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Research Paper

Groups with many quasihamiltonian-by-finite subgroups ☆

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A group G is said quasihamiltonian if $XY = YX$ for all subgroups X and Y of G. In this paper we study locally graded groups in which all proper subgroups are quasihamiltonian-by-finite. We prove that such groups are quasihamiltonian-by-finite or periodic. Moreover, we investigate the behaviour of groups of infinite rank in which all proper subgroups of infinite rank are quasihamiltonian-by-finite, within a suitable universe of generalised solubility condition. A result for uncountable groups of cardinality $\aleph$ without simple homomorphic images of cardinality $\aleph$ in which all proper large subgroups are quasihamiltonian-by-finite is also obtained.

1. Introduction

Let $\mathfrak{X}$ be a class of groups, a group G is said to be *minimal non-*$\mathfrak{X}$ if it is not an $\mathfrak{X}$-group but all its proper subgroups belong to $\mathfrak{X}$. Minimal non-$\mathfrak{X}$ groups have been investigated for several different choices of the group class $\mathfrak{X}$, especially in the case of finite groups. For instance, minimal non-abelian groups were analysed by G. A. Miller

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and H. C. Moreno [15], while O. J. Schmid [19] studied minimal non-nilpotent groups. On the other hand, the existence of *Tarski groups* (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) shows that a satisfactory description of infinite minimal non-$\mathfrak{X}$ groups may be in general quite difficult. In order to avoid Tarski groups and other similar pathologies, one can restrict the investigation to locally graded groups. Recall that a group is called *locally graded* if every finitely generated non-trivial subgroup contains a proper subgroup of finite index. Of course, all locally (soluble-by-finite) groups are locally graded, and so locally graded groups form a large class of generalised soluble groups.

If $\mathfrak{X}$ is a group class, it is natural to investigate the behaviour of locally graded groups whose proper subgroups belong to the class $\mathfrak{X}\mathfrak{F}$ of all groups admitting a (normal) $\mathfrak{X}$-subgroup of finite index. This point of view was first significantly adopted in [1], [2], by studying locally graded minimal non-(abelian-by-finite) groups and minimal non-(nilpotent-by-finite) groups. In particular, B. Bruno [1] proved that any locally graded minimal non-(abelian-by-finite) group that is periodic cannot be perfect and afterwards B. Bruno and R. E. Phillips [2] showed that the converse also holds. Moreover, B. Bruno gave an accurate description of locally graded periodic minimal non-(abelian-by-finite) groups when they are not primary (see [[1], Theorem 1]). Instead, in the case that such groups are p-groups, for some prime p, then they satisfy [[1], Theorem 2].

More recently, the class of abelian groups is replaced by another interesting group classes. For example, in the universe of locally graded groups, minimal non-(metahamiltonian-by-finite) groups were analysed by F. de Giovanni and M. Trombetti [12], where a group is called *metahamiltonian* if all its non-abelian subgroups are normal. Moreover, F. de Giovanni and F. Saccomanno [7] studied locally graded periodic minimal non-(quasihamiltonian-by-finite) groups, proving that such groups are exactly the minimal non-(abelian-by-finite) groups. Such result has been generalised in [13], replacing the class of quasihamiltonian groups with the class of groups having a modular subgroup lattice and assuming that the group is not perfect.

A lattice $\mathfrak{L}$ is said *modular* if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds for all elements x, y, z of $\mathfrak{L}$ such that $x \leq z$. Abelian groups and Tarski groups are obvious examples of groups with a modular subgroup lattice, and groups with a modular subgroup lattice may be considered as the natural lattice interpretation of abelian groups. We will denote by $\mathfrak{M}$ the class of groups having a modular subgroup lattice. A group G is said to be *quasihamiltonian* if all its subgroups permute pairwise, i.e. if $XY = YX$ for all subgroups X and Y of G. It is known that a group G is quasihamiltonian if and only if it is locally nilpotent and its subgroup lattice is modular.

