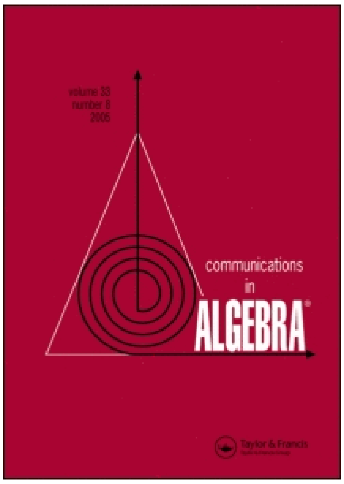

Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

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Online Publication Date: 01 June 2008

To cite this Article Polizzi, Francesco(2008)'On Surfaces of General Type with p<i>g</i> = q = 1 Isogenous to a Product of Curves',Communications in Algebra,36:6,2023 — 2053

To link to this Article: DOI: 10.1080/00927870801948676

URL: http://dx.doi.org/10.1080/00927870801948676

Communications in Algebra®, 36: 2023–2053, 2008

ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870801948676

ON SURFACES OF GENERAL TYPE WITH $p_g = q = 1$ ISOGENOUS TO A PRODUCT OF CURVES

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A smooth algebraic surface S is said to be* isogenous to a product of unmixed type *if there exist two smooth curves C, F and a finite group G, acting faithfully on both C and F and freely on their product, so that $S = (C \times F)/G$. In this article, we classify the surfaces of general type with $p_g = q = 1$ which are isogenous to an unmixed product, assuming that the group G is abelian. It turns out that they belong to four families, that we call surfaces of type I, II, III, IV. The moduli spaces $\mathfrak{M}_I, \mathfrak{M}_{II}, \mathfrak{M}_{IV}$ are irreducible, whereas $\mathfrak{M}_{III}$ is the disjoint union of two irreducible components. In the last section we start the analysis of the case where G is not abelian, by constructing several examples.

Key Words: Actions of finite groups on curves; Surfaces of general type.

2000 Mathematics Subject Classification: Primary 14J29; Secondary 14J10, 20F65.

INTRODUCTION

The problem of classification of surfaces of general type is of exponential computational complexity, see Catanese (1992), Chang (1996), Manetti (1997); nevertheless, one can hope to classify at least those with small numerical invariants. It is well-known that the first example of surface of general type with $p_g = q = 0$ was given by Godeaux (1931); later on, many other examples were discovered. On the other hand, any surface S of general type verifies $\chi(\mathcal{O}_S) > 0$, hence $q(S) > 0$ implies $p_g(S) > 0$. It follows that the surfaces with $p_g = q = 1$ are the irregular ones with the lowest geometric genus, hence it would be important to achieve their complete classification; so far, this has been obtained only in the cases $K_S^2 = 2, 3$

Received April 4, 2006; Revised November 16, 2006. Communicated by C. Pedrini.
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(see Catanese, 1981; Catanese and Ciliberto, 1991, 1993; Catanese and Pignatelli, to appear; Polizzi, 2005). As the title suggests, this article considers surfaces of general type with $p_g = q = 1$ which are *isogenous to a product*. This means that there exist two smooth curves C, F and a finite group G, acting freely on their product, so that $S = (C \times F)/G$. We have two cases: the *mixed* case, where the action of G exchanges the two factors (and then C and F are isomorphic) and the *unmixed* case, where G acts diagonally. In the unmixed case G acts separately on C and F, and the two projections $\pi_C : C \times F \longrightarrow C, \pi_F : C \times F \longrightarrow F$ induce two isotrivial fibrations $\alpha : S \longrightarrow C/G, \beta : S \longrightarrow F/G$, whose smooth fibers are isomorphic to F and C, respectively. If S is isogenous to a product, there exists a unique realization $S = (C \times F)/G$ such that the genera $g(C), g(F)$ are minimal (Catanese, 2000, Proposition 3.13); we will always work with minimal realizations. Surfaces of general type with $p_g = q = 0$ isogenous to a product appear in Beauville (1996), Pardini (2003) and Bauer and Catanese (2002); their complete classification has been finally obtained in Bauer et al. (2006). Some unmixed examples with $p_g = q = 1$ have been given in Polizzi (2006); so it seemed natural to attack the following problem.

Main Problem. Classify all surfaces of general type with $p_g = q = 1$ isogenous to a product, and describe the corresponding irreducible components of the moduli space.

In this article we fully solve the Main Problem in the unmixed case assuming that the group G is abelian. Our results are stated in the following theorem.

Theorem A (see Theorem 4.1). *If the group G is abelian, then there exist exactly four families of surfaces of general type with $p_g = q = 1$ isogenous to an unmixed product. In every case $g(F) = 3$, whereas the occurrences for $g(C)$ and G are*:

I. $g(C) = 3, G = (\mathbb{Z}_2)^2$;
II. $g(C) = 5, G = (\mathbb{Z}_2)^3$;
III. $g(C) = 5, G = \mathbb{Z}_2 \times \mathbb{Z}_4$;
IV. $g(C) = 9, G = \mathbb{Z}_2 \times \mathbb{Z}_8$.

Surfaces of type I *already appear in Polizzi* (2006), *whereas those of type* II, III, IV *provide new examples of minimal surfaces of general type with $p_g = q = 1, K^2 = 8$.*

Theorem B (see Theorem 5.1). *The moduli spaces $\mathfrak{M}_I, \mathfrak{M}_{II}, \mathfrak{M}_{IV}$ are irreducible of dimension* 5, 4, 2, *respectively. The moduli space $\mathfrak{M}_{III}$ is the disjoint union of two irreducible components $\mathfrak{M}_{III}^{(1)}, \mathfrak{M}_{III}^{(2)}$, both of dimension* 3.

The case where G is not abelian is more difficult, and a complete classification is still lacking (see Remark 7.4). However, we can shed some light on this problem, by proving

Theorem C (see Theorem 7.1). *Let $S = (C \times F)/G$ be a surface of general type with $p_g = q = 1$, isogenous to an unmixed product, and assume that the group G is not* *abelian. Then the following cases occur*:

G	$\lvert G\rvert$	$g(C)$	$g(F)$
S_3	6	3	4
D_4	8	3	5
D_6	12	7	3
A_4	12	4	5
S_4	24	9	4
A_5	60	21	4

The examples with $G = S_3$ and D_4 already appear in Polizzi (2006), whereas the others are new. It would be interesting to have a description of the moduli spaces for these new examples (see Remark 7.3).

While describing the organization of the article we shall now explain the steps of our classification procedure in more detail. The crucial point is that in the unmixed case the geometry of the surface $S = (C \times F)/G$ is encoded in the geometry of the two G-covers $h : C \longrightarrow C/G$, $f : F \longrightarrow F/G$. This allows us to "detopologize" the problem by transforming it into an equivalent problem about the existence of a pair of epimorphisms from two groups of Fuchsian type into G; this is essentially an application of the Riemann's existence theorem. These epimorphisms must satisfy some additional properties in order to get a free action of G on $C \times F$ and a quotient surface with the desired invariants (Proposition 3.1). The geometry of the moduli spaces can be also recovered from these algebraic data (Propositions 3.4 and 3.5).

In the nonabelian case we follow a similar approach (Proposition 7.2).

In Section 1 we fix the algebraic set up. The reader that is only interested in the proof of Theorems A and C might skip to Section 2 after reading Section 1.1. On the other hand, the content of Sections 1.2, 1.3, 1.4, 1.5 is essential in order to understand the proof of Theorem B. The results in 1.3 are well known, whereas for those in 1.4 and 1.5 we have not been able to find any complete reference; so we had to carry out "by hand" all the (easy) computations.

In Section 2 we establish some basic results about surfaces S of general type with $p_g = q = 1$ isogenous to a product. Such surfaces are always minimal and verify $K_S^2 = 8$. Moreover, we show that if G is abelian then the Albanese fibration of S is a genus 3 pencil with two double fibers.

The main results of Section 3 are Propositions 3.1 and 3.5, which play a central role in this article as they translate our Main Problem "from geometry to algebra".

Section 4 contains the proof of Theorem A, whereas Section 5 contains the proof of Theorem B.

In Section 6 we study the paracanonical system $\{K\}$ for surfaces of type I, II, III, IV, showing that in any case it has index 1 (Theorem 6.3). This section could appear as a digression with respect to the main theme of the article; however, since the index of $\{K\}$ is an important invariant of S (see Beauville, 1988; Catanese and Ciliberto, 1991, 1993) we thought worthwhile computing it.

Finally, Section 7 deals with the proof of Theorem C.

Notations and Conventions. All varieties and morphisms in this article are defined over the field $\mathbb{C}$ of complex numbers. By "surface" we mean a projective, non-singular surface S, and for such a surface K_S or ω_S denote the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler characteristic*. If S is a surface with $p_g = q = 1$, then $\alpha : S \longrightarrow E$ is the Albanese map of S and F denotes the general fiber of α.

1. TOPOLOGICAL BACKGROUND

Many of the result that we collect in this section are standard, so proofs are often omitted. We refer the reader to Broughton (1990, Section 2), Breuer (2000, Chapter 3), and Harvey (1971) for more details.

1.1. Admissible Epimorphisms

Let us denote by $\Gamma = \Gamma(g' \mid m_1, \dots, m_r)$ the abstract group of Fuchsian type with a presentation of the form

$$\begin{aligned} \text{generators:}\quad & a_1, \dots, a_{g'}, \qquad b_1, \dots, b_{g'}, \qquad c_1, \dots, c_r \\ \text{relations:}\quad & c_1^{m_1} = \cdots = c_r^{m_r} = 1 \\ & c_1 c_2 \dots c_r \prod_{i=1}^{g'} [a_i, b_i] = 1. \end{aligned} \tag{1}$$

The *signature* of Γ is the ordered set of integers $(g' \mid m_1, \dots, m_r)$, where without loss of generality we may suppose $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$. We will call g' the *orbit genus* of Γ and $\mathbf{m} := (m_1, \dots, m_r)$ the *branching data*. In fact the group Γ acts on the upper half-plane $\mathcal{H}$ so that the quotient space $\mathcal{H}/\Gamma$ is a compact Riemann surface of genus g' and the m_i are the ramification numbers of the branched covering $\mathcal{H} \longrightarrow \mathcal{H}/\Gamma$. For convenience, we make abbreviations such as $(2^3, 3^2)$ for $(2, 2, 2, 3, 3)$ when we write down the branching data. If the branching data are empty, the corresponding group $\Gamma(g' \mid -)$ is isomorphic to the fundamental group of a compact Riemann surface of genus g'; it will be denoted by $\Pi_{g'}$. The following result, which is essentially a reformulation of the Riemann's existence theorem, translates the problem of finding Riemann surfaces with automorphisms into the group theoretic problem of finding certain normal subgroups in a given group of Fuchsian type.

Proposition 1.1. *A finite group G acts as a group of automorphisms of some compact Riemann surface X of genus $g \geq 2$ if and only if there exist a group of Fuchsian type $\Gamma = \Gamma(g' \mid m_1, \dots, m_r)$ and an epimorphism $\theta : \Gamma \longrightarrow G$ such that* $\operatorname{Ker}\theta \cong \Pi_g$.

Since Π_g is torsion-free, it follows that θ preserves the orders of the elliptic generators $c_1, \dots, c_r$ of Γ. This motives the following definition.

