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Uncountable groups in which permutability is a transitive relation

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Abstract. Let G be a group. A subgroup H of G is called permutable if $HK = KH$ for all subgroups K of G. Permutability is not in general a transitive relation, and G is said to be a PT-group (or to have the PT-property) if, whenever K is a permutable subgroup of G and H is a permutable subgroup of K, H is permutable in G. The aim of this paper is to investigate the behaviour of uncountable soluble groups of cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property. We prove that such a group is a PT-group and so are all its subgroups.

1 Introduction

A group G is said to have *finite rank* if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has *infinite rank.* In a long series of papers, it has been shown that the structure of a group having infinite rank is strongly influenced by that of its proper subgroups of infinite rank (see for instance [4], where a full reference list on this subject can be found). Results of this type suggest that the behaviour of *small* subgroups in a *large* group is neglectable, at least for some definition of largeness and within a suitable universe. This point of view is adopted also in the papers [5,6,8], by considering uncountable groups G whose proper subgroups with the same cardinality as G belong to certain relevant group classes, such as the class of nilpotent groups, groups with finite conjugacy classes and groups with modular subgroup lattice. Moreover, groups G whose subgroups of the same cardinality of G are normal have been studied in [3]. In particular, uncountable groups G such that, in all proper subgroups equipotent to G, the relation of normality is transitive were considered in [7].

The aim of this paper is to give a further contribution to this topic, by investigating uncountable soluble groups G such that, in every proper subgroup equipotent to G, permutability is a transitive relation. The corresponding problem in the case of groups of infinite rank has been solved in [1].

Recall that a subgroup H of a group G is called *permutable* (or *quasinormal*) if $HK = KH$ for all subgroups K of G. Clearly, normal subgroups are permutable, permutable subgroups are modular and the set of permutable subgroups of a group G is not a sublattice of the subgroup lattice of G because the intersection of two permutable subgroups is not in general permutable. Actually, it was proved by Stonehewer in [18] that permutable subgroups of finitely generated groups are subnormal (in particular, the finite case was previously proved by Ore in [13]) and Iwasawa's example shows that this is not true for infinite groups (see [15, p. 396]). According to a well-known theorem by Stonehewer [18, Theorem A], permutable subgroups are always ascendant, although the converse is false (see for instance [2]).

The dihedral group of order 8 is the minimal example which shows that, like normality, the relation of being a permutable subgroup is not transitive. A group G is called a PT-*group* if, whenever K is a permutable subgroup of G and H is a permutable subgroup of K, we have that H is permutable in G. Obviously, groups in which all subgroups are permutable (i.e. the *quasihamiltonian* groups) are PT-groups, and in the finite case, a group G has the PT-property if and only if permutable and subnormal subgroups of G coincide. Note that the PT-property is inherited by homomorphic images and permutable subgroups, but not by arbitrary subgroups since any finite group G can be embedded into the alternating group $A_{|G|+2}$, which is simple and hence a PT-group if G has at least 3 elements. Furthermore, we say that a group G is a $\overline{\text{PT}}$-*group* if all its subgroups have the PT-property.

Finite soluble PT-groups were described by Zacher [19] in a similar way to Gaschütz's [9] characterization of finite soluble groups in which normality is a transitive relation (i.e. finite soluble T-*groups*). Clearly, finite T-groups are PT-groups, and it follows easily from the properties of T-groups that any hyperabelian finitely generated group with the T-property has the $\overline{\text{PT}}$-property. Menegazzo extended the description of soluble PT-groups to the periodic [10] and non-periodic cases [11]. It turns out in particular that hyperabelian groups with the PT-property are hypercyclic and it is known that subgroups of finite soluble PT-groups likewise have the PT-property. As a finite soluble PT-group is supersoluble, every finite $\overline{\text{PT}}$-group is supersoluble; in addition, it was proved in [1] that a finitely generated soluble PT-group is either finite or quasihamiltonian; thus every non-periodic locally soluble $\overline{\text{PT}}$-group is quasihamiltonian (see [1, Corollary 2.7]).