In the first part of this paper we will prove that locally graded minimal non-(quasihamiltonian-by-finite) groups are periodic, and then such groups are exactly the locally graded minimal non-(abelian-by-finite) groups.

Theorem A. *Let G be a locally graded group. If all proper subgroups of G are quasihamiltonian-by-finite, then G is either periodic or G is quasihamiltonian-by-finite.*

Theorem A should be compared with the results of F. Napolitani and E. Pegoraro [16] which show that there are no non-periodic locally graded group minimal non-(abelian-by-finite) groups (resp. minimal non-(nilpotent-by-finite) groups).

A group G is said to have *finite rank* if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements; if such an r does not exist, we say that the group G has *infinite rank*. The behaviour of groups of infinite rank in which every proper subgroup of infinite rank has a suitable property has been studied in many papers (see for instance [4] where a full reference list on this subject can be found). In particular, it was proved in [6] that if G is a group of infinite rank whose proper subgroups of infinite rank are abelian-by-finite, then all proper subgroups of G are abelian-by-finite, provided that G satisfies a suitable generalised solubility condition. The aim of the second part of this paper is to give a further contribution to this topic, by investigating groups of infinite rank in which all proper subgroups of infinite rank are quasihamiltonian-by-finite.

Theorem B. *Let G be a strongly locally graded group of infinite rank. If all proper subgroups of infinite rank are quasihamiltonian-by-finite, then G is either quasihamiltonian-by-finite or G is minimal non-(abelian-by-finite).*

We will work within the universe of strongly locally graded groups, a class of generalised soluble groups that can be defined as follows. Let $\mathfrak{D}$ be the class of all periodic locally graded groups, and let $\overline{\mathfrak{D}}$ be the closure of $\mathfrak{D}$ by the operators $\acute{\boldsymbol{P}}$, $\grave{\boldsymbol{P}}$, $\boldsymbol{R}$, $\boldsymbol{L}$ (see for instance [18]). It is easy to prove that any $\overline{\mathfrak{D}}$-group is locally graded and that the class $\overline{\mathfrak{D}}$ is closed with respect to forming subgroups. Moreover, since the class $\overline{\mathfrak{D}}$ is not closed with respect to homomorphic images (see [3]), we shall say that a group G is *strongly locally graded* if every section of G is a $\overline{\mathfrak{D}}$-group. Thus strongly locally graded groups form a large class of generalised soluble groups, which is closed with respect to subgroups and homomorphic images, and contains all locally (soluble-by-finite) groups.

In recent years, a long series of papers has been published on the structure of uncountable groups in which every proper subgroup having the same cardinality as the whole group has a given property (see for instance [8], [9]). We will investigate the behaviour of uncountable groups of cardinality $\aleph$ in which all proper subgroups of cardinality $\aleph$ are quasihamiltonian-by-finite.

Theorem C. *Let $\aleph$ be an uncountable regular cardinal and let G be a locally graded group of cardinality $\aleph$ such that G has no simple homomorphic images of cardinality $\aleph$. If all proper subgroups of G of cardinality $\aleph$ are quasihamiltonian-by-finite, then G itself is quasihamiltonian-by-finite.*

The main obstacle in the study of groups of large cardinality is the existence of uncountable groups G having all proper subgroups of cardinality strictly smaller than G, the so-called *Jónsson groups* (see for instance [9]). Relevant examples of Jónsson groups of cardinality $\aleph_1$ have been constructed by S. Shelah [21] and V. N. Obraztsov [17]. To avoid Jónsson groups and other similar obstructions, we will use the additional requirement that the group has no simple homomorphic images of cardinality $\aleph$.

Throughout this paper, if G is a group of cardinality $\aleph$, a subgroup X of G will be called *large* if it likewise has cardinality $\aleph$, and *small* otherwise. Most of our notation is standard and can be found in [18].