Definition 1.2. Let G be a finite group. An epimorphism $\theta : \Gamma \longrightarrow G$ is called *admissible* if $\theta(c_i)$ has order m_i for every $i \in \{1, \dots, r\}$. If an admissible epimorphism $\theta : \Gamma \longrightarrow G$ exists, then G is said to be $(g' \mid m_1, \dots, m_r)$-generated.

Proposition 1.3. *If an abelian group G is $(g' \mid m_1, \dots, m_r)$-generated, then $r \neq 1$.*

Proof. Suppose G abelian and $r = 1$. Then relation $x_1\Pi_{i=1}^{g'}[a_i, b_i] = 1$ yields $\theta(x_1) = 0$ for any epimorphism $\theta : \Gamma \longrightarrow G$, so θ cannot be admissible. □

If G is $(g' \mid m_1, \dots, m_r)$-generated, set

$$\begin{aligned} g_i &:= \theta(c_i) \qquad 1 \le i \le r; \\ h_j &:= \theta(a_j) \qquad 1 \le j \le g'; \\ h_{j+g'} &:= \theta(b_j) \qquad 1 \le j \le g'. \end{aligned}$$

The elements $g_1, \dots, g_r, h_1, \dots, h_{2g'}$ generate G and moreover one has

$$g_1 g_2 \dots g_r \prod_{i=1}^{g'} [h_i, h_{i+g'}] = 1 \tag{2}$$

and

$$o(g_i) = m_i. \tag{3}$$

Definition 1.4. An *admissible generating vector* (or, briefly, a *generating vector*) of G with respect to Γ is a $(2g' + r)$-ple of elements

$$\mathscr{V} = \{g_1, \dots, g_r; h_1, \dots h_{2g'}\}$$

such that $\mathscr{V}$ generates G and (2), (3) are satisfied.

If G is abelian, we use the additive notation and relation (2) becomes

$$g_1 + \cdots + g_r = 0. \tag{4}$$

It is evident that giving a generating vector for G with respect to Γ is equivalent to give an admissible epimorphism $\theta : \Gamma \longrightarrow G$; such an epimorphisms fixes the representation of G as a group of conformal automorphisms of a compact Riemann surface X of genus g and the quotient X/G has genus g', where g and g' are related by the Riemann–Hurwitz formula

$$2g - 2 = |G|\left(2g' - 2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right). \tag{5}$$

Hence, accordingly to Proposition 1.1, there is a short exact sequence

$$1 \longrightarrow \Pi_g \xrightarrow{\iota_\theta} \Gamma \xrightarrow{\theta} G \longrightarrow 1 \tag{6}$$

such that Γ can be viewed as the *orbifold fundamental group* of the branched cover $X \longrightarrow X/G$ (see Catanese, 2000). In particular, the cyclic subgroups $\langle g_i \rangle$ and their conjugates are the nontrivial stabilizers of the action of G on X.

1.2. Hurwitz Moves

Looking at exact sequence (6) it is important to remark that X is defined up to automorphisms not by the specific θ, but rather by its kernel $\iota_\theta(\Pi_g)$; this motives the following definition.

Definition 1.5. We set

$$\mathrm{Epi}(\Pi_g, \Gamma, G) := \left\{ \begin{array}{l} \text{Admissible epimorphims } \theta : \Gamma \longrightarrow G \\ \text{such that } \mathrm{Ker}\,\theta \cong \Pi_g \end{array} \right\} \Big/ \sim$$

where $\theta_1 \sim \theta_2$ if and only if $\mathrm{Ker}\,\theta_1 = \mathrm{Ker}\,\theta_2$.

Abusing notation we will often not distinguish between an epimorphism θ and its class in $\mathrm{Epi}(\Pi_g, \Gamma, G)$. An automorphism $\eta \in \mathrm{Aut}(\Gamma)$ is said to be orientation-preserving if, for all $i \in \{1, \ldots, r\}$, there exists j such that $\eta(c_i)$ is conjugated to c_j. This of course implies $o(c_i) = o(c_j)$. The subgroup of orientation-preserving automorphisms of Γ is denoted by $\mathrm{Aut}^+(\Gamma)$ and the quotient $\mathrm{Mod}(\Gamma) := \mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$ is called the *mapping class group* of Γ. There is a natural action of $\mathrm{Aut}(G) \times \mathrm{Mod}(\Gamma)$ on $\mathrm{Epi}(\Pi_g, \Gamma, G)$, namely,

$$(\lambda, \eta) \cdot \theta := \lambda \circ \theta \circ \eta.$$

Proposition 1.6. *Two admissible epimorphisms* $\theta_1, \theta_2 \in \mathrm{Epi}(\Pi_g, \Gamma, G)$ *define the same equivalence class of G-actions if and only if they lie in the same* $\mathrm{Aut}(G) \times \mathrm{Mod}(\Gamma)$*-class.*

The nontrivial part of the proof is to show that $\mathrm{Aut}(G) \times \mathrm{Mod}(\Gamma)$-equivalent epimorphisms give equivalent G-actions; this depends on Teichmüller theory and proofs can be found in Macbeath (1966) and Harvey (1971).

The action of $\mathrm{Aut}(G) \times \mathrm{Mod}(\Gamma)$ on $\mathrm{Epi}(\Pi_g, \Gamma, G)$ naturally induces an action on the set of generating vectors (up to inner automorphisms of G); in particular, if θ_1 and θ_2 are in the same $\{\mathrm{Id}\} \times \mathrm{Mod}(\Gamma)$-class, we say that the corresponding generating vectors are related by a *Hurwitz move*. If $\mathcal{V} = \{g_1, \ldots, g_r; h_1, \ldots h_{2g'}\}$ is a generating vector of G with respect to Γ, by definition of $\mathrm{Aut}^+(\Gamma)$ any Hurwitz move sends g_i to some conjugated of g_j, where $o(g_i) = o(g_j)$. In particular, if G is abelian then the Hurwitz moves permute the g_i having the same order. Moreover, in this case, the Hurwitz moves on $\mathcal{V}$ are unambiguously defined, since $\mathrm{Inn}(G)$ is trivial.

If $\Sigma_{g'}$ is a differentiable model of a compact Riemann surface of genus g' and $p_1, \ldots, p_r \in \Sigma_{g'}$, we define

$$\mathrm{Mod}_{g',[r]} := \pi_0 \mathrm{Diff}^+(\Sigma_{g'} - \{p_1, \ldots, p_r\}).$$

Given $\Gamma := \Gamma(g' \mid m^r)$, it is well known that $\mathrm{Mod}(\Gamma)$ is isomorphic to $\mathrm{Mod}_{g',[r]}$ (Schneps, 2003, Theorem 2.2.1). In the sequel of this article, we will deal with an abelian group G and with few types of signature, namely $(0 \mid \mathbf{m})$, $(1 \mid m)$ and $(1 \mid m^2)$. So let us explicitly describe the Hurwitz moves in these cases.

1.3. The Case $g' = 0$

For the sake of simplicity, let us suppose that all the m_i are equal, i.e., $m_1 = \cdots = m_r = m$. By the result mentioned above, the mapping class group $\mathrm{Mod}(\Gamma(0 \mid m^r))$ can be identified with

$$\mathrm{Mod}_{0,[r]} := \pi_0 \, \mathrm{Diff}^+(\mathbb{P}^1 - \{p_1, \dots p_r\}),$$

which is a quotient of the Artin braid group $\mathbf{B}_r$. Let σ_i be the positive-oriented Dehn twist about a simple closed curve in $\mathbb{P}^1$ containing p_i and none of the other marked points. Then it is well known (see for instance Schneps, 2003, Section 2.3) that $\mathrm{Mod}_{0,[r]}$ is generated by $\sigma_1, \dots, \sigma_r$ with the following relations:

$$\begin{aligned} &\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ &\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{if } |i-j| \geq 2 \\ &\sigma_{r-1}\sigma_{r-2}\cdots\sigma_1^2\cdots\sigma_{r-2}\sigma_{r-1} = 1. \end{aligned}$$

Now we can describe the Hurwitz moves in this case.

Proposition 1.7. *Up to inner automorphisms, the action of* $\mathrm{Mod}_{0,[r]}$ *on* $\Gamma(0 \mid m^r)$ *is given by*

$$\sigma_i : \begin{cases} y_i & \longrightarrow y_{i+1} \\ y_{i+1} & \longrightarrow y_{i+1}^{-1} y_i y_{i+1} \\ y_j & \longrightarrow y_j \quad \text{if } j \neq i, \quad i+1. \end{cases}$$

Proof. See Schneps (2003, Proposition 2.3.5) or Catanese (2005, Section 4). □

Corollary 1.8. *Let G be a finite abelian group and let* $\mathcal{V} = \{g_1, \dots, g_r\}$ *be a generating vector of G with respect to* $\Gamma(0 \mid m^r)$. *Then the Hurwitz moves coincide with the group of permutations of* $\mathcal{V}$.

The general case can be carried out in a similar way (see Broughton, 1990, Proposition 2.5.) and one obtains the following corollary.

Corollary 1.9. *Let G be a finite abelian group and let* $\mathcal{V} = \{g_1, \dots, g_r\}$ *be a generating vector of G with respect to* $\Gamma(0 \mid m_1, \dots, m_r)$. *Then the Hurwitz moves on* $\mathcal{V}$ *are generated by the transpositions of the* g_i *having the same order.*

1.4. The Case $g' = 1, r = 1$

Let $\Gamma = \Gamma(1 \mid m^1)$; then $\mathrm{Mod}(\Gamma)$ can be identified with

$$\mathrm{Mod}_{1,1} = \pi_0 \mathrm{Diff}^+(\Sigma_1 - \{p\}).$$

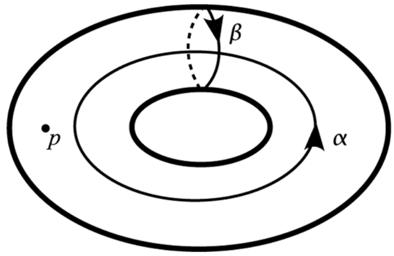

Figure 1 Generators of $\mathrm{Mod}_{1,1}$.

This group is generated by the positively-oriented Dehn twists t_α, t_β about the two simple closed curves α, β shown in Figure 1. The corresponding relations are the following (see Schneps, 2003):

$$t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta; \quad (t_\alpha t_\beta)^3 = 1.$$

Via the identifications

$$t_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad t_\beta = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

one verifies that $\mathrm{Mod}_{1,1}$ is isomorphic to $\mathrm{SL}_2(\mathbb{Z})$. The group $\Gamma(1 \mid m^1)$ is a quotient of $\pi_1(\Sigma_1 - \{p\})$, in fact it has the presentation

$$\Gamma(1 \mid m^1) = \langle a, b, x \mid x^m = x[a, b] = 1 \rangle.$$

Let us identify the torus Σ_1 with the topological space obtained by gluing the opposite sides of a square; then the generators a, b, x of $\Gamma(1 \mid m^1)$ and the two loops α, β are illustrated in Figure 2.

