{L^2}^2=\int_{\Omega_\varepsilon}\mathbf{g}(t)\cdot\mathbf{u}_{\varepsilon,k}(t)\,\mathrm{d}x. \tag{2.15}$$

Since $\mathbf{g}(t,x)=(\mathbf{g}_1(t,\overline{x}),0)$, we get

$$\begin{aligned}\left|\int_{\Omega_\varepsilon}\mathbf{g}(t)\cdot\mathbf{u}_{\varepsilon,k}(t)\,\mathrm{d}x\right| &\le C\varepsilon^{\frac12}\|\mathbf{g}_1(t)\|_{L^2(\Omega)^2}\|\mathbf{u}_{\varepsilon,k}(t)\|_{L^2(\Omega_\varepsilon)^3}\\ &\le C\varepsilon^{\frac32}\|\mathbf{g}_1(t)\|_{L^2(\Omega)^2}\|\nabla\mathbf{u}_{\varepsilon,k}(t)\|_{L^2(\Omega_\varepsilon)^{3\times3}}\ \text{by(2.11)}\\ &\le C\varepsilon\|\mathbf{g}_1(t)\|_{L^2(\Omega)^2}^2+\frac{\alpha}{2}\varepsilon^2\|\nabla\mathbf{u}_{\varepsilon,k}(t)\|_{L^2(\Omega_\varepsilon)^{3\times3}}^2.\end{aligned} \tag{2.16}$$

Integrating (2.15) over $(0,t)$ and using (2.16) and (1.4), we readily get

$$\begin{aligned}&\frac{\alpha}{2}\varepsilon^2\int_0^t\|\nabla\mathbf{u}_{\varepsilon,k}(s)\|_{L^2}^2\,\mathrm{d}s+\frac{\beta}{2}\|\nabla\varphi_{\varepsilon,k}(t)\|_{L^2}^2+\lambda\int_{\Omega_\varepsilon}F(\varphi_{\varepsilon,k}(t))\,\mathrm{d}x\\ &+m_1\int_0^t\|\nabla\mu_{\varepsilon,k}(s)\|_{L^2}^2\,\mathrm{d}s\le C\varepsilon+\frac{\beta}{2}\left\|\nabla\varphi_0^\varepsilon\right\|_{L^2}^2+\lambda\int_{\Omega_\varepsilon}F(\varphi_0^\varepsilon)\,\mathrm{d}x.\end{aligned} \tag{2.17}$$

It follows therefore from (1.5) that the following a priori estimates hold:

$$\varepsilon\|\nabla\mathbf{u}_{\varepsilon,k}\|_{L^2(Q_\varepsilon)^{3\times3}}\le C\varepsilon^{\frac12}, \tag{2.18}$$

$$\|\nabla\varphi_{\varepsilon,k}\|_{L^\infty(0,\infty;L^2(\Omega_\varepsilon)^3)}\le C\varepsilon^{\frac12}, \tag{2.19}$$

$$\|\nabla\mu_{\varepsilon,k}\|_{L^2(Q_\varepsilon)^3}\le C\varepsilon^{\frac12}, \tag{2.20}$$

$$\|F(\varphi_{\varepsilon,k})\|_{L^\infty(0,\infty;L^1(\Omega_\varepsilon))}\le C\varepsilon. \tag{2.21}$$

Next, we use (2.11) together with (2.18) to get

$$\|\mathbf{u}_{\varepsilon,k}\|_{L^2([0,\infty);L^2(\Omega_\varepsilon)^3)}\le C\varepsilon^{\frac12}. \tag{2.22}$$

This being so, the no-flux boundary condition $\frac{\partial\varphi_{\varepsilon,k}}{\partial\nu}=\frac{\partial\mu_{\varepsilon,k}}{\partial\nu}=0$ on $\partial\Omega_\varepsilon$ ensures the mass conservation of the following quantity

$$\left\langle\varphi_{\varepsilon,k}(t)\right\rangle=\fint_{\Omega_\varepsilon}\varphi_{\varepsilon,k}(t,x)\,\mathrm{d}x,$$

where $\fint_{\Omega_\varepsilon}=|\Omega_\varepsilon|^{-1}\int_{\Omega_\varepsilon}$ and $|\Omega_\varepsilon|$ denotes the Lebesgue measure of Ω_ε. This yields

$$\left\langle\varphi_{\varepsilon,k}(t)\right\rangle=\left\langle\varphi_{\varepsilon,k}(0)\right\rangle\quad\forall t>0. \tag{2.23}$$

Thus the Poincaré-Wirtinger inequality associated to (2.23) gives

$$\begin{aligned}\|\varphi_{\varepsilon,k}(t)\|_{L^2}&\le\left\|\varphi_{\varepsilon,k}(t)-\left\langle\varphi_{\varepsilon,k}(t)\right\rangle\right\|_{L^2}+\left\|\left\langle\varphi_{0,k}^\varepsilon\right\rangle\right\|_{L^2}\le C\|\nabla\varphi_{\varepsilon,k}(t)\|_{L^2}+\left\|\varphi_0^\varepsilon\right\|_{L^2}\\ &\le C\varepsilon^{\frac12},\end{aligned}$$

where the last inequality above is a consequence of (2.19) and (1.5). This, together with (2.19) gives

$$\|\varphi_{\varepsilon,k}\|_{L^\infty([0,\infty);H^1(\Omega_\varepsilon))}\le C\varepsilon^{\frac12}. \tag{2.24}$$

We need further estimates. First of all, in view of (1.7) one has

$$\int_{\Omega_\varepsilon}|F'(\varphi_{\varepsilon,k}(t))|\,\mathrm{d}x\le C\int_{\Omega_\varepsilon}(1+|\varphi_{\varepsilon,k}(t)|^3)\,\mathrm{d}x, \tag{2.25}$$

so that from the Sobolev embedding $H^1(\Omega_\varepsilon)\hookrightarrow L^3(\Omega_\varepsilon)$ we deduce that, for a.e. $t>0$,

$$\|\varphi_{\varepsilon,k}(t)\|_{L^3(\Omega_\varepsilon)}\le C\|\varphi_{\varepsilon,k}(t)\|_{H^1(\Omega_\varepsilon)}\le C\varepsilon^{\frac12}.$$

We infer from (2.25)

$$\int_{\Omega_\varepsilon}|F'(\varphi_{\varepsilon,k}(t))|\,\mathrm{d}x\le C(\varepsilon+\varepsilon^{\frac32})\le C\varepsilon, \tag{2.26}$$

that is,

$$\|F'(\varphi_{\varepsilon,k})\|_{L^\infty([0,\infty);L^1(\Omega_\varepsilon))} \leq C\varepsilon. \tag{2.27}$$

Inserting $v = b_j^k w_j$ into (2.6) and summing over $j = 1, \dots, k$ yields

$$\int_{\Omega_\varepsilon} |\mu_{\varepsilon,k}|^2 \,\mathrm{d}x = \int_{\Omega_\varepsilon} (\beta\nabla\varphi_{\varepsilon,k}\cdot\nabla\mu_{\varepsilon,k} + \lambda F'(\varphi_{\varepsilon,k})\mu_{\varepsilon,k})\,\mathrm{d}x.$$

We use Hölder's and Young's inequalities as well as the assumption on F (see especially the second inequality in $(1.7)_1$) together with the continuous embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^6(\Omega_\varepsilon)$ (the embedding constant here being independent of ε),

$$\begin{aligned}
\|\mu_{\varepsilon,k}\|_{L^2}^2 &\leq \beta\|\nabla\varphi_{\varepsilon,k}\|_{L^2}\|\nabla\mu_{\varepsilon,k}\|_{L^2} + \lambda\|F'(\varphi_{\varepsilon,k})\|_{L^2}\|\mu_{\varepsilon,k}\|_{L^2}\\
&\leq \frac{1}{2}\|\mu_{\varepsilon,k}\|_{L^2}^2 + \beta\|\nabla\varphi_{\varepsilon,k}\|_{L^2}\|\nabla\mu_{\varepsilon,k}\|_{L^2} + \frac{\lambda}{2}\|F'(\varphi_{\varepsilon,k})\|_{L^2}^2\\
&\leq \frac{1}{2}\|\mu_{\varepsilon,k}\|_{L^2}^2 + \beta\|\nabla\varphi_{\varepsilon,k}\|_{L^2}\|\nabla\mu_{\varepsilon,k}\|_{L^2} + C\int_{\Omega_\varepsilon}(1+|\varphi_{\varepsilon,k}|^6)\,\mathrm{d}x\\
&\leq \frac{1}{2}\|\mu_{\varepsilon,k}\|_{L^2}^2 + \beta\|\nabla\varphi_{\varepsilon,k}\|_{L^2}\|\nabla\mu_{\varepsilon,k}\|_{L^2} + C|\Omega_\varepsilon| + C\|\varphi_{\varepsilon,k}\|_{H^1}^6.
\end{aligned}$$

This leads to

$$\|\mu_{\varepsilon,k}\|_{L^2}^2 \leq C\|\nabla\varphi_{\varepsilon,k}\|_{L^2}\|\nabla\mu_{\varepsilon,k}\|_{L^2} + C|\Omega_\varepsilon| + C\|\varphi_{\varepsilon,k}\|_{H^1}^6. \tag{2.28}$$

Therefore, integrating (2.28) over $(t, t+1)$ (for any $t \geq 0$) and then taking the $\sup_{t\geq 0}$, we are led, and owing to (2.19), (2.20) and (2.24), to

$$\|\mu_{\varepsilon,k}\|_{L^2_{\mathrm{uloc}}([0,\infty);L^2(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}},$$

which together with (2.20) gives

$$\|\mu_{\varepsilon,k}\|_{L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}}. \tag{2.29}$$

Let us finally prove the following estimate on the time derivative:

$$\left\|\frac{\partial\varphi_{\varepsilon,k}}{\partial t}\right\|_{L^2([0,\infty);H^1(\Omega_\varepsilon)')} \leq C\varepsilon^{\frac{1}{2}}. \tag{2.30}$$

To that end, let $\zeta \in L^2([0,\infty); H^1(\Omega_\varepsilon))$ with coefficients $(\zeta_j^k)_{1\leq j\leq k}$ such that $\mathcal{P}_k v = \sum_{j=1}^k \zeta_j^k w_j$. Taking $v = \zeta_j^k w_j$ in (2.5) and summing over $j = 1, \dots, k$, next integrating in time over $[0,\infty)$ yields

$$\begin{aligned}
\left|\int_{Q_\varepsilon}\frac{\partial\varphi_{\varepsilon,k}}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \left|\int_{Q_\varepsilon}\mathbf{u}_{\varepsilon,k}\varphi_{\varepsilon,k}\cdot\nabla\mathcal{P}_k\zeta\,\mathrm{d}x\,\mathrm{d}t\right| + \left|\int_{Q_\varepsilon} m^\varepsilon(\cdot,\varphi_{\varepsilon,k})\nabla\mu_{\varepsilon,k}\cdot\nabla\mathcal{P}_k\zeta\,\mathrm{d}x\,\mathrm{d}t\right|\\
&\leq \|\mathbf{u}_{\varepsilon,k}\|_{L^2([0,\infty);L^4(\Omega_\varepsilon))}\|\varphi_{\varepsilon,k}\|_{L^\infty([0,\infty);L^4(\Omega_\varepsilon))}\|\nabla\mathcal{P}_k\zeta\|_{L^2(Q_\varepsilon)}\\
&\quad + m_2\|\nabla\mu_{\varepsilon,k}\|_{L^2(Q_\varepsilon)}\|\nabla\mathcal{P}_k\zeta\|_{L^2(Q_\varepsilon)}.
\end{aligned}$$

It follows by (2.12) in (Lemma 2.1) together with the embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^4(\Omega_\varepsilon)$ (with the Sobolev constant being independent of ε), (2.18), (2.24) and (2.29) that

$$\begin{aligned}
\left|\int_{Q_\varepsilon}\frac{\partial\varphi_{\varepsilon,k}}{\partial t}\zeta\,\mathrm{d}x\,\mathrm{d}t\right| &\leq \left(C\varepsilon^{\frac{1}{2}}\|\nabla\boldsymbol{u}_{\varepsilon,k}\|_{L^2(Q_\varepsilon)}\|\varphi_{\varepsilon,k}\|_{L^\infty([0,\infty);H^1(\Omega_\varepsilon))} + \|\nabla\mu_{\varepsilon,k}\|_{L^2(Q_\varepsilon)}\right)\|\nabla\mathcal{P}_k\zeta\|_{L^2(Q_\varepsilon)}\\
&\leq \left(C\varepsilon\|\nabla\mathbf{u}_{\varepsilon,k}\|_{L^2(Q_\varepsilon)} + \|\nabla\mu_{\varepsilon,k}\|_{L^2(Q_\varepsilon)}\right)\|\nabla_{\overline{x}}\zeta\|_{L^2(Q_\varepsilon)}\\
&\leq C\varepsilon^{\frac{1}{2}}\|\zeta\|_{L^2([0,\infty);H^1(\Omega_\varepsilon))}
\end{aligned}$$

where we have used the estimates (2.18) and (2.20). This leads us to

$$\left\|\frac{\partial\varphi_{\varepsilon,k}}{\partial t}\right\|_{L^2(0,\infty;H^1(\Omega_\varepsilon)')} \leq C\varepsilon^{\frac{1}{2}},$$

where C is independent of both ε and k.

We have just proved the following result.

Proposition 2.1. *Let $(\mathbf{u}_{\varepsilon,k}, \varphi_{\varepsilon,k}, \mu_{\varepsilon,k})$ be defined by (2.4)-(2.7). It holds that*

$$\|\mathbf{u}_{\varepsilon,k}\|_{L^2([0,\infty);L^2(\Omega_\varepsilon)^3)} \leq C\varepsilon^{\frac{1}{2}},$$

$$\varepsilon\|\nabla\mathbf{u}_{\varepsilon,k}\|_{L^2(Q_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{2}},$$

$$\|\varphi_{\varepsilon,k}\|_{L^\infty([0,\infty);H^1(\Omega_\varepsilon))} \le C\varepsilon^{\frac{1}{2}},$$

$$\|\mu_{\varepsilon,k}\|_{L^2_{\text{uloc}}([0,\infty);H^1(\Omega_\varepsilon))} \le C\varepsilon^{\frac{1}{2}},$$

$$\|F'(\varphi_{\varepsilon,k})\|_{L^\infty([0,\infty);L^1(\Omega_\varepsilon))} \le C\varepsilon,$$

$$\left\|\frac{\partial\varphi_{\varepsilon,k}}{\partial t}\right\|_{L^2([0,\infty);H^1(\Omega_\varepsilon)')} \le C\varepsilon^{\frac{1}{2}},$$

where $C > 0$ is a constant independent of both k and ε.