Our main result can be stated as follows.

Main Theorem 1.1. *Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT-*property. Then G is a soluble* $\overline{\text{PT}}$-*group.*

It is easy to see that the hypothesis of hyperabelianity is also enough to avoid the existence of uncountable groups G having all proper subgroups of cardinality strictly smaller than G, the so-called *Jónsson groups* (see for instance [6]), which represent the main obstacle in the study of groups of large cardinality. Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by S. Shelah [17] and V. N. Obraztsov [12].

Throughout this paper, $\aleph$ will denote an uncountable regular cardinal number, and if G is a group of cardinality $\aleph$, a subgroup X of G will be called *large* if it also has cardinality $\aleph$, and *small* otherwise. Most of our notation is standard and can be found in [15]. We refer to [16] for properties of permutable subgroups.

2 Some properties of PT-groups

Recall that if $\mathfrak{X}$ is a class of groups, a group G is said to be *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. It is known that finite minimal non-PT groups are soluble, and it was proved that a soluble group whose proper subgroups have the PT-property is either finite or a PT-group (see [1, Theorem 2.8]). As a group whose 2-generated subgroups have the PT-property is a $\overline{\mathrm{PT}}$-group (see [1, Corollary 2.4]), it follows immediately that a minimal non-PT group is 2-generated (note that the same result holds for several other classes of groups, such as the minimal non-quasihamiltonian groups and minimal non-T groups).

The next statement shows a more general local criterion for the PT-property, based on the concept of local system of subgroups.

Proposition 2.1. *Let G be a group and let $\mathfrak{L}$ be a local system of subgroups of G such that every $L \in \mathfrak{L}$ has the* PT*-property. Then G is a* PT*-group.*

Proof. Let H and K be subgroups of G such that H is permutable in K and K is permutable in G. It is enough to prove that $\langle g\rangle H = H\langle g\rangle$ for each element $g \in G$. Let $h \in H$; then there exists a subgroup $L \in \mathfrak{L}$ such that $g, h \in L$. In particular, $X = H \cap L \neq \{1\}$ is a permutable subgroup of L, and hence, for every $n \in \mathbb{Z}$, it follows that $gh = xg^m$ for some $x \in X \leq H$ and $m \in \mathbb{Z}$. Therefore, $\langle g\rangle H = H\langle g\rangle$, as required. □

While a locally finite minimal non-PT group is finite, it seems to be an open question whether there exist infinite minimal non-PT groups. Therefore, the following result about minimal non-PT groups holds for *locally graded* groups, which form a large class of generalized soluble groups. Recall that a group G is said to be *locally graded* if every finitely generated nontrivial subgroup of G contains a proper subgroup of finite index.

Proposition 2.2. *Let G be an infinite locally graded group and let J be the finite residual of G. If G is minimal non-*PT*, then G/J is quasihamiltonian.*

Proof. Since G is minimal non-PT, the group G is 2-generated, and hence it follows that the factor group G/J is infinite. Let N be any normal subgroup of finite index of G. Then the factor group G/N either is minimal non-PT or has the $\overline{\text{PT}}$-property so that, in any case, it is soluble and has derived length at most 3 since every finite soluble PT-group is metabelian. Therefore, also the infinite group G/J is soluble; thus it cannot be minimal non-PT, and hence it has the $\overline{\text{PT}}$-property. Moreover, G/J cannot be periodic, and so it is quasihamiltonian. □

In [1], it was proved that a hyperabelian PT-group is soluble of derived length at most 4. The next result will improve this bound, showing that the derived length cannot exceed 3. In particular, the description of soluble PT-groups by Menegazzo shows that such a bound is sharp.

Lemma 2.3. *Let G be a hyperabelian* PT*-group. Then G is soluble and has derived length at most* 3.