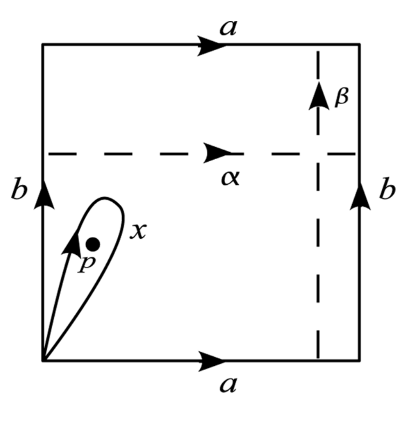

Figure 2 The generators a, b, x and the loops α, β.

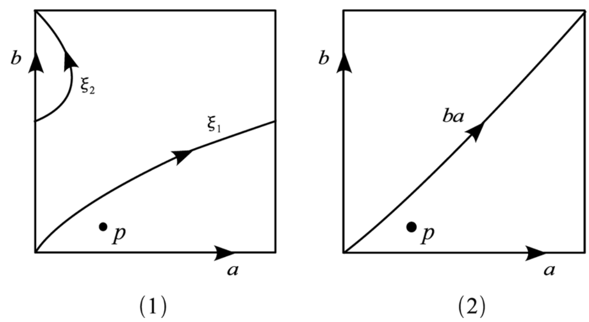

Figure 3 $t_\alpha(b) = \xi_1\xi_2 = ba$.

Proposition 1.10. *Up to inner automorphisms, the action of* $\mathrm{Mod}_{1,1}$ *on* $\Gamma(1 \mid m^1)$ *is given by*

$$t_\alpha : \begin{cases} x \longrightarrow x \\ a \longrightarrow a \\ b \longrightarrow ba \end{cases} \qquad t_\beta : \begin{cases} x \longrightarrow x \\ a \longrightarrow ab^{-1} \\ b \longrightarrow b. \end{cases}$$

Proof. It is sufficient to compute, up to inner automorphisms, the action of $\mathrm{Mod}_{1,1}$ on $\pi_1(\Sigma_1 - \{p\})$. Look at Figure 2. Evidently, $t_\alpha(a) = a$ and $t_\alpha(x) = x$, because α is disjoint from both a and x. Analogously, $t_\beta(b) = b$, $t_\beta(x) = x$. Hence we must only compute $t_\alpha(b)$ and $t_\beta(a)$. The pair (α, b) is positively oriented and the action of t_α on b is illustrated in Figure 3; so we obtain $t_\alpha(b) = \xi_1\xi_2$, which is homotopic to ba. Similarly, (β, a^{-1}) is a positively oriented pair and the action of t_β on a^{-1} is illustrated in Figure 4; so we obtain $t_\beta(a^{-1}) = \eta_1\eta_2 = ba^{-1}$, that is $t_\beta(a) = ab^{-1}$. □

Corollary 1.11. *Let G be a finite abelian group and let $\mathscr{W} = \{g; h_1, h_2\}$ be a generating vector for G with respect to $\Gamma(1 \mid m^1)$. Then the Hurwitz moves on $\mathscr{W}$ are*

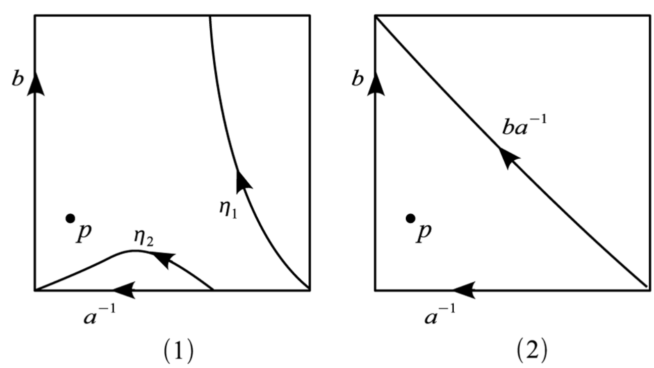

Figure 4 $t_\beta(a^{-1}) = \eta_1\eta_2 = ba^{-1}$.

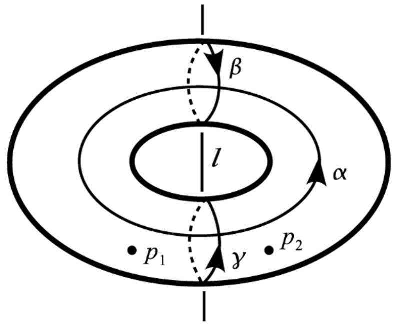

Figure 5 Generators of $\mathrm{Mod}_{1,2}$.

generated by

$$\mathbf{1}: \begin{cases} g \longrightarrow g \\ h_1 \longrightarrow h_1 \\ h_2 \longrightarrow h_1 + h_2 \end{cases} \qquad \mathbf{2}: \begin{cases} g \longrightarrow g \\ h_1 \longrightarrow h_1 - h_2 \\ h_2 \longrightarrow h_2. \end{cases}$$

Proof. This follows directly from Proposition 1.10. □

1.5. The Case $g' = 1$, $r = 2$, $m_1 = m_2 = m$

Let $\Gamma = \Gamma(1 \mid m^2)$; then $\mathrm{Mod}(\Gamma)$ can be identified with

$$\mathrm{Mod}_{1,[2]} = \pi_0 \mathrm{Diff}^+(\Sigma_1 - \{p_1, p_2\}).$$

This group is generated by the positively-oriented Dehn twists $t_\alpha, t_\beta, t_\gamma$ about the simple closed curves α, β, γ shown in Figure 5, and by the class of the rotation ρ of π radians around the line l, which exchanges the marked points. The relations defining $\mathrm{Mod}_{1,[2]}$ are the following (see Cattabriga and Mulazzani, 2004):

$$t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta; \qquad t_\alpha t_\gamma t_\alpha = t_\gamma t_\alpha t_\gamma;$$
$$t_\beta t_\gamma = t_\gamma t_\beta; \qquad (t_\alpha t_\beta t_\gamma)^4 = 1;$$
$$t_\alpha \rho = \rho t_\alpha; \qquad t_\beta \rho = \rho t_\beta; \qquad t_\gamma \rho = \rho t_\gamma.$$

The group $\Gamma(1 \mid m^2)$ is a quotient of $\pi_1(\Sigma_1 - \{p_1, p_2\})$, in fact its presentation is

$$\Gamma(1 \mid m^2) = \langle a, b, x_1, x_2 \mid x_1^m = x_2^m = x_1 x_2 [a, b] = 1 \rangle;$$

the generators a, b, x_1, x_2 and the loops α, β, γ are illustrated in Figure 6.

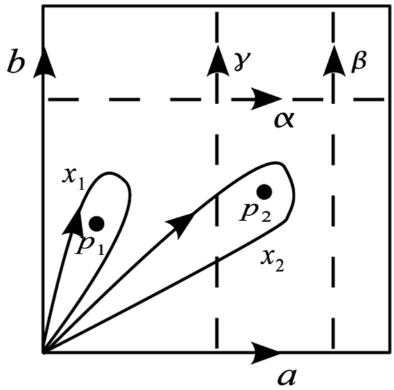

Figure 6

Proposition 1.12. *Up to inner automorphisms, the action of* $\mathrm{Mod}_{1,[2]}$ *on* $\Gamma(1 \mid m^2)$ *is given by*

$$t_\alpha : \begin{cases} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow x_2 \\ a \longrightarrow a \\ b \longrightarrow ba \end{cases} \qquad t_\beta : \begin{cases} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow x_2 \\ a \longrightarrow ab^{-1} \\ b \longrightarrow b \end{cases}$$

$$t_\gamma : \begin{cases} x_1 \longrightarrow x_1 \\ x_2 \longrightarrow ab^{-1}a^{-1}x_2aba^{-1} \\ a \longrightarrow b^{-1}x_1a \\ b \longrightarrow b \end{cases} \qquad \rho : \begin{cases} x_1 \longrightarrow b^{-1}a^{-1}x_2ab \\ x_2 \longrightarrow a^{-1}b^{-1}x_1ba \\ a \longrightarrow a^{-1} \\ b \longrightarrow b^{-1}. \end{cases}$$

Proof. It is sufficient to compute, up to inner automorphisms, the action of $\mathrm{Mod}_{1,[2]}$ on $\pi_1(\Sigma_1 - \{p_1, p_2\})$. Look at Figure 6 and consider the action of t_α. We have $t_\alpha(a) = a, t_\alpha(x_1) = x_1, t_\alpha(x_2) = x_2$ because α is disjoint from a, x_1, x_2; moreover $t_\alpha(b) = ba$ exactly as in the proof of Proposition 1.10. The computation of the action of t_β is similar. Next, let us consider the action of t_γ. The curve γ is disjoint from both b and x_1, then $t_\gamma(b) = b, t_\gamma(x_1) = x_1$. Moreover, since (γ, a^{-1}) is a positively-oriented pair, the action of t_γ on a^{-1} is as in Figure 7; this gives $t_\gamma(a^{-1}) = \xi_1\xi_2 = a^{-1}x_1^{-1}b$, hence $t_\gamma(a) = b^{-1}x_1a$. Using the computations above and the relation $x_1x_2[a, b] = 1$ we obtain $x_1t_\gamma(x_2)[b^{-1}x_1a, b] = 1$, hence

$$t_\gamma(x_2) = ab^{-1}a^{-1}x_1^{-1}b = ab^{-1}a^{-1}x_2aba^{-1}.$$

Finally, let us consider the action of ρ, which is illustrated in Figure 8. Evidently, $\rho(a) = a^{-1}$ and $\rho(b) = b^{-1}$. Moreover, the picture also shows that there is the relation

$$ba(\rho(x_1))^{-1}b^{-1}a^{-1} = x_1.$$

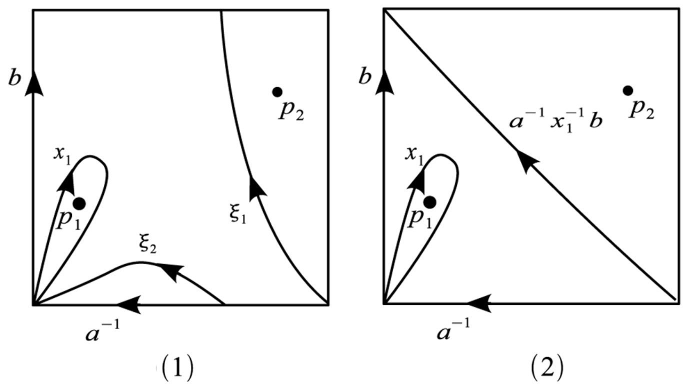

Figure 7 $t_\gamma(a^{-1}) = \xi_1\xi_2 = a^{-1}x_1^{-1}b$.