2.4. Passage to the limit

We fix ε. Owing to the estimates in Proposition 2.1, we derive the existence of functions $\mathbf{u}_\varepsilon \in L^2([0,\infty);\mathbb{V}_\varepsilon)$, $\varphi_\varepsilon \in L^\infty([0,\infty);H^1(\Omega_\varepsilon))$ with $\frac{\partial\varphi_\varepsilon}{\partial t} \in L^2([0,\infty);H^1(\Omega_\varepsilon)')$, $\mu_\varepsilon \in L^2_{\text{uloc}}([0,\infty);H^1(\Omega_\varepsilon))$ such that, up to a subsequence not relabeled,

$$\mathbf{u}_{\varepsilon,k} \to \mathbf{u}_\varepsilon \text{ in } L^2([0,\infty),\mathbb{V}_\varepsilon)\text{-weak},$$

$$\varphi_{\varepsilon,k} \to \varphi_\varepsilon \text{ in } L^\infty([0,\infty);H^1(\Omega_\varepsilon))\text{-weak } * \text{ and in } L^2_{\text{loc}}([0,\infty);L^2(\Omega_\varepsilon))\text{-strong},$$

$$\mu_{\varepsilon,k} \to \mu_\varepsilon \text{ in } L^2_{\text{uloc}}([0,\infty);H^1(\Omega_\varepsilon))\text{-weak } *,$$

when $k \to \infty$. We proceed as in [15] to prove that the triple $(\mathbf{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ satisfies (2.1), (2.2) and (2.3). Now, considering (2.1), we notice that it is equivalent to $\langle \mathbf{g}_\varepsilon(t), \mathbf{v}\rangle = 0$ for all $\mathbf{v} \in C_0^\infty(\Omega_\varepsilon)^3$ with $\operatorname{div}\mathbf{v} = 0$, where $\mathbf{g}_\varepsilon = \mathbf{g} + \alpha\varepsilon^2\Delta\mathbf{u}_\varepsilon + \mu_\varepsilon\nabla\varphi_\varepsilon$ and where $\langle,\rangle$ stands for the duality pairings between $\mathcal{D}'(\Omega_\varepsilon)^3$ and $\mathcal{D}(\Omega_\varepsilon)^3$. Arguing as in the proof of [33, Proposition 5], we derive the existence of a unique $p_\varepsilon \in L^2_{\text{loc}}([0,\infty);L^2_0(\Omega_\varepsilon))$ such that $\nabla p_\varepsilon = \mathbf{g}_\varepsilon$. The fact that $p_\varepsilon \in L^2_{\text{uloc}}([0,\infty);L^2_0(\Omega_\varepsilon))$ is shown below in a subsequent result.

2.5. Uniform estimates of the solutions of (1.1)

One may take the $\liminf_{k\to\infty}$ in (2.22), (2.18) and in (2.27) to get respectively (2.31), (2.32) and (2.35) below; we may also proceed as in Subsection 2.3 to check that the triple $(\mathbf{u}_\varepsilon, \varphi_\varepsilon, \mu_\varepsilon)$ further verifies (2.33), (2.34) and (2.36) below:

$$\|\mathbf{u}_\varepsilon\|_{L^2([0,\infty);L^2(\Omega_\varepsilon)^3)} \le C\varepsilon^{\frac{1}{2}}, \tag{2.31}$$

$$\varepsilon\|\nabla\mathbf{u}_\varepsilon\|_{L^2(Q_\varepsilon)^{3\times3}} \le C\varepsilon^{\frac{1}{2}}, \tag{2.32}$$

$$\|\varphi_\varepsilon\|_{L^\infty([0,\infty);H^1(\Omega_\varepsilon))} \le C\varepsilon^{\frac{1}{2}}, \tag{2.33}$$

$$\|\mu_\varepsilon\|_{L^2_{\text{uloc}}([0,\infty);H^1(\Omega_\varepsilon))} \le C\varepsilon^{\frac{1}{2}}, \tag{2.34}$$

$$\|F'(\varphi_\varepsilon)\|_{L^\infty([0,\infty);L^1(\Omega_\varepsilon))} \le C\varepsilon, \tag{2.35}$$

$$\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2([0,\infty);H^1(\Omega_\varepsilon)')} \le C\varepsilon^{\frac{1}{2}}, \tag{2.36}$$

where $C > 0$ is a constant independent of ε.

In order to deal with the strong convergence of the order parameter φ_ε, we need a further estimate. Before we can proceed any farther, let us, however define the partial integral $M_\varepsilon\varphi_\varepsilon$ of φ_ε as the average in the thin direction as follows:

$$\begin{aligned} M_\varepsilon\varphi_\varepsilon(t,\overline{x}) &= \fint_{\varepsilon h_1^\varepsilon(\overline{x})}^{\varepsilon h_2^\varepsilon(\overline{x})} \varphi_\varepsilon(t,\overline{x},x_3)\,\mathrm{d}x_3 \\ &= \int_0^1 \varphi_\varepsilon(t,\overline{x},\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x}))\,\mathrm{d}\tau, \quad (t,\overline{x}) \in Q. \end{aligned} \tag{2.37}$$

Then $M_\varepsilon\varphi_\varepsilon \in L^\infty(0,\infty;H^1(\Omega))$. Indeed, by the Lebesgue dominated convergence theorem, we have, for $1 \le i \le 2$,

$$\begin{aligned} \frac{\partial}{\partial x_I} M_\varepsilon\varphi_\varepsilon(t,\overline{x}) = \int_0^1 & \frac{\partial\varphi_\varepsilon}{\partial x_i}(t,\overline{x},\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x})) + \\ &+ \left[(1-\tau)\left(\frac{\partial h_1}{\partial y_i}\right)^\varepsilon(\overline{x}) + \tau\left(\frac{\partial h_2}{\partial y_i}\right)^\varepsilon(\overline{x})\right]\frac{\partial\varphi_\varepsilon}{\partial x_3}(t,\overline{x},\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x}))\,\mathrm{d}\tau, \end{aligned}$$

so that, with the help of (2.24) together with the fact that $\left\|\nabla_{\overline{y}} h_i\right\|_{L^\infty} \le C$,

$$\|M_\varepsilon\varphi_\varepsilon\|_{L^\infty(0,\infty;H^1(\Omega))} \le C. \tag{2.38}$$

With this in mind, we have the following result.

Proposition 2.2. *There holds*

$$\sup_{\varepsilon>0}\sup_{t\geq 0}\int_t^{t+1}\|M_\varepsilon\varphi_\varepsilon(\tau+h,\cdot)-M_\varepsilon\varphi_\varepsilon(\tau,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}\tau\leq C|h|^{\frac{1}{2}},\tag{2.39}$$

for any $0<|h|\ll 1$, where C is a positive constant independent of ε and h.

Proof. Assume without lost of generality that $0<h\ll 1$. Then

$$\begin{aligned}\|M_\varepsilon\varphi_\varepsilon(t+h,\cdot)-M_\varepsilon\varphi_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)}&=\int_\Omega\left|M_\varepsilon\varphi_\varepsilon(t+h,\overline{x})-M_\varepsilon\varphi_\varepsilon(t,\overline{x})\right|^2\mathrm{d}\overline{x}\\&=\int_\Omega\left|\fint_{\varepsilon h_1^\varepsilon(\overline{x})}^{\varepsilon h_2^\varepsilon(\overline{x})}(\varphi_\varepsilon(t+h,\overline{x},x_3)-\varphi_\varepsilon(t,\overline{x},x_3))\,\mathrm{d}x_3\right|^2\mathrm{d}\overline{x}\\&\leq\int_\Omega\fint_{\varepsilon h_1^\varepsilon(\overline{x})}^{\varepsilon h_2^\varepsilon(\overline{x})}\left|\varphi_\varepsilon(t+h,x)-\varphi_\varepsilon(t,x)\right|^2\mathrm{d}x\\&=\frac{1}{\varepsilon}\int_{\Omega_\varepsilon}\frac{1}{h_2^\varepsilon(\overline{x})-h_1^\varepsilon(\overline{x})}\left|\varphi_\varepsilon(t+h,x)-\varphi_\varepsilon(t,x)\right|^2\mathrm{d}x\\&\leq\frac{c_0}{\varepsilon}\int_{\Omega_\varepsilon}\left|\varphi_\varepsilon(t+h,x)-\varphi_\varepsilon(t,x)\right|^2\mathrm{d}x,\end{aligned}$$

where $c_0=1/(\min_Y h_2-\max_Y h_1)>0$. But

$$\begin{aligned}\int_{\Omega_\varepsilon}\left|\varphi_\varepsilon(t+h,x)-\varphi_\varepsilon(t,x)\right|^2\mathrm{d}x=&\left\langle\int_t^{t+h}\frac{\partial\varphi_\varepsilon}{\partial\tau}(\tau,\cdot)\,\mathrm{d}\tau,\varphi_\varepsilon(t+h,\cdot)-\varphi_\varepsilon(t,\cdot)\right\rangle\\&\leq h^{\frac{1}{2}}\left\|\varphi_\varepsilon(t+h,\cdot)-\varphi_\varepsilon(t,\cdot)\right\|_{H^1(\Omega_\varepsilon)}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(t,t+h;H^1(\Omega_\varepsilon)')}.\end{aligned}$$

Integrating the last inequality above over $(t,t+1)$ (for any $t\geq 0$) and then taking the $\sup_{t\geq 0}$, it follows that

$$\|M_\varepsilon\varphi_\varepsilon(\cdot+h,\cdot)-M_\varepsilon\varphi_\varepsilon\|^2_{L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega))}\leq\frac{C}{\varepsilon}h^{\frac{1}{2}}\|\varphi_\varepsilon\|_{L^\infty(0,\infty;H^1(\Omega_\varepsilon))}\left\|\frac{\partial\varphi_\varepsilon}{\partial t}\right\|_{L^2(0,\infty;H^1(\Omega_\varepsilon)')},$$

which, thanks to (2.24) and (2.30), yields (2.39). □

Remark 2.1. We infer from (2.39) that, for any $0<T<\infty$,

$$\sup_{\varepsilon>0}\int_0^{T-h}\|M_\varepsilon\varphi_\varepsilon(t+h,\cdot)-M_\varepsilon\varphi_\varepsilon(t,\cdot)\|^2_{L^2(\Omega)}\,\mathrm{d}t\leq C|h|^{\frac{1}{2}},\ \forall 0<|h|\ll 1,\tag{2.40}$$

where C is a positive constant independent of ε and h.

We close the section by the estimate of the pressure. To that end, we start with the following lemma that proves useful in estimating the pressure.

Proposition 2.3. *Let $p_\varepsilon\in L^2_{\mathrm{uloc}}(0,\infty;L^2_0(\Omega_\varepsilon))$ satisfying (1.1)$_1$. Then there exists $(p_\varepsilon^0,p_\varepsilon^1)\in L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega))\times L^2_{\mathrm{uloc}}([0,\infty);L^2(\Omega_\varepsilon))$ such that $p_\varepsilon=p_\varepsilon^0+\varepsilon p_\varepsilon^1$ and*

$$\left\|p_\varepsilon^0\right\|_{L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega))}\leq C,\quad\left\|p_\varepsilon^1\right\|_{L^2_{\mathrm{uloc}}([0,\infty);L^2(\Omega_\varepsilon))}\leq C\varepsilon^{\frac{1}{2}},\tag{2.41}$$

where $C>0$ is independent of ε.

Proof. Thanks to [13, Corollary 3.4] (recall that since Ω is Lipschitz and connected, the domain Ω_ε satisfies the assumptions of [13, Corollary 3.4]), there exists a couple $(p_\varepsilon^0,p_\varepsilon^1)\in L^2_{\mathrm{loc}}([0,\infty);H^1(\Omega))\times L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega_\varepsilon))$ such that

$$p_\varepsilon(t)=p_\varepsilon^0(t)+\varepsilon p_\varepsilon^1(t)\text{ in }\Omega_\varepsilon,\ \text{a.e. }t>0,\tag{2.42}$$

and

$$\varepsilon^{\frac{3}{2}}\left\|p_\varepsilon^0(t)\right\|_{H^1(\Omega)}+\varepsilon\left\|p_\varepsilon^1(t)\right\|_{L^2(\Omega_\varepsilon)}\leq C\left\|\nabla p_\varepsilon(t)\right\|_{H^{-1}(\Omega^\varepsilon)^3},\tag{2.43}$$

where the positive constant C in (2.43) is independent of ε and t. So we need to estimate $\|\nabla p_\varepsilon\|_{L^2_{\mathrm{uloc}}([0,\infty);H^{-1}(\Omega^\varepsilon)^3)}$. To that end, let $\mathbf{v}\in H^1_0(\Omega_\varepsilon)^3$; then appealing to (1.1)$_1$, we have, for a.e. $t>0$,

$$\begin{aligned}\left|\langle\nabla p_\varepsilon(t),\mathbf{v}\rangle\right|\leq&\left|\int_{\Omega_\varepsilon}\mathbf{g}\cdot\mathbf{v}\,\mathrm{d}x\right|+\alpha\varepsilon^2\left|\int_{\Omega_\varepsilon}\nabla\mathbf{u}_\varepsilon\cdot\nabla\mathbf{v}\,\mathrm{d}x\right|+\left|\int_{\Omega_\varepsilon}\mu_\varepsilon\nabla\varphi_\varepsilon\cdot\mathbf{v}\,\mathrm{d}x\right|\\&\leq C\varepsilon^2\left\|\nabla\mathbf{u}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)}+\|\mathbf{g}\|_{L^2(\Omega_\varepsilon)}\|\mathbf{v}\|_{L^2(\Omega_\varepsilon)}\\&+\left\|\mu_\varepsilon\right\|_{L^4(\Omega_\varepsilon)}\left\|\nabla\varphi_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\mathbf{v}\|_{L^4(\Omega_\varepsilon)}\\&\leq C\varepsilon^2\left\|\nabla\mathbf{u}_\varepsilon\right\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{\frac{3}{2}}\left\|\mathbf{g}_1\right\|_{L^2(\Omega)}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)}\end{aligned}$$

$$+ C\varepsilon^{\frac{1}{2}}\|\mu_\varepsilon\|_{H^1(\Omega_\varepsilon)}\|\nabla\varphi_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla\mathbf{v}\|_{L^2(\Omega_\varepsilon)},$$

where we have used (2.11) and (2.12) together with the embedding $H^1(\Omega_\varepsilon) \hookrightarrow L^4(\Omega_\varepsilon)$ (in which the embedding constant is independent of ε). This gives, for a.e. $t \geq 0$,

$$\|\nabla p_\varepsilon(t)\|_{H^{-1}(\Omega_\varepsilon)^3} \leq C\varepsilon^2\|\nabla\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega_\varepsilon)} + C\varepsilon^{\frac{3}{2}}\|\mathbf{g}_1(t)\|_{L^2(\Omega)} + C\varepsilon\|\mu_\varepsilon(t)\|_{H^1(\Omega_\varepsilon)}, \tag{2.44}$$

where we have now used (2.24) after having taken the $\sup_{\|\mathbf{v}\|_{H^1_0(\Omega_\varepsilon)^3}\leq 1}$ on both sides of the previous series of inequalities here above. Raising both members of (2.44) and integrating the resulting inequality over $(t, t+1)$ (for any $t \geq 0$) and then taking the $\sup_{t\geq 0}$, and relying on (2.18) and (2.29), we get

$$\|\nabla p_\varepsilon\|_{L^2_{\text{uloc}}([0,\infty);H^{-1}(\Omega_\varepsilon)^3)} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.45}$$

