Quantum entangling power of adiabatically connected Hamiltonians

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The space of quantum Hamiltonians has a natural partition in classes of operators that can be adiabatically deformed into each other. We consider parametric families of Hamiltonians acting on a bipartite quantum state space. When the different Hamiltonians in the family fall in the same adiabatic class, one can manipulate entanglement by moving through energy eigenstates corresponding to different values of the control parameters. We introduce an associated notion of adiabatic entangling power. This novel measure is analyzed for general $d\times d$ quantum systems, and specific two-qubit examples are studied

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I. INTRODUCTION

Adiabatic evolutions represent a very special class of quantum evolutions, nevertheless they allow for a broad set of quantum state manipulations. In particular, a great deal of activity has been devoted recently to the study of adiabatic techniques for quantum information processing (QIP) [1].

The notion of adiabatic quantum computing emerged as an novel intriguing paradigm for the development of efficient quantum algorithms [2–4]. In this approach, information, e.g., the solution of a hard combinatorial problem, is encoded in the ground state of a properly designed many-qubit Hamiltonian H_f. This ground state is then generated by letting the system evolve in an adiabatic fashion from the ground state of a simple initial Hamiltonian H_0 [2]. In view of the adiabatic theorem (see, e.g., [5]), the crucial property which governs the scaling behavior of the computational time is the spectral gap, i.e., the energy difference between the ground and the first excited state. The larger the gap, the faster the computation can be.

In adiabatic quantum computing as defined in Ref. [2], the parametric family of Hamiltonians has the simple form of a convex combination of H_0 and H_f; one can also consider a more general family of Hamiltonians and more complex paths in the control parameter space. For example, in the so-called *geometric quantum computation* [6], one considers *loops* in the control space of a nondegenerate set of Hamiltonians for the purpose of controlled Berry phase generation [7]. When even the nondegeneracy constraint is lifted and high-dimensional eigenspaces are allowed, one is led to consider nonAbelian holonomies which mix nontrivially the ground states of the system. This latter method, which provides a general approach to QIP as well, is termed *holonomic quantum computation* [8].

In this paper, we shall investigate how one can adiabatically generate quantum entanglement [9,10]. The idea is a simple one. One first prepares a bipartite quantum system in one of its eigenstates, e.g., the ground state, and then drives the control parameters of the system Hamiltonian along some path. If this path is adiabatic, the system will stand at any time in the corresponding eigenstate. In general, eigenstates associated with different control parameters will have different entanglement, therefore the described dynamical process will result in a protocol for entanglement manipulation. We would like to characterize a parametric family of Hamiltonians in terms of its capability of entanglement generation according to the above protocol. In this paper, we will focus on bipartite, e.g., two-qubit, quantum systems. The aim will be, given a Hamiltonian family, to characterize its entangling capabilities by means of adiabatic manipulations.

II. ADIABATIC CONNECTIBILITY

Let us start with a few simple general considerations about adiabatically connectible Hamiltonians. We would like to understand how the space of Hamiltonians over $\mathcal{H}\cong\mathbb{C}^D$ splits in classes of elements that can be adiabatically deformed into each other.

Definition. Two Hamiltonians H_0 and H_1 are adiabatically connectible if a continuous family of Hamiltonians $\{H_t\}_{t\in[0,1]}$ exist such that (i) $H(0)=H_0$ and $H(1)=H_1$, and (ii) the degeneracies of the spectra of the H_t's do not depend on t.

The notion of adiabatic deformability of Hamiltonians is an important concept in many-body and field theory quantum systems. Indeed, when two Hamiltonians can be connected in this way they share several properties, e.g., ground-state degeneracy, quasiparticle quantum numbers, etc., so that in many respects they can be regarded as belonging to the same kind of universality class [11]. On the other hand, an obstruction to such a process will be typically associated with some sort of quantum phase transition. Unconnectible Hamiltonians show qualitative different features. Since we will study how entanglement changes while remaining in the same adiabatic class, our analysis can be regarded as complimentary to that of entanglement behavior in quantum phase transitions [12].

In the simple finite-dimensional case we are interested in, one can prove the following

Proposition 1. Two Hamiltonians H_0 and H_1 over $\mathcal{H}\cong\mathbb{C}^D$ are adiabatically connectible if and only if they belong to the same connected component of the set of isodegenerate Hamiltonians.