Then

$$\begin{aligned}\rho(x_1) &= b^{-1}a^{-1}x_1^{-1}ba = b^{-1}a^{-1}(x_2aba^{-1}b^{-1})ba \\ &= b^{-1}a^{-1}x_2ab.\end{aligned}$$

Finally, using $x_1x_2[a, b] = 1$, we can write

$$\begin{aligned}\rho(x_2) &= (\rho(x_1))^{-1}(\rho([a, b]))^{-1} \\ &= (b^{-1}a^{-1}x_2ab)^{-1}(a^{-1}b^{-1}ab)^{-1} \\ &= b^{-1}a^{-1}x_2^{-1}ba = a^{-1}b^{-1}x_1ba.\end{aligned}$$

This completes the proof. □

Corollary 1.13. *Let G be a finite, abelian group and let $\mathscr{W} = \{g_1, g_2; h_1, h_2\}$ be a generating vector for G with respect to $\Gamma(1 \mid m^2)$. Then the Hurwitz moves on $\mathscr{W}$ are*

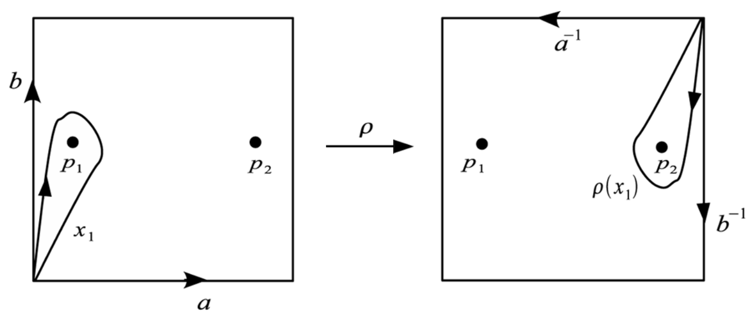

Figure 8 Action of ρ.

generated by

$$\mathbf{1}: \begin{cases} g_1 \longrightarrow g_1 \\ g_2 \longrightarrow g_2 \\ h_1 \longrightarrow h_1 \\ h_2 \longrightarrow h_1 + h_2 \end{cases} \qquad \mathbf{2}: \begin{cases} g_1 \longrightarrow g_1 \\ g_2 \longrightarrow g_2 \\ h_1 \longrightarrow h_1 - h_2 \\ h_2 \longrightarrow h_2 \end{cases}$$

$$\mathbf{3}: \begin{cases} g_1 \longrightarrow g_1 \\ g_2 \longrightarrow g_2 \\ h_1 \longrightarrow h_1 - h_2 + g_1 \\ h_2 \longrightarrow h_2 \end{cases} \qquad \mathbf{4}: \begin{cases} g_1 \longrightarrow g_2 \\ g_2 \longrightarrow g_1 \\ h_1 \longrightarrow -h_1 \\ h_2 \longrightarrow -h_2. \end{cases}$$

2. SURFACES OF GENERAL TYPE WITH $p_g = q = 1$ ISOGENOUS TO A PRODUCT

Definition 2.1. A surface S of general type is said to be *isogenous to a product* if there exist two smooth curves C, F and a finite group G, acting freely on their product, so that $S = (C \times F)/G$.

We have two cases: The *mixed* case, where the action of G exchanges the two factors (and then C, F are isomorphic) and the *unmixed* case, where G acts diagonally. If S is isogenous to a product, there exists a unique realization $S = (C \times F)/G$ such that the genera $g(C), g(F)$ are minimal (Catanese, 2000, Proposition 3.13). Our aim is to solve the following problem.

Main Problem. Classify the surfaces of general type $S = (C \times F)/G$ with $p_g = q = 1$, isogenous to an *unmixed* product, assuming that the group G is *abelian*. Describe the corresponding irreducible components of the moduli space.

Notice that, when S is of unmixed type, the group G acts separately on C and F and the two projections $\pi_C : C \times F \longrightarrow C$, $\pi_F : C \times F \longrightarrow F$ induce two isotrivial fibrations $\alpha : S \longrightarrow C/G$, $\beta : S \longrightarrow F/G$ whose smooth fibers are isomorphic to F and C, respectively. Moreover, we will always consider the minimal realization of S, so that the action of G will be faithful on both factors.

Proposition 2.2. *Let $S = (C \times F)/G$ be a surface of general type with $p_g = q = 1$, isogenous to an unmixed product. Then S is minimal and moreover*:

(i) $K_S^2 = 8$;
(ii) $|G| = (g(C) - 1)(g(F) - 1)$;
(iii) $F/G \cong \mathbb{P}^1$ *and* $C/G \cong E$, *where E is an elliptic curve isomorphic to the Albanese variety of S*;
(iv) *S contains no pencils of genus 2 curves*;
(v) $g(C) \geq 3$, $g(F) \geq 3$ (*hence* (ii) *implies* $|G| \geq 4$).

Proof. Since the projection $C \times F \longrightarrow S$ is étale, the pullback of any (-1)-curve of S would give rise to a (disjoint) union of (-1)-curves in $C \times F$, and this is impossible; then S is minimal. Now let us prove (i)–(v).

(i) Since S is isogenous to an unmixed product we have $K_S^2 = 2c_2(S)$ (Serrano, 1993, Proposition 3.5). Noether's formula gives $K_S^2 + c_2(S) = 12$, so it follows $K_S^2 = 8$.

(ii) We have

$$p_g(C \times F) = g(C) \cdot g(F) \quad \text{and} \quad q(C \times F) = g(C) + g(F),$$

see Beauville (1996, III.22). Since $C \times F \longrightarrow S$ is an étale covering, we obtain $|G| = \chi(\mathcal{O}_{C\times F})/\chi(\mathcal{O}_S) = (g(C)-1)(g(F)-1)$.

(iii) We have $q(S) = g(C/G) + g(F/G)$, then we may assume

$$g(C/G) = 1, \qquad g(F/G) = 0.$$

Setting $E = C/G$ it follows that $\alpha : S \longrightarrow E$ is a connected fibration with elliptic base, then it coincides with the Albanese morphism of S.

(iv) Let S be a minimal surface of general type with $p_g = q = 1$ which contains a genus 2 pencil. There are two cases:

(i) The pencil is rational, then either $K_S^2 = 2$ or $K_S^2 = 3$ (see Xiao, 1985, p. 51);
(ii) The pencil is irrational, therefore it must be the Albanese pencil, and in this case $2 \leq K_S^2 \leq 6$ (see Xiao, 1985, p. 17).

In both cases, part (i) implies that S cannot be isogenous to a product.

(v) This follows from part (iv). □

The two G-coverings $f : F \longrightarrow \mathbb{P}^1$, $h : C \longrightarrow E$ are induced by two admissible epimorphisms $\vartheta : \Gamma(0\,|\,\mathbf{m}) \longrightarrow G$, $\psi : \Gamma(1\,|\,\mathbf{n}) \longrightarrow G$, where $\mathbf{m} = (m_1, \ldots, m_r)$, $\mathbf{n} = (n_1, \ldots, n_s)$. The Riemann–Hurwitz formula gives

$$2g(F) - 2 = |G|\left(-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right)$$
$$2g(C) - 2 = |G|\sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \tag{7}$$

Proposition 2.3. *We have the following possibilities*:

(a) $g(F) = 3$, $\mathbf{n} = (2^2)$;
(b) $g(F) = 4$, $\mathbf{n} = (3^1)$;
(c) $g(F) = 5$, $\mathbf{n} = (2^1)$.

Moreover, if G is abelian only case (a) *may occur.*

Proof. Using (7) and part (ii) of Proposition 2.2 we obtain

$$2 = (g(F) - 1)\sum_{j=1}^{s}\left(1 - \frac{1}{n_j}\right). \tag{8}$$

Since $g(F) \geq 3$ and the sum in the right-hand side of (8) is $\geq \frac{1}{2}$, we have $g(F) \leq 5$. If $g(F) = 3$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = 1$, so $\mathbf{n} = (2^2)$; if $g(F) = 4$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{2}{3}$, so $\mathbf{n} = (3^1)$; if $g(F) = 5$ then $\sum_{j=1}^{s} \left(1 - \frac{1}{n_j}\right) = \frac{1}{2}$, so $\mathbf{n} = (2^1)$. Finally, if G is abelian then $s \geq 2$ (Proposition 1.3), so only case (a) may occur. □

Remark 2.4. If G is not abelian, then all possibilities (a), (b), (c) actually occur. Examples are given in Section 7.

Corollary 2.5. *Suppose that G is abelian. Then*:

(i) *The Albanese fibration of S is an isotrivial genus* 3 *pencil with two double fibers*;
(ii) $|G| = 2(g(C) - 1)$.

3. ABELIAN CASE: THE BUILDING DATA

In the sequel we will assume that G is abelian. By Proposition 2.3 the covering $h : C \longrightarrow E$ is induced by an admissible epimorphism $\psi : \Gamma(1 \mid 2^2) \longrightarrow G$. If $\mathcal{W} = \{g_1, g_2; h_1, h_2\}$ is the corresponding generating vector, we have $2g_1 = 2g_2 = g_1 + g_2 = 0$, hence $g_1 = g_2$. For the sake of simplicity we set $g_1 = g_2 = g$ and we denote the generating vector by $\{g; h_1, h_2\}$. Note that $\langle g \rangle$ is the only nontrivial stabilizer of the action of G on C. Analogously, if $\mathcal{V} := \{g_1, \ldots g_r\}$ is any generating vector of G with respect to $\Gamma(0 \mid \mathbf{m})$, the cyclic subgroups $\langle g_1 \rangle, \ldots, \langle g_r \rangle$ are the only nontrivial stabilizers of the action of G on F. Then the diagonal action of G on $C \times F$ is free if and only if

$$\left(\bigcup_{i=1}^{r} \langle g_i \rangle \right) \cap \langle g \rangle = \{0\}. \tag{9}$$

Using the results contained in the previous section, we obtain the following proposition.

Proposition 3.1. *Suppose that we have the following data*:

(i) *A finite abelian group G*;
(ii) *Two admissible epimorphisms*

$$\vartheta : \Gamma(0 \mid \mathbf{m}) \longrightarrow G, \qquad \mathbf{m} = (m_1, \ldots, m_r)$$
$$\psi : \Gamma(1 \mid 2^2) \longrightarrow G$$

with corresponding generating vectors $\mathcal{V} = \{g_1, \ldots, g_r\}$, $\mathcal{W} = \{g; h_1, h_2\}$.

Let

$$f : F \longrightarrow \mathbb{P}^1 = F/G$$
$$h : C \longrightarrow E = C/G$$

be the G-coverings induced by ϑ *and* ψ *and let* $g(C)$, $g(F)$ *be the genera of* C *and* F, *which are related on* G *and* $\mathbf{m}$ *by* (7). *Assume moreover that*:

(i) $g(C) \geq 3, \ g(F) = 3$;
(ii) $|G| = 2(g(C) - 1)$;
(iii) *Condition* (9) *is satisfied.*

Then the diagonal action of G *on* $C \times F$ *is free and the quotient* $S = (C \times F)/G$ *is a minimal surface of general type with* $p_g = q = 1$. *Conversely, any surface of general type with* $p_g = q = 1$, *isogenous to an unmixed product with* G *abelian, arises in this way.*

We will call the 4-ple $(G, \mathbf{m}, \vartheta, \psi)$ the *building data* of S.