2. Groups whose proper subgroups are quasihamiltonian-by-finite

The structure of groups with a modular subgroup lattice has been completely described by K. Iwasawa and R. Schmidt [20]. We refer to the monograph [20] for a detailed account of the results on groups with a modular subgroup lattice.

Let G be any non-periodic group with a modular subgroup lattice, then the set T of all elements of finite order of G is an abelian subgroup. Moreover, every subgroup of T is normal in G and the factor group G/T is abelian; if G is not abelian, the torsion-free abelian group G/T has rank 1 (see [[20], Theorem 2.4.11]). It follows that the commutator subgroup of any group with a modular subgroup lattice is periodic, therefore, such groups are abelian in the torsion-free case. Note also that non-periodic groups with a modular subgroup lattice are quasihamiltonian and both quasihamiltonian groups and locally graded groups with a modular subgroup lattice are metabelian.

The following result is an immediate consequence of [[16], Theorem B], where it is proved that any locally graded minimal non-(abelian-by-finite) group is periodic.

Corollary 1. *Let G be a torsion free locally graded group whose proper subgroups belong to $\mathfrak{MF}$, then G is abelian-by-finite.*

Lemma 2. *Let G be a group and let H be a subgroup of G whose subgroups are G-invariant. If G has no proper subgroups of finite index, then H is central.*

Proof. For each $x \in H$, its centraliser $C_G(x)$ is a subgroup of finite index in G, and since G has no proper subgroups of finite index it results that $x \in Z(G)$.

The next statement may be view as a generalisation of [[13], Lemma 2.1] to the case that the whole group G is non-periodic.

Lemma 3. *Let G be a locally graded $\mathfrak{M}$-group which has no proper subgroups of finite index. Then G is abelian.*

In particular, every quasihamiltonian group which has no proper subgroups of finite index is abelian.

Proof. Clearly, if G is either periodic or torsion free, then G is abelian (see [[13], Lemma 2.1]). Assume that G is non-periodic and suppose for a contradiction that G is not abelian. Let T the subgroup consisting of all elements of finite order of G, then all the subgroups of T are normal in G and so T is central in G by Lemma 2. Moreover, since G/T has rank 1, then $G/Z(G)$ is locally cyclic and so G is abelian, which is of course a contradiction.

Lemma 4. *Let G be a nilpotent minimal non-$\mathfrak{MF}$ group, then G is minimal non-(abelian-by-finite).*

Proof. Since G has no proper subgroups of finite index, the factor group G/G' is divisible and hence the subgroup T consisting of all elements of finite order of G is central in G (see [[18], Part 2, Theorem 9.23]). Moreover, G can be neither periodic nor torsion-free (see Corollary 1), otherwise G is abelian, which is a contradiction.

Now, let H be a proper subgroup of G, then there exists a normal $\mathfrak{M}$-subgroup M of H such that H/M is finite. Assume for contradiction that M is not abelian, then M can be neither periodic nor torsion-free. The subgroup T^* consisting of all elements of finite order of M is itself contained in the centre $Z(M)$ of M, then $M/Z(M)$ is locally cyclic and so M is abelian, which is of course a contradiction. It is proved that all proper subgroups of G are abelian-by-finite and then G is minimal non-(abelian-by-finite).

It is well known that any abelian-by-finite group contains an abelian characteristic subgroup of finite index. Consequently, a group class $\mathfrak{X}$ is said to be *F-characteristic* if any group containing an $\mathfrak{X}$-subgroup of finite index has also a characteristic subgroup of finite index belonging to the class $\mathfrak{X}$. It was proved in [11] that both the class $\mathfrak{M}$ and the class of quasihamiltonian groups are F-characteristic.