Proof. Let $H_\alpha=\Sigma_{i=1}^R \epsilon_\alpha^i \Pi_\alpha^i(\alpha=0,1)$ be the spectral resolution of H_0 and H_1. We now order their eigenvalues in ascending order, i.e., $\epsilon_\alpha^1 < \cdots < \epsilon_\alpha^R$. We define two vectors D_α $(\alpha=0,1)$ in $\mathbb{R}^R$ as follows: $D_\alpha:=(\mathrm{tr}\Pi_\alpha^1,\ldots,\Pi_\alpha^R)$, where the components are ordered according to the corresponding eigenvalues. The Hamiltonians H_0 and H_1 belong to the same connected component of the set of isodegenerate Hamiltonians if and only if $D_0=D_1$. Isodegeneracy is given by the weaker condition that a permutation P of R objects exists such that $(D_1)_i=(D_0)_{P(1)}$ $(i=1,\ldots,R)$. It is an elementary fact that, given the two systems of orthoprojectors $\{\Pi_\alpha^i\}_{i=1}^R$ $(\alpha=0,1)$ such that $\mathrm{Tr}\Pi_1^i=\mathrm{Tr}\Pi_2^i$ $(i=1,\ldots,R)$, a (nonunique) unitary W exists such that $W\,\Pi_0^i\,W^\dagger=\Pi_1^i$ $(i=1,\ldots,R)$. Let us introduce R real-valued functions $\epsilon^i:[0,1]\mapsto\mathbb{R}$ such that $\epsilon^i(0)=\epsilon_0^i$ and $\epsilon^i(1)=\epsilon_1^i(i=1,\ldots,R)$. In view of the ordering assumption, we can choose them to satisfy the no-crossing constraints $\epsilon^{i+1}(t)>\epsilon^i(t)$ $(i=1,\ldots,R-1)$. Consider now the following family of Hamiltonians: $H(t)=\Sigma_{i=1}^R \epsilon^i(t)U_t\Pi_0^iU_t^\dagger$, where the continuous unitary family $\{U_t\}_{t=0}^1$ is such that $U_0=\mathbb{1}$ and $U_1=U$. Clearly $H(0)=H_0$ and $H(1)=H_1$. Moreover, for the very way they have been constructed, all the $H(t)$ belong to the same connected component of the set of isodegenerate Hamiltonians of H_0 and H_1. This shows that the latter condition is sufficient in order that H_0 and H_1 are adiabatically connectible.

Isodegeneracy of H_0 and H_1 is also an obvious necessary condition for adiabatic connectibility because otherwise level crossing would necessarily occur. But level crossing would necessarily occur even if $D_0\neq D_1$ because, for some $t\in[0,1]$ and $1\leq i\leq R$, it would be $\epsilon^{i+1}=\epsilon^i$. This proves the necessity part of the Proposition.

The role of the functions $\epsilon^i(t)$ in the Proof above is to map the spectrum of H_0 onto that of H_1, whereas all the information about the eigenvectors is contained in the family of unitaries U_t. By setting all the connecting functions $\epsilon_t^i/\epsilon_0^i$ to 1, one gets a final Hamiltonian $\tilde{H}_1$ isospectral to H_0 having the same eigenvectors of H_1. This latter remark is important for the following in that it allows one to restrict to isospectral Hamiltonian families. The actual spectrum structure, e.g., the energy gaps, just imposes an upper bound over the speed at which the adiabatic deformation process can be carried on. Moreover, in order to have a one-to-one correspondence between eigenvalues and eigenstates, we shall assume that our Hamiltonians are *nondegenerate*, i.e., $d_i=1$ $(i=1,\ldots,R)$. Notice that in Hamiltonian space the condition of nondegeneracy is a *generic* one.

The simplest case one can consider is of course provided by two-level Hamiltonians with eigenvalues ϵ_1 and ϵ_2. Using the standard Pauli matrices, one can write $H=\epsilon_S\mathbb{1}+\epsilon_A\vec{n}\cdot\vec{\sigma}(\epsilon_S:=(\epsilon_1+\epsilon_2)/2),\epsilon_A:=(\epsilon_1-\epsilon_2)/2$. Here we have just two possibilities (i) $\epsilon_A=0$: the Hamiltonian is a rescaled identity and we have just one degree of freedom, and (ii) $\epsilon_A\neq 0$: all possible operators of this kind are then parametrized by a triple $(\epsilon_S,\epsilon_A,\vec{n})$, where $\epsilon_A\in\mathbb{R},\epsilon_A\in\mathbb{R}-\{0\}$, and $\vec{n}\in S^2\cong\mathrm{SU}(2)/\mathrm{U}(1)$. For each of the two isodegeneracy classes above there is just one connected component, i.e., any non (totally) degenerate Hamiltonian is adiabatically connectible to any other non (totally) degenerate Hamiltonian. Notice that this latter statement holds for any dimension of $\mathcal{H}$.