Proof. Firstly, assume that G is a periodic group. Since a hyperabelian p-group with the PT-property is metabelian, by [10, Theorem D], it follows that there exists an abelian normal subgroup N such that G/N is metabelian; thus, in particular, $G^{(3)} = \{1\}$. Suppose now that G is non-periodic. If the elements of finite order of G form a subgroup T, by [11, Theorem 1.3], the subgroup T is metabelian, and hence, in any case, G has derived length at most 3. Finally, if the elements of finite order of G do not form a subgroup, by [11, Theorem 2.3], there exists an abelian normal subgroup of index 2, and thus G is metabelian. □

3 Main Theorem

Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the PT-property. Of course, every quotient of G by a large normal subgroup N has all proper subgroups having the PT-property. Otherwise, if N is small, the factor group G/N has cardinality $\aleph$ and its proper subgroups of cardinality $\aleph$ have the PT-property.

The following result is a consequence of Lemma 2.3: it proves in particular that groups satisfying the hypotheses of the Main Theorem are soluble.

Lemma 3.1. *Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT*-property. Then G is soluble and has derived length at most* 3.

Proof. By Lemma 2.3, each proper subgroup of cardinality $\aleph$ of G is soluble of derived length at most 3. Then G is soluble and its derived length does not exceed 3 (see [8, Theorem D]). □

In the next result, we impose the PT-property only to large proper normal subgroups.

Lemma 3.2. *Let G be an uncountable soluble group of regular cardinality $\aleph$ whose proper normal subgroups of cardinality $\aleph$ have the* PT-*property. Then all permutable non-contranormal subgroups of G have the* PT-*property.*

Proof. Let H be a permutable non-contranormal subgroup of G. Obviously, if the normal closure H^G of H in G is a large subgroup, then H^G has the PT-property, and so also H is a PT-group. Suppose that H^G has cardinality strictly smaller than $\aleph$. As the factor group G/H^G has cardinality $\aleph$, it contains a proper normal subgroup N/H^G of cardinality $\aleph$. Therefore, N is a PT-group, and since H is permutable, it follows that H has the PT-property. □

Note that Lemma 3.2 shows that, under the same assumptions, all proper normal subgroups of G have the PT-property.

Lemma 3.3. *Let G be an uncountable group of regular cardinality $\aleph$ whose normal subgroups of cardinality $\aleph$ have finite index. Then every soluble normal subgroup of G has cardinality strictly smaller than $\aleph$.*

Proof. Firstly, assume for a contradiction that G contains an abelian normal subgroup A of cardinality $\aleph$. Since A has finite index in G, it follows that there exists a G-invariant subgroup H of A such that both H and G/H have cardinality $\aleph$ (see for instance [7, Lemma 3.3]), which is of course a contradiction.

Let S be a soluble normal subgroup of G of derived length n strictly larger than 1. For each $i = 0, \dots, n-1$, the quotient $S^{(i)}/S^{(i+1)}$ is an abelian normal subgroup of $G/S^{(i+1)}$; thus every $S^{(i)}$ has cardinality strictly smaller than $\aleph$, and so, in particular, S itself has cardinality strictly smaller than $\aleph$. □

It is known (see for instance [7, Lemma 3.2]) that if G is an uncountable group of regular cardinality $\aleph$ and it contains an abelian large subgroup A, then for each small subgroup E of G, there exists a large subgroup of A having trivial intersection with E. The following two statements will prove that the same property holds also if G contains a large abelian-by-finite subgroup or a large abelian-by-cyclic subgroup.

