THE WALDSCHMIDT CONSTANT FOR A SPECIAL PARTIAL INTERSECTION

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Abstract. We study the Waldschmidt constant of a partial intersection of type $(\underline{p}, \underline{q})$ with $\underline{p} = (3,1)$ and $\underline{q} = (2,1)$ in projective 2-space. The investigation is carried out via two approaches. The first approach is algebraic in nature by considering fixed components and dimensions. The second approach utilizes the underlying geometry of the points to repeatedly reduce fat point schemes by lines and smooth conics.

1. Introduction

In this paper, we investigate the invariant known as the *Waldschmidt constant* of a given homogeneous ideal $I = \oplus_{d\geq 0} I_d$ in the standard graded polynomial ring $R := K[x_0, \dots, x_n]$. We define the *initial degree* of I, denoted $\alpha(I)$, as the least integer d such that $I_d \neq (0)$ and let $I^{(m)}$ denote the m-th symbolic power of I. With this notation, the Waldschmidt constant of the ideal I is defined to be the real number

$$\widehat{\alpha}(I) = \lim_{m\to\infty} \frac{\alpha(I^{(m)})}{m}.$$

It is a well-known result that $\widehat{\alpha}(I) = \inf_{m\geq 1} \frac{\alpha(I^{(m)})}{m}$, see, for example, the proof of [3, Lemma 2.3.1].

In particular, we study two approaches for computing the Waldschmidt constant of special sets of points called *partial intersections* (see Definition 2.1 and Remark 2.3) in projective 2-space. Our goal is to compute the Waldschmidt constant for an example of this family of points, exhibited in Figure 1. By definition, we can see that the Waldschmidt constant is used to study the initial degree of the m-th symbolic power of I asymptotically. Introduced in [24] in a completely different setting, the Waldschmidt constant of the ideal I can also be used to bound the asymptotic and regular resurgences of I. Namely, [16, Theorem 1.2] gives the following bounds:

$$1 \leq \frac{\alpha(I)}{\widehat{\alpha}(I)} \leq \rho_a(I) \leq \rho(I),$$

where the *resurgence* of I is the real number

$$\rho(I) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\right\}.$$

With some progress from an asymptotic perspective, Guardo, Harbourne and Van Tuyl [16] introduced the *asymptotic resurgence* of I as

$$\rho_a(I) = \sup\left\{\frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} \text{ for all } t \gg 0\right\}.$$

Consequently, the Waldschmidt constant gives us information for various containments between symbolic and regular powers of ideals. Partial intersections display many nice properties. One consequence is that the minimal free resolution of a partial intersection is well-understood. As such, it is natural to consider their symbolic powers. In forthcoming work, we generalize this investigation to a larger family of partial intersections.

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Symbolic powers of homogeneous ideals in a polynomial ring have appeared in numerous projects across a variety of different branches of mathematics. However, symbolic powers often display unpredictable behaviour and so can be challenging to work with in concrete ways. The definition alone can be daunting to digest: if $I = \oplus_{d\geq 0} I_d$ is a homogeneous ideal then the m-th *symbolic power* of I is defined as

$$I^{(m)} = \bigcap_{\mathfrak{p}\in Ass(I)} (I^m R_{\mathfrak{p}} \cap R),$$

where I^m denotes the regular m-th power of I and $\mathrm{Ass}(I)$ denotes the set of associated primes of I.

The definition of a symbolic power becomes more palatable for certain families of ideals. For example, if I is a radical ideal (this includes, for instance, square-free monomial ideals and ideals of finite sets of points in projective space) then

$$I^{(m)} = \bigcap_{\mathfrak{p}\in \mathrm{Ass}(I)} \mathfrak{p}^m.$$

One way to gain traction when working with symbolic powers is to compare them with regular powers. For instance, we always have the containment $I^m \subseteq I^{(m)}$. For reverse containments, we have what is known as the *Containment Problem* in related literature. This is the difficult problem of determining for which m and d the containment $I^{(m)} \subseteq I^d$ is true. We refer the reader to [1, 2, 3, 4, 9, 10, 11, 17, 18, 19, 22, 23] for a partial survey of papers on this topic.

Another way to better understand symbolic powers is to relate their invariants with invariants of regular powers of the ideal.

This paper is structured as follows. In Section 2, we provide more of the necessary background and notation needed for the set-up of the investigation. We then consider the study of the initial degree invariant and the Waldschmidt constant from two different perspectives. Section 3 determines the Waldschmidt constant of the ideal in question by using fixed components. In Section 4, we look at a reduction process that helps to determine the initial degree of various symbolic powers of the ideal in question.

2. Preliminaries

In this section, we collect preliminary definitions and notation common to the two perspectives mentioned in the Introduction. We fix K to be an algebraically closed field. Consider the polynomial ring $R := K[\mathbb{P}^n] = K[x_0, \ldots, x_n]$. We first define some geometric objects in the projective n-space $\mathbb{P}^n$, but eventually we will work in $\mathbb{P}^2$. In this paper, we study objects called fat point schemes.