III. ADIABATIC ENTANGLING POWER

We move now to introduce our definition of adiabatic entangling power. Let $\mathcal{H}\cong\mathbb{C}^d\otimes\mathbb{C}^d$ be a bipartite quantum state space. We consider a family $\mathcal{F}$ of nondegenerate Hamiltonians over $\mathcal{H}$, $\mathcal{F}_H:=\{H(\lambda)/\lambda\in\mathcal{M}\}$, where $\mathcal{M}$ is an n-dimensional compact and connected manifold. The points of $\mathcal{M}$ are to be seen as dynamically controllable parameters. Let $E:\mathcal{H}\rightarrow\mathbb{R}_0^+$ be a measure of bipartite pure state entanglement over E, e.g., von Neumann entropy of the reduced density matrix. If $H(\lambda)=\Sigma_{i=1}^{d^2}\varepsilon_i|\Psi_i(\lambda)\rangle\langle\Psi_i(\lambda)|$ is the spectral resolution of an element of $\mathcal{F}$, we define the *adiabatic entangling power* of $\mathcal{F}$ by

$$e(\mathcal{F}_H):=\max_i \sup_{\lambda,\lambda'}\left|E(|\Psi_i(\lambda)\rangle)-E(|\Psi_i(\lambda')\rangle)\right| \tag{1}$$

$(i=1,\ldots,d^2,\lambda,\lambda'\in\mathcal{M})$.

We will assume that $H_{\lambda_0}\in\mathcal{F}_H$ such that the associated eigenvectors are all *product states*. Let us stress once again that the physical idea behind these definitions is quite simple: one starts from the (unentangled) eigenvectors of H_{λ_0}; then by adiabatically driving the control parameters λ, the states $|\Psi_i(\lambda)\rangle$ can be reached. If λ^* denotes the point at which the maximum (1) is achieved ($\mathcal{M}$ is compact), any adiabatic path connecting λ_0 to λ^* realizes an *optimal* entanglement generation procedure within the family $\mathcal{F}_H$.

An explicit evaluation of Eq. (1) is, for a general $\mathcal{F}$, quite a difficult task. In light of the observations after Proposition 1, we can, without loss of generality, consider only the case in which $\mathcal{F}$ is an *isospectral* family of nondegenerate Hamiltonians. Let $\mathcal{F}_U\subset\mathcal{U}(\mathbb{C}^d\otimes\mathbb{C}^d)$ be a set (compact and connected) of unitary transformations containing the identity. The isospectral family is

$$\mathcal{F}_H:=\{U\,H_0U^\dagger/U\in\mathcal{F}_U\}, \tag{2}$$

where $H_0=\Sigma_{i=1}^{d^2}\varepsilon_i|\Psi_i\rangle\langle\Psi_i|$, $i\neq j\Rightarrow\varepsilon_i\neq\varepsilon_j$, and the $|\Psi_i\rangle$'s are an orthonormal basis of product states. Moreover we can also restrict ourselves to *ground-state* entanglement, i.e., to consider the entanglement contents of just the eigenvector $|\Psi_0\rangle$ corresponding to the minimum energy eigenvalue. If this is the case, one can forget about the maximization over the eigenvalue index i in Eq. (1). The ground state of $H(\lambda)$ (H_0) will be denoted as $|\Psi_0(\lambda)\rangle$ $(|\Psi_0\rangle)$. For an isospectral family as in Eq. (2), we will use the notation $e(\mathcal{F}_U)$.