Proposition 5. *Let G be a non-periodic locally graded group which has no proper subgroups of finite index. If G can be decomposed into the product of two proper normal $\mathfrak{M}$-subgroups, then G is nilpotent of class at most 2.*

Proof. There exist two normal $\mathfrak{M}$-subgroups M_1 and M_2 of G such that $G = M_1M_2$. Of course, if M_1 and M_2 are both abelian, G is nilpotent of class at most 2 and hence we may suppose that G is not torsion free. Moreover, since

$$\frac{G}{M_1 \cap M_2} = \frac{M_1}{M_1 \cap M_2} \times \frac{M_2}{M_1 \cap M_2}$$

it follows by Lemma 3 that the factor group $G/M_1 \cap M_2$ is abelian and hence $G' \leq M_1 \cap M_2$.

Suppose now that neither M_1 nor M_2 is periodic, then G is locally nilpotent and let T be the subgroup consisting of all elements of finite order of G. Applying Corollary 1 to the factor group G/T it yields that G/T is abelian then G' is periodic. In particular, if either M_1 or M_2 is torsion free, then G must be abelian. Now suppose that neither M_1 nor M_2 is torsion free and for $i = 1, 2$ let $T_i = T \cap M_i$ be the subgroup consisting of all elements of finite order of M_i. Thus $G' \leq T_1 \cap T_2$ and for $i = 1, 2$ each subgroup of T_i is normal in M_i, so each subgroup of $T_1 \cap T_2$ is G-invariant, then $T_1 \cap T_2$ is central (see Lemma 2) and so G is nilpotent of class at most 2.

Suppose now that one of the factors, say M_1, is periodic. Since T_2 is abelian, it turns out that also $M_1 \cap M_2 = M_1 \cap T_2$ is abelian and put $A = M_1 \cap T_2$. Assume, for a contradiction, that M_1 is not abelian, then there exists a non-abelian direct factor U of such decomposition such that $M_1 = U \times V$ and $\pi(U) \cap \pi(V) = \emptyset$ (see [[20], Theorem 2.4.13, Theorem 2.4.14]). Then, U has finite exponent and since M_1/A is divisible, it turns out that the factor group M_1/VA is trivial and hence $M_1 = VA$. Then, the subgroup UA/A of M_1/A is both a $\pi(U)$-group and a $\pi(V)$-group and so $U \leq A$, which is of course a contradiction. This contradiction shows that M_1 is abelian. Hence every subgroup of $M_1 \cap T_2$ is G-invariant, so by Lemma 2 the subgroup $M_1 \cap T_2$ is central and G is still nilpotent of class at most 2 and the assumption is proved.

The next result is a special case of Proposition 5.

Corollary 6. *Let G be a non-periodic group such that G has no proper subgroups of finite index. Suppose that G can be decomposed into the product of two proper normal quasihamiltonian subgroups, then G is nilpotent of class at most* 2.

Corollary 7 and Corollary 8 will respectively describe the structure of the abelianisation of a locally graded non-perfect minimal non-$\mathfrak{MF}$ group (resp. of a non-perfect minimal non-(quasihamiltonian-by-finite) group).

Corollary 7. *Let G be locally graded minimal non-$\mathfrak{MF}$ group such that $G' \neq G$. Then G/G' is a group of type p^∞.*

Proof. Firstly, if G is periodic, then G is minimal non-(abelian-by-finite) (see [[13], Theorem A]) and hence it holds that G/G' is a group of type p^∞ (see [[1], Theorem 1]).

Now, suppose that G is non-periodic and assume for a contradiction that the divisible abelian group G/G' can be decomposed into the product of two proper subgroups X/G' and Y/G'. Then there exist a characteristic $\mathfrak{M}$-subgroup M_1 of X and a characteristic $\mathfrak{M}$-subgroup M_2 of Y such that the indices $|X : M_1|$ and $|Y : M_2|$ are finite. As the normal subgroup M_1M_2 has finite index in G, it follows that $G = M_1M_2$. Proposition 5 forces G to be nilpotent and then by Lemma 4 the group G is minimal non-(abelian-by-finite) and so the contradiction is proved (see [1], Lemma 3]).