Definition 2.1. *Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ and let $m_1, \ldots, m_s$ be positive integers. The subscheme $\mathbb{W}$ in $\mathbb{P}^n$ defined by*

$$I_{\mathbb{W}} = \wp_1^{m_1} \cap \cdots \cap \wp_s^{m_s} \subseteq R = K[x_0, \ldots, x_n],$$

where $\wp_i$ is the prime ideal corresponding to P_i, is called a **fat point scheme** *with* **support** *$\mathbb{X}$ and* **multiplicities** *$m_1, \ldots, m_s$. We denote this fat point scheme by $\mathbb{W} = m_1P_1 + \cdots + m_sP_s$.*

Note that if each multiplicity is the same integer m, then the defining ideal $I_{\mathbb{W}}$ is the m-th symbolic power of the ideal defining $\mathbb{X}$. Also, if $m_1 = \cdots = m_s = m$ then we refer to $\mathbb{W}$ as a *homogeneous fat point scheme* and write $\mathbb{W} = m\mathbb{X}$, otherwise $\mathbb{W}$ is said to be *non-homogeneous*. If $m_1 = \cdots = m_s = 1$, then $\mathbb{X} = \mathbb{W}$ is called *simple*. The *degree* of $\mathbb{W}$ is given by $\deg(\mathbb{W}) = \sum_{i=1}^{s} \binom{m_i+n-1}{n}$.

We now recall the definition of special families of reduced point sets called partial intersections as it was first given by Maggioni and Ragusa in [20].

Definition 2.2 ([20]). *Fix two sets of lines of $\mathbb{P}^2$, say $\{H_i\}$ for $i = 1, \ldots, a$ and $\{L_j\}$ for $j = 1, \ldots, b$ such that no three of the lines have a common point and set $P_{i,j} = H_i \cap L_j$. Let $\underline{p} = (p_1, \ldots, p_r)$ and $\underline{q} = (q_1, \ldots, q_r)$ be two sets of r positive integers with $b = p_1 > \cdots > p_r > 0$, $q_1 + \cdots + q_r = a$.*

Furthermore, let $r(i) = inf\{s \in \mathbb{N} \mid \sum_{j=1}^{s} q_j \geq i\}$ *for* $i = 1, \ldots, a$. *With this notation, we define the* **partial intersection** *of type* $(\underline{p}, \underline{q})$, *denoted* $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$, *to be the set of points*

$$\mathbb{X}_{stairs}^{\underline{p},\underline{q}} = \{P_{i,j(i)} \mid i = 1, \ldots, a, j(i) = 1, \ldots, p_{r(i)}\}. \tag{2.1}$$

(Here $r(i)$ *takes the meaning of the subscript of the* q *corresponding to the line* H_i*.)*

Remark 2.3. *To better understand what a partial intersection is and how it can be visualized, we now provide an alternate description. Start with a distinct lines* $H_1, H_2, \ldots, H_a \subset \mathbb{P}^2$ *and b distinct lines* $L_1, L_2, \ldots, L_b \subset \mathbb{P}^2$ *such that no three of the* $a + b$ *lines have a point in common. Assume we have two lists of positive integers* $\underline{p} = (p_1, p_2, \ldots, p_r)$ *such that* $b = p_1 > p_2 > \cdots > p_r > 0$, *and* $\underline{q} = (q_1, q_2, \ldots, q_r)$ *such that* $q_1 + q_2 + \cdots + q_r = a$. *This gives ab points*

$$P_{i,j} = H_i \cap L_j.$$

Order the points on each line H_i *as* $P_{i,1}, P_{i,2}, \ldots, P_{i,b}$. *The lines* H_i *are also ordered as* $H_1, H_2, \ldots, H_a$.

Then $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$ *is the set of points where we take the first* p_1 *points on each of the first* q_1 *lines, then the first* p_2 *points on each of the next* q_2 *lines, and so on. Since* $q_1 + \cdots + q_r = a$, *this uses up all of the lines* $H_1, \ldots, H_a$, *and the number of points in* $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$ *is the dot product* $p \cdot q$. *We write (2.1) as*

$$\mathbb{X}_{stairs}^{\underline{p},\underline{q}} = \bigcup_{i=1}^{a} \{P_{i,j} \ : \ j = 1, \ldots, p_{r(i)}\} = \bigcup_{k=1}^{r} \bigcup_{i=1+\sum_{s<k} q_s}^{\sum_{s\leq k} q_s} \{P_{i,j} : j = 1, \ldots, p_k\}.$$

Thus, we can visualize the points of a partial intersection $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$ *in a configuration as being made up of r levels of blocks as in Figure 1.*

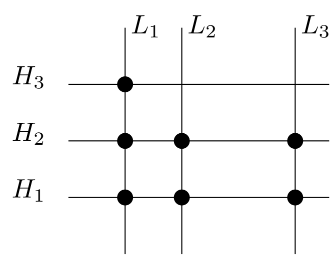

FIGURE 1. The partial intersection $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$ with $\underline{p} = (3, 1)$ and $\underline{q} = (2, 1)$.

