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# Uncountable groups with finitely many normalizers of large subgroups

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**Abstract.** In this paper, the structure of uncountable groups with finitely many normalizers of large subgroups is studied and the connections between this property and other natural finiteness conditions on large subgroups of uncountable groups are investigated. In particular, groups in which every large subgroup is close to be normal with the only obstruction of a finite section and groups with finitely many commutator subgroups of large subgroups are considered. Moreover, groups with a finite covering consisting of groups with normal large subgroups are studied.

**Mathematics Subject Classification.** 20F24.

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**1. Introduction.** In recent years, in a series of papers, it has been shown that the behaviour of the large subgroups of an uncountable group has a strong influence on the structure of the whole group (see, for instance, [1–3,5]). In particular, in [1] and in [5], the structure of uncountable groups, of cardinality $\aleph$ say, in which all subgroups of cardinality $\aleph$ are normal was investigated. The aim of this paper is to give a further contribution to this topic; here uncountable groups which have only finitely many normalizers of large subgroups are studied, and this property will be connected with the imposition of other relevant normality conditions to large subgroups.

The main obstacle in the study of groups of large cardinality $\aleph$ is the existence of infinite groups in which all proper subgroups have cardinality strictly smaller than $\aleph$ (the so-called Jònsson groups). Clearly, a countably infinite group is a Jònsson group if and only if all its proper subgroups are finite, so that Prüfer groups and Tarski groups (that is, infinite simple groups whose

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proper nontrivial subgroups have prime order) have the Jònsson property. On the other hand, it is known that if $G$ is an uncountable Jònsson group, then $G/Z(G)$ is simple and $G = G'$ (see, for instance, [1, Lemma 2.7]). Relevant examples of Jònsson groups of cardinality $\aleph_1$ have been constructed by Shelah [15] and Obraztsov [11].

Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number, and if $G$ is a group of cardinality $\aleph$, a subgroup $X$ of $G$ will be called large if it has likewise cardinality $\aleph$, and small otherwise.

The behaviour of finite-by-abelian groups (i.e., groups with a finite commutator subgroup) has been investigated by several authors. In particular, Neumann [10] proved that a group is finite-by-abelian if and only if every subgroup has finite index in its normal closure. A further interesting result on this topic states that locally graded groups with a finite commutator subgroup can be characterized as those groups in which the set $\{X' \mid X \leq G\}$ is finite [7]. In the frame of the study of the behaviour of large subgroups in uncountable groups, in [4], groups with only finitely many commutator subgroups of large subgroups have been studied, while in [5] and in [6], the structure of groups in which all large subgroups have finite index in their normal closure has been investigated.

In [10], Neumann proved that a group $G$ is finite over its centre if and only if every subgroup of $G$ has finitely many conjugates; moreover, in [12], Polovickǐ proved that groups with this property are precisely those with finitely many normalizers of (abelian) subgroups, and by a celebrated theorem of Schur [14], they are also finite-by-abelian. Here some kind of translations of these results will be obtained by restricting the attention only to the set of large subgroups of an uncountable group. Our notation is mostly standard and can be found in [13].

**2. Statements and proofs.** Our first result shows that the commutator subgroup of a group $G$ of cardinality $\aleph$ with finitely many normalizers of large subgroups must be small, provided that $G$ has no simple sections of cardinality $\aleph$; on the other hand, notice that even uncountable groups whose large subgroups are normal may have an infinite commutator subgroup. In fact, Ehrenfeucht and Faber [8] exhibited an uncountable nilpotent group $G$ of class 2 whose commutator subgroup is infinite such that $X' = G'$ for each large subgroup $X$; in particular, such a group does not contain large abelian subgroups. Groups whose large subgroups all have the same commutator subgroup, which in [6] have been called $C^1_\aleph$-groups, naturally occur in the description of groups in which all large subgroups are normal (see [5]), and they also play an important role in the present paper.

**Proposition 2.1.** *Let $G$ be a group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

*If $G$ contains only finitely many normalizers of large subgroups, then the commutator subgroup $G'$ of $G$ is small.*

*Proof.* We will proceed by induction on the number $k$ of the proper normalizers of large subgroups.