Lemma 3.4. *Let G be an uncountable group of regular cardinality $\aleph$ and let E be a subgroup of G having cardinality strictly smaller than $\aleph$. If G contains an abelian-by-finite subgroup N of cardinality $\aleph$, then there exists an N-invariant abelian subgroup C of cardinality $\aleph$ such that $C \cap E = \{1\}$ and the factor group N/C has cardinality $\aleph$.*

Proof. Let A be an abelian normal subgroup of N such that the factor group N/A is finite. Since each subgroup of A has only finitely many conjugates in N, there exists a collection $(a_i)_{i\in I}$ of non-trivial elements of A such that

$$\langle a_i \mid i \in I\rangle^N = \operatorname*{Dr}_{i\in I} \langle a_i\rangle^N$$

and the index set I has cardinality $\aleph$ (see for instance [7, Lemma 3.3]). Put

$$B = \operatorname*{Dr}_{i\in I} \langle a_i\rangle^N;$$

clearly, I has a subset J of cardinality strictly smaller than $\aleph$ such that $\mathrm{Dr}_{i\in J} \langle a_i\rangle^N$ contains $E \cap B$. Therefore, $D = \mathrm{Dr}_{i\in I\setminus J} \langle a_i\rangle^N$ is an N-invariant subgroup of A of cardinality $\aleph$ such that $D \cap E = \{1\}$. Therefore, there exist two subsets I_1 and I_2 of $I\setminus J$ such that $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I\setminus J$ both having cardinality $\aleph$. Then we may consider $D = C_1 \times C_2$, where, for each $k = 1, 2$, both the N-invariant subgroup $C_k = \mathrm{Dr}_{i\in I_k} B_i$ and its quotient D/C_k have cardinality $\aleph$ and $C_k \cap E = \{1\}$, so each subgroup C_k satisfies the required conditions. □

Lemma 3.5. *Let G be an uncountable group of regular cardinality $\aleph$ and let E be a subgroup of G having cardinality strictly smaller than $\aleph$. If G contains an abelian-by-cyclic subgroup N of cardinality $\aleph$, then there exists an N-invariant abelian subgroup C of cardinality $\aleph$ such that $C \cap E = \{1\}$ and the factor group N/C has cardinality $\aleph$.*

Proof. We may write $N = A\langle x\rangle$, where A is an abelian normal subgroup of N and x is an element of N. In particular, A can be regarded as a $\mathbb{Z}\langle x\rangle$-module, where $\mathbb{Z}\langle x\rangle$ is the group ring of $\langle x\rangle$ and an arbitrary subgroup of A is a $\mathbb{Z}\langle x\rangle$-submodule if and only if it is normal in N. Application of [3, Lemma 2.3] yields that A contains a subgroup of the form $B = \mathrm{Dr}_{i\in I} B_i$, where the index set I has cardinality $\aleph$ and each B_i is a non-trivial N-invariant subgroup. Applying the same argument as in Lemma 3.4, the statement is proved. □

Lemma 3.6. *Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT*-property. If G contains normal subgroups M and N such that N is contained in M, G/N is soluble, M/N is abelian of cardinality $\aleph$ and every subgroup of M/N has only finitely many conjugates in G/N, then G is a* PT*-group.*

Proof. Assume for a contradiction that G is not a PT-group. As M/N is an abelian group of cardinality $\aleph$ whose subgroups are almost normal in G/N, it follows that M/N contains a G-invariant subgroup L/N such that both L/N and M/L have cardinality $\aleph$ (see for instance [7, Lemma 3.3]). Therefore, all proper subgroups of G containing L have the PT-property, and hence G/L is finitely generated (see [1, Lemma 3.2]), which is of course a contradiction. □

By choosing $M = G$ and $N = G'$ in the statement of Lemma 3.6, it turns out in particular that if G is an uncountable group whose proper large subgroups have the PT-property, then G itself is a PT-group, provided that the factor group G/G' is large.

