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Design and Development of a Teaching–Learning Sequence about Deterministic Chaos Using Tracker Software

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Abstract: In this paper, we present the design, development, and pilot implementation of a Teaching–Learning Sequence (TLS) about the physics of deterministic chaos. The main aim of the activities is to let students become aware of two key ideas about deterministic chaos: (1) the role of initial conditions and (2) the graphical representation in a momentum–position graph. To do so, the TLS is based on the observation and analysis of the trajectory of the free end of a double pendulum through the modeling software Tracker. In particular, the Tracker-based activities help students understand that, by modifying the well-known simple pendulum dynamic system into a double pendulum, long-time-scale predictability is lost, and a completely new behavior appears. The TLS was pilot tested in a remote teaching setting with about 70 Italian high school students (16–17 years old). The pretest analysis shows that before participating in the activities, students held typical misconceptions about chaotic behavior. Analysis of the written responses collected during and after implementation shows that the proposed activities allowed students to grasp the two key ideas about nondeterministic chaos. A possible integration of the TLS with an online simulation is finally discussed.

Keywords: chaotic systems; tracker software; secondary school

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1. Introduction

Experiments and simulations aimed at introducing the study of deterministic chaotic systems at the secondary school level are widely used in physics education [1–7]. However, examples of classroom-based validation of these didactical materials are limited to specific cases, such as the behavior of Lorenz's water wheel [8], nonlinear systems in a musical context [9], fractal structures [10], and chaotic motion [11]. A more structured sequence of activities on chaotic systems was proposed by Komorek and Duit [12], who designed a Teaching–Learning Sequence (TLS) about nonlinear systems for 15–16-year-old students. The activities focused on the chaotic behavior of a magnetic pendulum, i.e., a simple pendulum suspended from a base with three magnets placed at the vertices of an equilateral triangle, which the authors proposed as a paradigmatic example of a chaotic system. The results of the implementation of the TLS show that only a limited number of students correctly understood the key concepts of chaotic systems, such as, e.g., the physical sense of strange attractors. Building on the work in [12], Stavrou et al. [13] designed a TLS based on four experiments to address the behavior of nonlinear systems, such as the spherical pendulum, the chaotic pendulum, Benard cells, and Dendrite. The TLS was implemented with 30 students. The results show that some students described the systems in terms of either a deterministic or a random process, while other students were able to use probabilistic and deterministic laws to explain the systems' behavior. However, the authors do not provide enough details about the collected data to be able to infer the physical reasoning underlying the students' responses.

The above brief review shows that the evidence on how students construct mental models, i.e., internal representations that individuals construct to make sense of a natural phenomenon [14], of chaotic phenomena is limited both in terms of the content addressed and the size of the sample. In addition, it is not clear to what extent modeling activities, i.e., activities that allow students to construct a scientific model of the phenomenon under study [15], can help students understand key concepts of chaotic systems. Furthermore, there is limited evidence on which approach (e.g., inquiry-based, demonstrative, etc.) may be most appropriate to achieve learning goals related to the physics of chaos. Finally, it is not clear to what extent knowledge of classical physics influences students' understanding of deterministic chaos. In this paper, we addressed these issues by answering the following research questions:

1. What are the secondary school students' spontaneous models when interpreting simple physical phenomena that show a deterministic chaotic behavior?
2. How are secondary school students' models of deterministic chaotic systems affected by their models of classical deterministic systems?
3. To what extent is a computer-based TLS effective in familiarizing secondary school students with the key concepts of deterministic chaos?

To answer the above research questions, we designed a TLS about a simple chaotic system, namely a double pendulum [16], using Tracker software v.5 [17] to derive a mathematical model of its behavior.

2. Description of the Teaching–Learning Sequence

Building on the prior reviewed work, we chose two key ideas about deterministic chaos that should be addressed at the secondary school level:

1. Key Idea 1: Slight variations in the initial conditions affect the motion of chaotic deterministic systems.
2. Key Idea 2: Representation of the unpredictability of chaotic deterministic systems.

The first key idea accounts for small deviations and perturbations that can randomly affect the motion of a deterministic system in an unpredictable way. The second key idea accounts for the different appearance of the trajectories of a deterministic chaotic system with respect to a classical deterministic system in the momentum–position space (i.e., strange attractors vs. fixed points) [18]. The three phases of the TLS are described below.