Putting together (2.43) and (2.45), we are led to

$$\left\|p^0_\varepsilon\right\|_{L^2_{\text{uloc}}([0,\infty);H^1(\Omega))} \leq C \text{ and } \left\|p^1_\varepsilon\right\|_{L^2_{\text{uloc}}([0,\infty);L^2(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}},$$

thereby completing the proof. □

3. Two-scale convergence for thin heterogeneous domains

The two-scale convergence for thin heterogeneous domains with flat parallel boundaries were introduced in [26]. In this section, we extend it to thin heterogeneous domains with highly oscillating boundaries. Prior to that, we define some useful sets and spaces. Any x in $\mathbb{R}^3$ writes $(\overline{x}, x_3)$ where $\overline{x} = (x_1, x_2)$. The domain Ω_ε being defined as in Section 1, when $\varepsilon \to 0$, Ω_ε shrinks to the "interface" $\Omega_0 = \Omega \times \{0\}$ (we recall that $0 \in [h_1^-, h_2^+]$). For any positive number $T \in (0, \infty]$, we define the spatiotemporal domain $Q^T_\varepsilon = (0, T) \times \Omega_\varepsilon$, and we set $Q^T = (0, T) \times \Omega_0 \equiv (0, T) \times \Omega$. We identify Ω_0 with Ω so that the generic element in Ω_0 is also denoted by $\overline{x}$ instead of $(\overline{x}, 0)$. If $T = \infty$, we shall often merely write Q_ε (resp. Q) instead of Q^∞_ε (resp. Q^∞).

This being so, let

$$J = \left\{y = (\overline{y}, y_3) \in \mathbb{R}^3 : \overline{y} \in \mathbb{R}^2 \text{ and } h_1(\overline{y}) < y_3 < h_2(\overline{y})\right\},$$

$$Z = \left\{y \in J : \overline{y} \in Y\right\}, \quad Y = (0,1)^2 \text{ and } \Gamma = \left\{y \in \partial J : \overline{y} \in Y\right\}. \tag{3.1}$$

We start with the following space:

$$C_\#(Z) = \left\{u \in C(J) : u(\overline{y} + k, y_3) = u(y) \ \ \forall k \in \mathbb{Z}^2, \ \forall y \in J\right\},$$

together with its smooth subspaces

$$C^\ell_\#(Z) = \left\{u \in C^\ell(J) : D^\alpha_y u \in C_\#(Z)\ \forall \alpha \in \mathbb{N}^3, \ |\alpha| \leq \ell\right\}, (\ell \in N),$$
$$C^\infty_\#(Z) = \cap_{\ell\in\mathbb{N}} C^\ell_\#(Z).$$

$C_\#(Z)$ is equipped with the norm $\|u\|_\infty = \sup_{y\in Z}|u(y)|$, which makes it a Banach space. Next, for $1 \leq p < \infty$, $L^p_\#(Z)$ stands for the completion of $C_\#(Z)$ with respect to the norm $\|u\|_{L^p_\#(Z)} = \left(\int_Z |u(y)|^p\, dy\right)^{1/p}$ $(u \in L^p_{\text{loc}}(J))$. It is a fact that

$$L^p_\#(Z) = \left\{u \in L^p_{\text{loc}}(J) : \int_Z |u(y)|^p\, dy < \infty, \ \forall k \in \mathbb{Z}^2, \text{ a.e. } y \in J\right\}.$$

The following Sobolev-type spaces are also in order:

$$W^{1,p}_\#(J) = \left\{u \in L^p_\#(Z) : \nabla_y u \in L^p_\#(Z)^3\right\},\ W^{1,p}_\#(J)/\mathbb{R} = \left\{u \in W^{1,p}_\#(J) : \int_Z u\, dy = 0\right\},$$
$$W^{1,p}_{0,\#}(J) = \left\{u \in W^{1,p}_\#(J) : u = 0 \text{ on } \Gamma\right\}.$$

We endow $W^{1,p}_\#(J)$ with the relative norm.

Bearing all this in mind, here below is the definition of the concept.

Definition 3.1. A sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q^T_\varepsilon)$ $(1 \leq p < \infty)$ is said to

(i) weakly two-scale converge in $L^p(Q^T_\varepsilon)$ to $u_0 \in L^p(Q^T; L^p_\#(Z))$ if as $\varepsilon \to 0$,

$$\frac{1}{\varepsilon}\int_{Q^T_\varepsilon} u_\varepsilon(t, x) f\left(t, \overline{x}, \frac{x}{\varepsilon}\right) dx\, dt \to \frac{1}{|Z|}\iint_{Q^T\times Z} u_0(t, \overline{x}, y) f(t, \overline{x}, y)\, dy\, d\overline{x}\, dt$$

for any $f \in L^{p'}(Q^T; C_\#(Z))$ $(1/p' = 1 - 1/p)$; we denote this by "$u_\varepsilon \to u_0$ in $L^p(Q^T_\varepsilon)$-weak $2s$";

(ii) strongly two-scale converge in $L^p(Q_\varepsilon^T)$ to $u_0 \in L^p(Q^T; L^p_\#(Z))$ if it is weakly two-scale convergent and further

$$\varepsilon^{-\frac{1}{p}} \|u_\varepsilon\|_{L^p(Q_\varepsilon^T)} \to \|u_0\|_{L^p(Q^T;L^p_\#(Z))}; \tag{3.2}$$

we denote this by "$u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon^T)$-strong $2s$".

Remark 3.1. It is easy to see that if:

(1) $u_0 \in L^p(Q^T; C_\#(Z))$ then (3.2) is equivalent to

$$\varepsilon^{-\frac{1}{p}} \left\| u_\varepsilon - u_0^\varepsilon \right\|_{L^p(Q_\varepsilon^T)} \to 0 \text{ as } \varepsilon \to 0, \tag{3.3}$$

where $u_0^\varepsilon(t,x) = u_0(t,\overline{x},x/\varepsilon)$ for $(t,x) \in Q_\varepsilon^T$.

(2) $h_1 := -1$ and $h_2 := 1$, then we recover the definition stated in [26] in the case of thin domains with flat parallel lateral boundaries.

(3) If "$u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon^T)$-weak $2s$", then one easily sees that $M_\varepsilon u_\varepsilon \to \frac{1}{|Z|}\int_Z u_0 \,\mathrm{d}y$ in $L^p(Q_\varepsilon^T)$-weak, where $M_\varepsilon u_\varepsilon$ is defined by (2.37).

(4) It is worth noting that Definition 3.1 has been generalized in [22] to more deterministic settings.

Before proceeding further, we need to compare Definition 3.1 with the usual definition of the two-scale convergence. Prior to that, for any $u \in L^p_{\mathrm{loc}}(J)$ $(1 \le p < \infty)$, we define the transform $u^b \in L^p_{\mathrm{loc}}(\mathbb{R}^2; L^2(0,1))$ as follows:

$$u^b(\overline{y},\tau) = u(\overline{y},(1-\tau)h_1(\overline{y}) + \tau h_2(\overline{y})), \quad (\overline{y},\tau) \in \mathbb{R}^2 \times (0,1). \tag{3.4}$$

Then, define the mapping $L : C_\#(Z) \to C_{\mathrm{per}}(Y; C(\overline{I}))$ by $Lu = u^b$, where $I = (0,1)$. Clearly L is linear and satisfies

$$\|u\|_{L^p_\#(Z)} = \left\| (h_2 - h_1)^{1/p} Lu \right\|_{L^p_{\mathrm{per}}(Y;L^p(I))} \text{ for } u \in C_\#(Z).$$

Since $h_2 - h_1$ is a bounded continuous positive function, there are $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 \le h_2 - h_1 \le \alpha_2$ in Y, so that

$$\alpha_1^{1/p} \|Lu\|_{L^p_{\mathrm{per}}(Y;L^p(I))} \le \|u\|_{L^p_\#(Z)} \le \alpha_2^{1/p} \|Lu\|_{L^p_{\mathrm{per}}(Y;L^p(I))}.$$

Moreover, L is surjective as

$$(L^{-1}v)(y) = v\left(\overline{y}, \frac{y_3 - h_1(\overline{y})}{h_2(\overline{y}) - h_1(\overline{y})}\right) \text{ for } y \in Z.$$

This shows that L can be extended to an isomorphism still denoted by L, of $L^p_\#(Z)$ onto $L^p_{\mathrm{per}}(Y; L^p(I))$.

For a sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q_\varepsilon^T)$, we also define the transform u_ε^b accordingly:

$$u_\varepsilon^b(t,\overline{x},\tau) = u_\varepsilon(t,\overline{x},\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x})) \text{ for } (t,\overline{x},\tau) \in Q^T \times I. \tag{3.5}$$

Let us now recall the usual two-scale convergence concept in $L^p(Q^T \times I)$: a sequence $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q^T \times I)$ is said to weakly two-scale converge in $L^p(Q^T \times I)$ towards $u_0 \in L^p(Q^T \times I; L^p_{\mathrm{per}}(Y)) \equiv L^p(Q^T; L^p_{\mathrm{per}}(Y; L^p(I)))$ if

$$\int_{Q^T\times I} u_\varepsilon(t,\overline{x},\tau) g(t,\overline{x},\overline{x}/\varepsilon,\tau)\,\mathrm{d}\overline{x}\,\mathrm{d}t\,\mathrm{d}\tau \to \int_{Q^T\times I}\int_Y u_0(t,\overline{x},\overline{y},\tau) g(t,\overline{x},\overline{y},\tau)\,\mathrm{d}\overline{y}\,\mathrm{d}\overline{x}\,\mathrm{d}t\,\mathrm{d}\tau, \tag{3.6}$$

for all $g \in L^{p'}(Q^T \times I; C_{\mathrm{per}}(Y))$. We denote this by $u_\varepsilon \to u_0$ in $L^p(Q^T \times I)$-weak $2s$.

With this in mind, the following result shows that the two-scale convergence in the thin domain Q_ε^T is equivalent to the two-scale convergence in the sense of (3.6).

Proposition 3.1. *Let $(u_\varepsilon)_{\varepsilon>0}$ be a sequence in $L^p(Q_\varepsilon^T)$ $(1 \le p < \infty)$. Then*

$u_\varepsilon \to u_0$ in $L^p(Q_\varepsilon^T)$-weak $2s$ iff $u_\varepsilon^b \to (h_2 - h_1)u_0^b$ in $L^p(Q^T \times I)$-weak $2s$.

Proof. The result arises from the obvious equality

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon^T} u_\varepsilon(t,x) f(t,\overline{x},x/\varepsilon)\,\mathrm{d}t\,\mathrm{d}x = \int_{Q^T\times I} u_\varepsilon^b(t,\overline{x},\tau)(h_2^\varepsilon(\overline{x}) - h_1^\varepsilon(\overline{x}))(f^b)^\varepsilon(t,\overline{x},\tau)\,\mathrm{d}\overline{x}\,\mathrm{d}t\,\mathrm{d}\tau,$$

where we have made the change of variable $x_3 = \varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x})$, so that

$$(f^b)^\varepsilon(t,\overline{x},\tau) = f(t,\overline{x},\overline{x}/\varepsilon,(1-\tau)h_1^\varepsilon(\overline{x}) + \tau h_2^\varepsilon(\overline{x})) \text{ for } (t,\overline{x},\tau) \in Q^T \times I.$$

□

Proposition 3.1 permits us to state without proof, the following two compactness results. Throughout the work, the letter E will stand for any ordinary sequence $(\varepsilon_n)_{n\ge1}$ with $0 < \varepsilon_n \le 1$ and $\varepsilon_n \to 0$ when $n \to \infty$. The generic term of E will be merely denote by ε and $\varepsilon \to 0$ will mean $\varepsilon_n \to 0$ as $n \to \infty$. This being so, the following compactness result holds true.

Theorem 3.1. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(Q_\varepsilon^T)$ $(1 < p < \infty)$ such that*

$$\sup_{\varepsilon\in E} \varepsilon^{-1/p} \|u_\varepsilon\|_{L^p(Q_\varepsilon^T)} \le C$$

where C is a positive constant independent of ε. Then there exists a subsequence E' of E such that the sequence $(u_\varepsilon)_{\varepsilon\in E'}$ weakly two-scale converges in $L^p(Q_\varepsilon^T)$ to some $u_0 \in L^p(Q^T; L^p_\#(Z))$.

We also state a compactness result dealing with the convergence of gradient.

Theorem 3.2. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T;W^{1,p}(\Omega_\varepsilon))$ $(1<p<\infty)$ such that*

$$\sup_{\varepsilon\in E}\left(\varepsilon^{-1/p}\|u_\varepsilon\|_{L^p(Q_\varepsilon^T)}+\varepsilon^{-1/p}\|\nabla u_\varepsilon\|_{L^p(Q_\varepsilon^T)}\right)\le C, \tag{3.7}$$

where $C>0$ is independent of ε. Then there exist a subsequence E' of E and a couple (u_0,u_1) with $u_0\in L^p(0,T;W^{1,p}(\Omega))$ and $u_1\in L^p(Q^T;W_\#^{1,p}(Z)/\mathbb{R})$ such that, as $E'\ni\varepsilon\to 0$,

$$u_\varepsilon\to u_0 \text{ in } L^p(Q_\varepsilon^T)\text{-weak } 2s, \tag{3.8}$$

$$\frac{\partial u_\varepsilon}{\partial x_i}\to\frac{\partial u_0}{\partial x_i}+\frac{\partial u_1}{\partial y_i} \text{ in } L^p(Q_\varepsilon^T)\text{-weak } 2s,\ i=1,2, \tag{3.9}$$

$$\frac{\partial u_\varepsilon}{\partial x_3}\to\frac{\partial u_1}{\partial y_3} \text{ in } L^p(Q_\varepsilon^T)\text{-weak } 2s. \tag{3.10}$$

Remark 3.2. If we set

$$\nabla_{\overline{x}}u_0=\left(\frac{\partial u_0}{\partial x_1},\frac{\partial u_0}{\partial x_2},0\right),$$

then (3.9) and (3.10) are equivalent to

$$\nabla u_\varepsilon\to\nabla_{\overline{x}}u_0+\nabla_y u_1 \text{ in } L^p(Q_\varepsilon^T)^3-weak2s.$$

The following result provides us with sufficient conditions for which the convergence result in (3.8) is strong.