Corollary 8. *Let G be a minimal non-(quasihamiltonian-by-finite) group such that $G' \neq G$. Then G/G' is a group of type p^∞.*

Proof. Firstly, if G is periodic, the factor group G/G' is a group of type p^∞ (see [[7], Lemma 4]). Now, suppose that G is not periodic and assume for a contradiction that the divisible abelian group G/G' can be decomposed into the product of two proper subgroups. Applying the same argument above, Corollary 6 and Lemma 4, it turns out that G is minimal non-(abelian-by-finite), which is again a contradiction.

We are now in position to prove the following two important results. Let $\mathfrak{X}$ be a class of groups. Note that if G is a group without proper subgroups of finite index such that G' has a characteristic $\mathfrak{X}$-subgroup N of finite index, then the conjugacy classes of G/N are finite, thus it is abelian and hence $G' = N$ is an $\mathfrak{X}$-group.

Proposition 9. *If G is a non-perfect locally graded group whose all proper subgroups belong to $\mathfrak{MF}$, then G is either periodic or G belongs to $\mathfrak{MF}$.*

Proof. Assume that the group G is minimal non-$\mathfrak{MF}$ and suppose for a contradiction that G is not periodic. By Corollary 7 it turns out that G/G' is a group of type p^∞. Moreover, the commutator subgroup G' of G contains a characteristic $\mathfrak{M}$-subgroup M of finite index in G' and since G has no proper subgroups of finite index, it follows that $G' = M$ and so the group G is soluble.

Note that G' cannot be periodic, then the subgroup T consisting of all elements of finite order of G' is properly contained in G'. Put $\overline{G} = G/T$. Now $\overline{G}'$ and $\overline{G}/\overline{G}'$ satisfy the hypotheses of [[2], Lemma 2.3] (with $M = \overline{G}'$ and $G = \overline{G}/\overline{G}'$); hence there is a G-invariant subgroup N such that $T < N < G'$, G'/N is periodic and it contains elements of orders p_1 and p_2 for two distinct primes p_1 and p_2 different from p. Clearly G/N is a periodic group such that all proper subgroups belong to $\mathfrak{MF}$. If G/N is an $\mathfrak{M}$-group, then G/N is abelian (see Lemma 3) and so $G' = N$, which is not possible. Then G/N is a periodic minimal non-$\mathfrak{MF}$, then G/N is minimal non-(abelian-by-finite) (see [[13], Theorem A]) and it satisfies the hypotheses of [[1], Theorem 1]. Thus, by that Theorem, $(G/N)' = G'/N$ must be an abelian q-subgroup of G/N, for some prime q. This contradiction proves the assert.

Proposition 10. *Let G be a locally graded minimal non-(quasihamiltonian-by-finite) group. G is periodic if and only if $G' \neq G$.*

Proof. Clearly, if G is periodic, then G is minimal non-(abelian-by-finite) (see [[7], Theorem]) and the assert is consequence of [[1], Theorem 1].

Assume now that $G' < G$. By Lemma 3 it results that G cannot be an $\mathfrak{M}$-group, otherwise G is abelian. Therefore G is a minimal non-$\mathfrak{MF}$ group, then Proposition 9 proves the assert. Note that in this implication, we can remove the hypothesis that G is locally graded. Indeed, the commutator subgroup G' of G contains a characteristic quasihamiltonian subgroup of finite index in G', therefore G' is quasihamiltonian and then G is soluble.

Proposition 9 and Proposition 10 can be compared with the results [[2], Theorem 2.1, Theorem 2.5] which show that a locally graded non-perfect minimal non-(abelian-by-finite) group (resp. minimal non-(nilpotent-by-finite) group) is periodic.