If $k = 0$, all large subgroups are normal, and the statement follows from [1, Lemma 3.2].

Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup $N_i$ has less than $k$ proper normalizers of large subgroups, and hence, by the induction hypothesis, $N_i'$ is small. On the other hand, the subgroup $N_i$ has finitely many conjugates in $G$, so that the conjugacy class of $N_i'$ is also finite and hence the subgroup $N = \langle N_1', \dots, N_k' \rangle^G$ is small.

Let $H/N$ be any non-abelian large subgroup of $G/N$; then $H$ is not contained in any $N_i$, so that $H$ is normal in $G$. Therefore $G'/N$ is small (see [1, Theorem 4.3]), so that $G'$ is small. □

**Proposition 2.2.** *Let $G$ be a group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$ such that $G$ contains only finitely many normalizers of large subgroups.*

*If $G$ contains a large abelian subgroup, then $G/Z(G)$ is finite.*

*Proof.* We will proceed by induction on the number $k$ of the proper normalizers of large subgroups.

If $k = 0$, all large subgroups are normal, and the statement follows from [1, Theorem 2.5].

Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup $N_i$ has less than $k$ proper normalizers of large subgroups, and hence, by the induction hypothesis, $N_i/Z(N_i)$ is finite.

If $G = N_1 \cup \dots \cup N_k$, then $G/Z(G)$ is finite.

Assume that $N_1 \cup \dots \cup N_k$ is properly contained in $G$, and let $g$ be an element of $G \backslash (N_1 \cup \dots \cup N_k)$. It follows from Proposition 2.1 that $|G : C_G(g)| < \aleph$, so that $C_G(g)$ contains a large abelian subgroup; therefore $C_G(g)$ contains a subgroup of the form $A_1 \times A_2$ where each $A_i$ is a large abelian group (see [5, Corollary 3.2]). Clearly, for each $i = 1, 2$, $g \in N_G(A_i)$, so that $A_i$ is normal in $G$; moreover, $G/A_i$ has finitely many normalizers, so that its centre $Z_i/A_i$ has finite index in $G/A_i$. It follows that $Z_1 \cap Z_2$ has finite index in $G$, so that $Z(G)$ has likewise finite index in $G$. □

If $\mathcal{X}$ is a class of groups, a group $G$ is called residually $\mathcal{X}$ if it has a collection of normal subgroups $(N_i)_{i \in I}$ such that $G/N_i$ belongs to $\mathcal{X}$ for each $i$ and

$$\bigcap_{i \in I} N_i = \{1\}.$$

Here, a group will be called residually small if the intersection of all its normal subgroups of index strictly smaller than $\aleph$ is trivial.

Recall also that a subgroup $H$ of a group $G$ is said to be *nearly normal* in $G$ if it has finite index in its normal closure $H^G$, while $H$ is said to be *almost normal* in $G$ if it has finitely many conjugates; we shall say that all large subgroups of $G$ are boundedly nearly (respectively almost) normal if there exists a positive integer $k$ such that $|H^G : H| \leq k$ (respectively $|G : N_G(H)| \leq k$) for every large subgroup $H$ of $G$. Clearly these conditions are satisfied in the above quoted example of Ehrenfeucht and Faber; on the other hand, our next result shows that the behaviour of small subgroups can be neglected in the case of residually small groups.

**Proposition 2.3.** *Let $G$ be a residually small uncountable group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) *If all large subgroups of $G$ are boundedly nearly normal, then $G$ contains an abelian large subgroup and is finite-by-abelian.*
(2) *If all large subgroups of $G$ are boundedly almost normal, then $G/Z(G)$ is finite of bounded order.*

*Proof.*

(1) Let $K$ be a $C^1_{\aleph}$-subgroup of $G$ (see [6, Theorem 2.6]); it follows from [1, Theorem 3.1] that $K$ is a Dedekind group. Therefore $G$ contains an abelian large subgroup and is finite-by-abelian (see [5, Theorem 3.6]).
(2) It follows from Propositions 2.6 and 2.7 that all large subgroups of $G$ are boundedly nearly normal, so that $G$ contains a large abelian subgroup. Therefore $G/Z(G)$ is finite by [5, Theorem 3.3]. □