Theorem 3.3. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T;W^{1,p}(\Omega_\varepsilon))$ $(1<p<\infty)$ such that*

$$\sup_{\varepsilon>0}\varepsilon^{-\frac{1}{p}}\|u_\varepsilon\|_{L^p(0,T;W^{1,p}(\Omega_\varepsilon))}\le C, \tag{3.11}$$

where C is a positive constant. Assume further that

$$\sup_{\varepsilon>0}\left(\int_0^{T-h}\|M_\varepsilon u_\varepsilon(t+h,\cdot)-M_\varepsilon u_\varepsilon(t,\cdot)\|_{L^p(\Omega)}^p\,\mathrm{d}t\right)^{\frac{1}{p}}\le C|h|^\rho, \tag{3.12}$$

for any $0<|h|\ll 1$ and for some $\rho\in(0,1]$, where M_ε is defined by (2.37). Finally, suppose that the embedding $W^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ is compact. Let (u_0,u_1) and E' be as in Theorem 3.2. Then, as $E'\ni\varepsilon\to 0$, the conclusion of Theorem 3.2 holds and further

$$u_\varepsilon\to u_0 \text{ in } L^p(Q_\varepsilon^T)\text{-strong } 2s. \tag{3.13}$$

Proof. We recall that

$$M_\varepsilon u_\varepsilon(t,\overline{x})=⨍_{\varepsilon h_1^\varepsilon(\overline{x})}^{\varepsilon h_2^\varepsilon(\overline{x})}u_\varepsilon(t,\overline{x},x_3)\,\mathrm{d}x_3,\ \text{for a.e. } (t,\overline{x})\in Q^T.$$

Since $(M_\varepsilon u_\varepsilon)_{\varepsilon\in E}$ satisfies (3.11)-(3.12) and further the embedding $W^{1,p}(\Omega)\hookrightarrow L^p(\Omega)$ is compact, we apply [32, Theorem 1] to derive the existence of a subsequence of E' (E' being as in Theorem 3.2) still denoted by E' and of a function $v_0\in L^p(Q^T)$ such that, as $E'\ni\varepsilon\to 0$,

$$M_\varepsilon u_\varepsilon\to v_0 \text{ in } L^p(Q^T)\text{-strong}. \tag{3.14}$$

Next, (3.8) yields $M_\varepsilon u_\varepsilon\to u_0$ in $L^p(Q^T)$-weak (consider test functions in $C_0^\infty(Q^T)$), so that $v_0=u_0$.

Let us now show that

$$\varepsilon^{-\frac{1}{p}}\|u_\varepsilon-M_\varepsilon u_\varepsilon\|_{L^p(Q_\varepsilon^T)}\to 0 \text{ when } E'\ni\varepsilon\to 0. \tag{3.15}$$

To see this, we have

$$\begin{aligned}u_\varepsilon(t,x)-M_\varepsilon u_\varepsilon(t,\overline{x})&=\int_0^1[u_\varepsilon(t,\overline{x},x_3)-u_\varepsilon(t,\overline{x},\varepsilon(1-\tau)h_1^\varepsilon(\overline{x})+\varepsilon\tau h_2^\varepsilon(\overline{x}))]\,\mathrm{d}\tau\\&=\int_0^1\left\{\int_0^1\frac{\partial u_\varepsilon}{\partial x_3}(t,\overline{x},x_3+\zeta(\varepsilon(1-\tau)h_1^\varepsilon(\overline{x})+\varepsilon\tau h_2^\varepsilon(\overline{x})-x_3))\times\right.\\&\qquad\left.\times[\varepsilon(1-\tau)h_1^\varepsilon(\overline{x})+\varepsilon\tau h_2^\varepsilon(\overline{x})-x_3]\,\mathrm{d}\zeta\right\}\mathrm{d}\tau.\end{aligned}$$

Thus,

$$
\begin{aligned}
&\left|u_\varepsilon(t,x) - M_\varepsilon u_\varepsilon(t,\overline{x})\right|^p \leq \\
&\leq \int_0^1 \left\{ \left|\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x}) - x_3\right|^p \times \right. \\
&\left. \times \int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_3}(t,\overline{x},x_3 + \zeta(\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x}) - x_3))\right|^p \mathrm{d}\zeta \right\} \mathrm{d}\tau \\
&\leq \varepsilon^p (h_2^+ - h_1^-)^p \int_0^1 \int_0^1 \left|\frac{\partial u_\varepsilon}{\partial x_3}(t,\overline{x},x_3 + \zeta(\varepsilon(1-\tau)h_1^\varepsilon(\overline{x}) + \varepsilon\tau h_2^\varepsilon(\overline{x}) - x_3))\right|^p \mathrm{d}\zeta\,\mathrm{d}\tau \\
&\leq \varepsilon^{p+1}(h_2^+ - h_1^-)^{p+1} \int_{h_1^\varepsilon(\overline{x})}^{h_2^\varepsilon(\overline{x})} \left|\frac{\partial u_\varepsilon}{\partial x_3}(t,\overline{x},\eta)\right|^p \mathrm{d}\eta.
\end{aligned}
$$

Integrating over Q_ε^T the last series of inequalities above yields

$$
\|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^p(Q_\varepsilon^T)} \leq C\varepsilon^{1+\frac{1}{p}} \left\|\frac{\partial u_\varepsilon}{\partial x_3}\right\|_{L^p(Q_\varepsilon^T)},
$$

from which we get (3.15), taking into account (3.7).

Finally, from the triangle inequality

$$
\varepsilon^{-\frac{1}{p}}\|u_\varepsilon - u_0\|_{L^p(Q_\varepsilon^T)} \leq \varepsilon^{-\frac{1}{p}}\|u_\varepsilon - M_\varepsilon u_\varepsilon\|_{L^p(Q_\varepsilon^T)} + \varepsilon^{-\frac{1}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(Q_\varepsilon^T)}
$$

associated to the inequality

$$
\begin{aligned}
\varepsilon^{-\frac{1}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(Q_\varepsilon^T)} &= \left\|(h_2^\varepsilon - h_1^\varepsilon)^{\frac{1}{p}}(M_\varepsilon u_\varepsilon - u_0)\right\|_{L^p(Q^T)} \\
&\leq (h_2^+ - h_1^-)^{\frac{1}{p}}\|M_\varepsilon u_\varepsilon - u_0\|_{L^p(Q^T)},
\end{aligned}
$$

we infer from both (3.14) and (3.15) that, when $E' \ni \varepsilon \to 0$,

$$
\varepsilon^{-\frac{1}{p}}\|u_\varepsilon - u_0\|_{L^p(Q_\varepsilon^T)} \to 0.
$$

This gives the result. □

The next result and its corollary are proved exactly as their homologues in [31, Theorem 6 and Corollary 5] (see also [39]).

Theorem 3.4. *Let $1 < p, q < \infty$ and $r \geq 1$ be such that $1/r = 1/p + 1/q \leq 1$. Assume $(u_\varepsilon)_{\varepsilon\in E} \subset L^q(Q_\varepsilon^T)$ is weakly two-scale convergent in $L^q(Q_\varepsilon^T)$ to some $u_0 \in L^q(Q^T; L^q_\#(Z))$, and $(v_\varepsilon)_{\varepsilon\in E} \subset L^p(Q_\varepsilon^T)$ is strongly two-scale convergent in $L^p(Q_\varepsilon^T)$ to some $v_0 \in L^p(Q^T; L^p_\#(Z))$. Then the sequence $(u_\varepsilon v_\varepsilon)_{\varepsilon\in E}$ is weakly two-scale convergent in $L^r(Q_\varepsilon^T)$ to $u_0 v_0$.*

Corollary 3.1. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p(Q_\varepsilon^T)$ and $(v_\varepsilon)_{\varepsilon\in E} \subset L^{p'}(Q_\varepsilon^T) \cap L^\infty(Q_\varepsilon^T)$ ($1 < p < \infty$ and $p' = p/(p-1)$) be two sequences such that:*

(i) $u_\varepsilon \to u_0$ *in* $L^p(Q_\varepsilon^T)$*-weak* $2s$;
(ii) $v_\varepsilon \to v_0$ *in* $L^{p'}(Q_\varepsilon^T)$*-strong* $2s$;
(iii) $(v_\varepsilon)_{\varepsilon\in E}$ *is bounded in* $L^\infty(Q_\varepsilon^T)$.

Then $u_\varepsilon v_\varepsilon \to u_0 v_0$ in $L^p(Q_\varepsilon^T)$-weak $2s$.

Another important result is the following proposition.

Proposition 3.2. *Let $(u_\varepsilon)_{\varepsilon\in E}$ be a sequence in $L^p(0,T; W^{1,p}(\Omega_\varepsilon))$ such that*

$$
\sup_{\varepsilon\in E}\left(\varepsilon^{-1/p}\|u_\varepsilon\|_{L^p(Q_\varepsilon^T)} + \varepsilon^{1-1/p}\|\nabla u_\varepsilon\|_{L^p(Q_\varepsilon^T)}\right) \leq C
$$

where $C > 0$ is independent of ε. Then there exist a subsequence E' of E and a function $u_0 \in L^p(Q^T; W^{1,p}_\#(Z))$ such that, as $E' \ni \varepsilon \to 0$,

$$
u_\varepsilon \to u_0 \text{ in } L^p(Q_\varepsilon^T)\text{-weak } 2s,
$$

and

$$
\varepsilon\nabla u_\varepsilon \to \nabla_y u_0 \text{ in } L^p(Q_\varepsilon^T)^3\text{-weak } 2s.
$$

Proof. From Theorem 3.1, we can find a subsequence E' from E and a couple $(u_0, u_1) \in L^p(Q^T; L^p_\#(Z)) \times L^p(Q^T; L^p_\#(Z))^3$ such that, as $E' \ni \varepsilon \to 0$,

$$
u_\varepsilon \to u_0 \text{ in } L^p(Q_\varepsilon^T)\text{-weak } 2s \quad \text{and} \quad \varepsilon\nabla u_\varepsilon \to u_1 \text{ in } L^p(Q_\varepsilon^T)^3\text{-weak } 2s.
$$

Let us characterize u_1 in terms of u_0. To that end, let $\Phi \in (C_0^\infty(Q^T) \otimes C_\#^\infty(Z))^3$; then we have

$$
\varepsilon^{-1}\int_{Q_\varepsilon^T} \varepsilon\nabla u_\varepsilon \cdot \Phi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = -\varepsilon^{-1}\int_{Q_\varepsilon^T} u_\varepsilon\left[\varepsilon(\mathrm{div}_{\overline{x}}\Phi)^\varepsilon + (\mathrm{div}_y\Phi)^\varepsilon\right]\mathrm{d}x\,\mathrm{d}t.
$$

Letting $E' \ni \varepsilon \to 0$, we get

$$\int_{Q^T}\int_Z u_1(t,\overline{x},y)\cdot\Phi(t,\overline{x},y)\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t = -\int_{Q^T}\int_Z u_0(t,\overline{x},y)\,\mathrm{div}_y\,\Phi(t,\overline{x},y)\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t. \tag{3.16}$$

This shows that $u_1 = \nabla_y u_0$, so that $u_0 \in L^p(Q^T; W^{1,p}_{\#}(Z))$. □

The following result whose proof is straightforward and is therefore left to the reader, will be useful in the sequel.

Proposition 3.3. *Let $1 \le p < \infty$ and let $(u_\varepsilon)_{\varepsilon>0} \subset L^p(Q^T_\varepsilon)$ be such that $u_\varepsilon \to u_0$ in $L^p(Q^T_\varepsilon)$-weak $2s$, where $u_0 \in L^p(Q^T; L^p_{\#}(Z))$. Then*

$$\|u_0\|_{L^p(Q^T\times Z)} \le \liminf_{\varepsilon\to 0} \varepsilon^{-\frac{1}{p}}\|u_\varepsilon\|_{L^p(Q^T_\varepsilon)}. \tag{3.17}$$

We close this section with an important compactness result.

Theorem 3.5. *Let $(u_\varepsilon)_{\varepsilon\in E} \subset L^p_{\mathrm{loc}}(0,\infty; L^p(\Omega_\varepsilon))$ ($1 < p < \infty$) be a sequence satisfying*

$$\|u_\varepsilon\|_{L^p_{\mathrm{uloc}}(0,\infty;L^p(\Omega_\varepsilon))} \le C\varepsilon^{\frac{1}{p}}, \tag{3.18}$$

where C is a positive constant independent of ε. Then there exist a subsequence E' of E and a function $u_0 \in L^p_{\mathrm{uloc}}(0,\infty; L^p(\Omega; L^p_{\#}(Z)))$ such that, for all positive real number T,

$$u_\varepsilon \to u_0 \text{ in } L^p(Q^T_\varepsilon)\text{-weak } 2s, \text{ as } E' \ni \varepsilon \to 0. \tag{3.19}$$

Proof. For each nonnegative integer n, we set $Q_{\varepsilon,n} = (n, n+1)\times\Omega_\varepsilon$ and $Q_n = (n, n+1)\times\Omega$. We denote by u^n_ε the restriction of u_ε to $Q_{\varepsilon,n}$. Then $\|u_\varepsilon\|_{L^p_{\mathrm{uloc}}(0,\infty;L^p(\Omega_\varepsilon))} = \sup_{n\in\mathbb{N}}\|u^n_\varepsilon\|_{L^p(Q_{\varepsilon,n})}$, so that $\|u^n_\varepsilon\|_{L^p(Q_{\varepsilon,n})} \le C\varepsilon^{\frac{1}{p}}$. Appealing to Theorem 3.1, we derive the existence, for each n, of a subsequence E_n of E and of a function $u^n_0 \in L^p(Q_n; L^p_{\#}(Z))$ such that $u^n_\varepsilon \to u^n_0$ in $L^p(Q_{\varepsilon,n})$-weak $2s$ when $E_n \ni \varepsilon \to 0$. By a diagonal process, we find that there exists a subsequence E' of E independent of n, such that, as $E' \ni \varepsilon \to 0$,

$$u_\varepsilon \to u^n_0 \text{ in } L^p(Q_{\varepsilon,n})\text{-weak } 2s \text{ for each integer } n. \tag{3.20}$$

This being so, we define the function $u_0 : (0,\infty) \to L^p(\Omega; L^p_{\#}(Z))$ by $u_0 := u^n_0$ in Q_n. Then, by virtue of (3.17), we have

$$\|u_0\|_{L^p(Q_n;L^p_{\#}(Z))} \le \liminf_{\varepsilon\to 0}\varepsilon^{-\frac{1}{p}}\|u_\varepsilon\|_{L^p(Q_{\varepsilon,n})} \le C, \text{ for each } n,$$

where the last inequality above stems from (3.18).