Corollary 3.2. *Let* $S = (C \times F)/G$ *be a surface of general type with* $p_g = q = 1$, *isogenous to an unmixed product. Then the group* G *cannot be cyclic.*

Proof. By contradiction suppose that G is cyclic; then $G = \mathbb{Z}_{2m}$ for some integer m. Let $\{g_1, \dots, g_r\}$ and $\{g; h_1, h_2\}$ be generating vectors of G as in Proposition 3.1. The group G contain exactly one subgroup of order 2, namely, $\langle g \rangle$. On the other hand, since $\langle g_1, \dots, g_r \rangle = G$, we have l.c.m.$(m_1, \dots, m_r) = 2m$; hence 2 divides some of the m_i, say m_1. This implies $g \in \langle g_1 \rangle$, which violates condition (9). □

Let $\mathfrak{M}_{a,b}$ be the moduli space of smooth minimal surfaces of general type with $\chi(\mathcal{O}_S) = a$, $K_S^2 = b$; by a result of Gieseker, we know that $\mathfrak{M}_{a,b}$ is a quasiprojective variety for all $a, b \in \mathbb{N}$ (see Gieseker, 1977). Obviously, our surfaces are contained in $\mathfrak{M}_{1,8}$ and we want to describe their locus there.

Proposition 3.3. *For fixed* G *and* $\mathbf{m}$, *denote by* $\mathfrak{M}(G, \mathbf{m})$ *the moduli space of surfaces with* $p_g = q = 1$ *described by the remaining building data* (ϑ, ψ). *Then* $\mathfrak{M}(G, \mathbf{m})$ *consists of a finite number of irreducible components of* $\mathfrak{M}_{1,8}$, *all of dimension* $r - 1$.

Proof. The fact that $\mathfrak{M}(G, \mathbf{m})$ consists of finitely many irreducible components of $\mathfrak{M}_{1,8}$ follows by general results of Catanese on surfaces isogenous to a product (see Catanese, 2000). The dimension of each component is $r - 1$ because we take r points on $\mathbb{P}^1$ modulo projective equivalence and 2 points on E modulo projective equivalence. □

Let us define

$$\Phi(G, \ \mathbf{m}) := \mathrm{Epi}(\Pi_3, \Gamma(0 \,|\, \mathbf{m}), G) \times \mathrm{Epi}(\Pi_{g(C)}, \Gamma(1 \,|\, 2^2), G);$$
$$\mathfrak{G} := \mathrm{Aut}(G) \times \mathrm{Mod}(\Gamma(0 \,|\, \mathbf{m})) \times \mathrm{Mod}(\Gamma(1 \,|\, 2^2)).$$

The group $\mathfrak{G}$ naturally acts on the set $\Phi(G, \mathbf{m})$ in the following way:

$$(\lambda, \eta_0, \eta_1) \cdot (\vartheta, \psi) := (\lambda \circ \vartheta \circ \eta_0, \lambda \circ \psi \circ \eta_1).$$

Proposition 3.4. *Let S_1, S_2 be two surfaces defined by building data $(\vartheta_1, \psi_1), (\vartheta_2, \psi_2) \in \Phi(G, \mathbf{m})$. Then S_1 and S_2 belong to the same connected component of $\mathfrak{M}(G, \mathbf{m})$ if and only if (ϑ_1, ψ_1) and (ϑ_2, ψ_2) are in the same $\mathfrak{G}$-class.*

Proof. We can use the same argument of Bauer and Catanese (2002, Theorem 1.3). In fact, Proposition 1.6 allows us to substitute the pair of braid group actions considered in that article with the two actions of $\mathrm{Mod}(\Gamma(0\,|\,\mathbf{m}))$ and $\mathrm{Mod}(\Gamma(1\,|\,2^2))$. □

Now let $\mathfrak{B}(G, \mathbf{m})$ be the set of pairs of generating vectors $(\mathcal{V}, \mathcal{W})$ such that the hypotheses of Proposition 3.1 are satisfied (in particular (9) must hold). Let us denote by $\mathfrak{R}$ the equivalence relation on $\mathfrak{B}(G, \mathbf{m})$ generated by:

(i) Hurwitz moves on $\mathcal{V}$;
(ii) Hurwitz moves on $\mathcal{W}$;
(iii) Simultaneous conjugation of $\mathcal{V}$ and $\mathcal{W}$ by an element of $\lambda \in \mathrm{Aut}(G)$, i.e., we let $(\mathcal{V}, \mathcal{W})$ be equivalent to $(\lambda(\mathcal{V}), \lambda(\mathcal{W}))$.

Proposition 3.5. *The number of irreducible components in $\mathfrak{M}(G, \mathbf{m})$ equals the number of $\mathfrak{R}$-classes in $\mathfrak{B}(G, \mathbf{m})$.*

Proof. Immediate consequence of Proposition 3.4. □

4. ABELIAN CASE: THE CLASSIFICATION

We have the following result.

Theorem 4.1. *If the group G is abelian, then there exist exactly four families of surfaces of general type with $p_g = q = 1$, isogenous to an unmixed product. In any case $g(F) = 3$, whereas the possibilities for $g(C)$ and G are the following:*

I. $g(C) = 3$, $G = (\mathbb{Z}_2)^2$;
II. $g(C) = 5$, $G = (\mathbb{Z}_2)^3$;
III. $g(C) = 5$, $G = \mathbb{Z}_2 \times \mathbb{Z}_4$;
IV. $g(C) = 9$, $G = \mathbb{Z}_2 \times \mathbb{Z}_8$.

Surfaces of type I already appear in Polizzi (2006), whereas those of type II, III, IV provide new examples of minimal surfaces of general type with $p_g = q = 1$, $K^2 = 8$. The remainder of this section deals with the proof of Theorem 4.1. Let S be defined by building data $(G, \mathbf{m}, \vartheta, \psi)$ as in Proposition 3.1. Using (7) and Corollary 2.5 we obtain

$$2 = (g(C) - 1)\left(-2 + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)\right). \tag{10}$$

Proposition 4.2. *We have $3 \leq r \leq 6$. Moreover, if $r = 6$ the only possibility is*

$$\mathbf{m} = (2^6), \qquad G = (\mathbb{Z}_2)^2.$$

Proof. Using equation (10) we can write

$$(g(C)-1)\left(\frac{r}{2}-2\right) \leq 2 < (g(C)-1)(r-2). \tag{11}$$

Part (v) of Proposition 2.2 yields $g(C)-1 \geq 2$, hence (11) implies $3 \leq r \leq 6$. Moreover, we have $r=6$ if and only if $\mathbf{m}=(2^6)$, and in this case $|G|=4$. Then Corollary 3.2 implies $G=(\mathbb{Z}_2)^2$. □

Proposition 4.3. *If $r=5$ the only possibility is*

$$\mathbf{m}=(2^5), \qquad G=(\mathbb{Z}_2)^3.$$

Proof. If $r=5$ formula (10) gives

$$2=(g(C)-1)\left(3-\sum_{i=1}^{5}\frac{1}{m_i}\right) \geq (g(C)-1)\left(3-\frac{5}{m_1}\right), \tag{12}$$

hence

$$g(C)-1 \leq \frac{2m_1}{3m_1-5}\,.$$

If $m_1 \geq 3$ then $g(C) \leq 2$, a contradiction. Then $m_1=2$ and $g(C)-1 \leq 4$, hence $|G| \leq 8$ (Corollary 2.5) with equality if and only if $\mathbf{m}=(2^5)$. If $\mathbf{m} \neq (2^5)$, then G would be a noncyclic group of order smaller than 8 which contains some element of order greater than 2, a contradiction. Therefore, $\mathbf{m}=(2^5)$ is actually the only possibility. Then G is an abelian group of order 8 generated by elements of order 2, hence $G=(\mathbb{Z}_2)^3$. □

Proposition 4.4. *If $r=4$ the only possibility is*

$$\mathbf{m}=(2^2,4^2), \qquad G=\mathbb{Z}_2\times\mathbb{Z}_4.$$

Proof. If $r=4$ formula (10) gives

$$2=(g(C)-1)\left(2-\sum_{i=1}^{4}\frac{1}{m_i}\right) \geq (g(C)-1)\left(2-\frac{4}{m_1}\right), \tag{13}$$

hence

$$g(C)-1 \leq \frac{m_1}{m_1-2}\,.$$

If $m_1 \geq 3$ then $g(C)-1 \leq 3$, which implies $|G| \leq 6$. Since G is not cyclic the only possibility would be $G=\mathbb{Z}_2\times\mathbb{Z}_2$, which contains no elements of order ≥ 3, a contradiction. It follows $\mathbf{m}=(2,m_2,m_3,m_4)$. Applying again formula (10) we get

$$2=(g(C)-1)\left(\frac{3}{2}-\sum_{i=2}^{4}\frac{1}{m_i}\right) \geq (g(C)-1)\left(\frac{3}{2}-\frac{3}{m_2}\right), \tag{14}$$

hence

$$g(C) - 1 \leq \frac{4m_2}{3m_2 - 6} \cdot$$

If $m_2 \geq 3$ then $|G| \leq 8$, and equality occurs if and only if $\mathbf{m} = (2, 3^3)$. Since a group of order 8 contains no elements of order 3, we must have $|G| < 8$ and the only possibility would be $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which contradicts $m_2 \geq 3$. Then $m_2 = 2$, so that $\mathbf{m} = (2^2, m_3, m_4)$. Now suppose $m_3 = 2$ and consider the generating vector $\mathcal{V} = \{g_1, g_2, g_3, g_4\}$; since $g_1 + g_2 + g_3 + g_4 = 0$ it follows $m_4 = 2$, hence $\mathbf{m} = (2^4)$ which violates (10). Therefore, $m_3 \geq 3$ and an easy computation using (10) shows that there are only the following possibilities:

(i) $\mathbf{m} = (2^2, 3^2)$, $|G| = 12$;
(ii) $\mathbf{m} = (2^2, 3, 6)$, $|G| = 8$;
(iii) $\mathbf{m} = (2^2, 4^2)$, $|G| = 8$;
(iv) $\mathbf{m} = (2^2, 4, 12)$, $|G| = 6$;
(v) $\mathbf{m} = (2^2, 6^2)$, $|G| = 6$.

In case (i), equality $g_1 + g_2 + g_3 + g_4 = 0$ yields $g_3 + g_4 = g_1 + g_2$, hence $g_1 + g_2 = g_3 + g_4 = 0$. This implies $G = \langle g_1, g_3 \rangle = \mathbb{Z}_2 \times \mathbb{Z}_3$, which is a contradiction. Case (ii) must be excluded since a group of order 8 contains no elements of order 3. Finally, cases (iv) and (v) must be excluded since an abelian group of order 6 is cyclic. Then the only possibility is (iii), so $|G|$ is a noncyclic group of order 8 which contains some elements of order 4. It follows $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. □

Proposition 4.5. *If $r = 3$ the only possibility is*

$$\mathbf{m} = (2, 8^2), \qquad G = \mathbb{Z}_2 \times \mathbb{Z}_8.$$

Proof. If $r = 3$ formula (10) gives

$$2 = (g(C) - 1)\left(1 - \sum_{i=1}^{3} \frac{1}{m_i}\right) \geq (g(C) - 1)\left(1 - \frac{3}{m_1}\right), \tag{15}$$

hence

$$g(C) - 1 \leq \frac{2m_1}{m_1 - 3} \cdot$$

Let us consider now the generating vector $\mathcal{V} = \{g_1, g_2, g_3\}$.

Case 1. Suppose $m_1 \geq 4$. Then $g(C) - 1 \leq 8$, so $|G| \leq 16$ and equality holds if and only if $\mathbf{m} = (4^3)$. In this case the abelian group G is generated by two elements of order 4, thus $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Without loss of generality, we may suppose $g_1 = e_1$, $g_2 = e_2$, $g_3 = -e_1 - e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Therefore $\langle g_1 \rangle \cup \langle g_2 \rangle \cup \langle g_3 \rangle$ contains all the elements of order 2 in G, and condition (9) cannot be satisfied; hence $\mathbf{m} = (4^3)$ must be excluded. It follows $|G| < 16$; since G is not cyclic and $|G|$ is even, we are left with few possibilities.