**Corollary 2.4.** *Let $G$ be an uncountable group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) *If all large subgroups of $G$ are boundedly nearly normal, then $G/Z_3(G)$ is finite.*
(2) *If all large subgroups of $G$ are boundedly almost normal, then $G/Z_2(G)$ is finite.*

*Proof.*

(1) The commutator subgroup $G'$ is small by [6, Theorem 2.7], so that $|G : C_G(g)| < \aleph$ for every $g \in G$. It follows now from Proposition 2.3 that $G/Z(G)$ is finite-by-abelian, and hence $Z_3(G)$ has finite index in $G$.
(2) As above, it follows from Proposition 2.3 that $G/Z(G)$ is central-by-finite, so that $Z_2(G)$ has finite index in $G$. □

If $G$ is a group of cardinality $\aleph$, following [4], we shall denote by $\mathcal{C}_{\aleph}(G)$ the set of all commutator subgroups of the large subgroups of $G$ and by $I_{\aleph}(G)$ the intersection of all members of the set $\mathcal{C}_{\aleph}(G)$.

**Theorem 2.5.** *Let $G$ be a group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

*Then the following conditions are equivalent:*

(i) *$G$ contains only finitely many normalizers of large subgroups.*
(ii) *All large subgroups are boundedly almost normal.*
(iii) *$G$ contains a subgroup $N$ of finite index in $G$ such that $N$ normalizes all large subgroups of $G$ and all large subgroups of $N$ are normal in $G$.*

*Proof.*

(i) ⇒ (ii) We will proceed by induction on the number $k$ of the proper normalizers of large subgroups.

If $k = 0$, all large subgroups are normal. Assume that $k > 0$, and let $N_1, \dots, N_k$ be the proper normalizers of large subgroups; for every $i = 1, \dots, k$, the subgroup $N_i$ has less than $k$ proper normalizers of large subgroups, and hence, by the induction hypothesis, all large subgroups of $N_i$ are boundedly almost normal in $N_i$.

If $G = N_1 \cup \dots \cup N_k$, then there exists $\{i_1, \dots, i_t\} \subseteq \{1, \dots, k\}$ such that $G = N_{i_1} \cup \dots \cup N_{i_t}$ and $|G : Ni_j| < \infty$ for every $j = 1, \dots t$, so that for every large subgroup $H$ of $G$, $H = (H \cap Ni_1) \cup \dots \cup (H \cap N_{i_t})$ is almost normal in $G$.

Assume that $N_1 \cup \dots \cup N_k$ is properly contained in $G$, and let $H$ be any large subgroup of $G$. Let $g$ be an element of $G \backslash (N_1 \cup \dots \cup N_k)$; it follows from Proposition 2.1 that $|G : C_G(g)| < \aleph$, so that $L = H \cap C_G(g)$ is large. Clearly, $g \in N_G(L)$, so that $L$ is normal in $G$; moreover, $G/L$ has finitely many normalizers, so that it is central-by-finite, and hence $H$ is almost normal in $G$. In particular, the normalizers of the large subgroups of $G$ have finite index, so that all large subgroups are boundedly almost normal.

(ii) ⇒ (iii) All large subgroups of $G$ are boundedly nearly normal (see [5, proof of Corollary 3.4]), so that the set $\mathcal{C}_\aleph(G)$ satisfies the minimal condition and $G'$ is small (see [6, Theorem 2.6 and Theorem 2.7]); it follows that $G/Z(G)$ is residually small, and hence $Z_2(G)$ is large by Proposition 2.3. Let $K$ be a large subgroup such that $K'$ is a minimal element of $\mathcal{C}_\aleph(G)$ and $K' \leq Z(G)$; the factor group $G/K'$ contains the large abelian subgroup $K/K'$ and hence its centre $N/K'$ has bounded finite index (see [5, proof of Theorem 3.3]).

If $Y$ is any large subgroup of $N$, then $Y' \leq N' \leq K'$, so that $Y' = K'$ by the minimality of $K'$; in particular $K' \leq Y \leq N$, so that $Y$ is normal in $G$. Finally, let $H$ be any large subgroup of $G$. Then $H \cap N$ is large, so that, as above, $(H \cap N)' = K'$, and hence the subgroup $H/K'$ is normalized by $N/K'$; therefore $H$ is normalized by $N$.