2.1. Phase 1: Review of Classical Oscillating Systems and Introduction to the Double Pendulum Mechanics (3 h)

The main objective of this phase is to reflect on the concept of the predictability of a phenomenon, in particular on the dependence of the time evolution of the variables describing the phenomenon starting from certain initial conditions. To achieve this aim, the students are guided to compare the time evolution of the position, velocity, and acceleration of a mass attached to the free end of a spring when the initial conditions of the motion are slightly changed using an online simulation (https://www.myphysicslab.com/springs/single-spring-en.html, accessed on 1 August 2024). The same analysis is carried out for a simple pendulum (https://www.myphysicslab.com/pendulum/pendulum-en.html, accessed on 1 August 2024). The teacher hence guides the students to understand that, in a classical nonchaotic phenomenon, a small variation in the initial conditions does not change the way in which the variables that describe the phenomenon depend on time. The students are then asked to hypothesize the behavior of a double pendulum, schematically represented in Figure 1. Finally, the students are asked to identify the forces acting on the system by tracing the free-body diagram and to qualitatively predict the motion of the system in two limit cases: $m_2 << m_1$ and $l_2 << l_1$.

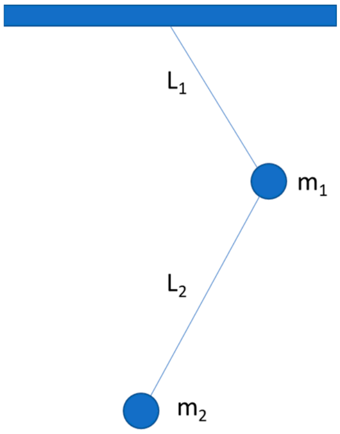


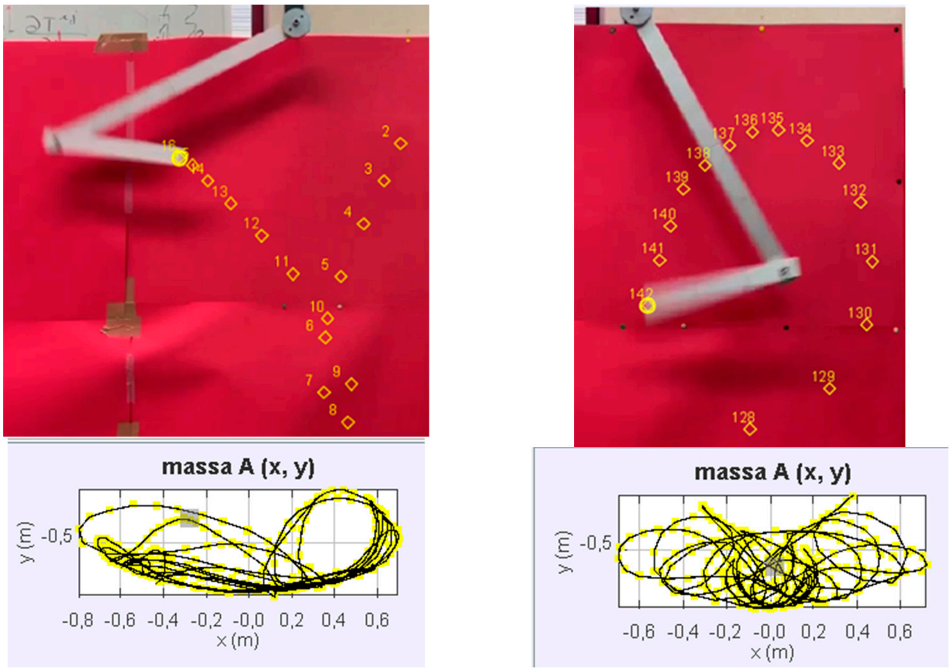
Figure 1. Schematic representation of a double pendulum.

2.2. Phase 2: Analysis of the Double Pendulum Motion Using Tracker Software (3 h)

The aim of this phase is to develop a qualitative description of the motion of the double pendulum system in the momentum–position space using the Tracker software and to investigate the conditions under which the double pendulum behaves like a physical pendulum with an appropriate moment of inertia. The double pendulum used for this study is shown in Figure 2.