The adiabatic entangling power (1) induces, for the class of Hamiltonian families (2), the following real-valued function over the subsets $\mathcal{F}_U$ of $\mathcal{U}(\mathbb{C}^d\otimes\mathbb{C}^d)$:

$$e(\mathcal{F}_U)=\max_i \sup_{U\in\mathcal{F}_U} E[U|\Psi_i\rangle]. \tag{3}$$

It is important to stress that this expression has the physical meaning of entanglement achievable by adiabatically manipulating the parameters found on a manifold, say $\mathcal{M}$, upon which the U's in $\mathcal{F}_U$ depend. Indeed, for an isospectral Hamiltonian family (2), the adiabatic evolution operator corresponding to the path $\gamma:[0,T]\mapsto\mathcal{M}$ is given by the product of three different kinds of contributions $U_{\mathrm{ad}}(\gamma) = U(\gamma(T))e^{-iH_0T}U_B(\gamma)$. The first term $U(\gamma(T))$ is simply the unitary corresponding to the end-point of the path γ. Due to the adiabatic theorem, an initial eigenstate $|\Psi_i\rangle$ is indeed mapped, up to a phase, onto the final eigenstate $U(\gamma(T))|\Psi_i\rangle$. The second factor in U_{ad} is clearly just the dynamical phase associated with H_0, whereas the third is an operator taking into account the geometric contribution to the phase accumulated by the eigenvectors $U_B(\gamma)=\sum_{i=1}^{d^2} e^{i\phi_g(\gamma)}|\Psi_i\rangle\langle\Psi_i|$, in which $\phi_g(\gamma)=i\int_\gamma \langle\Psi_i(\lambda)|d|\Psi_i(\lambda)\rangle$ are Berry's phases associated with γ. Notice in passing that when γ is a loop, i.e., $\gamma(0)=\gamma(T)=\lambda_0$, then $U(\gamma(T))=\mathbb{1}$. As far as the adiabatic entangling power (1) is concerned, the phases can be obviously neglected.

The adiabatic entangling power is invariant under left (and not right in general) multiplication by bilocal unitary operators, i.e., $e(\mathcal{F}_U)=e((U_1\otimes U_2)\mathcal{F}_U)$, $\forall\, U_1,U_2\in\mathcal{U}(d)$. This implies that, as far as adiabatic entangling capabilities are concerned, a unitary family $\mathcal{F}_U$ can always be considered closed under the left multiplication by local unitary operators [13].

We want now to establish a connection between the adiabatic entangling power (3) and a variation of entangling power $e_p^{(\mathrm{av})}$ of bipartite unitaries introduced in Ref. [14] (for a different definition, based on *average* entanglement production, see also [15]). In this paper, we define $e_p(U)$ as the *maximum* entanglement obtainable by the action of U over all possible product states, i.e., $e_p(U)=\sup_{\psi_1,\psi_2}E[U|\psi_1\rangle\otimes|\psi_1\rangle]$.

Since the $|\Psi_i\rangle$'s are by hypothesis product states, one clearly has $E[U|\Psi_i\rangle]\leqslant\sup_{\psi_1,\psi_2}E[U|\psi_1\rangle\otimes|\psi_1\rangle]$. Therefore, one obtains the upper bound

$$e(\mathcal{F}_U)\leqslant \sup_{U\in\mathcal{F}_U} e_p(U). \tag{4}$$

In some circumstances one can get the equality.

Proposition 2. Suppose that the unitary family $\mathcal{F}_U$ is such that for all $U_1,U_2\in U(d)$ one has $\mathcal{F}_U(U_1\otimes U_2)\subset\mathcal{F}_U$, i.e., the family is closed also under right multiplication of bilocal operators. It follows that the adiabatic entangling power coincides with the supremum over $\mathcal{F}_U$ of the entangling power $e_p(U)$.

Proof. It is straightforward that

$$\begin{aligned} e(\mathcal{F}) &= \max_i \sup_{U\in\mathcal{F}_U,U_1,U_2} E[U(U_1\otimes U_2)|\Psi_i\rangle] \\ &= \sup_{U\in\mathcal{F}_U,\psi_1,\psi_2} E[U|\psi_1\rangle\otimes|\psi_2\rangle] \geqslant \sup_{U\in\mathcal{F}_U} e_p(U). \end{aligned} \tag{5}$$

Therefore, using Eq. (4) one obtains $e(\mathcal{F})=\sup_{U\in\mathcal{F}_U}e_p(U_\lambda)$. Notice also that for such a family the maximization over the eigenvalue index i in Eq. (1) is irrelevant. ■