Remark 2.4. *Note that the choice of the two sets of lines* H_i *and* L_j *can affect the underlying geometry of the partial intersection. We can construct a partial intersection choosing the set of lines to be "general" such that no three points lie on a diagonal, and in this case we call the scheme a "general partial intersection". Or we can choose the set of lines such that the partial intersection can be drawn as a "lattice of points" where we allow three or more points to be on diagonals. In this case we refer to the scheme as a "standard partial intersection". In this paper, the methods used in Sections 3 and 4 work for both types of partial intersections. Hence, we simply use the term "partial intersection".*

Partial intersections are known to exhibit nice properties. For example, work of Maggioni and Ragusa [20] completely describes the minimal free resolution of a partial intersection.

Lemma 2.5 (Maggioni, Ragusa)**.** *Let* $\mathbb{X}_{stairs}^{\underline{p},\underline{q}}$ *be a partial intersection of type* $(\underline{p}, \underline{q})$, *with* $\underline{p} = (p_1, \ldots, p_r)$ *and* $\underline{q} = (q_1, \ldots, q_r)$. *Set* $b = p_1$ *and* $a = \sum_{i=1}^{r} q_i$. *Then* $I_{\mathbb{X}_{stairs}^{\underline{p},\underline{q}}}$ *has a minimal free resolution of the* *following form*

$$0 \to \bigoplus_{t=1}^{r} R(-b_t) \to \bigoplus_{t=0}^{r} R(-a_t) \to R \to R/I_{\mathbb{X}^{\underline{p},\underline{q}}_{stairs}} \to 0$$

where

- $a_0 = a$;
- $a_t = p_t + \sum_{k=0}^{t-1} q_k$ *(with* $q_0 = 0$*) for* $t = 1, \ldots, r$*; and*
- $b_t = p_t + \sum_{k=1}^{t} q_k$ *for* $t = 1, \ldots, r$.

Proof. See [20, Proposition 2.1]. □

With the minimal free resolutions, and hence initial degree invariants, determined for partial intersections, it is natural to study their regular and symbolic powers. In particular, we ask:

Question 1. *What can we say about the Waldschmidt constant for partial intersections in* $\mathbb{P}^2$*?*

As previously stated, in this paper we give an answer in the case that the partial intersection is of type $(\underline{p}, \underline{q})$, with $\underline{p} = (3, 1)$ and $\underline{q} = (2, 1)$. In particular, we show two approaches on how to compute its Waldschmidt constant. In forthcoming work, we generalize this investigation to a larger family of partial intersections.

Notation 2.6. *From now on, we denote by* $\mathbb{X}$ *the partial intersection* $\mathbb{X}^{\underline{p},\underline{q}}_{stairs}$ *of type* $(\underline{p}, \underline{q})$*, with* $\underline{p} = (3, 1)$ *and* $\underline{q} = (2, 1)$*. We denote by* $I_{m\mathbb{X}}$ *the ideal associated to the homogeneous fat point scheme* $m\mathbb{X}$ *whose support is* $\mathbb{X}$*. In this case we have that* $I_{\mathbb{X}}^{(m)} = I_{m\mathbb{X}}$*.*

3. Fixed Components Approach

We recall some known results that are useful in determining the Waldschmidt constant of the defining ideal of a point set $\mathbb{Y}$ in $\mathbb{P}^2$.

Lemma 3.1 ([6, Lemma 2.7])**.** *Let* $m, d \in \mathbb{N}$*. Let* $P_1, \ldots, P_5 \in \mathbb{P}^2$ *be points lying on an irreducible plane conic* $\mathcal{C}$ *and let* $\mathbb{Y} \subset \mathbb{P}^2$ *be the scheme* $mP_1 + \cdots + mP_5$ *with* $m \geq 1$*. Let*

$$\tau = \max\{5m - 2d, 0\}.$$

(i) *If* $d \geq 2\tau$*, then the plane conic* $\mathcal{C}$ *is a fixed component of multiplicity at least* τ *for the curves defined by the forms of the ideal* $[I_{\mathbb{Y}}]_d$*.*
(ii) *If* $d < 2\tau$*, then* $[I_{\mathbb{Y}}]_d = \{0\}$*.*