Figure 2. Double pendulum used for the measurements in Tracker. Length of the longer bar is L_1 = 0.66 m, length of shorter bar is L_2 = 0.33 m, total mass is M = 3 kg.

At the beginning of the activity, the students watch a short clip of the oscillations of the double pendulum (Clip 1, see Supplementary Materials). Students are asked to compare their predictions with the actual motion of the system and write down similarities and differences. Then, they import into the Tracker software Clip 1 and a second short clip (Clip 2, see Supplementary Materials) of the same double pendulum, in which the initial conditions of the motion are as similar as possible. Then, using the autotracking function of the software, they are asked to reconstruct autonomously the trajectory of the free end of the double pendulum in the vertical plane. The teacher then guides the students to notice that the trajectories shown in the two clips are clearly different, even though the initial conditions are very similar. Examples of measurements using the "tracking" function are reported in Figure 3a,b.



(**a**) (**b**)

Figure 3. Example measurements of the trajectory of the double pendulum of Figure 2 using Tracker. The two motions correspond to slightly different initial conditions. Figure (**a**) refers to Clip 1; Figure (**b**) refers to Clip 2 (see Supplementary Materials).

Finally, the students are asked to verify through Tracker that, for very small oscillations (Clip 3, see Supplementary Materials), the double pendulum behaves as a physical pendulum, with an inertia moment which is given by $I = \frac{1}{3}\,ML^2 = 1$ kg m^2 where $L = L_1 + L_2 = 1$ m and equivalent length = $\frac{2}{3}\,L = 0.67$ m in agreement with the theoretical model (see Figure 4). To further explore the role of initial conditions, the students are asked to compare the trajectories in the momentum–position space of the double pendulum for small and ample oscillations by visualizing them using Tracker and an online simulation (https://www.myphysicslab.com/pendulum/double-pendulum-en.html, accessed on 1 August 2024).

A summary of the TLS phases is reported in Table 1.

Table 1. Time schedule and corresponding description of the TLS activities.

Phase of the TLS	Time	Description of Students' Activities
1	2 h	Review of classical oscillating systems: initial conditions in spring-mass system and simple pendulum
1	1 h	Introduction to the double pendulum: qualitative description of its motion, role of initial conditions
2	3 h	Analysis of the double pendulum motion using Tracker software: construction of position vs. time, trajectory, and momentum vs. position graphs

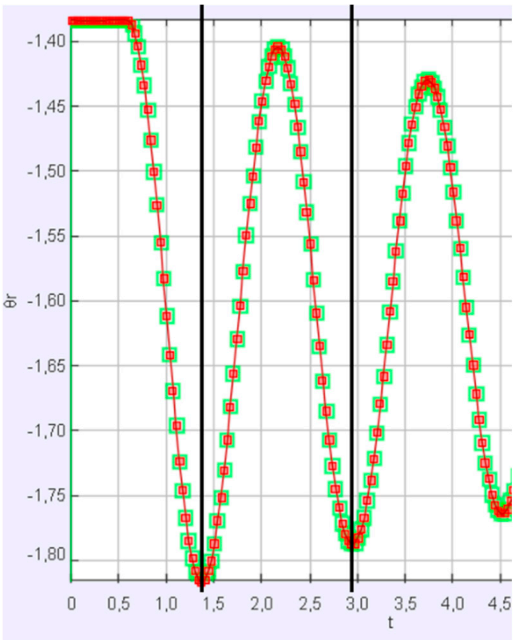


Figure 4. Measurement of the pseudoperiod of the small oscillations of the double pendulum of Figure 2 by means of the Tracker software (Clip 3). Approximately, the double pendulum behaves as a physical pendulum whose period T is given by $T = 2\pi\sqrt{\frac{I}{Mgd}}$, where I is the inertial moment, M is the mass. and d is the distance between the suspension point and the center of mass of the double pendulum. In our case, $d = \frac{L}{2}$, since both bars are supposed to be homogenous, $L = L_1 + L_2$ and $I = \frac{1}{3}ML^2$ so that $T = 2\pi\sqrt{\frac{2L}{3g}}$. The estimate is $T = (1.6 \pm 0.2)s$, in agreement with the expected value.