(iii) ⇒ (i) This claim is obvious. □

**Proposition 2.6.** *Let $G$ be a group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

(1) *If $G$ has finitely many commutator subgroups of large subgroups, then $G$ is finite-by-$C^1_\aleph$.*
(2) *If $G$ is finite-by-$C^1_\aleph$, then all large subgroups are boundedly nearly normal.*

*Proof.*

(1) Since $G'/I_\aleph(G)$ is finite (see [4]), for every large subgroup $H$, we have that $|G' : H'| \leq |G'/I_\aleph(G)|$, and hence $|H^G : H| \leq |G'/I_\aleph(G)|$; therefore we can assume that $G$ is nilpotent of class 2 and $G'$ has prime exponent (see [6, Theorem 2.7]). It follows that there exists a finite subgroup $E$ of $G$ such that $G' = I_\aleph(G) \times E$; clearly $G/E$ is a $C^1_\aleph$-group.

(2) Let $N$ be a finite normal subgroup such that $G/N$ is a $C^1_{\aleph}$-group; for every large subgroup $H$, we have that $H'N = G'N$, so that $|G' : H'| \leq |N|$, and hence $|H^G : H| \leq |N|$. □

**Theorem 2.7.** *Let $G$ be a group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

*If $G$ contains only finitely many normalizers of large subgroups, then $G$ contains only finitely many commutator subgroups of large subgroups.*

*Proof.* It follows from Theorem 2.5 that $G$ contains a normal subgroup $N$ of finite index whose large subgroups are normal in $G$. Then $N/I_{\aleph}(N)$ contains a large abelian subgroup (see [5, Theorem 2.3]), so that it follows from Proposition 2.2 that $G/I_{\aleph}(N)$ is central-by-finite; in particular, $G'/I_{\aleph}(N)$ is finite.

Finally, if $H$ is any large subgroup of $G$, then $H \cap N$ is large, so that $(H \cap N)' \geq I_{\aleph}(N)$ and so $I_{\aleph}(N) \leq H' \leq G'$. □

It was proved by Neumann [9] that a group $G$ is covered by finitely many abelian subgroups if and only if its centre $Z(G)$ has finite index (that is equivalent to say that $G$ has finitely many normalizers of subgroups); it follows easily from this theorem that the same result holds if $G$ is covered by finitely many Dedekind subgroups (i.e., subgroups with only normal subgroups). Our last result describes the behaviour of groups with a finite covering consisting of subgroups with large normal subgroups. The following Neumann lemma is crucial in all arguments concerning finite coverings of groups.

**Lemma 2.8.** *Let the group $G = X_1 \cup \cdots \cup X_n$ be the union of finitely many subgroups $X_1, \ldots, X_n$. Then any $X_i$ of infinite index can be omitted from this decomposition; in particular, at least one of the subgroups $X_1, \ldots, X_n$ has finite index in $G$.*

**Theorem 2.9.** *Let $G$ be an uncountable group of cardinality $\aleph$ having no simple sections of cardinality $\aleph$.*

*If $G = X_1 \cup \cdots \cup X_n$ is the union of finitely many subgroups $X_1 \ldots, X_n$ where each $X_i$ has all large subgroups that are normal, then $G$ has finitely many normalizers of large subgroups.*

*Proof.* It follows from Lemma 2.8 that there exists a subset $\{i_1, \ldots, i_m\}$ of $\{1, \ldots, n\}$ such that $G = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_m}$ and $|G : X_{i_j}| < \infty$ for every $j = 1, \ldots m$. Put

$$N = \bigcap_{j=1}^{m} (X_{i_j});$$

then $|G : N|$ is finite.

Let $H$ be any large subgroup of $G$. Then for every $j = 1, \ldots m$, $H \cap X_{i_j}$ is large and hence is normalized by $N$; therefore

$$H = (H \cap X_{i_1}) \cup (H \cap X_{i_2}) \cup \cdots \cup (H \cap X_{i_m})$$

is normalized by $N$. □

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