The first step to prove Theorem A is to show that there are no non-periodic locally graded minimal non-(quasihamiltonian-by-finite) groups in the class of groups in which all simple quotients are finite.

Lemma 11. *Let G be a group without proper subgroup of finite index and without infinite simple homomorphic images. If all proper normal subgroups of G are quasihamiltonian-by-finite, then the commutator subgroup G' of G is locally nilpotent.*

Proof. Let N be a proper normal subgroup of G, then N contains a quasihamiltonian G-invariant subgroup Q of finite index and so N/Q lies in the centre of G/Q. Therefore $[N, G]$ is contained in Q. In particular, $[N, G]$ is a locally nilpotent normal subgroup of G and hence it is contained in the Hirsch–Plotkin radical H of G. It follows that all proper normal subgroups of $\overline{G} = G/H$ lie in the centre, then $\overline{G}/Z(\overline{G})$ is a simple group and so the factor group G/H is abelian.

Lemma 12. *Let G be a locally graded group which has no infinite simple homomorphic images. If all proper subgroups of G are quasihamiltonian-by-finite, then G is either periodic or G is quasihamiltonian-by-finite.*

Proof. Assume that G is a minimal non-(quasihamiltonian-by-finite) group. Clearly, if G is not perfect, by Proposition 10 we know that G is periodic. Assume now that G is a perfect locally graded minimal non-(quasihamiltonian-by-finite) group whose simple homomorphic images cannot be infinite. We want to prove that such a group does not exist.

By [[7], Theorem] the group G is non-periodic, and of course G cannot be torsion free, otherwise by Corollary 1 the group G is abelian. Since $G = G'$, applying Lemma 11 it turns out that G is locally nilpotent. Let T be the subgroup consisting of all elements of finite order of G, then T is a normal proper subgroup of G and hence G/T is a torsion free group in which all proper subgroups are quasihamiltonian-by-finite, then by Corollary 1, the factor group G/T is abelian, which is not possible.

We are now in position to prove Theorem A.

Proof of Theorem A. Assume that G is a minimal non-(quasihamiltonian-by-finite) group. Clearly, the group G satisfy the hypotheses of [[5], Theorem C], then if G is either locally soluble or soluble, then G is periodic as a consequence of Lemma 12. Assume now that G is either soluble-by-$PSL(2, F)$ or G is soluble-by-$S_z(F)$, where F is an infinite locally finite field with no infinite proper subfields. Let A be the normal soluble subgroup of G such that either $G/A \simeq PSL(2, F)$ or $G/A \simeq S_z(F)$. However, it cannot happen because the groups $PSL(2,F)$ and $S_z(F)$ are not minimal non-(quasihamiltonian-by-finite) (see [[7], Lemma 8]). □

It is an open question whether a corresponding result to Theorem A holds also for the class of $\mathfrak{MF}$ groups, i.e. whether Proposition 9 holds even without the non-perfectness hypothesis.

3. Groups with many proper large quasihamiltonian-by-finite subgroups

It is known that an arbitrary strongly locally graded group of infinite rank, and hence an arbitrary locally (soluble-by-finite) group of infinite rank, contains a proper subgroup of infinite rank (see for instance [[3], Lemma 2.3]). However, it is an open question whether this is also true in the case of a locally graded group and this seems to be the main obstacle to extend Lemma 13 to the case of locally graded groups. However, we will point out that this is at least possible for the class of strongly locally graded groups.

Lemma 13. *Let G be a locally (soluble-by-finite) group of infinite rank whose proper subgroups of infinite rank are quasihamiltonian-by-finite. Then G is either quasihamiltonian-by-finite or G is minimal non-(abelian-by-finite).*

Proof. Assume that G is not quasihamiltonian-by-finite. Clearly, if G is periodic then G is locally finite and it follows that G is minimal non-(abelian-by-finite) (see [[7], Corollary]). Then assume for a contradiction that G is not periodic. Note that G cannot have a normal subgroup H of infinite rank such that G/H also has infinite rank. Indeed, if a such H exists, it implies that every subgroup X of finite rank of G is quasihamiltonian-by-finite, since XH is properly contained in G. Therefore G is minimal non-(quasihamiltonian-by-finite) and then it is periodic (see Theorem A), which is not possible.