This shows that $u_0 \in L^p_{\mathrm{uloc}}(0,\infty; L^p(\Omega; L^p_{\#}(Z)))$. It remains to check (3.19). Fix $T > 0$ and denote by $[T]$ the integer part of T. We assume without lost of generality that $T \ge 1$. It holds from (3.18) that

$$\sup_{\varepsilon\in E'}\left(\varepsilon^{-\frac{1}{p}}\|u_\varepsilon\|_{L^p(Q^T_\varepsilon))}\right) \le C.$$

So, to prove (3.19), we may only choose test functions in $\mathcal{C}^\infty_0(Q^T)\otimes\mathcal{C}_{\#}(Z)$. Let $f \in \mathcal{C}^\infty_0(Q^T)\otimes\mathcal{C}_{\#}(Z)$, then

$$\frac{1}{\varepsilon}\int_{Q^T_\varepsilon} u_\varepsilon(t,x)f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t = \frac{1}{\varepsilon}\sum_{n=0}^{[T]-1}\int_{Q_{\varepsilon,n}} u_\varepsilon(t,x)f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{[T]}^{T}\int_{\Omega_\varepsilon} u_\varepsilon(t,x)f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t.$$

Since $f = 0$ in $(T, [T]+1)\times\Omega\times Z$, we have

$$\int_{[T]}^{T}\int_{\Omega_\varepsilon} u_\varepsilon(t,x)f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t = \int_{Q_{\varepsilon,[T]}} u_\varepsilon(t,x)f\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t.$$

Also, the restriction of f to each $Q_{\varepsilon,n}$ belongs to $\mathcal{C}(\overline{Q}_{\varepsilon,n})\otimes\mathcal{C}_{\#}(Z)$, so that it can be taken as test function in the two-scale convergence. Therefore, using (3.20) together with the definition of u_0, we get readily (3.19). □

4. Homogenized system

Throughout this section, we assume that the mobility coefficient $m : \mathbb{R}^3\times\mathbb{R} \to \mathbb{R}$, is constrained as follows:

$$m(\overline{y}+k, y_3, r) = m(y, r) \text{ for all } y = (\overline{y}, y_3) \in \mathbb{R}^3 \text{ and all } r \in \mathbb{R}. \tag{4.1}$$

Then we see that $m(\cdot, r) \in \mathcal{C}_{\#}(Z)$ for all $r \in \mathbb{R}$.

4.1. Preliminaries

According to Propositions 2.1, 2.2 and 2.3, the following uniform estimates hold: there exists a positive constant C such that for all $\varepsilon > 0$,

$$\begin{aligned}
&\|\mathbf{u}_\varepsilon\|_{L^2(0,\infty;L^2(\Omega_\varepsilon)^3)} \leq C\varepsilon^{\frac{1}{2}},\ \varepsilon\|\nabla \mathbf{u}_\varepsilon\|_{L^2(Q_\varepsilon)^{3\times 3}} \leq C\varepsilon^{\frac{1}{2}},\ \|\varphi_\varepsilon\|_{L^\infty(0,\infty;H^1(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}},\\
&\|\mu_\varepsilon\|_{L^2_{\mathrm{uloc}}(0,\infty;H^1(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}},\ \|F'(\varphi_\varepsilon)\|_{L^\infty(0,\infty;L^1(\Omega_\varepsilon))} \leq C\varepsilon,\\
&\sup_{\varepsilon>0}\|M_\varepsilon\varphi_\varepsilon(\cdot+h,\cdot) - M_\varepsilon\varphi_\varepsilon(t,\cdot)\|^2_{L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega))} \leq C|h|^{\frac{1}{2}}\ \forall 0<|h|\ll 1,\\
&p_\varepsilon = p^0_\varepsilon + \varepsilon p^1_\varepsilon \text{ with } \left\|p^0_\varepsilon\right\|_{L^2_{\mathrm{uloc}}(0,\infty;H^1(\Omega))} \leq C, \left\|p^1_\varepsilon\right\|_{L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega_\varepsilon))} \leq C\varepsilon^{\frac{1}{2}}.
\end{aligned} \tag{4.2}$$

Bearing this in mind, we have the first preliminary result.

Proposition 4.1. *Up to subsequences, the following convergence results hold true: for every $T > 0$, we have*

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \text{ in } L^2(Q^T_\varepsilon)^3\text{-weak } 2s \tag{4.3}$$

$$\varepsilon\nabla\mathbf{u}_\varepsilon \to \nabla_y\mathbf{u}_0 \text{ in } L^2(Q^T_\varepsilon)^{3\times 3}\text{-weak } 2s \tag{4.4}$$

$$\varphi_\varepsilon \to \varphi_0 \text{ in } L^2(Q^T_\varepsilon)\text{-strong } 2s \tag{4.5}$$

$$\nabla\varphi_\varepsilon \to \nabla_{\overline{x}}\varphi_0 + \nabla_y\varphi_1 \text{ in } L^2(Q^T_\varepsilon)^3\text{-weak } 2s \tag{4.6}$$

$$\mu_\varepsilon \to \mu_0 \text{ in } L^2(Q^T_\varepsilon)\text{-weak } 2s \tag{4.7}$$

$$\nabla\mu_\varepsilon \to \nabla_{\overline{x}}\mu_0 + \nabla_y\mu_1 \text{ in } L^2(Q^T_\varepsilon)^3\text{-weak } 2s, \tag{4.8}$$

$$p^0_\varepsilon \overset{2s}{\to} p_0 \text{ in } L^2(Q^T)\text{-weak}, \tag{4.9}$$

$$\nabla_{\overline{x}}p^0_\varepsilon \overset{2s}{\to} \nabla_{\overline{x}}p_0 + \nabla_{\overline{y}}p^1_0 \text{ in } L^2(Q^T)^2, \tag{4.10}$$

$$p^1_\varepsilon \to p_1 \text{ in } L^2(Q^T_\varepsilon)\text{-weak } 2s, \tag{4.11}$$

where

$\mathbf{u}_0 \in L^2(Q;H^1_{0,\#}(Z))^3$,
$(\varphi_0,\varphi_1),(\mu_0,\mu_1) \in L^2_{\mathrm{uloc}}(0,\infty;H^1(\Omega)) \times L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega;H^1_\#(Z)/\mathbb{R}))$,
$(p_0,p^1_0) \in L^2_{\mathrm{uloc}}(0,\infty;H^1(\Omega))) \times L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega;H^1_{\mathrm{per}}(Y)/\mathbb{R}))$ *and*
$p_1 \in L^2_{\mathrm{uloc}}(0,\infty;L^2(\Omega;L^2_\#(Z)))$,

and where in (4.2),

$\nabla_{\overline{x}}\varphi_0 = \left(\frac{\partial\varphi_0}{\partial x_1}, \frac{\partial\varphi_0}{\partial x_2}, 0\right)$ *(and the same for $\nabla_{\overline{x}}\mu_0$)*

while in (4.10),

$\nabla_{\overline{x}}p_0 = \left(\frac{\partial p_0}{\partial x_1}, \frac{\partial p_0}{\partial x_2}\right)$ *(and the same for $\nabla_{\overline{y}}p^1_0$).*

Proof. By virtue of Theorem 3.5, we use Proposition 3.2 together with Theorems 3.1, 3.2 and 3.3 to get that, given an ordinary sequence E and owing to (4.2), there exist a subsequence E' of E and functions $\mathbf{u}_0, \varphi_0, \varphi_1, \mu_0, \mu_1, p_0, p^1_0, p_1$ satisfying the convergence results in the above proposition when $E' \ni \varepsilon \to 0$, for all $T > 0$. We emphasize on the fact that the strong convergence (4.5) stems from Theorem 3.3 where we have used Remark 2.1. □

In the passage to the limit, we shall also need the following preliminary result.

Lemma 4.1. *Assume that (4.1) holds true. Let φ_0 be determined by Proposition 4.1 and satisfying (4.5). Then, for any $T>0$, we have, as $E'\ni\varepsilon\to 0$,*

$$m^{\varepsilon}(\cdot,\varphi_{\varepsilon})\to m(\cdot,\varphi_0)\text{ in }L^2(Q_{\varepsilon}^T)\text{-strong }2s. \tag{4.12}$$

Proof. First, define the function $g(t,\overline{x},y)=m(y,\varphi_0(t,\overline{x}))$, $(t,\overline{x},y)\in Q^T\times J$. Then, $g\in L^2(Q^T;\mathcal{C}_{\#}(Z))$. Therefore, it is sufficient to show that

$$\varepsilon^{-\frac{1}{2}}\left\|m^{\varepsilon}(\cdot,\varphi_{\varepsilon})-m^{\varepsilon}(\cdot,\varphi_0)\right\|_{L^2(Q_{\varepsilon}^T)}\to 0\text{ when }E'\ni\varepsilon\to 0.$$

But in view of the properties of m ($m(y,\cdot)\in\mathcal{C}^{0,1}_{\mathrm{loc}}(\mathbb{R})$), we have

$$\varepsilon^{-\frac{1}{2}}\left\|m^{\varepsilon}(\cdot,\varphi_{\varepsilon})-g^{\varepsilon}\right\|_{L^2(Q_{\varepsilon}^T)}\le C\varepsilon^{-\frac{1}{2}}\left\|\varphi_{\varepsilon}-\varphi_0\right\|_{L^2(Q_{\varepsilon}^T)}\to 0$$

as $E'\ni\varepsilon\to 0$ (recall that $\varphi_{\varepsilon}\to\varphi_0$ in $L^2(Q_{\varepsilon}^T)$-strong $2s$ and that φ_0 does not depend on y). This concludes the proof. □

Now, since $\operatorname{div}\mathbf{u}_{\varepsilon}=0$ in Q_{ε}^T, it follows that $\operatorname{div}_y\mathbf{u}_0=0$ in $Q^T\times Z$. Indeed, setting $\overline{\mathbf{u}}_{\varepsilon}=(u_{\varepsilon,1},u_{\varepsilon,2})$, we have, for $\phi\in\mathcal{C}_0^{\infty}(Q^T)\otimes\mathcal{C}_{\#}^{\infty}(Z)$,

$$\begin{aligned}0&=\int_{Q_{\varepsilon}^T}\operatorname{div}\mathbf{u}_{\varepsilon}(t,x)\phi\left(t,\overline{x},\frac{x}{\varepsilon}\right)\mathrm{d}x\,\mathrm{d}t\\&=-\int_{Q_{\varepsilon}^T}\overline{\mathbf{u}}_{\varepsilon}\cdot(\nabla_{\overline{x}}\phi)^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_{Q_{\varepsilon}^T}\mathbf{u}_{\varepsilon}\cdot(\nabla_y\phi)^{\varepsilon}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\phi^{\varepsilon}(t,x)=\phi\left(t,\overline{x},\frac{x}{\varepsilon}\right)$ for $(t,x)\in Q_{\varepsilon}^T$. Letting $E'\ni\varepsilon\to 0$ yields

$$\int_{Q^T}\int_Z\mathbf{u}_0(t,\overline{x},y)\cdot\nabla_y\phi(t,\overline{x},y)\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t=0.$$

This amounts to $\operatorname{div}_y\mathbf{u}_0=0$ in $Q^T\times Z$, where $\operatorname{div}_y\mathbf{u}_0=\operatorname{div}_{\overline{y}}\overline{\mathbf{u}}_0+\frac{\partial u_{0,3}}{\partial y_3}$ with $\overline{\mathbf{u}}_0=(u_{0,i})_{1\le i\le 2}$.

Setting

$$\begin{aligned}\mathbf{u}(t,\overline{x})&=\frac{1}{|Z|}\int_Z\mathbf{u}_0(t,\overline{x},y)\,\mathrm{d}y\text{ for }(t,\overline{x})\in Q^T\\&=(u_i(t,\overline{x}))_{1\le i\le 3}\text{ and }\overline{\mathbf{u}}=(u_i)_{1\le i\le 2},\end{aligned} \tag{4.13}$$

we have $\mathbf{u}\in L^2(Q^T)^3$. Moreover

$$\operatorname{div}_{\overline{x}}\overline{\mathbf{u}}=0\text{ in }Q\text{ and }\overline{\mathbf{u}}\cdot\boldsymbol{n}=0\text{ on }(0,\infty)\times\partial\Omega, \tag{4.14}$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$. Indeed, let $\varphi\in\mathcal{D}(\overline{Q})$. Using the Stokes formula together with the equality $\operatorname{div}\mathbf{u}_{\varepsilon}=0$ in Q_{ε}, we obtain

$$\int_{Q_{\varepsilon}}\overline{\mathbf{u}}_{\varepsilon}(t,x)\cdot\nabla_{\overline{x}}\varphi(t,\overline{x})\,\mathrm{d}x\,\mathrm{d}t=0.$$

Dividing the last equality above by ε and letting $E'\ni\varepsilon\to 0$, we are led to

$$\int_Q\overline{\mathbf{u}}(t,x)\cdot\nabla_{\overline{x}}\varphi(t,\overline{x})\,\mathrm{d}\overline{x}\,\mathrm{d}t=0.$$

This yields at once (4.14).