(i) $G = \mathbb{Z}_2 \times \mathbb{Z}_6$. This gives $m_1 = 6$, hence $g(C) - 1 \leq 4$ and $|G| \leq 8$, a contradiction.
(ii) $G = \mathbb{Z}_2 \times \mathbb{Z}_4$. This implies that the highest order of an element of G is 4, so $\mathbf{m} = (4^3)$, again a contradiction.
(iii) $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. Impossible because $m_1 \geq 4$.

Therefore $m_1 \geq 4$ does not occur.

Case 2. Suppose $m_1 = 3$. Since $G = \langle g_1, g_2 \rangle$ and G is not cyclic, it follows $G = \mathbb{Z}_3 \times \mathbb{Z}_{m_2}$ with $3 \mid m_2$. Moreover, $g_1 + g_2 + g_3 = 0$ implies $m_3 = o(g_1 + g_2) = m_2$. Set $m_2 = m_3 = m$; by using (10) we obtain

$$2 = (g(C) - 1)\left(1 - \frac{1}{3} - \frac{2}{m}\right) = (g(C) - 1)\frac{2m - 6}{3m}. \tag{16}$$

On the other hand

$$g(C) - 1 = \frac{1}{2}|G| = \frac{3m}{2}. \tag{17}$$

From (16) and (17) it follows $m = 5$, a contradiction. Then $m_1 = 3$ cannot occur.

Case 3. Suppose $m_1 = 2$. Exactly as before we get $\mathbf{m} = (2, m, m)$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_m$, with $2 \mid m$. Therefore, we have

$$2 = (g(C) - 1)\left(1 - \frac{1}{2} - \frac{2}{m}\right) = (g(C) - 1)\frac{m - 4}{2m} \tag{18}$$

and

$$g(C) - 1 = \frac{1}{2}|G| = m. \tag{19}$$

Relations (18) and (19) imply $m = 8$, hence $\mathbf{m} = (2, 8^2)$ and $G = \mathbb{Z}_2 \times \mathbb{Z}_8$. □

This completes the proof of Theorem 4.1.

5. ABELIAN CASE: THE MODULI SPACES

Now we provide the effective construction of surfaces of type I, II, III, IV and the description of their moduli spaces $\mathfrak{M}_I$, $\mathfrak{M}_{II}$, $\mathfrak{M}_{III}$, $\mathfrak{M}_{IV}$.

Theorem 5.1. *The moduli spaces* $\mathfrak{M}_I, \mathfrak{M}_{II}, \mathfrak{M}_{IV}$ *are irreducible of dimension* $5, 4, 2$, *respectively. The moduli space* $\mathfrak{M}_{III}$ *is the disjoint union of two irreducible components* $\mathfrak{M}_{III}^{(1)}, \mathfrak{M}_{III}^{(2)}$, *both of dimension* 3.

The rest of this section deals with the proof of Theorem 5.1. By Proposition 3.3 we only have to compute the number of irreducible components in each case; this will be done by using Proposition 3.5. Let $(\mathcal{V}, \mathcal{W}) \in \mathfrak{B}(G, \mathbf{m})$; then the Hurwitz moves on $\mathcal{V} = \{g_1, \ldots, g_r\}$ are generated by the transpositions of elements having the same order (Corollary 1.9), whereas the Hurwitz moves on $\mathscr{W} = \{g; h_1, h_2\}$ are generated by

$$\{g; h_1, h_2\} \xrightarrow{\mathbf{1}} \{g; h_1, h_1 + h_2\}, \qquad \{g; h_1, h_2\} \xrightarrow{\mathbf{2}} \{g; h_1 - h_2, h_2\},$$
$$\{g; h_1, h_2\} \xrightarrow{\mathbf{3}} \{g; g + h_1 - h_2, h_2\}, \qquad \{g; h_1, h_2\} \xrightarrow{\mathbf{4}} \{g; -h_1, -h_2\}$$

(see Corollary 1.13). Moreover, we will often use the Hurwitz move obtained by successively applying **1**, **2**, **1**, and that for the sake of shortness will be denoted by **5**:

$$\{g; h_1, h_2\} \xrightarrow{\mathbf{5}} \{g; -h_2, h_1\}.$$

5.1. Surfaces of Type I. $G = (\mathbb{Z}_2)^2$, $\mathbf{m} = (2^6)$, $g(C) = 3$

Let $\{e_1, e_2\}$ be the canonical basis of G and consider the generating vector $\mathscr{W} := \{g; h_1, h_2\}$. Up to Hurwitz move **5** we may assume $\langle g, h_1 \rangle = G$. Modulo automorphisms of G, we have $g = e_1$ and $h_1 = e_2$, so there are four possibilities:

$$\mathscr{W}_1 = \{e_1; e_2, 0\}, \qquad \mathscr{W}_2 = \{e_1; e_2, e_1\},$$
$$\mathscr{W}_3 = \{e_1; e_2, e_2\}, \qquad \mathscr{W}_4 = \{e_1; e_2, e_1 + e_2\}.$$

These vectors are all equivalent to $\mathscr{W}_1$ via a finite sequence of Hurwitz moves:

$$\mathscr{W}_2 \xrightarrow{\mathbf{1,3,5,3}} \mathscr{W}_1; \qquad \mathscr{W}_3 \xrightarrow{\mathbf{1}} \mathscr{W}_1; \qquad \mathscr{W}_4 \xrightarrow{\mathbf{1,1,3,5,3}} \mathscr{W}_1.$$

Now let us consider $\mathscr{V} = \{g_1, \ldots, g_6\}$. Condition (9) implies $g_i \neq e_1$, so there are two possibilities up to permutations:

$$\mathscr{V}_1 = \{e_2, e_2, e_2, e_2, e_1 + e_2, e_1 + e_2\},$$
$$\mathscr{V}_2 = \{e_1 + e_2, e_1 + e_2, e_1 + e_2, e_1 + e_2, e_2, e_2\}.$$

The automorphism of G given by $e_1 \to e_1, e_2 \to e_1 + e_2$ sends $\mathscr{V}_1$ to $\mathscr{V}_2$ and $\mathscr{W}_1$ to $\{e_1; e_1 + e_2, 0\}$, which is equivalent to $\mathscr{W}_1$ via the Hurwitz move **3**. This shows that the elements of $\mathfrak{B}(G; \mathbf{m})$ are all $\mathfrak{R}$-equivalent, hence $\mathfrak{M}_I$ is irreducible.

5.2. Surfaces of Type II. $G = (\mathbb{Z}_2)^3$, $\mathbf{m} = (2^5)$, $g(C) = 5$

In this case the generating vector $\mathscr{W} := \{g; h_1, h_2\}$ must be a basis of G as a $\mathbb{Z}_2$-vector space. Therefore, up to automorphisms we may assume $\mathscr{W} = \{e_1; e_2, e_3\}$, where the e_i form the canonical basis of G. Now let us consider the generating vector $\mathscr{V} = \{g_1, \ldots, g_5\}$; notice that condition (9) implies $g_i \neq e_1$. Let λ_0 be the automorphism of G given by $\lambda_0(e_1) := e_1$, $\lambda_0(e_2) := e_3$, $\lambda_0(e_3) := e_2$. It sends $\mathscr{W}$ to $\{e_1; e_3, e_2\}$, which is equivalent to $\mathscr{W}$ via the Hurwitz move **5**. Since $\langle g_1, \ldots, g_5 \rangle = G$ and $g_1 + \cdots + g_5 = 0$, up to λ_0 and permutations $\mathscr{V}$ must be one of the following:

$$\mathscr{V}_1 = \{e_2, e_2, e_3, e_1 + e_2, e_1 + e_2 + e_3\},$$
$$\mathscr{V}_2 = \{e_2, e_2, e_1 + e_3, e_1 + e_2, e_2 + e_3\},$$

$$\mathscr{V}_3 = \{e_1 + e_2 + e_3, e_1 + e_2 + e_3, e_3, e_2 + e_3, e_2\},$$
$$\mathscr{V}_4 = \{e_1 + e_2 + e_3, e_1 + e_2 + e_3, e_1 + e_2, e_2 + e_3, e_1 + e_3\},$$
$$\mathscr{V}_5 = \{e_1 + e_2, e_1 + e_2, e_3, e_2, e_2 + e_3\},$$
$$\mathscr{V}_6 = \{e_1 + e_2, e_1 + e_2, e_1 + e_3, e_2, e_1 + e_2 + e_3\},$$
$$\mathscr{V}_7 = \{e_2 + e_3, e_2 + e_3, e_2, e_1 + e_2 + e_3, e_1 + e_3\}.$$

Set $\mathscr{V}_i = \{\alpha_i, \alpha_i, \beta_i, \gamma_i, \delta_i\}$. One checks that, for every $i \in \{1, \dots, 7\}$, the element $\lambda_i \in \text{Aut}(G)$ defined by $\lambda_i(e_1) := e_1$, $\lambda_i(e_2) := \alpha_i$, $\lambda_i(e_3) := \beta_i$ sends $\mathscr{V}_1$ to $\mathscr{V}_i$. To prove that $\mathfrak{M}_{II}$ is irreducible it is therefore sufficient to show that, for every i, the generating vector $\lambda_i(\mathscr{W})$ is equivalent to $\mathscr{W}$ via a sequence of Hurwitz moves. But this is a straightforward computation:

	$\lambda_i(\mathscr{W})$	$\lambda_i(\mathscr{W}) \to \mathscr{W}$
λ_1	$\{e_1; e_2, e_3\}$	
λ_2	$\{e_1; e_2, e_1 + e_3\}$	**5, 3, 2, 5**
λ_3	$\{e_1; e_1 + e_2 + e_3, e_3\}$	**3**
λ_4	$\{e_1; e_1 + e_2 + e_3, e_1 + e_2\}$	**2, 5, 3, 2**
λ_5	$\{e_1; e_1 + e_2, e_3\}$	**3, 2**
λ_6	$\{e_1; e_1 + e_2, e_1 + e_3\}$	**1, 3, 5, 2**
λ_7	$\{e_1; e_2 + e_3, e_2\}$	**2, 5**

5.3. Surfaces of Type III. $G = \mathbb{Z}_2 \times \mathbb{Z}_4$, $m = (2^2, 4^2)$, $g(C) = 5$

Consider the generating vector $\{g; h_1, h_2\}$; condition (9) implies $g \neq (0, 2)$, so up to automorphisms of G and Hurwitz moves of type **5** we may assume $g = (1, 0)$, $h_1 = (0, 1)$. Therefore, modulo the Hurwitz move **1** we have two possibilities:

$$\{(1, 0); (0, 1), (0, 0)\} \quad \text{and} \quad \{(1, 0); (0, 1), (1, 0)\},$$

that are equivalent via the sequence **1**, **3**, **5**, **4**; so we may assume $\mathscr{W} = \{(1, 0); (0, 1), (1, 0)\}$. Now look at the generating vector $\mathscr{V} = \{g_1, g_2, g_3, g_4\}$; here the Hurwitz moves are generated by the transposition of g_1 and g_2 and the transposition of g_3 and g_4. Condition (9) now implies $g_i \neq (1, 0)$; since $\langle g_1, \dots, g_4 \rangle = G$ and $g_1 + g_2 + g_3 + g_4 = 0$, there are four possibilities up to permutations:

$$\mathscr{V}_1 = \{(1, 2), (0, 2), (0, 1), (1, 3)\}, \qquad \mathscr{V}_2 = \{(1, 2), (0, 2), (0, 3), (1, 1)\},$$
$$\mathscr{V}_3 = \{(1, 2), (1, 2), (0, 1), (0, 3)\}, \qquad \mathscr{V}_4 = \{(1, 2), (1, 2), (1, 3), (1, 1)\}.$$