Lemma 3.2 ([5, Lemma 2.5])**.** *Let* $m_1, \ldots, m_s$ *and* d *be positive integers and let* $P_1, \ldots, P_s$ *be* s *points in* $\mathbb{P}^2$ *lying on a line* $\mathcal{L}$ *with* $s > 1$*. Let* $\mathbb{Y}$ *be the scheme* $m_1P_1 + \cdots + m_sP_s$*. Set*

$$\mu = \left\lceil \frac{m_1 + \cdots + m_s - d}{s - 1} \right\rceil, \tag{3.1}$$

and assume $[I_{\mathbb{Y}}]_d \neq \{0\}$*. Then*

(i) $\mu \leq d$*;*
(ii) *the line* $\mathcal{L}$ *is a fixed component of multiplicity at least* μ *for the plane curves of degree* d *defined by the forms of the ideal* $[I_{\mathbb{Y}}]_d$*.*

Our computation of the Waldschmidt constant $\widehat{\alpha}(I_{\mathbb{X}})$, where $\mathbb{X}$ is from Notation 2.6, is structured according to the following method described in [5, Section 3].

Step 1. We look for a curve $\mathcal{F}$ of degree d, which contains each point of $\mathbb{X}$ with multiplicity exactly ν, so that, for each $m > 0$, $m\mathcal{F}$ is a curve in the linear system $\left[I_{m\nu\mathbb{X}}\right]_{md}$ and so $\left[I_{m\nu\mathbb{X}}\right]_{md} \neq \{0\}$.

Step 2. We prove, by contradiction, that $\left[I_{m\nu\mathbb{X}}\right]_{md-1} = \{0\}$ for each $m \geq 1$. For this purpose we define

$$\overline{m} = \min\{m \,:\, [I_{m\nu\mathbb{X}}]_{md-1} \neq \{0\}\}.$$

We prove, mostly directly, that $\overline{m} \neq 1$. For $\overline{m} > 1$, applying Lemma 3.2 several times, we show that $\mathcal{F}$ is a fixed component for the linear system $\left[I_{\overline{m}\nu\mathbb{X}}\right]_{\overline{m}d-1}$. Thus, by removing $\mathcal{F}$, we get

$$\dim\left[I_{\overline{m}\nu\mathbb{X}}\right]_{\overline{m}d-1} = \dim\left[I_{\overline{m}\nu\mathbb{X}-\mathcal{F}}\right]_{\overline{m}d-1-d}$$

and, since $\mathcal{F}$ contains each point of $\mathbb{X}$ with multiplicity exactly ν, we have

$$\left[I_{\overline{m}\nu\mathbb{X}-\mathcal{F}}\right]_{\overline{m}d-1-d} = \left[I_{(\overline{m}-1)\nu\mathbb{X}}\right]_{(\overline{m}-1)d-1}.$$

The contradiction follows from the minimality of $\overline{m}$.

Step 3. Since the initial degree of $\left[I_{m\nu\mathbb{X}}\right]$ is md, by [5, Lemma 2.2], we have

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{d}{\nu}.$$

We are now ready to determine $\widehat{\alpha}(I_{\mathbb{X}})$.

Theorem 3.3. *If $\mathbb{X}$ is a partial intersection as in Notation 2.6 with associated ideal $I_{\mathbb{X}}$, then the Waldschmidt constant is*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{5}{2}.$$

Proof. Set $P_{ij} = H_i \cap L_j$. We follow the method as previously described.

Step 1. We claim that for all $m \geq 1$ there exists a curve $\mathcal{C}_{5m} \in [I_{(2m\mathbb{X})}]_{5m}$.

Consider the fat point scheme where each point has multiplicity 2, that is $2\mathbb{X}$. It is known that there exists a unique irreducible conic $\mathcal{C}_2 \in [I_{\mathbb{Y}}]_5$ where

$$\mathbb{Y} := P_{31} + P_{22} + P_{23} + P_{12} + P_{13}$$

is described in Figure 2 (we also could use [5, Lemma 2.7]). The curve

$$\mathcal{F} := H_1 + H_2 + L_1 + \mathcal{C}_2$$

contains each point of $\mathbb{X}$ with multiplicity exactly 2 and hence, for all $m > 0$ we have

$$m\mathcal{F} \in [I_{(2m\mathbb{X})}]_{5m}.$$

Step 2. We now prove that for each $m > 0$, $[I_{(2m\mathbb{X})}]_{5m-1} = \{0\}$. We prove this claim by contradiction. To this aim, we will use Lemma 3.2 many times in order to get a fixed component for the curves defined by the forms of $[I_{(2m\mathbb{X})}]_{5m}$.