Corollary 3.7. *Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT*-property. If G/G' has cardinality $\aleph$, then G is a* PT*-group.*

Lemma 3.8. *Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT*-property. If G contains a subgroup N of cardinality $\aleph$ whose subgroups are G-invariant, then G is a* $\overline{\mathrm{PT}}$*-group.*

Proof. Clearly, we may assume that N is abelian, and since the class of $\overline{\mathrm{PT}}$-groups is local, it is enough to prove that every finitely generated subgroup E of G has the PT-property. The subgroup N contains a large subgroup C such that $C \cap E = \{1\}$ and EC is a proper subgroup of G. Then the subgroup EC has the PT-property and hence $E \simeq EC/C$ is itself a PT-group. □

Proposition 3.9. *Let G be an uncountable group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT*-property. If G is abelian-by-cyclic or abelian-by-finite, then G is a* $\overline{\mathrm{PT}}$*-group.*

Proof. Assume that G is abelian-by-cyclic and let E be a finitely generated subgroup of G. It follows from Lemma 3.5 that G contains a G-invariant subgroup C of cardinality $\aleph$ such that $C \cap E = \{1\}$ and EC is a proper subgroup of G. Then the subgroup EC has the PT-property, and hence $E \simeq EC/C$ is itself a PT-group.

Applying Lemma 3.4, the same argument used above works also for the abelian-by-finite case. □

We recall here the following result of Menegazzo (see [10, Theorem C2]), which describes the hyperabelian 2-groups having the PT-property.

Theorem 3.10 ([10, Theorem C2]). *Let G be a hyperabelian 2-group. Then G is a* PT*-group if and only if one of the following holds:*

(a) *G is a nilpotent group with modular subgroup lattice;*

(b) *G is a hyperabelian T-group;*

(c) *$G = \langle C, z\rangle$, where $C = A \times B_1 \times B_2$ and z has order 4, A is an abelian divisible group such that $a^z = a^{-1}$ for every $a \in A$, B_1 is an elementary abelian group such that $b^z = b$ for every $b \in B_1$, and $B_2 = \langle z^2\rangle$;*

(d) *$G = \langle C, z\rangle$, where $C = A \times B_1 \times B_2$ and z has order $2^t > 4$, A is an abelian divisible group such that $a^z = a^{-1}$ for every $a \in A$, B_1 is an elementary abelian group such that $b^z = bz^{2^{t-1}}$ for every $b \in B_1$, and $B_2 = \langle z^2\rangle$;*

(e) *$G = \langle C, z\rangle$, where $C = A \times B_1 \times B_2$, A is an abelian divisible group such that $a^z = a^{-1}$ for every $a \in A$, B_1 is an elementary abelian group such that $b^z = b$ for every $b \in B_1$, $B_2 = \langle b_2\rangle$ is a cyclic group of order 2 such that $b_2^z = z^4 b_2$, $z^2 = ab_2$ (for some $a \in A$), $z^4 = a^2 \neq 1$, $z^8 = 1$.*

Remark 3.11. Note that if G is a hyperabelian 2-group having the PT-property, then G contains an abelian normal subgroup N of finite index whose subgroups are normal in G, provided that G does not satisfy condition (d) of Theorem 3.10. Indeed, this is clear if G is a locally finite group with modular subgroup lattice (see for instance [16, Theorem 2.4.14]) or if G has the T-property (see for instance [14, Lemma 4.2.1 and Theorem 3.1.1]). Otherwise, if G satisfies conditions (c) or (e) of Theorem 3.10, using the same notation, we conclude that C, respectively $A \times B_1$, is the subgroup N with the required properties.

Lemma 3.12. *Let G be a hyperabelian non-periodic* PT-*group whose elements of finite order do not form a subgroup. Then G contains a characteristic abelian subgroup N of finite index whose subgroups are normal in G.*

Proof. The group G has the structure described in [11, Theorem 2.3]: using the same notation, we may write $G = \langle z\rangle(A \times B)$, where in particular A is characteristic in G. Furthermore, in each case of the theorem previously mentioned, there exists a central subgroup D of B such that D is an elementary abelian group with finite index in B. Then the subgroup $N = A \times D$ satisfies the required conditions. □

The next theorem is the first step in the proof of the Main Theorem.