3. Educational Context and Sample

The study described in this paper was carried out between February and March 2022. Due to the ongoing measures against the COVID-19 pandemic at the time of this study, all phases of the designed TLS were tested in a remote modality as physics-focused extracurricular activities. Seventy-two secondary school Italian students (16–17 years old) attended the activities for a total of 6 h, evenly subdivided into two meetings over one week. The students were attending either the Scientific or Applied Sciences stream from ten different schools in the same suburban area of the authors' university. The participating students chose to follow these extracurricular activities, as they were motivated to enroll in a STEM-related undergraduate course in university. As students were underage, a brief informed consent document to be signed by both the students' parents in order to obtain permission to use the collected responses for research purposes was sent to the participating schools before the beginning of the activities. An identification number was assigned by the schools' teachers to the students in order to ensure anonymity. The schools' teachers returned the identification numbers of only the students who had agreed to use their responses for research purposes. The students used their identification number during the activities.

The first meeting was dedicated to phase 1 of the TLS, while the second meeting was dedicated to phase 2 of the TLS. The activities were carried out by the first author while the students attended remotely via Microsoft Teams for Windows 10 in a synchronous manner. Instruction in each phase was supported by online worksheets implemented through Microsoft Forms (see below for details). Students were asked to use dual cameras in order to prevent cheating or to use web-based search tools when answering the questions of the pretest and the worksheets.

4. Educational Context and Sample

To answer the first research question (RQ1), we submitted a brief open-answer pretest before the beginning of the activities. Here, we report the results of the analysis of students' answers to two emblematic questions adapted from [19]:

I1. When do we say that a system does not follow deterministic laws? Explain by discussing some examples.

I2. What does the term chaos make you think of?

The first question aimed at investigating the students' ideas about the possibility that the evolution of certain mechanical systems is the result of the interplay of chance and determinism. The second question aimed at investigating students' use of the term chaos in their everyday language. The complete pretest is available in Appendix A.

To answer RQ2, we analyzed the responses to the following prompts of the interactive worksheets:

P1. Describe in words the role of initial conditions in the temporal evolution of the motion of oscillating systems as a spring–mass system or a simple pendulum.

P2. Describe in words the role of the initial conditions in the temporal evolution of the motion of the double pendulum.

P3. What are the main similarities and differences between the motion of a simple pendulum and that of a double pendulum for small oscillations? And for ample ones?

These questions aimed to investigate whether the students were able to identify how the role of initial conditions becomes crucial when describing chaotic systems. Moreover, the prompts aimed to investigate whether the students were able to identify the differences between seemingly similar classical systems and deterministic chaotic systems as a physical or double pendulum.

To answer RQ3, we analyzed the responses to the following prompts:

P4. Qualitatively describe the motion of the double pendulum.

P5. Using the tracking function of Tracker, describe how the motion of the double pendulum evolves over time choosing a suitable graphical representation.

P6. What happens when the oscillations of the double pendulum become small?

These questions aimed to investigate whether the students were able to use Tracker to inform their descriptions and interpretation of the motion of the double pendulum and to identify differences and similarities with their predictions. The complete set of student worksheets is available in Appendix B.

Students' answers were analyzed using a constant comparison method [20]. The reason for such a choice is that this method makes it possible to describe the qualitatively different ways in which people conceptualize, perceive, and understand different aspects of phenomena in the world around them. The constant comparison method helped us to derive mutually exclusive categories of students' responses to the above questions and prompts.

5. Results

Overall, the response rate was 75%, namely, 55 out of 72 students answered the pretest and the interactive worksheets.

5.1. Pretest Analysis

Collected data show that students, before the activities, held naïve ideas about chaotic systems, confusing chaos and chance and using chaos as a term that characterizes disordered systems. For instance, when asked about the possibility of describing the evolution of a system through chance laws (I1), about half of the students (28/55) reported examples of random events such as lightning bolts or the throwing of dice, thus confusing chaos with disorder or probability. About 30% (17/55) claimed that an event can be described by chance laws if it is not possible to describe it through mathematical laws. For instance, S23 wrote the following:

A physical phenomenon is regulated by chance laws if, despite knowing all the initial conditions, it is impossible to predict its temporal evolution since there is no physical-mathematical law that describes it.

The remaining students (10/55) emphasized the impossibility of knowing the causes of chance events. For instance, S11 wrote the following:

Not having experimental data for a certain phenomenon is equivalent to not knowing its causes; therefore, we use nondeterministic laws.