Assume that for some $m > 0$ we have $[I_{(2m\mathbb{X})}]_{5m-1} \neq \{0\}$. Applying Lemma 3.2 H_1 is a fixed component of multiplicity at least 1 for the plane curves of the linear system $[I_{(2m\mathbb{X})}]_{5m-1}$. By removing mH_1 for those curves, we get that

$$\dim[I_{(2m\mathbb{X})}]_{5m-1} = \dim[I_{(2m\mathbb{X})-mH_1}]_{5m-1-m} = \dim[I_{(2m\mathbb{X})-mH_1}]_{4m-1}.$$

If the above dimension is zero, we have a contradiction. If the above dimension is non-zero, then H_2 is a fixed component of multiplicity at least 1 for the plane curves of the linear system $[I_{(2m\mathbb{X})-mH_1-mH_2}]_{4m-1}$. By removing mH_2 for those curves, we have

$$\dim[I_{(2m\mathbb{X})}]_{5m-1} = \dim[I_{(2m\mathbb{X})-mH_1-mH_2}]_{3m-1}.$$

If the above dimension is zero, we obtain a contradiction. If the above dimension is different from zero, then L_1 is a fixed component of multiplicity at least 1 for the plane curves of the linear system $[I_{(2m\mathbb{X})-mH_1-mH_2}]_{3m-1}$.

We get that

$$\dim[I_{(2m\mathbb{X})-mH_1-mH_2}]_{3m-1} = \dim[I_{(2m\mathbb{X})-mH_1-mH_2-mL_1}]_{2m-1}.$$

We observe that the residual scheme $\mathbb{Y}' = (2m\mathbb{X}) - mH_1 - mH_2 - mL_1$ is described as in Figure 2 where each point has multiplicity m.

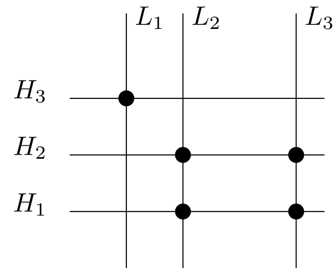

Figure 2. The scheme $\mathbb{Y}$.

Applying Lemma 3.1 we have that the unique irreducible conic through the five points has multiplicity $\tau = \max\{5m-2(2m-1), 0\} = m+2$. Since it is always true that $2m-1 < 2(m+2)$ we have that $\dim[I_{(2m\mathbb{X})}]_{5m-1} = \dim[I_{\mathbb{Y}'}]_{2m-1} = 0$, a contradiction.

Step 3. Since the initial degree of $\left[I_{2m\mathbb{X}}\right]$ is $5m$, by [5, Lemma 2.2] we have

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{5}{2}. \qquad \square$$

4. Reduction Approach

In this section, we use a generalization of a reduction process introduced in [8] that results in lower and upper bounds to the Hilbert function of the ideal of a fat point scheme. The original approach involves iteratively applying *Castelnuovo sequences* obtained by restricting the ideal sheaf of a fat point scheme $\mathbb{W}$ supported on a union of points $\mathbb{X} \subset \mathbb{P}^n$ to a hyperplane, then taking the kernel sheaf of such a sequence and restricting it to another hyperplane, etc. Taking cohomology, one obtains an upper bound to the number of global sections of the ideal sheaf tensored with $\mathcal{O}_{\mathbb{P}^n}(t)$, for any positive integer t, that is, an upper bound for $h^0(\mathbb{P}^n, \mathcal{I}_{\mathbb{W}}(t))$.

In this note, we consider the fat point scheme $\mathbb{W} = m\mathbb{X}$, where $\mathbb{X} \subseteq \mathbb{P}^2$ is a partial intersection from Notation 2.6. The generalization consists of restricting not only to lines but also to a smooth conic. Specifically, starting from the point configuration $\mathbb{X}$ of Figure 1, we will iterate restrictions on the following curves: the three lines H_1, H_2, L_1 and the smooth conic C determined by the five points marked in Figure 2. In this first example, the reduction procedure is similar to the approach used in Theorem 3.3 and is essentially an application of Bézout's Theorem. However, it will differ for more complicated partial intersections. This idea will be worked out and a general criterion, similar to that of [8], will be explored in forthcoming work through reductions to curve configurations consisting of lines and irreducible conics.

4.1. **Reduction in the even case.**

First, assume that $m = 2k$, for $k \geq 1$ and consider the scheme $\mathbb{W}_0 := \mathbb{W} = m\mathbb{X}$. Consider the following periodic sequence consisting of $4k$ curves:

$$l_1, \ldots, l_{4k} = (H_1, H_2, L_1, C)^k = H_1, H_2, L_1, C, H_1, H_2, L_1, C, \ldots, H_1, H_2, L_1, C.$$