Now, since each proper subgroup of G is soluble-by-finite rank, the group G satisfies the hypotheses of [[5], Theorem A]. Clearly, if G contains a proper soluble normal subgroup S such that G/S has finite rank it follows that $G' \neq G$ (see [[6], Lemma 2.5]). Therefore, assume that G contains a soluble proper normal subgroup K such that G/K has infinite rank and it is isomorphic either to $PSL(2,F)$ or to $S_z(F)$ for some infinite locally field F with no infinite proper subfields. On the other hand, each of the last two groups contains a proper subgroup of infinite rank which is not quasihamiltonian-by-finite (see [[7], Lemma 8]), which is a contradiction. Assume now that G is locally soluble and let N be a normal subgroup of G of finite rank, then G/N has infinite rank and so it contains a proper subgroup M/N of infinite rank, then N is contained in the proper subgroup M of infinite rank of G, which is quasihamiltonian-by-finite. Therefore each proper normal subgroup of G is quasihamiltonian-by-finite and so applying Lemma 11 it turns out that G' is locally nilpotent.

Clearly, if G is not locally nilpotent, then $G' \neq G$. On the other hand, assume that G is locally nilpotent. Let T be the subgroup consisting of all elements of finite order of G. If T has finite rank, then G/T is a torsion free group of infinite rank in which all proper subgroups of infinite rank are abelian-by-finite, then all proper subgroups of G/T are abelian-by-finite (see [[6], Theorem]) and hence G/T must be abelian (see [[16], Theorem B]), so $G' \leq T < G$. If T has infinite rank, G/T has still all the proper subgroups abelian-by-finite and hence again $G' \leq T < G$. In each case it turns out that $G' \neq G$.

If G/G' has infinite rank, it turns out that every proper subgroup of G is quasihamiltonian-by-finite (see [[3], Lemma 2.7]) and hence by Theorem A the whole group G must be periodic, which is a contradiction. Assume that G/G' has finite rank, then G' has infinite rank and it contains a characteristic quasihamiltonian subgroup of finite index, then G' is itself quasihamiltonian. If the divisible abelian group G/G' can be decomposed into the product of two proper subgroups Y/G' and Z/G', it follows by Proposition 5 that G is nilpotent, which is of course a contradiction (see [[18], Part 1, Theorem 2.26, Theorem 2.27]). Then G/G' is a group of type p^∞ for some prime number p.

Note that G' cannot be periodic. Let T^* the subgroup consisting of all elements of finite order of G'. If T^* has infinite rank, then all proper subgroups of G/T^* are quasihamiltonian-by-finite and hence the non-periodic group G/T^* is quasihamiltonian (see Theorem A), then G/T^* is abelian by Lemma 3, a contradiction, because T^* is properly contained in G'. Therefore T^* has finite rank and hence one can assume that G' is a torsion free group, hence G' is abelian. An application of [[2], Lemma 2.3] to the G/G'-module G' yields that there exists a G-invariant subgroup B of G' such that $B < G'$, G'/B is periodic and the set $\pi(G'/B)$ contains more than two distinct prime numbers. As G' is torsion free, the subgroup B has infinite rank and hence G/B is a periodic minimal non-(quasihamiltonian-by-finite) group (see [[13], Lemma 2.1]), then G/B is minimal non-(abelian-by-finite) (see [[7], Theorem]). This is not possible by [[1], Theorem]. The contradiction proves G must be periodic and then the assert is a consequence of [[7], Corollary].