Also since $\int_{\Omega_{\varepsilon}}p_{\varepsilon}\,\mathrm{d}x=0$, we have, for any $\chi\in\mathcal{C}_0^{\infty}(0,T)$,

$$0=\frac{1}{\varepsilon}\int_{Q_{\varepsilon}^T}p_{\varepsilon}\chi\,\mathrm{d}t\,\mathrm{d}x=\int_{Q^T}(h_2^{\varepsilon}(\overline{x})-h_1^{\varepsilon}(\overline{x}))\chi(t)p_{\varepsilon}^0\,\mathrm{d}\overline{x}\,\mathrm{d}t+\int_{Q_{\varepsilon}^T}\chi p_{\varepsilon}^1\,\mathrm{d}x\,\mathrm{d}t,$$

so that letting $E'\ni\varepsilon\to 0$ and using (4.9) and (4.11) we obtain

$$\int_{Q^T}\int_Y(h_2(\overline{y})-h_1(\overline{y}))\chi(t)p_0(t,\overline{x})\,\mathrm{d}\overline{y}\,\mathrm{d}\overline{x}\,\mathrm{d}t=0,$$

that is, $\int_{\Omega}p_0\,\mathrm{d}\overline{x}=0$, and so, $p_0\in L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega)\cap L^2_0(\Omega))$.

4.2. Passage to the limit in (1.1)

The following global homogenized result holds.

Proposition 4.2. *The* 8*-tuplet* $(\mathbf{u}_0, \varphi_0, \varphi_1, \mu_0, \mu_1, p_0, p_0^1, p_1)$ *determined by Proposition 4.1, solves the system (4.15)-(4.18) below:*

$$\begin{cases} \dfrac{\alpha}{|Z|}\displaystyle\iint_{Q^T\times Z} \nabla_y \mathbf{u}_0 \cdot \nabla_y \Psi \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ -\dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \varphi_0 \left[(\nabla_{\overline{x}}\mu_0 + \nabla_y \mu_1)\Psi + \mu_0 \operatorname{div}_{\overline{x}} \Psi\right] \mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ +\dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \nabla p_0 \cdot \Psi \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t - \iint_{Q^T\times Z} (p_0^1 + p_1) \operatorname{div}_y \Psi \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ \quad = \dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \mathbf{g}\Psi \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t; \end{cases} \tag{4.15}$$

$$\begin{cases} -\dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \varphi_0 \frac{\partial \phi_0}{\partial t} \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t - \dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \varphi_0 \mathbf{u}_0 (\nabla_{\overline{x}}\phi_0 + \nabla_y \phi_1) \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ +\dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} m(\cdot,\varphi_0)(\nabla_{\overline{x}}\mu_0 + \nabla_y \mu_1)(\nabla_{\overline{x}}\phi_0 + \nabla_y\phi_1) \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t = 0; \end{cases} \tag{4.16}$$

$$\begin{cases} \dfrac{1}{|Z|}\displaystyle\iint_{Q^T\times Z} \mu_0 \chi_0 \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t = \dfrac{\lambda}{|Z|}\displaystyle\iint_{Q^T\times Z} F'(\varphi_0)\chi_0 \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t \\ +\dfrac{\beta}{|Z|}\displaystyle\int_{Q^T}\int_Z (\nabla_{\overline{x}}\varphi_0 + \nabla_y \varphi_1)(\nabla_{\overline{x}}\chi_0 + \nabla_y \chi_1) \,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t; \end{cases} \tag{4.17}$$

$$\varphi_0(0,\overline{x}) = \varphi^0(\overline{x}) \text{ for a.e. } \overline{x} \in \Omega, \tag{4.18}$$

for all $\Psi \in (C_0^\infty(Q^T) \otimes H^1_{0,\#}(Z))^3$ *and all* $(\phi_0, \phi_1), (\chi_0, \chi_1) \in C_0^\infty(Q^T) \times (C_0^\infty(Q^T) \otimes H^1_\#(Z))$.

Proof. Let $T > 0$ be freely fixed. Let $\Psi \in (C_0^\infty(Q^T) \otimes H^1_{0,\#}(Z))^3$, and let $(\phi_0, \phi_1), (\chi_0, \chi_1) \in C_0^\infty(Q^T) \times (C_0^\infty(Q^T) \otimes H^1_\#(Z))$. We define, for $(t, x) \in Q_\varepsilon^T$

$$\Psi^\varepsilon(t,x) = \Psi\left(t, \overline{x}, \frac{x}{\varepsilon}\right), \quad \phi_\varepsilon(t,x) = \phi_0(t,\overline{x}) + \varepsilon\phi_1\left(t,\overline{x},\frac{x}{\varepsilon}\right)$$
$$\chi_\varepsilon(t,x) = \chi_0(t,\overline{x}) + \varepsilon\chi_1\left(t,\overline{x},\frac{x}{\varepsilon}\right).$$

Taking $(\Psi^\varepsilon, \phi_\varepsilon, \chi_\varepsilon)$ as test function in the variational form (2.1), (2.2) and (2.3), we obtain, after dividing each member of the resulting equalities by ε,

$$\begin{aligned} &\frac{\alpha}{\varepsilon}\int_{Q_\varepsilon^T} \varepsilon \nabla \mathbf{u}_\varepsilon \cdot \left((\nabla_{\overline{x}}\Psi)^\varepsilon + \frac{1}{\varepsilon}(\nabla_y \Psi)^\varepsilon\right) \mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{Q_\varepsilon^T} \nabla_{\overline{x}} p_\varepsilon^0 \cdot \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &-\int_{Q_\varepsilon^T} p_\varepsilon^1 \left((\operatorname{div}_{\overline{x}}\Psi)^\varepsilon + \frac{1}{\varepsilon}(\operatorname{div}_y \Psi)^\varepsilon\right) \mathrm{d}x\,\mathrm{d}t - \frac{1}{\varepsilon}\int_{Q_\varepsilon^T} \mu_\varepsilon \nabla\varphi_\varepsilon \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &\quad = \frac{1}{\varepsilon}\int_{Q_\varepsilon^T} \mathbf{g}\Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t; \end{aligned} \tag{4.19}$$

$$\begin{aligned} &-\frac{1}{\varepsilon}\int_{Q_\varepsilon^T} \varphi_\varepsilon \frac{\partial\phi_\varepsilon}{\partial t} \,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_{Q_\varepsilon^T} (\mathbf{u}_\varepsilon \cdot \nabla\varphi_\varepsilon)\phi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &+\frac{1}{\varepsilon}\int_{Q_\varepsilon^T} m^\varepsilon(\cdot,\varphi_\varepsilon)\nabla\mu_\varepsilon \cdot (\nabla_{\overline{x}}\phi_0 + \varepsilon(\nabla_{\overline{x}}\phi_1)^\varepsilon + (\nabla_y\phi_1)^\varepsilon) \,\mathrm{d}x\,\mathrm{d}t = 0; \end{aligned} \tag{4.20}$$

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon^T} \mu_\varepsilon \chi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t = \frac{\beta}{\varepsilon}\int_{Q_\varepsilon^T} \nabla\varphi_\varepsilon \cdot \nabla\chi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t + \frac{\lambda}{\varepsilon}\int_{Q_\varepsilon^T} F'(\varphi_\varepsilon)\chi_\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \tag{4.21}$$

Let us first deal with (4.19). We use the identity

$$\int_{Q_\varepsilon^T} \mu_\varepsilon \nabla\varphi_\varepsilon \Psi^\varepsilon \,\mathrm{d}x\,\mathrm{d}t = -\int_{Q_\varepsilon^T} \varphi_\varepsilon(\nabla\mu_\varepsilon \Psi^\varepsilon + \mu_\varepsilon(\operatorname{div}_{\overline{x}}\Psi)^\varepsilon) \,\mathrm{d}x\,\mathrm{d}t.$$

in (4.19) and then let therein $E' \ni \varepsilon \to 0$ to get

$$\begin{aligned}
&\frac{\alpha}{|Z|}\iint_{Q^T\times Z}\nabla_y\mathbf{u}_0\cdot\nabla_y\Psi\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t\\
&-\frac{1}{|Z|}\iint_{Q^T\times Z}\varphi_0\left[(\nabla_{\overline{x}}\mu_0+\nabla_y\mu_1)\Psi+\mu_0\,\mathrm{div}_{\overline{x}}\Psi\right]\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t\\
&+\frac{1}{|Z|}\iint_{Q^T\times Z}\nabla p_0\cdot\Psi\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t-\iint_{Q^T\times Z}(p_0^1+p_1)\,\mathrm{div}_y\Psi\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t\\
&=\frac{1}{|Z|}\iint_{Q^T\times Z}\mathbf{g}\Psi\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t,
\end{aligned}\tag{4.22}$$

that is, (4.15). We recall that to obtain the second term of the left-hand side of (4.22), we have used the strong sigma-convergence (4.5) associated to the weak sigma-convergence (4.8) in light of Corollary 3.1.

Let us now consider (4.20). We use the equality

$$\int_{Q_\varepsilon^T}(\mathbf{u}_\varepsilon\nabla\varphi_\varepsilon)\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t=-\int_{Q_\varepsilon^T}\varphi_\varepsilon\mathbf{u}_\varepsilon\nabla\phi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t,$$

and then pass to the limit when $E' \ni \varepsilon \to 0$ in the resulting equality to get (4.16). We recall that in getting (4.16), we have also used Lemma 4.1.

Let us finally deal with (4.21). Therein the limit passage in $\int_{Q_\varepsilon^T} F'(\varphi_\varepsilon)\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t$ needs a careful treatment. Indeed we need to check that

$$\frac{1}{\varepsilon}\int_{Q_\varepsilon^T}F'(\varphi_\varepsilon)\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\to\frac{1}{|Z|}\iint_{Q^T\times Z}F'(\varphi_0)\chi_0\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t.\tag{4.23}$$

First of all, from (4.5) we have $\varepsilon^{-\frac{1}{2}}\left\|\varphi_\varepsilon-\varphi_0\right\|_{L^2(Q_\varepsilon^T)}\to 0$ as $E' \ni \varepsilon \to 0$. But

$$\begin{aligned}
&\varepsilon^{-1}\int_{Q_\varepsilon^T}\left|\varphi_\varepsilon(t,x)-\varphi_0(t,\overline{x})\right|^2\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_{Q^T}\int_0^1\left|\varphi_\varepsilon^b(t,\overline{x},\tau)-\varphi_0(t,\overline{x})\right|^2(h_2^\varepsilon(\overline{x})-h_1^\varepsilon(\overline{x}))\,\mathrm{d}\overline{x}\,\mathrm{d}\tau\,\mathrm{d}t\to 0\text{ as }E'\ni\varepsilon\to 0,
\end{aligned}$$

where φ_ε^b has been defined by (3.5). Since $h_2^\varepsilon(\overline{x})-h_1^\varepsilon(\overline{x})\geq\min_Y h_2-\max_Y h_1=\alpha_1>0$, it follows that

$$\alpha_1\int_{Q^T}\int_0^1\left|\varphi_\varepsilon^b(t,\overline{x},\tau)-\varphi_0(t,\overline{x})\right|^2\mathrm{d}\overline{x}\,\mathrm{d}\tau\,\mathrm{d}t\leq\int_{Q^T}\int_0^1\left|\varphi_\varepsilon^b(t,\overline{x},\tau)-\varphi_0(t,\overline{x})\right|^2(h_2^\varepsilon(\overline{x})-h_1^\varepsilon(\overline{x}))\,\mathrm{d}\overline{x}\,\mathrm{d}\tau\,\mathrm{d}t.$$

This shows that the sequence $(\varphi_\varepsilon^b)_{\varepsilon\in E'}$ defined on $Q^T\times(0,1)$ converges strongly to φ_0 in $L^2(Q^T\times(0,1))$, and so, $\varphi_\varepsilon^b\to\varphi_0$ a.e. in $Q^T\times(0,1)$. The continuity of F' entails $F'(\varphi_\varepsilon^b)\to F'(\varphi_0)$ a.e. in $Q^T\times(0,1)$. Now, the uniform bound $\|F'(\varphi_\varepsilon)\|_{L^1(Q_\varepsilon^T)}\leq C\varepsilon$ yields $\|F'(\varphi_\varepsilon^b)\|_{L^1(Q^T\times(0,1))}\leq C$ for all $\varepsilon>0$. The Lebesgue dominated convergence theorem leads to

$$F'(\varphi_\varepsilon^b)\to F'(\varphi_0)\text{ in }L^1(Q^T\times(0,1))\text{-strong},$$

so that (4.23) is proved.

With this in mind, we pass to the limit in (4.21) and get (4.17). Finally, since $\varphi_0^\varepsilon\to\varphi^0$ in $L^2(\Omega_\varepsilon)$-strong $2s$, we conclude by integration by parts that $\varphi_0(0)=\varphi^0$. The proof is complete. □

4.3. Derivation of the homogenized system

We intend to derive the system whose solution is $(\overline{\mathbf{u}},\varphi_0,\mu_0,p_0)$. It is worth noticing that $\overline{\mathbf{u}}$ is defined by (4.13) and satisfies (4.14). To that end, we first consider (4.17); it is equivalent to the system consisting of (4.24) and (4.25) below:

$$\left\{\begin{aligned}
&\frac{1}{|Z|}\iint_{Q^T\times Z}\mu_0\chi_0\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t=\frac{\beta}{|Z|}\iint_{Q^T\times Z}(\nabla_{\overline{x}}\varphi_0+\nabla_y\varphi_1)\cdot\nabla_{\overline{x}}\chi_0\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t\\
&+\frac{\lambda}{|Z|}\iint_{Q^T\times Z}F'(\varphi_0)\chi_0\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t\text{ for all }\chi_0\in C_0^\infty(Q^T);
\end{aligned}\right.\tag{4.24}$$

$$\iint_{Q^T\times Z}(\nabla_{\overline{x}}\varphi_0+\nabla_y\varphi_1)\cdot\nabla_y\chi_1\,\mathrm{d}y\,\mathrm{d}\overline{x}\,\mathrm{d}t=0,\text{ all }\chi_1\in C_0^\infty(Q^T)\otimes H_\#^1(Z).\tag{4.25}$$

In (4.25) we take χ_1 under the form $\chi_1(t,\overline{x},y)=\chi_1^0(t,\overline{x})\theta(y)$ with $\chi_1^0\in C_0^\infty(Q^T)$ and $\theta\in H_\#^1(Z)$. Then (4.25) becomes

$$\int_Z(\nabla_{\overline{x}}\varphi_0+\nabla_y\varphi_1)\cdot\nabla_y\theta\,\mathrm{d}y=0\ \ \forall\theta\in H_\#^1(Z),$$

which equation is easily seen to possess a unique solution $\varphi_1 \equiv 0$.