IV. EXAMPLES

We will now illustrate the use of the general notions introduced so far by considering in a detailed fashion some concrete Hamiltonian families acting on a two-qubit space. Before doing that, let us recall a few basic facts about two-qubit entanglement in pure states. We denote the standard product basis by $|\Psi_i\rangle$ $(i=1,\ldots,4)$ and consider a generic two-qubit state $|\Phi\rangle=U|\Psi\rangle=\sum_{i=1}^4 a_i|\Psi_i\rangle$. The eigenvalues of the associated reduced density matrix are given by $\lambda=(1+\sqrt{1-4C^2})/2$ and $1-\lambda$, where $C^2=|a_1a_2-a_3a_4|^2$ and $2\,C$ is the so-called "concurrence." The entanglement measure is given by $E=-[\lambda\log_2\lambda+(1-\lambda)\log_2(1-\lambda)]$. Since $dE/d\lambda<0$, finding the maximum possible entanglement for the output state $|\Phi\rangle$ means minimizing λ, or, which is the same, maximizing C^2. The state $|\Phi\rangle$ is maximally entangled for $\lambda=\frac{1}{2}$ or $C^2=\frac{1}{4}$.

Example 1. It is useful to start with an example of a two-qubit Hamiltonian family with *zero* adiabatic entangling power. Let $H(\lambda)=\sum_{\alpha=x,y,z}\lambda_\alpha\sigma_\alpha\otimes\sigma_\alpha$, where the λ's are such that the corresponding Hamiltonian is always nondegenerate. One has that $[H(\lambda),H(\lambda')]=0(\forall\lambda,\lambda')$. Then all the elements of the family can be diagonalized simultaneously. The joint eigenvectors are clearly given by Bell's basis $|\Phi^\pm\rangle:=1/\sqrt{2}(|00\rangle\pm|11\rangle)$, $|\Psi^\pm\rangle:=1/\sqrt{2}(|10\rangle\pm|01\rangle)$. Entanglement in the eigenstates is therefore maximal and cannot be changed by varying the control parameters λ. Analogously one can easily build examples of Hamiltonian families having joint constant eigenvectors given by products.

Example 2. The nondegenerate Hamiltonian we consider is the following:

$$H_0=\lambda_1\sigma_z\otimes\mathbb{1}+\lambda_2\mathbb{1}\otimes\sigma_z \quad (\lambda_1\neq\lambda_2). \tag{6}$$

The eigenvectors are given by the standard product basis. We introduce the family of unitaries $U(\mu,\mu_z)=\exp[iK(\mu,\mu_z)]$, where

$$K(\mu,\mu_z):=\mu\sigma^+\otimes\sigma^-+\bar{\mu}\sigma^-\otimes\sigma^++\mu_z(\sigma_z\otimes\mathbb{1}-\mathbb{1}\otimes\sigma_z) \tag{7}$$

and the associated isospectral family $H(\mu,\mu_z):=U(\mu,\mu_z)H_0U(\mu,\mu_z)^\dagger$. The Hilbert space is given by $\mathcal{H}=\mathrm{span}\{|00\rangle,|01\rangle,|10\rangle,|11\rangle\}$ and we can split it in the two subspaces $\mathcal{H}_0=\mathrm{span}\{|00\rangle,|11\rangle\}$ and $\mathcal{H}_1=\mathrm{span}\{|01\rangle,|10\rangle\}$, where obviously $\mathcal{H}=H_0\oplus H_1$.

The evolution operator U is the identity on $\mathcal{H}_0$, while it is a straightforward exercise to verify that on $\mathcal{H}_1$ it yields $U|01\rangle\equiv|\xi\rangle\equiv a|01\rangle+b|10\rangle$ and $U|10\rangle\equiv|\zeta\rangle\equiv-\bar{b}|01\rangle+\bar{a}|10\rangle$, where $a=\cos\theta+(2i\sin\theta/\theta)\mu_z$, $b=(4i\sin\theta/\theta)\bar{\mu}$ and $\vec{\theta}\equiv2(\mu+\bar{\mu},i(\mu-\bar{\mu}),\mu_z)$. For the generic state $|\Psi\rangle=\alpha|01\rangle+\beta|10\rangle+\gamma|00\rangle+\delta|11\rangle$ one has $C^2=|xy-\gamma\delta|^2$, where $x=\alpha a-\beta\bar{b}$ and $y=\alpha b+\beta\bar{a}$.