For $j = 1, \dots, 4k$, denote with $\overline{\mathbb{W}}_{j-1} := \mathbb{W}_{j-1} \cap l_j$ the scheme theoretic intersection and let $\mathbb{W}_j$ be the residual scheme. For every $i = 0, \dots, k-1$, we consider the following Castelnuovo sequences:

$$\begin{array}{llll}
0 \to \mathcal{I}_{\mathbb{W}_{4i+1}}(t-5i-1) & \to \mathcal{I}_{\mathbb{W}_{4i}}(t-5i) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4i},l_{4i+1}}(t-5i) & \to 0 \\
0 \to \mathcal{I}_{\mathbb{W}_{4i+2}}(t-5i-2) & \to \mathcal{I}_{\mathbb{W}_{4i+1}}(t-5i-1) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4i+1},l_{4i+2}}(t-5i-1) & \to 0 \\
0 \to \mathcal{I}_{\mathbb{W}_{4i+3}}(t-5i-3) & \to \mathcal{I}_{\mathbb{W}_{4i+2}}(t-5i-2) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4i+2},l_{4i+3}}(t-5i-2) & \to 0 \\
0 \to \mathcal{I}_{\mathbb{W}_{4i+4}}(t-5i-5) & \to \mathcal{I}_{\mathbb{W}_{4i+3}}(t-5i-3) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4i+3},l_{4i+4}}(2(t-5i-3)) & \to 0
\end{array}$$

Notice that $\mathbb{W}_{4i+4} = (m-i-1)\mathbb{X}$. In particular, if $i = k-1$ we obtain $\mathbb{W}_{4k} = \emptyset$. Hence, after the iterated application of the $4k$ sequences, we will have completely reduced the scheme $\mathbb{W}$ to the empty set and the kernel of the $4k$-th sequence is just a twisting sheaf on $\mathbb{P}^2$:

$$\mathcal{I}_{\mathbb{W}_{4k}}(t-5k) = \mathcal{O}_{\mathbb{P}^2}(t-5k).$$

Taking cohomology, we obtain an upper bound for the dimension of the space of global sections of $\mathcal{I}_{\mathbb{W}_0}(t)$, for every t:

$$\begin{aligned}
h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(t)) \leq\ & h^0(\mathcal{O}_{\mathbb{P}^2}(t-5k)) + \\
& + \sum_{i=0}^{k-1} \Big(h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i},H_1}(t-5i)) + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+1},H_2}(t-5i-1)) + \\
& \qquad + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+2},L_1}(t-5i-2)) + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+3},C}(2(t-5i-3))) \Big).
\end{aligned}$$

Recalling that for a rational curve $l_i \cong \mathbb{P}^1$, such as a line H_1, H_2, L_1 or a conic C, the number of global sections of the sheaf $\mathcal{I}_{\mathbb{Y}}(\delta)$ on l_i, where $\mathbb{Y}$ is a fat point scheme of degree μ, is

$$h^0(\mathbb{P}^1, \mathcal{I}_{\mathbb{Y}}(\delta)) = \max\{0, \delta - \mu + 1\} = \binom{\delta - \mu + 1}{1},$$

we can rewrite the above bound as:

$$\begin{aligned}
h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(t)) \leq\ & \binom{t-5k+2}{2} + \\
& + \sum_{i=0}^{k-1} \left(\binom{t-5i-\deg(\overline{\mathbb{W}}_{4i})+1}{1} + \binom{t-5i-\deg(\overline{\mathbb{W}}_{4i+1})}{1} + \right. \\
& \left. + \binom{t-5i-1-\deg(\overline{\mathbb{W}}_{4i+2})}{1} + \binom{2(t-5i-3)-\deg(\overline{\mathbb{W}}_{4i+3})+1}{1} \right).
\end{aligned} \tag{4.1}$$

By construction, for $i = 0, \dots, k-1$, we have

$$\begin{aligned}
\deg(\overline{\mathbb{W}}_{4i}) &= 3(m-i) = 6k - 3i, \\
\deg(\overline{\mathbb{W}}_{4i+1}) &= 3(m-i) = 6k - 3i, \\
\deg(\overline{\mathbb{W}}_{4i+2}) &= (m-i) + 2(m-i-1) = 6k - 3i - 2, \\
\deg(\overline{\mathbb{W}}_{4i+3}) &= 5(m-i-1) = 10k - 5i - 5.
\end{aligned}$$

One can now easily check that for every value of $t \leq 5k$, all terms in the summation of (4.1) vanishes. In particular we obtain that

$$\begin{aligned}
h^0(\mathcal{I}_{\mathbb{W}}(t)) &= 0, \text{ if } t < 5k, \\
h^0(\mathcal{I}_{\mathbb{W}}(t)) &\leq 1, \text{ if } t = 5k.
\end{aligned}$$

Moreover, from the exact sequences in cohomology, one obtains a lower bound:

$$\begin{aligned}
h^0(\mathcal{I}_{\mathbb{W}}(5k)) &\geq h^0(\mathcal{I}_{\mathbb{W}_{4i}}(t-5i)) \\
&\geq h^0(\mathcal{I}_{\mathbb{W}_{4i+1}}(5k-5i-1)) \\
&\geq h^0(\mathcal{I}_{\mathbb{W}_{4i+2}}(5k-5i-2)) \\
&\geq h^0(\mathcal{I}_{\mathbb{W}_{4i+3}}(5k-5i-3)) \\
&\geq h^0(\mathcal{I}_{\mathbb{W}_{4i+4}}(5k-5i-5)) \\
&\geq \cdots \\
&\geq h^0(\mathcal{I}_{\mathbb{W}_{4k}}(5k-5k)) \\
&= 1.
\end{aligned}$$

We conclude that $t = 5k$ is the minimum degree for which $h^0(\mathcal{I}_{\mathbb{W}}(t)) \neq 0$ for the fat point scheme $\mathbb{W} = 2k\mathbb{X}$.