Theorem 3.13. *Let G be an uncountable hyperabelian group of regular cardinality $\aleph$ whose proper subgroups of cardinality $\aleph$ have the* PT-*property. Then G is a soluble* PT-*group.*

Proof. The group G is soluble by Lemma 3.1. Assume for a contradiction that G has not the PT-property. If G is periodic, every normal subgroup of G of cardinality $\aleph$ has finite index (see [1, Lemma 3.2]), which is not possible by Lemma 3.3. This contradiction shows that G is non-periodic. By Corollary 3.7, the quotient G/G' is forced to be small, and hence G/G' is finitely generated (see [1, Lemma 3.2]). Therefore, G contains a proper normal subgroup H of finite index.

Firstly, assume that the commutator subgroup G' of G is periodic and let T be the torsion subgroup of G. Then G/T is a non-trivial abelian finitely generated torsion-free group and we may consider $G/T = \langle gT\rangle \times K/T$, where g is an element of G of infinite order and K is a proper subgroup of G. Moreover, $\langle g\rangle \cap K \leq T$ and $G = \langle g\rangle K$, so K cannot be a central subgroup of G. It follows that there exist two distinct prime numbers p_1 and p_2 such that neither $\langle g^{p_1}\rangle K$ nor $\langle g^{p_2}\rangle K$ are abelian. Consequently, for $i = 1, 2$, $\langle g^{p_i}\rangle K$ is a proper normal large subgroup of G; thus it has the PT-property, and so it satisfies the conditions in the statement of [11, Theorem 1.3]. Therefore, each group $\langle g^{p_i}\rangle K$ contains a periodic characteristic abelian subgroup L_i such that the following hold:

(1) L_i is a Hall subgroup of $\langle g^{p_i}\rangle K$ and $2 \notin \pi(L_i)$;

(2) every subgroup of L_i is normal in $\langle g^{p_i}\rangle K$;

(3) either $\langle g^{p_i}\rangle K/L_i$ is abelian or $\langle g^{p_i}\rangle K/T$ is abelian of rank 1; T/L_i is abelian and, for each prime number p, every element of $\langle g^{p_i}\rangle K$ induces on the unique Sylow p-subgroup $(T/L_i)_p$ of T/L_i a power automorphism that fixes all elements of order p (of order 4 if $p = 2$ and $(T/L_i)_2^4 \neq (T/L_i)_2^2$);

(4) for each element $x \in \langle g^{p_i}\rangle K$ of infinite order and for each element $a \in L_i$ of prime order, $[a, x] \neq 1$ holds.

Put $L = L_1 \cap L_2$; it follows that L is a periodic normal subgroup of G whose subgroups are G-invariant. Moreover, the torsion subgroup T/L of G/L is abelian, $2 \notin \pi(L)$ and $\pi(L) \cap \pi(T/L) = \emptyset$. The factor group $\langle g^{p_i}\rangle K/T$ has torsion-free rank $r_0(\langle g^{p_i}\rangle K/T)$ equal to $r_0(K) + r_0(\langle g^{p_i}\rangle)$. If $T < K$, it follows that $r_0(\langle g^{p_i}\rangle K/T) > 1$, and so $\langle g^{p_i}\rangle K/L_i$ is abelian; hence, in any case, the quotient K/L is abelian, too. Note also that no element of G of infinite order commutes with any element of L having prime order.