Coherent with the above ideas, most of the students initially taught that the behavior of a chaotic system cannot be described in mathematical terms or described only by very complicated mathematical functions. For instance, when asked about the term chaos (I2), most of the students (42/55) answered that chaos means the absence of deterministic rules, disorder, or, more generally, is a synonym of a phenomenon that occurs without any reason. For instance, S44 wrote the following:

The term chaos makes me think of a disordered event, not regulated by any law and lacking any symmetry.

About 14% (8/55) associated the term chaos with phenomena that can only be described through transcendent functions. An example of an answer given by S67 is as follows:

Chaos makes me think of all those physical, chemical, and biological phenomena that cannot be modeled by a simple function, but only by more "difficult" functions as sine, cosine, exponential, logarithm, etc.

Only 5 out of 55 students did not answer the question.

5.2. Analysis of Students' Responses to Worksheet Prompts

In the following, we report the analysis of the students' responses to the worksheet prompts relevant to our research questions.

When asked about the role of initial conditions in classical systems as a spring–mass system or a simple pendulum (P1), the majority (34/55) of the students in the sample described the role of the initial conditions as system parameters that do not affect the temporal evolution of the systems. For instance, S43 wrote the following:

In both systems, the initial conditions are represented by the values assumed by the position and speed variables at the initial instant of time. They do not in any way affect the oscillations but are related to the amplitude and initial phase of the motion.

Only 24% (13/55) of the students in the sample described in a more exhaustive way the role of the initial conditions in the temporal evolution of a system. For example, S2 wrote the following:

The initial conditions allow us to define the state in which the system is at a given instant of time in order to predict future evolution through the fundamental law of dynamics. Something similar happens in the charging process of a capacitor in the RC circuit: when the switch is closed, at time t = 0, the initial condition is represented by the fact that the charge accumulated by the capacitor is zero.

About 14% (8/55) claimed that they did not know how to describe the role of the initial conditions in the temporal evolution of the two given systems.

When asked about the role of the initial conditions in the temporal evolution of the double pendulum in Figure 1 (P2), about half of the students (26/55) claimed that a small variation in the initial positions of the two masses leads only to a small variation in the trajectory of the free end of the double pendulum. For instance, S18 wrote the following:

Even if the trajectories of the two masses of the double pendulum will be more complicated than an arc of circumference, the phenomenon remains deterministic, and it is possible to write the fundamental equation of the dynamics for both masses. It follows those small changes in the initial *conditions will not be able to alter the trajectories of the two masses constituting the pendulum, just as in the case of the mass-spring system and the simple pendulum.*

About 33% (18/55) of the students correctly claimed that even if the system in Figure 1 starts from two very similar configurations, the resulting trajectories are completely different. For example, S30 claimed the following:

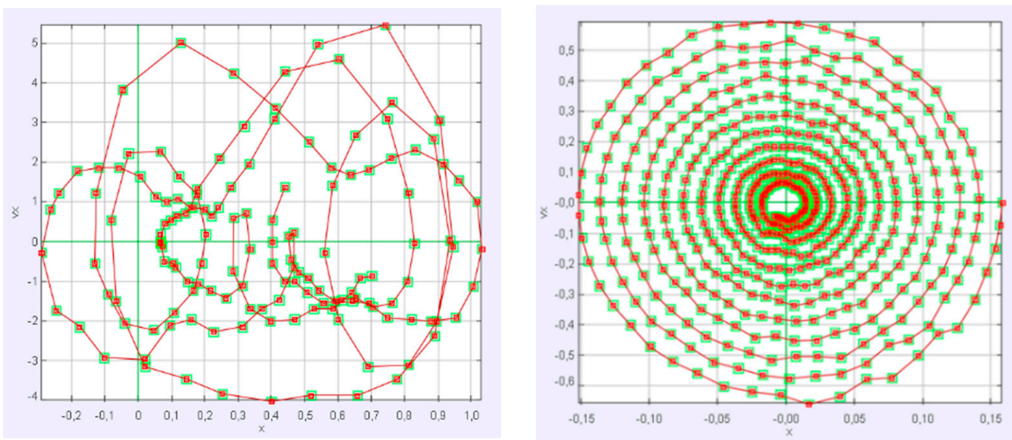
If it was possible to reduce the inevitable errors that are made when fixing the initial positions of the two pendulums, the trajectories would be different in any case.