Proof of Theorem B. Since all proper subgroups of infinite rank of G are soluble-by-finite, it turns out that the group G is locally (soluble-by-finite) (see [[6], Lemma 3.1]), and so the statement is a direct consequence of Lemma 13. □

Recall that if $\mathfrak{X}$ is a class of groups, $\mathfrak{X}$ is said to be *countably recognisable* if a group G belongs to $\mathfrak{X}$ whenever all its countable subgroups are $\mathfrak{X}$-groups.

Since the class $\mathfrak{M}$ is local, then applying [[10], Theorem 3.2] it turns out that the class $\mathfrak{MF}$ is countably recognisable. For the same result also the class of quasihamiltonian-by-finite groups is countably recognisable.

Theorem 14. *Let $\aleph$ be an uncountable regular cardinal and let G be a locally graded group of cardinality $\aleph$ such that $G' \neq G$. If all proper subgroups of G of cardinality $\aleph$ belong to $\mathfrak{MF}$, then G itself is an $\mathfrak{MF}$-group.*

Proof. Assume for a contradiction that the group G does not contain a subgroup of finite index having a modular subgroup lattice, thus the group G cannot be periodic (see [[13], Theorem B]). Moreover the abelian group G/G' cannot have cardinality $\aleph$ (see [[9], Lemma 2.3]). Therefore, it turns out that G/G' must have cardinality strictly smaller than $\aleph$, hence G' is a large proper subgroup of G and so it contains a characteristic $\mathfrak{M}$-subgroup M of finite index having a modular subgroup lattice. Since G has no proper subgroups of finite index, it follows that $G' = M$ has a modular subgroup lattice.

If the divisible abelian group G/G' can be decomposed into the product of two proper subgroups X/G' and Y/G', it follows by Proposition 5 that G is nilpotent, which is of course a contradiction (see [[18], Part 1, Theorem 2.26, Theorem 2.27]). Then G/G' is a group of type p^∞ for some prime number p.

Note that G' cannot be periodic, then the subgroup T consisting of all elements of finite order of G' is a proper subgroup of G'. An application of [[2], Lemma 2.3] to the G/G'-module G'/T yields that there exists a normal subgroup N of G such that $T < N < G'$, G'/N is periodic and the set $\pi(G'/N)$ contains more than two distinct prime numbers. Moreover, N has cardinality $\aleph$ (see for instance [[9], Lemma 2.2]), so that G/N is a periodic minimal non-$\mathfrak{M}\mathfrak{F}$ group (see [[13], Lemma 2.1]) and then G/N is minimal non-(abelian-by-finite) (see [[13], Theorem A]). This is not possible by [[1], Theorem 1]. The contradiction proves the assert.

Proof of Theorem C. Assume for a contradiction that the group G does not contain a quasihamiltonian subgroup of finite index. Since the class of quasihamiltonian-by-finite groups is countably recognisable, then G contains a small subgroup X which is not quasihamiltonian-by-finite and it is easy to prove that G has a proper normal large subgroup N (see the Proof of [[13], Corollary 2.7]), and G/N has cardinality strictly smaller than $\aleph$ since $G = XN$. As N is quasihamiltonian-by-finite, there exists a G-invariant subgroup Q of N such that Q is quasihamiltonian and the index $|N : Q|$ is finite. It follows that Q is contained in the Hirsch–Plotkin radical of G and thus the factor group G/Q is still locally graded (see [[14], Theorem]) all its proper subgroups are quasihamiltonian-by-finite. If G/Q is likewise quasihamiltonian-by-finite, then G/Q is quasihamiltonian and hence $G' \neq G$. If G/Q is minimal non-(quasihamiltonian-by-finite), by Theorem A it turns out that G/Q is periodic and hence G/Q is still not perfect and so also G is not perfect. Furthermore, in both case it turns out that $G' \neq G$.

Applying Theorem 14, it results that the group G has modular subgroup lattice and thus G is abelian by Lemma 3. The contradiction proves the result. $\square$

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