The subgroup L has cardinality strictly smaller than $\aleph$ by Lemma 3.8, so that the quotient $\overline{G} = G/L$ is not abelian, it has cardinality $\aleph$ and its proper large subgroups have the PT-property. Let $\overline{X}$ be a small subgroup of $\overline{G}$; then there exists a large subgroup $\overline{V} = V/L$ of T/L such that $\overline{V} \cap \overline{X} = \{1\}$. Since V is normal in T and every proper subgroup of G containing T has the PT-property, V is permutable in G (see [1, Lemma 3.6]). It follows that $V = \langle g\rangle V \cap T$; thus V is normal in $\langle g \rangle V$; moreover, since K/L is abelian, V is K-invariant, and hence V is normal in G. As the factor group G/V is infinite and V has cardinality $\aleph$, it follows that $G/V \simeq \overline{G}/\overline{V}$ is quasihamiltonian (see [1, Lemma 3.2]), and in particular, the subgroup $\overline{X} \simeq \overline{XV}/\overline{V}$ is quasihamiltonian. This proves that all proper subgroups of $\overline{G}$ have the PT-property, so $\overline{G}$ is a $\overline{\text{PT}}$-group.

Application of [11, Lemma 1.2] now yields that $\overline{G}/\overline{T} \simeq G/T$ is abelian of rank 1 and, for each prime number p, every element of G of infinite order, actually every element of G since G is generated by its elements of infinite order, induces on the unique Sylow p-subgroup $(T/L)_p$ of T/L a power automorphism that fixes all elements of order p (of order 4 if $p = 2$ and $(T/L)_2^4 \neq (T/L)_2^2$). Therefore, G satisfies the conditions in the statement of [11, Theorem 1.3], and so we conclude that G has the PT-property.

This contradiction shows that G' is not periodic. Suppose first that the elements of finite order of H do not form a subgroup. It follows from Corollary 3.12 that H contains a characteristic abelian subgroup U of finite index in H such that every subgroup of U is H-invariant. In particular, U is normal in G and the factor group G/U is finite; thus G is a PT-group by Lemma 3.6, which is a contradiction.

Therefore, the elements of finite order of H form a subgroup T_1, and H satisfies the conditions in the statement of [11, Theorem 1.3] which yields the existence of an abelian periodic characteristic abelian subgroup N whose subgroups are H-invariant. Moreover, T_1/N is an abelian group whose subgroups are normal in H/N. Since all subgroups of N are almost normal in G, by Lemma 3.6, the subgroup N has cardinality strictly smaller than $\aleph$. Furthermore, [11, Theorem 1.3] claims that H/T_1 is abelian: indeed, if H/N is abelian, by Lemma 3.6, the quotient H/N is forced to be small, which is clearly impossible. Therefore, all subgroups of T_1/N and H/T_1 have only finitely many conjugates under the action of G, and an iterated application of Lemma 3.6 yields that T_1/N and H/T_1 have cardinality strictly smaller than $\aleph$. In particular, H itself has cardinality strictly smaller than $\aleph$, which is of course a contradiction. Thus the group G has the PT-property. □

Proof of Main Theorem 1.1. The group G is a soluble PT-group by Theorem 3.13. In the first place, assume that G is periodic. Since G satisfies the conditions in the statement of [10, Theorem D], G contains an abelian characteristic subgroup N whose subgroups are normal in G; moreover, $\pi(N) \cap \pi(G/N) = \emptyset$, $2 \notin \pi(N)$ and the factor group G/N is the direct product of its Sylow subgroups, which have the PT-property.