This being so, going back to (4.24), we readily see that it is the variational form of the following equation

$$\mu_0 = -\beta\Delta_{\overline{x}}\varphi_0 + \lambda F'(\varphi_0) \text{ in } Q^T. \tag{4.26}$$

Now we consider (4.16) and we see that it is equivalent to the system

$$\begin{cases} -\int_{Q^T} \varphi_0 \dfrac{\partial \phi_0}{\partial t}\, \mathrm{d}\overline{x}\, \mathrm{d}t - \dfrac{1}{|Z|}\iint_{Q^T\times Z} \varphi_0 \boldsymbol{u}_0 \cdot \nabla_{\overline{x}}\phi_0)\, \mathrm{d}y\, \mathrm{d}\overline{x}\, \mathrm{d}t \\ +\dfrac{1}{|Z|}\iint_{Q^T\times Z} m(\cdot,\varphi_0)(\nabla_{\overline{x}}\mu_0 + \nabla_y\mu_1)\cdot\nabla_{\overline{x}}\phi_0\, \mathrm{d}y\, \mathrm{d}\overline{x}\, \mathrm{d}t = 0, \\ \text{for all } \phi_0 \in C_0^\infty(Q^T), \end{cases} \tag{4.27}$$

$$\begin{cases} -\dfrac{1}{|Z|}\iint_{Q^T\times Z} \varphi \boldsymbol{u}_0 \cdot \nabla_y\phi_1\, \mathrm{d}y\, \mathrm{d}\overline{x}\, \mathrm{d}t \\ +\dfrac{1}{|Z|}\iint_{Q^T\times Z} m(\cdot,\varphi)(\nabla_{\overline{x}}\mu_0 + \nabla_y\mu_1)\nabla_y\phi_1\, \mathrm{d}y\, \mathrm{d}\overline{x}\, \mathrm{d}t = 0, \\ \text{for all } \phi_1 \in C_0^\infty(Q^T)\otimes H^1_\#(Z). \end{cases} \tag{4.28}$$

We continue with (4.28), where we choose ϕ_1 under the form $\phi_1(t,\overline{x},y) = \phi_1^0(t,\overline{x})\theta(y)$ with $\phi_1^0 \in C_0^\infty(Q^T)$ and $\theta \in H^1_\#(Z)$. Then, we obtain

$$\int_Z \varphi \boldsymbol{u}_0 \cdot \nabla_y\theta\, \mathrm{d}y + \int_Z m(\cdot,\varphi)(\nabla_{\overline{x}}\mu_0 + \nabla_y\mu_1)\nabla_y\theta\, \mathrm{d}y = 0,$$

which, using the fact that $\int_Z \varphi_0\boldsymbol{u}_0\nabla_y\theta\, \mathrm{d}y = \int_Z \varphi_0 \operatorname{div}_y(\boldsymbol{u}_0\theta)\, \mathrm{d}y = 0$ (recall that φ_0 does not depend on y), amounts to

$$\int_Z m(\cdot,\varphi_0(t,\overline{x}))(\nabla_{\overline{x}}\mu_0(t,\overline{x}) + \nabla_y\mu_1(t,\overline{x},\cdot))\nabla_y\theta\, \mathrm{d}y = 0,\ \theta \in H^1_\#(Z) \tag{4.29}$$

for a.e. $(t,\overline{x}) \in Q^T$, which besides, is the weak formulation of the equation

$$-\operatorname{div}_y(m(\cdot,\varphi_0(t,\overline{x}))(\nabla_{\overline{x}}\mu_0(t,\overline{x}) + \nabla_y\mu_1(t,\overline{x},\cdot)) = 0 \text{ in } Z.$$

So, for $\xi \in \mathbb{R}^2\times\{0\}$ and $r \in \mathbb{R}$ arbitrarily fixed, we consider the corrector problem

$$\begin{cases} \text{Find } \pi_{\xi,r} \equiv \pi_{\xi,r}(t,\overline{x},\cdot) \in H^1_\#(Z)/\mathbb{R} \text{ such that} \\ -\operatorname{div}_y(m(\cdot,r)(\xi + \nabla_y\pi_{\xi,r}(t,\overline{x},\cdot)) = 0 \text{ in } Z. \end{cases} \tag{4.30}$$

Then, since $m(\cdot,r) \geq m_1 > 0$, (4.30) possesses a unique solution $\pi_{\xi,r}(t,\overline{x},\cdot) \in H^1_\#(Z)/\mathbb{R}$ for a.e. $(t,\overline{x}) \in Q^T$. This being so, choosing $\xi = \nabla_{\overline{x}}\mu_0(t,\overline{x})$ and $r = \varphi_0(t,\overline{x})$ in (4.30), we infer from the uniqueness of the solution to (4.30) that

$$\mu_1(t,\overline{x},y) = \pi_{\nabla_{\overline{x}}\mu_0(t,\overline{x}),\varphi_0(t,\overline{x})}(t,\overline{x},y) \text{ for a.e. } (t,\overline{x},y) \in Q^T\times Z.$$

Now, choosing $\xi = e_j$ (the jth vector of the canonical basis $\mathbb{R}^2$; remember that we view it as the vector $(e_j,0) \in \mathbb{R}^3$) and $r = \varphi_0(t,\overline{x})$, and denoting by $\omega_j(t,\overline{x},\cdot)$ the corresponding solution of (4.30), we easily get that

$$\mu_1(t,\overline{x},y) = \nabla_{\overline{x}}\mu_0(t,\overline{x})\cdot\omega(t,\overline{x},y) \text{ with } \omega(t,\overline{x},\cdot) = (\omega_j(t,\overline{x},\cdot))_{1\leq j\leq 2}. \tag{4.31}$$

Bearing this in mind, we define the homogenized mobility term (matrix) as follows

$$\widehat{m}(\varphi_0)(t,\overline{x}) = \frac{1}{|Z|}\int_Z m(y,\varphi_0(t,\overline{x}))(I_2 + \nabla_{\overline{y}}\omega(t,\overline{x},y))\, \mathrm{d}y,\ (t,\overline{x}) \in Q^T, \tag{4.32}$$

where I_2 denotes the 2×2 identity matrix.

Finally, substituting in (4.27) the expression for μ_1 given by (4.31), we arrive at

$$\frac{\partial\varphi_0}{\partial t} + \overline{\mathbf{u}}\cdot\nabla_{\overline{x}}\varphi_0 - \operatorname{div}_{\overline{x}}(\widehat{m}(\varphi_0)\nabla_{\overline{x}}\mu_0) = 0 \text{ in } Q^T. \tag{4.33}$$

Setting $q = p_0^1 + p_1$, (4.15) is equivalent to

$$-\alpha\Delta_y\mathbf{u}_0 + \nabla_y q = \mathbf{g} - \nabla_{\overline{x}}p_0 + \mu_0\nabla_{\overline{x}}\varphi_0 \text{ in } Q^T\times Z. \tag{4.34}$$

In order to deal with (4.34), let us consider the following cell problem

$$\begin{cases} -\alpha\Delta_y\omega^j + \nabla_y\pi^j = e_j \text{ in } Z,\ \operatorname{div}_y\omega^j = 0 \text{ in } Z, \\ \int_Z \omega_3^j(y)\, \mathrm{d}y = 0, \end{cases} \tag{4.35}$$

where e_j ($j = 1, 2$) is the jth vector of the canonical basis in $\mathbb{R}^3$ and $\omega^j = (\omega_i^j)_{1\le i\le 3}$. It is important to note that the condition $\int_Z \omega_3^j(y)\,dy = 0$ is imposed owing to the fact that $\int_Z u_{0,3}\,dy = 0$. For $j = 3$, we still consider the system (4.35), but now without the above-mentioned condition.

This being so, it is a fact that (4.35) possesses a unique solution $\omega^j \in (H^1_{0,\#}(Z))^3$. Next, define

$$a_{ij} = \frac{1}{|Z|}\int_Z \omega^i(y)e_j\,dy,\quad 1 \le i, j \le 2. \tag{4.36}$$

From (4.35) we see that $a_{ij} = \frac{\alpha}{|Z|}\int_Z \nabla_y\omega^i \cdot \nabla_y\omega^j\,dy$, so that the matrix $(a_{ij})_{1\le i,j\le 3}$ is symmetric and positive definite. Moreover, from the equality $\int_Z \omega_3^j(y)\,dy = 0$, we get that $a_{j3} = 0$ and so, $a_{3j} = 0$ (recall that $(a_{ij})_{1\le i,j\le 3}$ is symmetric) for $j = 1, 2$.

With this in mind, we set $A = (a_{ij})_{1\le i,j\le 2}$; then A is a 2×2 symmetric and positive definite matrix.

This being so, fix $(t, \overline{x}) \in Q^T$ and choose $v(y) = \mathbf{u}_0(t, \overline{x}, y)$ $(y \in Z)$ as test function in (4.35) to obtain

$$\frac{\alpha}{|Z|}\int_Z \nabla_y\omega^j \cdot \nabla_y\mathbf{u}_0\,dy = \frac{1}{|Z|}\int_Z \mathbf{u}_0 e_j\,dy. \tag{4.37}$$

Next, in the variational formulation of (4.34), we take the test function $\Psi(t, \overline{x}, y) = \varphi(t, \overline{x})\omega^j(y)$ with $\varphi \in C_0^\infty(Q^T)$ and obtain, after simplifications, the following equality, which holds in the distributional sense in Q^T:

$$\frac{\alpha}{|Z|}\int_Z \nabla_y\mathbf{u}_0 \cdot \nabla_y\omega^j\,dy + \frac{1}{|Z|}\int_Z \nabla_{\overline{x}}p_0 \cdot \omega^j\,dy - \frac{1}{|Z|}\int_Z \mu_0\nabla_{\overline{x}}\varphi_0\omega^j\,dy = \frac{1}{|Z|}\int_Z \omega^j\mathbf{g}\,dy. \tag{4.38}$$

Comparing (4.37) and (4.38) yields

$$\frac{1}{|Z|}\int_Z \mathbf{u}_0 e_j\,dy = \frac{1}{|Z|}\left(\int_Z \omega^j\,dy\right)(h - \nabla_{\overline{x}}p_0 + \nabla_{\overline{x}}\varphi_0) \text{ for } j = 1, 2,$$

that is, using the fact that $u_j = \frac{1}{|Z|}\int_Z \mathbf{u}_0 e_j\,dy$,

$$\overline{\mathbf{u}} = A(\mathbf{g}_1 - \nabla_{\overline{x}}p_0 + \mu_0\nabla_{\overline{x}}\varphi_0) \text{ in } Q^T. \tag{4.39}$$

We have therefore proved the following result.

Theorem 4.1. *The quadruplet $(\overline{\mathbf{u}}, \varphi_0, \mu_0, p_0)$ defined by (4.13), (4.5), (4.7) and (4.11) solves the homogenized system (4.39), (4.33), (4.26) with appropriate boundary and initial conditions, namely,*

$$\begin{cases} \overline{\mathbf{u}} = A(\mathbf{g}_1 + \mu_0\nabla_{\overline{x}}\varphi_0 - \nabla_{\overline{x}}p_0) \text{ in } Q = (0,\infty)\times\Omega, \\ \operatorname{div}_{\overline{x}}\overline{\mathbf{u}} = 0 \text{ in } Q \text{ and } \overline{\mathbf{u}}\cdot\boldsymbol{n} = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ \dfrac{\partial\varphi_0}{\partial t} + \overline{\mathbf{u}}\cdot\nabla_{\overline{x}}\varphi_0 - \operatorname{div}_{\overline{x}}(\widehat{m}(\varphi_0)\nabla_{\overline{x}}\mu_0) = 0 \text{ in } Q, \\ \mu_0 = -\beta\Delta_{\overline{x}}\varphi_0 + \lambda F'(\varphi_0) \text{ in } Q, \\ \dfrac{\partial\varphi_0}{\partial\boldsymbol{n}} = \dfrac{\partial\mu_0}{\partial\boldsymbol{n}} = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ \varphi_0(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{4.40}$$

Eq. $(4.40)_1$ is a Hele-Shaw equation, so that the homogenized system is a *Hele-Shaw-Cahn-Hilliard* (HSCH) system arising from flow through thin domains, and modeling in particular tumor growth.

5. Proof of the main result

We begin this section with the definition of some function spaces. For $\psi \in (H^1(\Omega))'$ we set $\overline{\psi} = |\Omega|^{-1}\langle\psi, 1\rangle$. We have $\overline{\psi} = \fint_\Omega \psi$ for $\psi \in L^2(\Omega)$. This being so, let $H^1_{(0)}(\Omega) = \{v \in H^1(\Omega) : \overline{v} = 0\}$ and $H^s_N(\Omega) = \{v \in H^s(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$ for $s = 1, 2$. It is well-known that the unbounded operator $-\Delta : L^2(\Omega) \to L^2(\Omega)$ with dense domain $H^2_N(\Omega)$ is self-adjoint and nonnegative, becoming strictly positive on $H^1_{(0)}(\Omega)$. Moreover, it maps $H^1_{(0)}(\Omega)$ onto $H^1_{(0)}(\Omega)' = \{\psi \in H^1(\Omega)' : \overline{\psi} = 0\}$.

5.1. Analysis of the homogenized system

In this subsection, we are concerned with the $2D$ HSCH system (4.40) derived from the upscaling of the micro-model (1.1) in $3D$. We aim at deriving some regularity properties for the solution of (4.40). Before proceeding further, let us recall the statement of

(4.40) below. We drop the subscripts on the unknown functions and we assume without loss of generality that $\beta = \lambda = 1$. Then (4.40) reads as follows

$$\begin{cases} \mathbf{u} = A(\mathbf{g}_1 + \mu\nabla\varphi - \nabla p) \text{ in } Q, \\ \operatorname{div}\mathbf{u} = 0 \text{ in } Q \text{ and } \mathbf{u}\cdot\boldsymbol{n} = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ \dfrac{\partial\varphi}{\partial t} + \mathbf{u}\cdot\nabla\varphi - \operatorname{div}(\widehat{m}(\varphi)\nabla\mu) = 0 \text{ in } Q, \\ \mu = -\Delta\varphi + F'(\varphi) \text{ in } Q, \\ \dfrac{\partial\varphi}{\partial\boldsymbol{n}} = \dfrac{\partial\mu}{\partial\boldsymbol{n}} = 0 \text{ on } (0,\infty)\times\partial\Omega, \\ \varphi(0) = \varphi^0 \text{ in } \Omega. \end{cases} \tag{5.1}$$

In (5.1), $\boldsymbol{n}$ denotes the outward unit normal to $\partial\Omega$. We know from the homogenization process that there exists at least a quadruplet $(\mathbf{u},\varphi,\mu,p)$ solving (5.1) such that $\mathbf{u}\in L^2(0,\infty;\mathbb{H})$, $\varphi\in L^\infty([0,\infty);H^1_{(0)}(\Omega))$, $F'(\varphi)\in L^\infty([0,\infty);L^1(\Omega))$, $\mu\in L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega))$ and $p\in L^2_{\mathrm{uloc}}([0,\infty);H^1(\Omega)\cap L^2_0(\Omega))$, where

$$\mathbb{H} = \{\mathbf{u}\in L^2(\Omega)^2 : \operatorname{div}\mathbf{u} = 0 \text{ in } \Omega \text{ and } \mathbf{u}\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

We recall that in (5.1) the mobility coefficient $\widehat{m}(\varphi)$ is a 2×2 matrix which is symmetric and positive definite.