For $|01\rangle$ the evolved state is $|\xi\rangle=a|01\rangle+b|10\rangle$ and its reduced density matrix is obviously $\rho=\mathrm{diag}(|a|^2,|b|^2)$ whose eigenvalues are $|a|^2$ and $1-|a|^2$. The condition to obtain a maximally entangled state is hence $|a|^2=\frac{1}{2}$, that is, $\sin^2\theta=\frac{1}{2}[1+(\mu_z/2|\mu|)^2]$. This equation admits (at least) one solution iff $|\mu_z|\leqslant2|\mu|$. Thus a maximally entangled state can be reached starting from either $|01\rangle,|10\rangle$. In Fig. 1 is shown the reachable entanglement from the input state $|01\rangle$ as a function of the parameters μ,μ_z. We see how moving in the

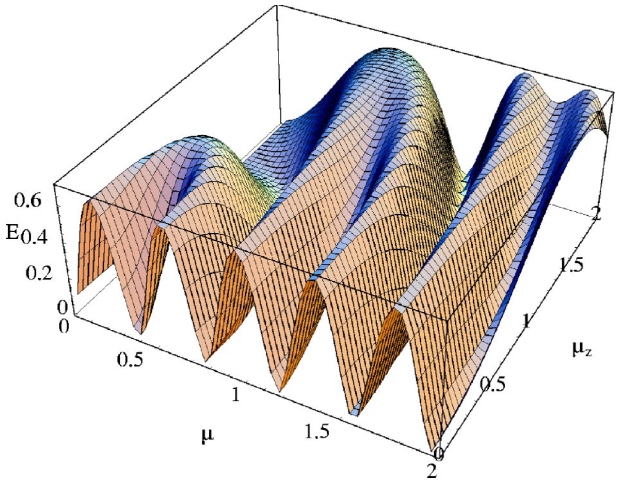

FIG. 1. (Color online) Entanglement generated by the Hamiltonian of the example 2 for the input state $|01\rangle$ as a function of the parameters μ, μ_z.

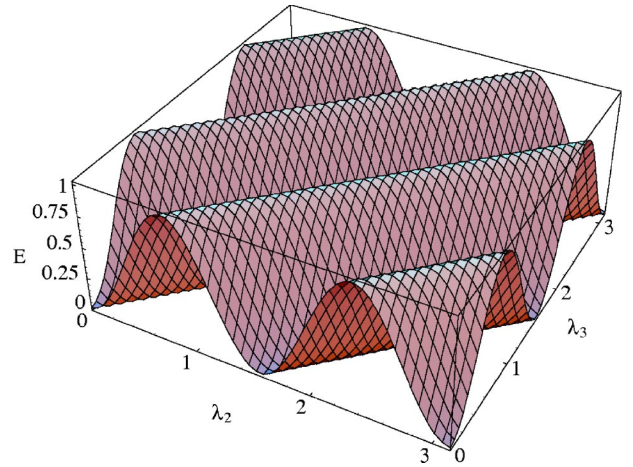

FIG. 2. (Color online) Entanglement for the unitary $U = e^{i\Sigma_{i=1}^{3}\lambda_i\sigma^i\otimes\sigma^i}$, with the input state $(1/\sqrt{2})(|\Psi_1\rangle + i|\Psi_2\rangle)$ as a function of λ_2, λ_3, with $\lambda_1 = 1$.

parameter space to higher values of μ_z spoils the reachability of a maximally entangled state.

Example 3. Let us examine now the following unitary family: $U = \exp(i\Sigma_{j=1}^{3}\lambda_j\sigma_j\otimes\sigma_j)$. In the so-called magic basis $[|\Psi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}, |\Psi_2\rangle = -i(|00\rangle - |11\rangle)/\sqrt{2}, |\Psi_3\rangle = (|01\rangle - |10\rangle)/\sqrt{2}, |\Psi_4\rangle = -i(|01\rangle + |10\rangle)/\sqrt{2}]$ (as well in the Bell basis) these unitaries are diagonal and read $U = \Sigma_{k=1}^{4} e^{ih_k}|\Psi_k\rangle\langle\Psi_k|$ where $\{h_1 = \lambda_1 - \lambda_2 + \lambda_3, h_2 = \lambda_1 + \lambda_2 - \lambda_3, h_3 = -\lambda_1 + \lambda_2 + \lambda_3, h_4 = -\lambda_1 - \lambda_2 - \lambda_3\}$. So in this basis the input state is $|\Psi\rangle = \Sigma_k w_k|\Psi_k\rangle$ and the output state is $|\Phi\rangle = \Sigma_k w_k e^{-ih_k}|\Psi_k\rangle$. The concurrence is given by $C^2 = \Sigma_{k,l}(w_k e^{-ih_k})^2(w_l^* e^{ih_l})^2$. Following Ref. [16], we find that the maximum reachable concurrence is $C = \max_{k,l}|\sin(h_k - h_l)|$ and the product input state which gives the best entangling capability as a function of the parameters λ_k is then $(1/\sqrt{2})(|\Psi_k\rangle + i|\Psi_l\rangle)$. So, for instance, a maximally entangled state can be reached from the input state $(1/\sqrt{2})(|\Psi_1\rangle + i|\Psi_2\rangle) = |00\rangle$ for parameters such that $\lambda_3 - \lambda_2 = \pi/4$ (see Fig. 2).