Clearly, if N has cardinality $\aleph$, then G is a $\overline{\text{PT}}$-group by Lemma 3.8. Therefore, we may assume that N is a small subgroup of G, so that $\overline{G} = G/N$ has cardinality $\aleph$. Let $\overline{P} = P/N$ be a Sylow p-subgroup of $\overline{G}$ having cardinality $\aleph$, for some prime p. If p is odd, by [10, Theorem C1] and [16, Theorem 2.4.14], there exists a subgroup of finite index in $\overline{P}$ whose subgroups are normal in $\overline{P}$, and thus in $\overline{G}$. On the other hand, suppose that $p = 2$. If $\overline{P}$ does not satisfy condition (d) of Theorem 3.10, there exists a subgroup of finite index in $\overline{P}$ whose subgroups are normal in $\overline{P}$. In any case, by Lemma 3.8, we get that $\overline{G}$ is a $\overline{\mathrm{PT}}$-group. Finally, suppose that $\overline{P}$ satisfies condition (d) of Theorem 3.10 and let $\overline{F}$ be a finitely generated subgroup of $\overline{G}$. It follows that $\overline{P}$ is abelian-by-finite and contains a $\overline{P}$-invariant abelian subgroup $\overline{A}$ of cardinality $\aleph$ such that $\overline{A} \cap \overline{F} = \{1\}$ (see Lemma 3.4). Actually, $\overline{A}$ is normal in $\overline{G}$, and since $\overline{A}\,\overline{F}$ has the PT-property, it follows that $\overline{F} \simeq \overline{A}\,\overline{F}/\overline{A}$ has the PT-property. This proves that, also in this case, $\overline{G}$ is a $\overline{\mathrm{PT}}$-group. Let E be a finitely generated subgroup of G; thus $E/E \cap N \simeq EN/N$ has the PT-property, and it is the direct product of its Sylow subgroups; moreover, $2 \notin \pi(E \cap N)$, $\pi(E \cap N) \cap \pi(E/E \cap N) = \emptyset$ and all the subgroups of $E \cap N$ are E-invariant. By [10, Theorem D], we conclude E has the PT-property, and so the statement is proved if G is periodic.

Assume now that G is non-periodic. Obviously, if the elements of finite order of G do not form a subgroup, it follows from Corollary 3.12 that G contains a subgroup of cardinality $\aleph$ whose subgroups are normal in G, and hence G is a $\overline{\mathrm{PT}}$-group by Lemma 3.8. Suppose finally that the elements of finite order of G form a proper subgroup T. Applying [11, Theorem 1.3], the group G contains a periodic characteristic abelian subgroup L such that the following hold:

(1) $\pi(L) \cap \pi(G/L) = \emptyset$, $2 \notin \pi(L)$ and every subgroup of L is normal in G;

(2) either G/L is abelian or G/T is abelian of rank 1, T/L is abelian and, for each prime number p, every element of G induces on the unique Sylow p-subgroup $(T/L)_p$ of T/L a power automorphism that fixes all elements of order p (of order 4 if $p = 2$ and $(T/L)_2^4 \neq (T/L)_2^2$);

(3) for each element $g \in G$ of infinite order and for each element $a \in L$ of prime order, $[a, g] \neq 1$ holds.

Again, Lemma 3.8 allows us to suppose that L has cardinality strictly smaller than $\aleph$. Note that the factor group G/L has cardinality $\aleph$ and is a $\overline{\mathrm{PT}}$-group; indeed, if G/T has rank 1, then T/L is abelian of cardinality $\aleph$ and there exists a Sylow subgroup of T/L of cardinality $\aleph$ whose subgroups are normal in G/L; thus G/L is a $\overline{\mathrm{PT}}$-group by Lemma 3.8. Let X be a finitely generated subgroup of G. If X is periodic, then $X/X \cap L$ has the PT-property and it is the direct product of its Sylow subgroups; moreover, $2 \notin \pi(X \cap L)$, $\pi(X \cap L) \cap \pi(X/X \cap L) = \emptyset$ and all the subgroups of $X \cap L$ are normal in X. By [10, Theorem D], we conclude X has the PT-property.

In order to complete the proof, we have to assume that X is non-periodic. Let T_1 be the subgroup of elements of finite order of X. Clearly, if $X/X \cap L$ is non-abelian, it follows from [11, Lemma 1.2] that, for each prime number p, every element of infinite order of X induces on the unique Sylow p-subgroup $(T_1/X \cap L)_p$ of $T_1/X \cap L$ a power automorphism that fixes all elements of order p (of order 4 if $p = 2$ and $(T_1/X \cap L)_2^4 \neq (T_1/X \cap L)_2^2$). This proves that X has the PT-property (see [11, Theorem 1.3]). Therefore, G is a $\overline{\text{PT}}$-group and the proof is complete. □

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