About 20% of the students (11/55) claimed that they were unable to predict the effects of small perturbances in the initial conditions.

Finally, when asked about similarities and differences between the classical simple pendulum and the double pendulum in Figure 1 (P3), the majority of the students (26/55) identified total mechanical energy as a conserved quantity in both systems in the absence of dissipative forces. Moreover, these students also correctly claimed that there would be no analogy between the oscillations of the two systems, regardless of the amplitude of the oscillations. For instance, S27 wrote the following:

As an analogy, the principle of conservation of mechanical energy must exist for both the simple pendulum and the double pendulum (in the absence of friction). However, the oscillations of the two systems will generally be different since, even in the case where the two angles are small, the equations of the first and second mass cannot be separated.

About one-third of the students (16/55) claimed that, as in the case of the simple pendulum, if the oscillations of the two bars are small, the system evolves towards a deterministic behavior. For instance, S11 wrote the following (see Figure 5):



(**a**) (**b**)

Figure 5. Momentum vs. distance representation of the double pendulum motion for ample oscillations (**a**) and small oscillations (**b**) produced by S11.

If the two angles that each bar forms with the vertical are large, the oscillations of the double pendulum will show no analogy with those of the simple pendulum. Contrarily, if the angles are small, the forces acting on the two bars will be proportional to the displacements, as for the small oscillations of the simple pendulum. This is clearly visible when analyzing the trajectories of the two systems with Tracker

About one-fourth of the students (13/55) did not answer P3.

When requested to provide a qualitative description of the motion of the double pendulum in Figure 2 (P4), about 60% (33/55) of the students claimed that oscillations of the two bars' ends are harmonic and independent of each other. In particular, half of these students claimed that the trajectory of the free end of the second bar never repeats, with sudden and seemingly random changes in direction.

For instance, S19 answered the following:

The oscillations of this new system are very different from those of the simple pendulum: the trajectory of the second end of the double pendulum is not traceable back to any known curve *and there is no periodicity. I believed that the systems described by chaotic models could only be constituted by very complicated mechanical structures.*

About one-third (16/55) claimed that it would not be possible to establish at any given time the positions and the speeds of the two bars. Only 6 out of 55 were unable to make any description of the motion.

After using Tracker (P5), half of the students (31/55) of the students in the sample observed that the trajectories of the free end of the double pendulum in Figure 2 were very different in the two clips (see Figure 3). For instance, S60 claimed that:

The trajectories seem to be differentiating more and more. Evidently, no matter how accurately the initial conditions have been reproduced, it will never be possible to reproduce them with infinite precision.

About one-third (17/55) of the students, in addition to the observations made by the other students, claimed that if it was possible to reduce the inevitable errors made in reproducing the initial positions of the two pendulums, the trajectories would remain the same only a little longer. Only 7 out of 55 encountered difficulties in using Tracker.

When asked to describe what happens when the oscillations of the double pendulum in Figure 2 become small (P6), about 67% (37/55) of the students, using the position vs. time graph produced by the Tracker software, observed that the oscillations appeared to be harmonic and dampened as in the case of the simple pendulum in the presence of friction. For instance, S42 claimed wrote the following:

The graph of the function $x = x(t)$ is oscillating with an amplitude that decreases exponentially over time. We then obtain the same trend of the time law of a simple pendulum in the presence of friction or, similarly, of the time law of the mass–spring system in the presence of friction.

About a fifth of the students (12/55) also used the Tracker to construct the momentum–position space of the double pendulum from Clip 2 (see Figure 5a). Then, after analyzing Clip 3, one of these students also noticed a similar representation between the momentum–position space of the double pendulum and the corresponding graph for the mass–spring system in the presence of friction (see Figure 5b), thus identifying a fixed-point attractor of the system.

6. Conclusions

This study describes the development and pilot implementation of a modeling-based TLS on deterministic chaos. Overall, this study contributes to the field by showing how students can be successfully engaged in modeling nonlinear systems with complex behavior in secondary school physics education. A specific contribution of this study is the description of TLS activities accessible to high school students using modeling software.