4.2. **Reduction in the odd case.** Assume that $m = 2k+1$, with $k \geq 0$, and consider the fat point scheme $\mathbb{W}_0 := \mathbb{W} = m\mathbb{X}$. We consider the following sequence of curves:

$$l_1, \dots, l_{4k+3} = (H_1, H_2, L_1, C)^k, H_1, H_2, L_1.$$

For $j = 1, \dots, 4k+3$, let $\overline{\mathbb{W}}_{j-1} := \mathbb{W}_{j-1} \cap l_j$ be the scheme theoretic intersection, and let $\mathbb{W}_j$ denote the residual scheme. After applying the same series of $4k$ Castelnuovo sequences as in the even case, we obtain as the last kernel the sheaf $\mathcal{I}_{\mathbb{W}_{4k}}(t-5k)$. Like in the even case, observe that $\mathbb{W}_{4i+4} = (m-i-1)\mathbb{X}$, but in this case $\mathbb{W}_{4k} = \mathbb{X}$. We proceed with the reduction by means of the last three lines in the sequence:

$$\begin{array}{llll}
0 \to \mathcal{I}_{\mathbb{W}_{4k+1}}(t-5k-1) & \to \mathcal{I}_{\mathbb{W}_{4k}}(t-5k) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4k},H_1}(t-5k) & \to 0 \\
0 \to \mathcal{I}_{\mathbb{W}_{4k+2}}(t-5k-2) & \to \mathcal{I}_{\mathbb{W}_{4k+1}}(t-5k-1) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4k+1},H_2}(t-5k-1) & \to 0 \\
0 \to \mathcal{I}_{\mathbb{W}_{4k+3}}(t-5k-3) & \to \mathcal{I}_{\mathbb{W}_{4k+2}}(t-5k-2) & \to \mathcal{I}_{\overline{\mathbb{W}}_{4k+2},L_1}(t-5k-2) & \to 0
\end{array}$$

Notice that $\mathbb{W}_{4k+3} = \emptyset$, so that

$$\mathcal{I}_{\mathbb{W}_{4k+3}} = \mathcal{O}_{\mathbb{P}^2}(t-5k-3).$$

Like in the even case, we obtain an upper bound for the dimension of the space of global sections of $\mathcal{I}_{\mathbb{W}_0}(t)$, for every t:

$$\begin{aligned}
h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(t)) \leq{}& h^0(\mathcal{O}_{\mathbb{P}^2}(t-5k-3)) + \\
&+ \sum_{i=0}^{k-1} \Big(h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i},H_1}(t-5i)) + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+1},H_2}(t-5i-1)) + \\
&\qquad + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+2},L_1}(t-5i-2)) + h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4i+3},C}(2(t-5i-3))) \Big) \\
&+ h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4k},H_1}(t-5k)) \\
&+ h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4k+1},H_2}(t-5k-1)) \\
&+ h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4k+2},L_1}(t-5k-2)).
\end{aligned}$$

It is easy to check that all terms on the right-hand side of the above inequality vanish if $t < 5k+3$, while for $t = 5k+3$, omitting the vanishing terms, the above inequality becomes

$$\begin{aligned}
h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(5k+3)) \leq{}& h^0(\mathcal{O}_{\mathbb{P}^2}) + \\
&+ \sum_{i=1}^{k-1} h^0(\mathcal{I}_{\overline{W}_{4i+3},C}(2(t-5i-3))) \\
&+ h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4k},H_1}(3)) \\
&+ h^0(\mathcal{I}_{\overline{\mathbb{W}}_{4k+2},L_1}(1)).
\end{aligned}$$

The degrees of $\overline{\mathbb{W}}_{4i+3}$, $\overline{\mathbb{W}}_{4k}$ and $\overline{\mathbb{W}}_{4k+2}$ are respectively equal to $10(k-i)$, 3 and 1, so we conclude that

$$h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(5k+3)) \leq k+3.$$

Since the lower bound

$$h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(t)) \geq h^0(\mathcal{I}_{\mathbb{W}_{4k+3}}(t-5k-3)) = h^0(\mathcal{O}_{\mathbb{P}^2}(t-5k-3))$$

always holds, for $t = 5k+3$ we obtain

$$h^0(\mathbb{P}^2, \mathcal{I}_{\mathbb{W}_0}(5k+3)) \geq 1.$$

We conclude that $t = 5k+3$ is the minimum integer for which $h^0(\mathcal{I}_{\mathbb{W}}(t)) \neq 0$ for the fat point scheme $\mathbb{W} = (2k+1)\mathbb{X}$.