Before proceeding to the conclusions, we would like to show that the first two-qubit Hamiltonian family associated with the unitaries (7) can be used to generate a nontrivial entangling gate in an adiabatic fashion.

Proposition 3. An adiabatic loop in the parameter space $(\mu, \mu_z)(|\mu|^2 + \mu_z^2 = \text{const})$ gives rise to the diagonal unitary mapping $|\alpha\beta\rangle \to \exp(i\phi_{\alpha\beta})|\alpha\beta\rangle$, where, if γ denotes the geometric contribution, $E_{\alpha\beta}$ the eigenvelues, and T is the operation time, one has $\phi_{01} = E_{01}T + \gamma$, $\phi_{10} = E_{01}T - \gamma$, $\phi_{00} = E_{00}T$, and $\phi_{11} = E_{11}T$. For $\phi_{01} + \phi_{10} - (\phi_{00} + \phi_{11}) = -4T \neq 0 \bmod 2\pi$, the obtained transformation is equivalent to a controlled-phase-shift.

Proof. Indeed it is easy to check that (i) by the adiabatic theorem the evolution has to be diagonal in the product basis, (ii) the geometric contribution of the states $|\alpha\alpha\rangle$ $(\alpha = 0, 1)$ is zero $[\Leftarrow U(\mu, \mu_z)|\alpha\alpha\rangle = |\alpha\alpha\rangle]$, and (iii) in the one-qubit subspace spanned by $|0\rangle := |01\rangle$ and $|1\rangle := |10\rangle$, the unitaries e^{iK} with the K defined in Eq. (7) look like $U(\mu, \mu_z) = \exp[i(\mu\tilde{\sigma}^+ + \bar{\mu}\tilde{\sigma}^- + \mu_z\tilde{\sigma}_z)]$. This latter equation can of course be written as $\mathbf{B}\cdot\dot{\boldsymbol{\sigma}}$, where a fictitious magnetic field $\mathbf{B}$ has been introduced. One can then use the standard Berry-phase argument for a spin-$\frac{1}{2}$ particle in an adiabatically changing magnetic field to claim that under a $\mathbf{B}$ going along an adiabatic loop, one has $|\mathbf{0}\rangle \mapsto e^{i\gamma}|\mathbf{0}\rangle$ and $|\mathbf{1}\rangle \mapsto e^{-i\gamma}|\mathbf{1}\rangle$. Here γ denotes the standard geometric phase, i.e., proportional to the solid angle swept by $\mathbf{B}$. The final equivalence claim stems from a known result in the literature [18]. ■

Of course the general fact that entangling gates can be obtained via adiabatic manipulations is not new; see, e.g., [8,17]. The point of proposition 4 is to show explicitly how the particular two-qubit Hamiltonian family associated with the untaries (7) can be exploited to enact a controlled phase via a simple adiabatic protocol.

V. CONCLUSION

In this paper we analyzed the entanglement generation capabilities of a parametric family of adiabatically connected nondegenerate Hamiltonians. One prepares the system in a separable eigenstate of a distinguished Hamiltonian H_0 in the family and then the space of parameters is adiabatically explored. The system remains then in an energy eigenstate and the (bipartite) entanglement contained in such an eiegenstate can be maximized over the manifold of control parameters. We introduced an associated measure e of adiabatic entangling power and discussed its properties and relations with a previously introduced measure for the case of isospectral families of Hamiltonians. We illustrated the general ideas by studying explicitly the adiabatic entangling power of concrete two-qubit Hamiltonian families. We also showed how to generate a nontrivial two-qubit entangling gate by means of adiabatic loops.

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