Regarding RQ1, the analysis of the students' answers to the pretest confirmed the results of recent studies carried out in Italy [19] and worldwide [21] on the topic of deterministic chaos. In particular, our results confirm that students had no clear idea about the role of initial conditions in the time evolution of a classical system and that this lack of knowledge also affected their reasoning about the time evolution of a chaotic system. This may be due to a lack of familiarity with nonlinear phenomena, which are rarely addressed in high school curricula.

Regarding RQ2, we found that students initially used the single pendulum as a starting point for describing and interpreting the chaotic behavior of the double pendulum, ignoring the role of unexpected behavior due to changes in the initial conditions of the system. It was only after the qualitative discussions about the differences between a simple pendulum and the double pendulum in Figure 1 that the students were able to grasp how small variations in the initial conditions affect the motion of chaotic systems and to identify the differences between classical and chaotic systems. We also note that even after completing the Tracker activity, very few students were able to use the model of a physical pendulum to describe small oscillations of a double pendulum, suggesting that transferring knowledge of classical mechanics to chaotic systems may be challenging for most students, in line with other studies on the same topic [21]. Therefore, although focusing on small changes in the initial condition plays an important role in developing students' understanding of chaotic phenomena, teaching should also focus on simplified models that can explain the behavior of complex systems.

Finally, with regard to RQ3, analysis of the responses shows that most students were able to use the Tracker software to support their description of the motion of the double pendulum in Figure 2 using either the position vs. time graph or the momentum vs. position graph. In addition, the use of the Tracker software probably helped most students to interpret the graphical representations of the trajectories of a simple chaotic system as the double pendulum under very different regimes of motion (small and large oscillations). Thus, the reported evidence supports that the proposed TLS activities were effective in helping students to grasp, at least qualitatively, the two key ideas of deterministic chaos.

Although this exploratory study supports the effectiveness of the designed TLS to incorporate chaotic phenomena into high school physics curricula, the activities can be further improved. For example, we plan to integrate the Tracker-based activities with online simulations in order to allow students to quickly analyze the behavior of the double pendulum in Figure 1 by comparing the evolution of the system when the initial conditions, masses, and length of the bars are changed. In such a way, students can reinforce their understanding of the two key ideas of deterministic chaos. Moreover, we plan to design a suitable multiple-choice instrument to assess the effectiveness of the TLS, building on the students' responses collected in this pilot study.

Supplementary Materials: The following supporting information can be downloaded at: Video S1: http://youtube.com/watch?v=1N-L-nxRJ-I; Video S2: http://youtube.com/watch?v=EJezLVuieIQ; Video S3: http://youtube.com/watch?v=loGz_ImIVV4.

Author Contributions: Conceptualization, I.T. and A.P.; methodology, A.P.; software, A.P. validation, I.T., A.P. and G.G.; formal analysis, A.P.; investigation, A.P.; resources, G.G.; data curation, A.P.; writing—original draft preparation, I.T. and A.P.; writing—review and editing, G.G.; visualization, A.P.; supervision, I.T.; funding acquisition, I.T. All authors have read and agreed to the published version of the manuscript.

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Institutional Review Board Statement: The research with human subjects described in this study was conducted in accordance with the Declaration of Helsinki on Ethical Principles for Medical Research Involving Human Subjects and the ICMJE guidelines on Protection of Research Participants. Approval by local research ethics committee was not required at the time when the study was carried out since no medical treatment was carried out and participants were anonymized. Informed consent forms were signed before the beginning of the activities by both the students' parents for study participation and research purposes, in order to fulfill the requirements of Italian law. Refer to the following document for the Italian regulation of this matter: Regolamento+UE+2016+679.+Arricchito+con+riferimenti+ai+Considerando+Aggiornato+alle+rettifiche+pubblicate+sulla+Gazzetta+Ufficiale++dell'Unione+europea+127+del+23+maggio+2018 (garanteprivacy.it) (accessed on 1 August 2024).

Informed Consent Statement: Informed consent was obtained from all subjects involved in the study.