Remark 4.1. *If $\mathbb{X}$ is a partial intersection from Notation 2.6, then the above $m = 2k$ or $m = 2k+1$ cases prove that*

$$\widehat{\alpha}(I_{\mathbb{X}}) = \frac{5}{2}.$$

4.3. **Reduction vector and graphical representation.** We now present a graphical method for determining the upper bounds (4.1) for $h^0(\mathcal{I}_{\mathbb{W}}(t))$, for any $t \geq 1$, generalizing the idea of [8].

In the even case, for the fat point scheme $\mathbb{W} = 2k\mathbb{X}$ and the sequence of curves $l_1, \ldots, l_{4k}$, we associate a *reduction vector* $\mathbf{d} = (d_1, \ldots, d_{4k})$, where for every $j = 1, \ldots, 4k$ we set $d_j = \deg(\overline{\mathbb{W}}_{j-1})$. From this reduction vector, we construct a configuration consisting of a subset of the leftmost lattice points in the first quadrant of $\mathbb{Z} \times \mathbb{Z}$. This configuration takes into account whether l_j is a line or a conic. For $1 \leq j \leq 4k$, let ℓ_j be the largest positive integer such that $4\ell_j - 3 \leq j \leq 4\ell_j$. If l_j is a line, then we include in the configuration the integer lattice points $(i, j+\ell_j-2)$ with $0 \leq i < d_j$. If l_j is a conic, then we include in the configuration the integer lattice points $(i, j+\ell_j-2)$ with $0 \leq i \leq \lceil \frac{d_j}{2} \rceil - 1$ and $(i, j+\ell_j-1)$ with $0 \leq i \leq \lfloor \frac{d_j}{2} \rfloor - 1$. The configurations are shown in Figure 3 for $m = 2, 4$ and 6. The empty circles represent the lattice points resulting when a line is used in the reduction, while the circles with dots in the middle represent the lattice points resulting when a conic was used in the reduction. Now intersecting the lattice points with lines of slope -1 and counting these lines starting from 0, we observe that the first line of slope -1 that contains lattice points not in the constructed configuration corresponds to the least degree in which $I_{\mathbb{W}}$ has a non-zero element.

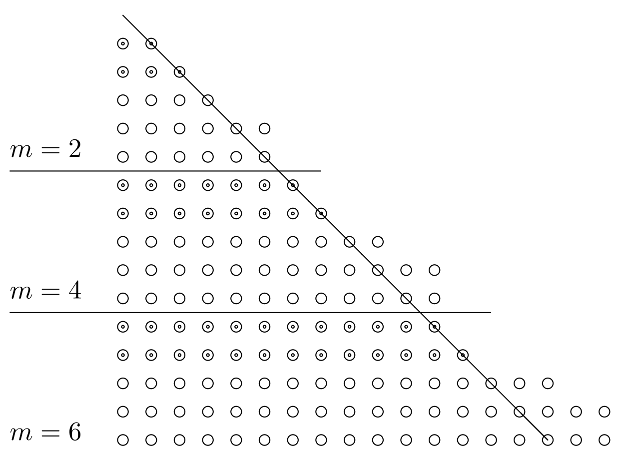

FIGURE 3. Representing the reduction vector for $m = 2, 4, 6$.

Similarly, in the odd case, to the fat point scheme $\mathbb{W} = (2k+1)\mathbb{X}$ and to the sequence of curves $l_1, \ldots, l_{4k+3}$, we associate a *reduction vector* $\mathbf{d} = (d_1, \ldots, d_{4k+3})$, where for every $j = 1, \ldots, 4k+3$ we set $d_j = \deg(\overline{\mathbb{W}}_{j-1})$. Figure 4 displays the associated configurations for $m = 1, 3$ and 5.

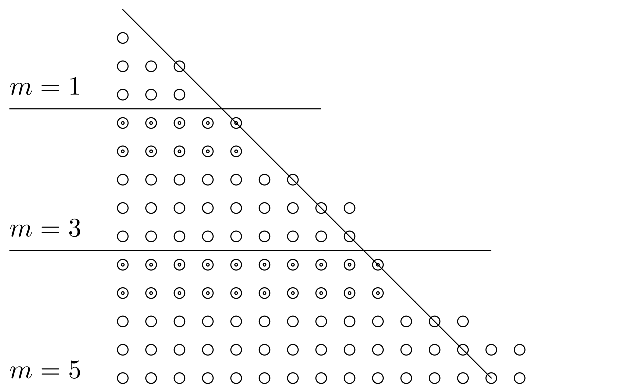

FIGURE 4. Representing the reduction vector for $m = 1, 3, 5$.

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