Notice that:

(i) The automorphism of G given by $(1, 0) \to (1, 0)$, $(0, 1) \to (0, 3)$ sends $\mathscr{V}_1$ to $\mathscr{V}_2$ and $\mathscr{W}$ to $\{(1, 0); (0, 3), (1, 0)\}$, that is equivalent to $\mathscr{W}$ via the Hurwitz move **4**. So the pair $(\mathscr{V}_1, \mathscr{W})$ is $\mathfrak{R}$-equivalent to $(\mathscr{V}_2, \mathscr{W})$;
(ii) The automorphism of G given by $(1, 0) \to (1, 0)$, $(0, 1) \to (1, 3)$ sends $\mathscr{V}_3$ to $\mathscr{V}_4$ and $\mathscr{W}$ to $\{(1, 0); (1, 3), (1, 0)\}$, that is equivalent to $\mathscr{W}$ via the sequence of two Hurwitz moves **2**, **4**. So $(\mathscr{V}_3, \mathscr{W})$ is $\mathfrak{R}$-equivalent to $(\mathscr{V}_4, \mathscr{W})$.

On the other hand $(\mathscr{V}_1, \mathscr{W})$ and $(\mathscr{V}_3, \mathscr{W})$ are not $\mathfrak{R}$-equivalent, since every automorphism of G leaves $(0, 2)$ invariant. It follows that $\mathfrak{M}_{III}$ contains exactly two irreducible components.

5.4. Surfaces of Type IV. $G = \mathbb{Z}_2 \times \mathbb{Z}_8$, $m = (2, 8^2)$, $g(C) = 9$

Consider the generating vector $\mathscr{W} = \{g_1; h_1, h_2\}$; condition (9) implies $g \neq (0, 4)$, so exactly as in the previous case we may assume, up to automorphisms of G and Hurwitz moves, $\mathscr{W} = \{(1, 0); (0, 1), (1, 0)\}$. Now look at the generating vector $\mathscr{V} = \{g_1, g_2, g_3\}$; here the only Hurwitz move is the transposition of g_2 and g_3. Condition (9) now implies $g_i \neq (1, 0)$; since $\langle g_1, g_2, g_3 \rangle = G$ and $g_1 + g_2 + g_3 = 0$, there are four possibilities up to permutations:

$$\mathscr{V}_1 = \{(1,4), (0,1), (1,3)\}, \qquad \mathscr{V}_2 = \{(1,4), (1,1), (0,3)\},$$
$$\mathscr{V}_3 = \{(1,4), (1,7), (0,5)\}, \qquad \mathscr{V}_4 = \{(1,4), (0,7), (1,5)\}.$$

Notice that

(i) The automorphism of G given by $(1, 0) \to (1, 0)$, $(0, 1) \to (1, 1)$ sends $\mathscr{V}_1$ to $\mathscr{V}_2$ and $\mathscr{W}$ to $\{(1, 0); (1, 1), (1, 0)\}$, which is equivalent to $\mathscr{W}$ via the Hurwitz move **2**;
(ii) The automorphism of G given by $(1, 0) \to (1, 0)$, $(0, 1) \to (1, 7)$ sends $\mathscr{V}_1$ to $\mathscr{V}_3$ and $\mathscr{W}$ to $\{(1, 0); (1, 7), (1, 0)\}$, which is equivalent to $\mathscr{W}$ via the sequence of two Hurwitz moves **2**, **4**;
(iii) The automorphism of G given by $(1, 0) \to (1, 0)$, $(0, 1) \to (0, 7)$ sends $\mathscr{V}_1$ to $\mathscr{V}_4$ and $\mathscr{W}$ to $\{(1, 0); (0, 7), (1, 0)\}$, which is equivalent to $\mathscr{W}$ via the Hurwitz move **4**.

It follows that $(\mathscr{V}_1, \mathscr{W}), \ldots, (\mathscr{V}_4, \mathscr{W})$ are all $\mathfrak{R}$-equivalent, hence $\mathfrak{M}_{IV}$ is irreducible.

This completes the proof of Theorem 5.1.

6. ABELIAN CASE: THE PARACANONICAL SYSTEM

Now we want to study the paracanonical system of surfaces constructed in the previous sections. We start by recalling some definitions and results; we refer the reader to Catanese and Ciliberto (1991) for omitted proofs and further details. Let S be a minimal surface of general type with $p_g = q = 1$, let $\alpha : S \longrightarrow E$ be its Albanese fibration and denote by F_t the fiber of α over the point $t \in E$. Moreover, define $K_S + t := K_S + F_t - F_0$, where 0 is the zero element in the group structure of E. By Riemann–Roch, we obtain

$$h^0(S, K_S + t) = 1 + h^1(S, K_S + t)$$

for all $t \in E - \{0\}$. Since $p_g = 1$, by semicontinuity there is a Zariski open set $E' \subset E$, containing 0, such that for any $t \in E'$ we have $h^0(S, K_S + t) = 1$; we denote by C_t the unique curve in $|K_S + t|$. The *paracanonical incidence correspondence* is the surface $Y \subset S \times E$ which is the schematic closure of the set $\{(x, t) \in S \times E' \mid x \in C_t\}$. Then we can define C_t for any $t \in E$ as the fiber of $Y \longrightarrow E$ over t, and Y

provides in this way a flat family of curves on S, that we denote by $\{K\}$ or by $\{C_t\}$ and we call the *paracanonical system* of S. According to Beauville (1988), $\{K\}$ is the irreducible component of the Hilbert scheme of curves on S algebraically equivalent to K_S which dominates E. Let $\mathscr{P}$ be a Poincaré sheaf on $S \times E$; then we call $\mathscr{K} = \pi_S^*(\omega_S) \otimes \mathscr{P}$ the *paracanonical system* on $S \times E$. Let $\Lambda_i := R^i(\pi_E)_*\mathscr{K}$. By the base change theorem, Λ^0 is an invertible sheaf on E, Λ^2 is a skyscraper sheaf of length 1 supported at the origin and Λ^1 is zero at the origin, and supported on the set of points $\{t \in E \mid h^0(S, K_S + t) > 1\}$; set $\lambda := \text{length}(\Lambda^1)$.

Definition 6.1. The *index* $\iota = \iota(K)$ of the paracanonical system is the intersection number $Y \cdot (\{x\} \times E)$. Roughly speaking, ι is the number of paracanonical curves through a general point of S.

If F is a smooth Albanese fiber of S, then the following relation holds:

$$\iota = g(F) - \lambda. \tag{20}$$

Set $V := \alpha_* \omega_S$. Then V is a vector bundle of rank $g(F)$ over E, such that any locally free quotient Q of V verifies $\deg(Q) \geq 0$ (this is a consequence of Fujita's theorem, see Fujita, 1978). Moreover, we have

$$h^0(E, V) = 1; \qquad h^1(E, V) = 0; \qquad \deg(V) = 1. \tag{21}$$

By Krull–Schmidt theorem (see Atiyah, 1956) there is a decomposition of V into irreducible summands:

$$V = \bigoplus_{i=1}^{k} W_i \tag{22}$$

which is unique up to isomorphisms. Set $d_i := \deg(W_i)$; by (21) we may assume $d_1 = 1$, and $d_i = 0$ for $2 \leq i \leq k$. The following result shows that decomposition (22) is strongly related on the behavior of the paracanonical system $\{K\}$.

Proposition 6.2. *Let* $V = \bigoplus_{i=1}^{k} W_i$ *as above. Then the following holds*:

(i) $k = \lambda + 1$;
(ii) $\text{rank}(W_1) = \iota$;
(iii) $\text{rank}(W_i) = 1$ *for* $2 \leq i \leq k$. *Hence* W_i *is a line bundle of degree* 0 *for* $i > 1$;
(iv) *Let* L *be a line bundle over* E; *then* $h^0(S, \omega_S \otimes \alpha^* L) > 1$ *if and only if* $L = W_i^{-1}$ *for some* $i > 1$.

Proof. See Catanese and Ciliberto (1991). □

Now we can prove the main result of this section.

Theorem 6.3. *Let* $S = (C \times F)/G$ *be a surface of general type with* $p_g = q = 1$, *isogenous to an unmixed product. If* G *is abelian, then* $\iota(K) = 1$.

Proof. We start with a lemma.

Lemma 6.4. *Let* $\pi_C : C \times F \longrightarrow C$, $\pi_F : C \times F \longrightarrow F$ *be the two projections. Then* $(\pi_C)_* \pi_F^* \omega_F = \mathcal{O}_C^{\oplus g(F)}$.

Proof. Being $C \times F$ a product, if we fix one fiber F_o of the map π_C then *any* fiber of the bundle $(\pi_C)_* \pi_F^* \omega_F$ can be canonically identified with the vector space $H^0(F_o, (\pi_F^* \omega_F)|_{F_o})$, which in turn is isomorphic to $H^0(F_o, \omega_{F_o}) = \mathbb{C}^{g(F)}$ by the adjunction formula. This ends the proof. □

Now consider the commutative diagram

$$\begin{array}{ccc} C \times F & \xrightarrow{p} & S \\ \downarrow{\scriptstyle \pi_C} & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow{h} & E. \end{array} \tag{23}$$

Since flatness commutes with the base change (see Hartshorne, 1977), we have

$$\alpha_* p_* \omega_{C \times F} = h_* (\pi_C)_* \omega_{C \times F}.$$

On the other hand, by using projection formula and Lemma 6.4, we can write

$$\begin{aligned} (\pi_C)_* \omega_{C \times F} &= (\pi_C)_* (\pi_F^* \omega_F \otimes \pi_C^* \omega_C) \\ &= (\pi_C)_* \pi_F^* \omega_F \otimes \omega_C = \omega_C^{\oplus g(F)}. \end{aligned}$$

Hence we obtain

$$\alpha_* p_* \omega_{C \times F} = (h_* \omega_C)^{\oplus g(F)}. \tag{24}$$

Since G is abelian, the structure theorem for abelian covers proven in Pardini (1991) implies that the sheaves $p_* \mathcal{O}_{C \times F}$ and $h_* \mathcal{O}_C$ split in the following way:

$$\begin{aligned} p_* \mathcal{O}_{C \times F} &= \mathcal{O}_S \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} \mathcal{L}_\chi^{-1} \\ h_* \mathcal{O}_C &= \mathcal{O}_E \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} L_\chi^{-1}, \end{aligned} \tag{25}$$

where G^* is the group of irreducible characters of G and $\mathcal{L}_\chi$, L_χ are line bundles. More precisely, $\mathcal{L}_\chi^{-1}$ and L_χ^{-1} are the eigensheaves corresponding to the non-zero character $\chi \in G^*$. Moreover, since the map $p : C \times F \longrightarrow S$ is étale, the degree of each $\mathcal{L}_\chi$ is zero. From (25) we obtain

$$\begin{aligned} p_* \omega_{C \times F} &= \omega_S \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} (\omega_S \otimes \mathcal{L}_\chi), \\ h_* \omega_C &= \mathcal{O}_E \oplus \bigoplus_{\chi \in G^* \setminus \{0\}} L_\chi, \end{aligned}$$

that is, using relation (24),

$$\alpha_*\omega_S \oplus \bigoplus_{\chi \in G^*\setminus\{0\}} \alpha_*(\omega_S \otimes \mathscr{L}_\chi) = \mathscr{O}_E^{\oplus g(F)} \bigoplus_{\chi\in G^*\setminus\{0\}} L_\chi^{\oplus g(F)}. \tag{26}$$

The right-hand side of (26) is a direct sum of line bundles; since the decomposition of a vector bundle into irreducible summands is unique up to isomorphisms, we deduce that $\alpha_*\omega_S$ decomposes as a direct sum of line bundles. Then $\mathrm{rank}(W_1) = 1$, which implies $\iota(K) = 1$ by Proposition 6.2(ii). This concludes the proof of Theorem 6.3. □

If S is any minimal surface of general type with $p_g = q = 1$, let us write $\{K\} = Z + \{M\}$, where Z is the fixed part and $\{M\}$ is the movable part of the paracanonical system.