Our first goal here is to improve the regularity on φ and μ. Prior to that, we gather below some classical results such as the Galiardo-Nirenberg and Agmon inequalities. They will be used throughout the current section.

Lemma 5.1 (Temam [34])**.** *Let $\Omega\subset\mathbb{R}^2$ be any bounded smooth domain. Then*

(i) $\|f\|_{L^4} \le C(\|f\|_{L^2}^{1/2}\|\nabla f\|_{L^2}^{1/2} + \|f\|_{L^2})$ *for any* $f\in H^1(\Omega)$,
(ii) $\|f\|_{L^p} \le C\|f\|_{H^1}$ *for any* $1\le p<\infty$ *and for any* $f\in H^1(\Omega)$,
(iii) $\|f\|_{L^\infty} \le C\|f\|_{L^2}^{1/2}\|f\|_{H^2}^{1/2}$ *for any* $f\in H^2(\Omega)$,
(iv) $\left\|f - \fint_\Omega f\right\|_{H^2} \le C\|\Delta f\|_{L^2}$ *for any* $f\in H^2(\Omega)$ *with* $\nabla f\cdot\boldsymbol{n} = 0$ *on* $\partial\Omega$,

where $C=C(p,\Omega)>0$.

Remark 5.1. Putting together (iii) and (iv) of Lemma 5.1, we obtain

$$\|f\|_{L^\infty} \le C\|f\|_{L^2}^{\frac12}\|\Delta f\|_{L^2}^{\frac12} \text{ for any } f\in H^2(\Omega) \text{ with } \nabla f\cdot\boldsymbol{n} = 0 \text{ on } \partial\Omega \text{ and } \fint_\Omega f = 0. \tag{5.2}$$

where $C=C(\Omega)>0$.

We denote by $\mathrm{BC}([0,\infty);H^1(\Omega))$ the space of bounded continuous functions from $[0,\infty)$ into $H^1(\Omega)$. The following result holds.

Proposition 5.1. *Let $\varphi^0\in H^1(\Omega)$ be given. Let $(\mathbf{u},\varphi,\mu,p)$ be a solution of (5.1) determined by Theorem 4.1. Then we have $\mu\in L^4_{\mathrm{uloc}}([0,\infty);L^2(\Omega))$. If further Ω is of class C^3, then $\varphi\in L^2_{\mathrm{uloc}}([0,\infty);H^3(\Omega))\cap L^4_{\mathrm{uloc}}([0,\infty);H^2_N(\Omega))\cap \mathrm{BC}([0,\infty);H^1_N(\Omega))$. Moreover, for any $r\ge 2$, we have $\varphi\in L^2_{\mathrm{uloc}}([0,\infty);W^{2,r}(\Omega))$.*

Proof. Let us check that $\mu\in L^4_{\mathrm{uloc}}([0,\infty);L^2(\Omega))$. Let $T>0$ be freely fixed. For $\eta\in L^1(0,T;H^1(\Omega))$, we have

$$\begin{aligned}\int_0^T(\mu(t),\eta(t))\,\mathrm{d}t &= \int_{Q^T}\nabla\varphi\cdot\nabla\eta\,\mathrm{d}x\,\mathrm{d}t + \int_{Q^T}\eta F'(\varphi)\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T\|\nabla\varphi\|_{L^2(\Omega)}\|\nabla\eta\|_{L^2(\Omega)}\,\mathrm{d}t + \int_0^T\left\|F'(\varphi)\right\|_{L^{6/5}(\Omega)}\|\eta\|_{L^6(\Omega)}\,\mathrm{d}t \\ &\le \left(\|\varphi\|_{L^\infty(0,T;H^1(\Omega))} + C(1+\|\varphi\|_{L^\infty(0,T;L^2(\Omega))}^{18/5})\right)\|\eta\|_{L^1(0,T;H^1(\Omega))} \\ &\le C\|\eta\|_{L^1(0,T;H^1(\Omega))},\end{aligned}$$

where we have used the embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$ together with the inequality $(1.7)_1$ on F'. This shows that

$$\|\mu\|_{L^\infty(0,\infty;H^1(\Omega)')} \le C. \tag{5.3}$$

This being so, we have

$$\begin{aligned}\|\mu\|_{L^2(\Omega)}^2 &= \langle\mu,\mu\rangle_{H^1(\Omega)',H^1(\Omega)} \le \|\mu\|_{H^1(\Omega)'}\|\mu\|_{H^1(\Omega)} \\ &\le \|\mu\|_{H^1(\Omega)'}(\|\mu\|_{L^2(\Omega)} + \|\nabla\mu\|_{L^2(\Omega)}) \\ &\le \frac12\|\mu\|_{L^2(\Omega)}^2 + \|\mu\|_{H^1(\Omega)'}^2 + \|\nabla\mu\|_{L^2(\Omega)}\|\mu\|_{H^1(\Omega)'}.\end{aligned}$$

It follows that

$$\|\mu\|^4_{L^2(\Omega)} \le C\Big(\|\mu\|^4_{H^1(\Omega)'} + \|\nabla\mu\|^2_{L^2(\Omega)}\|\mu\|^2_{H^1(\Omega)'}\Big) \\ \le C\Big(1 + \|\nabla\mu\|^2_{L^2(\Omega)}\Big),$$

where we have used (5.3). This yields at once $\mu \in L^4_{\text{uloc}}([0,\infty); L^2(\Omega))$.

Let us proceed with the regularity on φ. To that end, let us note that, for a.e. $t \in (0,\infty)$, $\varphi(t)$ solves the problem

$$-\Delta\varphi = \mu - F'(\varphi) \text{ in } \Omega, \quad \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega.$$

The properties of μ and F associated to the fact that $\varphi \in L^\infty(0,\infty; H^1(\Omega))$ together with $\Omega \in C^3$ show that $\varphi \in L^2_{\text{loc}}(0,\infty; H^3(\Omega))$. This show that $\mu = -\Delta\varphi + F'(\varphi)$ a.e. in Ω. Thus, we multiply $(5.1)_4$ by $-\Delta^2\varphi$ and use $(5.1)_5$ to get

$$(\nabla\mu, \nabla\Delta\varphi) = -\|\nabla\Delta\varphi\|^2_{L^2} + \big(F''(\varphi)\nabla\varphi, \nabla\Delta\varphi\big).$$

First, we have

$$\Big|\big(F''(\varphi)\nabla\varphi, \nabla\Delta\varphi\big)\Big| \le \|F''(\varphi)\|_{L^3(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)}.$$

Recalling the first inequality of $(1.7)_1$ together with the Sobolev embedding $H^1(\Omega) \hookrightarrow L^3(\Omega)$, we have

$$\|F''(\varphi)\|^3_{L^3(\Omega)} \le C\int_\Omega (1 + |\varphi|^3)\,\mathrm{d}x \le C + C\|\varphi(t)\|^3_{H^1(\Omega)} \\ \le C.$$

It follows that

$$\Big|\big(F''(\varphi)\nabla\varphi, \nabla\Delta\varphi\big)\Big| \le C\|\nabla\varphi\|^{\frac12}_{L^2(\Omega)}\|\nabla\Delta\varphi\|^{\frac12}_{L^2(\Omega)}\|\nabla\Delta\varphi\|_{L^2(\Omega)} \\ \le C\|\nabla\Delta\varphi\|^{\frac32}_{L^2(\Omega)} \le C + \frac14\|\nabla\Delta\varphi\|^2_{L^2(\Omega)}.$$

Next, one has

$$|(\nabla\mu, \nabla\Delta\varphi)| \le \|\nabla\mu\|^2_{L^2(\Omega)} + \frac14\|\nabla\Delta\varphi\|^2_{L^2(\Omega)},$$

so that

$$\frac12\|\nabla\Delta\varphi\|^2_{L^2(\Omega)} \le C + \|\nabla\mu\|^2_{L^2(\Omega)}. \tag{5.4}$$

This yields

$$\|\nabla\Delta\varphi\|^2_{L^2_{\text{uloc}}([0,\infty);L^2(\Omega))} \le C, \tag{5.5}$$

or,

$$\|\varphi\|_{L^2_{\text{uloc}}([0,\infty);H^3(\Omega))} \le C. \tag{5.6}$$

By interpolation we have

$$\int_t^{t+1} \|\varphi_\varepsilon(\tau)\|^4_{H^2(\Omega)}\,\mathrm{d}\tau \le \int_t^{t+1} \|\varphi(\tau)\|^2_{H^1(\Omega)}\|\varphi(\tau)\|^2_{H^3(\Omega)}\,\mathrm{d}\tau \\ \le C\int_t^{t+1} \|\varphi_\varepsilon(t)\|^2_{H^3(\Omega_\varepsilon)}\,\mathrm{d}t \quad \forall t \ge 0.$$

It follows at once from (5.6) that $\varphi_\varepsilon \in L^4_{\text{uloc}}([0,\infty); H^2(\Omega))$.

Now, let $\phi \in L^2(0,\infty; H^2_N(\Omega))$; we have

$$\left|\left\langle \frac{\partial\varphi}{\partial t}, \phi\right\rangle\right| \le \int_0^\infty \big(\|\mathbf{u}\|_{L^2(\Omega)}\|\varphi\|_{L^4(\Omega)}\|\nabla\phi\|_{L^4(\Omega)} + C\|\nabla\mu\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}\big)\,\mathrm{d}t \\ \le C\int_0^\infty \big(\|\mathbf{u}\|_{L^2(\Omega)}\|\varphi\|_{H^1(\Omega)}\|\phi\|_{H^2(\Omega)} + C\|\nabla\mu\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}\big)\,\mathrm{d}t \\ \le C\big(\|\mathbf{u}\|_{L^2(Q)} + \|\nabla\mu\|_{L^2(Q)}\big)\|\phi\|_{L^2(0,\infty;H^2(\Omega))},$$

which amounts to $\partial\varphi/\partial t \in L^2(0,\infty; H^2_N(\Omega)')$. Taking into account that $\varphi \in L^4_{\text{uloc}}([0,\infty); H^2_N(\Omega)) \cap L^\infty([0,\infty); H^1_N(\Omega))$, we are led to $\varphi \in \mathrm{BC}([0,\infty); H^1_N(\Omega))$, the space of bounded continuous functions from $[0,\infty)$ into $H^1_N(\Omega)$.

Let finally check that $\varphi \in L^2_{\text{uloc}}([0,\infty); W^{2,r}(\Omega))$ for any $r \ge 2$. Let $r \ge 2$. From inequality $(1.7)_1$ on F', we easily see that

$$\|F'(\varphi)\|_{L^r(\Omega)} \le C(1 + \|\varphi\|^3_{L^{3r}(\Omega)}) \le C(1 + \|\varphi\|^3_{H^1(\Omega)}),$$

where for the last inequality above, we used the continuous embedding $H^1(\Omega) \hookrightarrow L^{3r}(\Omega)$.

Now, we come back to $(5.1)_{4-5}$ and write under the form

$$-\Delta\varphi + \varphi = \mu + \varphi + F'(\varphi) \text{ in } \Omega, \ \frac{\partial\varphi}{\partial \boldsymbol{n}} = 0 \text{ on } \partial\Omega.$$

We infer from the elliptic regularity estimates that there is a positive constant C depending on r and Ω such that

$$\begin{aligned}\|\varphi\|_{W^{2,r}(\Omega)} &\leq C(1 + \|\mu + \varphi + F'(\varphi)\|_{L^r(\Omega)}) \\ &\leq C(1 + \|\mu\|_{H^1(\Omega)} + \|\varphi\|_{H^1(\Omega)} + \|\varphi\|^3_{H^1(\Omega)}).\end{aligned}$$

This shows that $\varphi \in L^2_{\mathrm{uloc}}([0,\infty); W^{2,r}(\Omega))$. □

We are now able to prove the main result of the work.

5.2. *Proof of Theorem 1.1*

Given any ordinary sequence E of positive real numbers converging to zero, we have derived the existence of a subsequence E' from E and of a quadruple $(\mathbf{u}_0, \varphi_0, \mu_0, p_0)$ with $\mathbf{u}_0 \in L^2(Q; H^1_{0,\#}(Z)^3)$, $\varphi_0 \in L^\infty(0,\infty; H^1(\Omega))$, $\mu_0 \in L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega))$ and $p_0 \in L^2_{\mathrm{uloc}}([0,\infty); H^1(\Omega) \cap L^2_0(\Omega))$ such that, for any $T > 0$, we have, when $E' \ni \varepsilon \to 0$,

$$\mathbf{u}_\varepsilon \to \mathbf{u}_0 \text{ in } L^2(Q^T_\varepsilon)^3\text{-weak } 2s \quad \text{and} \quad \varepsilon\nabla\mathbf{u}_\varepsilon \to \nabla_y\mathbf{u}_0 \text{ in } L^2(Q^T_\varepsilon)^{3\times3}\text{-weak } 2s,$$

$$\varphi_\varepsilon \to \varphi_0 \text{ in } L^2(Q^T_\varepsilon)\text{-strong } 2s$$

$$\mu_\varepsilon \to \mu_0 \text{ in } L^2(Q^T_\varepsilon)\text{-weak } 2s$$

$$p_\varepsilon \to p_0 \text{ in } L^2(Q^T)\text{-weak } 2s.$$

Next, defining the operator M_ε as in (2.37) and setting $\mathbf{u}(t,\overline{x}) = \int_Z \mathbf{u}_0(t,\overline{x},y)\,\mathrm{d}y = (\overline{\mathbf{u}}(t,\overline{x}), u_3(t,\overline{x}))$, we see from [part (3) of] Remark 3.1 that we have the convergence results (1.8). We also note that the quadruplet $(\overline{\mathbf{u}}, \varphi_0, \mu_0, p_0)$ solves the system (1.9). Furthermore assuming that Ω is of class C^3, we have that $\varphi_0 \in \mathrm{BC}([0,\infty); H^1_N(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); H^3(\Omega)) \cap L^4_{\mathrm{uloc}}([0,\infty); H^2(\Omega)) \cap L^2_{\mathrm{uloc}}([0,\infty); W^{2,r}(\Omega))$ for any $r \geq 2$. This completes the proof of Theorem 1.1.

Declaration of competing interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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