Data Availability Statement: Data are available from the authors upon request. The data are not publicly available due to containing information that could compromise the privacy of the participants. Clips are available as Supplementary Materials.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Pretest

In the following, we report the pretest used in our study:

1. What does it mean to study and/or know the temporal evolution of a phenomenon? Explain by giving concrete examples.
2. What is necessary to know in order to be able to predict the temporal evolution of a phenomenon?
3. When do we say that an event is predictable and when unpredictable? Explain using examples from everyday life.
4. When do we say that a system does not follow deterministic laws? Explain by discussing some examples.
5. When do we say that a body is subject to linear stress and when it is not? Explain this by giving studied examples.
6. Do you think the terms random and chaotic have a similar meaning or not? Can you find similarities and differences between the two terms through concrete examples?
7. Have you ever heard of random natural phenomena? Explain with concrete examples.
8. Have you ever heard of chaotic natural phenomena? Explain with concrete examples.
9. What does the term chaos make you think of? How would you explain the term deterministic chaos? Can a chaotic event be deterministic? Explain by giving concrete examples.
10. Have you ever heard on television that, for example, a rainy and cold day is forecasted and then, instead, there is a beautiful day with bright sunshine and blue skies? Have you ever wondered why? Why do weather conditions not really seem ‘predictable’?

Appendix B. Interactive Worksheets

In the following, we report the interactive worksheets used during the online sessions.

Phase 1

1. Consider the following two systems: a body of mass *m* resting on a frictionless surface and attached to the free end of a spring and a simple pendulum in the absence of friction. Identify the forces at play in the two systems by drawing the force diagram in correspondence with at least four points of the trajectory of your choice. Finally, trace the acceleration vector at the previously chosen points. Import the available image in the team in Microsoft PowerPoint to draw the force diagram.
2. What do you expect if we slightly vary the initial position? Briefly explain.
3. If friction is considered, would it still be possible to determine the time evolution of the systems?
4. Observe the *“myPhysicsLab Single Spring”* simulation of the motion of a simple harmonic oscillator. Vary slightly the initial conditions several times and observe the corresponding motions. Observe how the x coordinate varies as a function of time, x(t) and how the velocity of the mass varies as a function of the x coordinate, v(x). Briefly explain the trends observed.
5. Observe the *“myPhysicsLab Simple Pendulum”* simulation of the motion of a simple pendulum. Vary slightly the initial conditions several times and observe the corresponding motions. Observe the *x(t)* and *v(x)* graphs. Briefly explain the observed trends.
6. Based on the previous observations, describe in words the role of initial conditions in the temporal evolution of the motion of oscillating systems as a spring-mass system or a simple pendulum.
7. Look at the double pendulum in the figure. Predict the motion of the system when it is released from a generic configuration of the two masses.
8. Identify the forces at play in the system under examination by tracing the force diagram in the figure shown on this sheet. Import the available image in the team in PowerPoint to draw the force diagram.

9. Qualitatively describe the motion of the double pendulum.
10. Describe the motion of the system in the following two cases: $m_2 \ll m_1$; $l\_2 \ll l_1$
11. What are the main similarities and differences between the motion of a simple pendulum and that of a double pendulum for small oscillations? And for ample ones?
12. Suppose you repeatedly start the system consisting of the two pendulums from rest varying slightly the initial positions ϑ_1 and ϑ_2. What do you expect to happen? Justify your reasoning.

Phase 2

1. Watch the Video S1 available on YouTube and compare it with your predictions. What are the main differences and similarities? In particular, how does the amplitude of the angles ϑ_1(t) and ϑ_2(t) change with time?
2. The Tracker software functionalities allow to import a video, in our case the oscillations of the double pendulum, and to quantitatively measure those quantities that are difficult to measure with "traditional" instrumentation. Import the two Videos S1 and S2 provided to you (also available on YouTube) into the software, in which we have reproduced the same initial conditions (position and speed) as accurately as possible. By labeling the free end of the double pendulum, it is possible to reconstruct its trajectory. What do you observe when comparing the trajectories of the pendulum in the two videos?
3. Using the tracking function of the Tracker software, describe how the motion of the double pendulum evolves over time choosing a suitable graphical representation.
4. Import the Video S3 (also available on YouTube) that shows the small oscillations of the double pendulum. What happens when the oscillations of the double pendulum become small?
5. Try to model the double pendulum as a physical pendulum (i.e., a rigid body) with mass $M = 3\ Kg$ and length $L = 1\ m$ and verify with the Tracker software that the period of the small oscillations is in agreement with the theoretical model of a physical pendulum.
6. Observe the *"myPhysicsLab Double Pendulum"* simulation and change the masses and length of the two pendulums. Compare the behavior of the double pendulum in the simulation with what you observed in the previous videos. What are the main similarities and differences?

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