Corollary 6.5. *Let S as in Theorem 6.3. Then $\{M\}$ coincides with the Albanese pencil $\{F\}$.*

Proof. Since $\iota = 1$, through the general point of S passes only one paracanonical curve, hence $M^2 = 0$. By Catanese and Ciliberto (1991, Lemma 3.1), the general member of $\{M\}$ is irreducible, hence $\{M\}$ provides a connected, irrational pencil on S. By the universal property of the Albanese morphism, it follows $\{M\} = \{F\}$. □

7. THE NONABELIAN CASE

The classification of surfaces of general type with $p_g = q = 1$, isogenous to a product of unmixed type, is still lacking when the group G is not abelian. The following theorem sheds some light on this problem, by providing several examples.

Theorem 7.1. *Let $S = (C \times F)/G$ be a surface of general type with $p_g = q = 1$, isogenous to an unmixed product, and suppose that the group G is not abelian. Then the following cases occur*:

G	$\|G\|$	$g(C)$	$g(F)$
S_3	6	3	4
D_4	8	3	5
D_6	12	7	3
A_4	12	4	5
S_4	24	9	4
A_5	60	21	4

The remainder of Section 7 deals with the proof of Theorem 7.1. Let $S = (C \times F)/G$ be a minimal surface of general type with $p_g = q = 1$, isogenous to an unmixed product, and let $f : F \longrightarrow \mathbb{P}^1$, $h : C \longrightarrow E$ be the two quotient maps. Therefore f, h are induced by two admissible epimorphisms

$$\vartheta : \Gamma(0\,|\,\mathbf{m}) \longrightarrow G, \qquad \psi : \Gamma(1\,|\,\mathbf{n}) \longrightarrow G,$$

where $\mathbf{m} = (m_1, \ldots, m_r)$, $\mathbf{n} = (n_1, \ldots, n_s)$. Let $\mathscr{V} = \{g_1, \ldots, g_r\}$ and $\mathscr{W} = \{\ell_1, \ldots, \ell_s; h_1, h_2\}$ be the generating vectors defined by ϑ and ψ, respectively. By definition we have

$$g_1^{m_1} = \cdots = g_r^{m_r} = g_1 g_2 \ldots g_r = 1,$$
$$\ell_1^{n_1} = \cdots = \ell_s^{n_s} = \ell_1 \ell_2 \ldots \ell_s [h_1, h_2] = 1,$$
$$G = \langle g_1, \ldots, g_r \rangle = \langle \ell_1, \ldots, \ell_s, h_1, h_2 \rangle.$$

The cyclic subgroups $\langle g_1 \rangle, \ldots, \langle g_r \rangle$ and their conjugates are the nontrivial stabilizers of the action of G on F, whereas $\langle \ell_1 \rangle, \ldots, \langle \ell_s \rangle$ and their conjugates are the nontrivial stabilizers of the actions of G on C; then the diagonal action of G on $C \times F$ is free if and only if

$$\left(\bigcup_{h \in G} \bigcup_{i=1}^{r} \langle h g_i h^{-1} \rangle \right) \cap \left(\bigcup_{h \in G} \bigcup_{j=1}^{s} \langle h \ell_j h^{-1} \rangle \right) = \{1\}. \tag{27}$$

Summing up, we obtain the following generalization of Proposition 3.1 to the nonabelian case.

Proposition 7.2. *Let us suppose that we have the following data:*

(i) *A finite group G;*
(ii) *Two admissible epimorphisms*

$$\vartheta : \Gamma(0 \mid \mathbf{m}) \longrightarrow G, \qquad \mathbf{m} = (m_1, \ldots, m_r)$$
$$\psi : \Gamma(1 \mid \mathbf{n}) \longrightarrow G, \qquad \mathbf{n} = (n_1, \ldots, n_s)$$

with corresponding generating vectors $\mathscr{V} = \{g_1, \ldots, g_r\}$ and $\mathscr{W} = \{\ell_1, \ldots, \ell_s; h_1, h_2\}$.

Let

$$f : F \longrightarrow \mathbb{P}^1 = F/G$$
$$h : C \longrightarrow E = C/G$$

be the G-coverings induced by ϑ and ψ and let $g(F)$, $g(C)$ be the genera of F and C, that are related on $|G|, \mathbf{m}, \mathbf{n}$ by (7). *Assume moreover that*

(i) $g(C) \geq 3$, $g(F) \geq 3$;
(ii) $|G| = (g(C) - 1)(g(F) - 1)$;
(iii) *Condition* (27) *is satisfied.*

Then the diagonal action of G on $C \times F$ is free and the quotient $S = (C \times F)/G$ is a minimal surface of general type with $p_g = q = 1$. Conversely, any surface of general type with $p_g = q = 1$, isogenous to an unmixed product, arises in this way.

Remark 7.3. We could also generalize Proposition 3.4 to the nonabelian case, in order to study the moduli spaces of surfaces listed in Theorem 7.1, but we will not develop this point here.

Remark 7.4. By Proposition 2.3 we have $g(F) \leq 5$, so $|\mathrm{Aut}(G)| \leq 192$ (see Broughton, 1990, p. 91). We believe that the classification of the unmixed, nonabelian case is not out of reach and we hope to achieve it on a forthcoming article.

Now let us construct our examples. In the case of symmetric groups, we will write the composition of permutations from the right to the left; for instance, $(13)(12) = (123)$.

7.1. $G = S_3, g(C) = 3, g(F) = 4$

Take $\mathbf{m} = (2^6), \mathbf{n} = (3^1)$ and set

$$g_1 = g_2 = (12), \quad g_3 = g_4 = (13), \quad g_5 = g_6 = (23)$$
$$h_1 = (12), \quad h_2 = (123), \quad \ell_1 = (132).$$

Condition (27) is satisfied, hence Proposition 7.2 implies that this case occurs.

7.2. $G = D_4, g(C) = 3, g(F) = 5$

G is the group of order 8 with presentation

$$\langle \rho, \sigma \mid \rho^4 = \sigma^2 = 1, \rho\sigma = \sigma\rho^3 \rangle.$$

Take $\mathbf{m} = (2^6), \mathbf{n} = (2^1)$ and set

$$g_1 = g_2 = g_3 = g_4 = \sigma, \quad g_5 = g_6 = \rho\sigma$$
$$h_1 = \sigma, \quad h_2 = \rho, \quad \ell_1 = \rho^2.$$

Condition (27) is satisfied, so this case occurs. This example and the previous one were already described in Polizzi (2006).

7.3. $G = D_6, g(C) = 7, g(F) = 3$

G is the group of order 12 with presentation

$$\langle \rho, \sigma \mid \rho^6 = \sigma^2 = 1, \rho\sigma = \sigma\rho^5 \rangle.$$

Take $\mathbf{m} = (2^3, 6^1), \mathbf{n} = (2^2)$ and set

$$g_1 = \rho^3, \quad g_2 = \rho\sigma, \quad g_3 = \rho^5\sigma, \quad g_4 = \rho$$
$$h_1 = h_2 = \rho, \quad \ell_1 = \ell_2 = \sigma.$$

Condition (27) is satisfied, so this case occurs.

7.4. $G = A_4, g(C) = 4, g(F) = 5$

Take $\mathbf{m} = (3^4), \mathbf{n} = (2^1)$ and set

$$g_1 = (234), \quad g_2 = (123), \quad g_3 = (124), \quad g_4 = (134)$$
$$h_1 = (123), \quad h_2 = (124), \quad \ell_1 = (12)(34).$$

Condition (27) is satisfied, so this case occurs.

7.5. $G = S_4, g(C) = 9, g(F) = 4$

Take $\mathbf{m} = (2^3, 4^1), \mathbf{n} = (3^1)$ and set

$$g_1 = (23), \quad g_2 = (24), \quad g_3 = (12), \quad g_4 = (1234)$$
$$h_1 = (12), \quad h_2 = (1234), \quad \ell_1 = (132).$$

Condition (27) is satisfied, so this case occurs.

7.6. $G = A_5, g(C) = 21, g(F) = 4$

Take $\mathbf{m} = (2, 5^2), \mathbf{n} = (3^1)$ and set

$$g_1 = (24)(35), \quad g_2 = (13452), \quad g_3 = (12345)$$
$$h_1 = (345), \quad h_2 = (15432), \quad \ell_1 = (235).$$

One checks by direct computation that $g_1g_2g_3 = \ell_1[h_1, h_2] = 1$. Since $g_3g_1g_3 = (152)$, it follows that the subgroup generated by g_1, g_2, g_3 has order at least $2 \cdot 3 \cdot 5 = 30$. On the other hand, G is simple, so it cannot contain a subgroup of order 30; therefore $\langle g_1, g_2, g_3\rangle = G$. Analogously, $\ell_1h_1\ell_1 = (24)(35)$ which implies that the subgroup $\langle \ell_1, h_1, h_2\rangle$ has order at least 30 and so must be equal to G too. Condition (27) is verified, hence this case occurs.

This completes the proof of Theorem 7.1.

ACKNOWLEDGMENTS

Part of the present research was done during the author's visits to the Universities of Bayreuth, Cambridge, and Warwick, where he was supported by EU Research Training Network EAGER, no. HPRN-CT-2000-00099, and by a Marie Curie pre-doc fellowship. The author is grateful to C. Ciliberto for useful advice and constructive remarks during the preparation of this work and to F. Catanese and I. Bauer for suggesting the problem and sharing the ideas contained in Bauer and Catanese (2002). He also wishes to thank A. Corti, M. Reid, G. Infante for helpful conversations, and R. Pardini for pointing out some mistakes contained in the first version of this article. Finally, he is indebted with the referee for several detailed comments that considerably improved the presentation of these results.

To the memory of my colleague and friend Giulio Minervini.

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