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A Priori Estimates for the Motion of Charged Liquid Drop: A Dynamic Approach via Free Boundary Euler Equations

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Abstract. We study the motion of charged liquid drop in three dimensions where the equations of motions are given by the Euler equations with free boundary with an electric field. This is a well-known problem in physics going back to the famous work by Rayleigh. Due to experiments and numerical simulations one may expect the charged drop to form conical singularities called Taylor cones, which we interpret as singularities of the flow. In this paper, we study the well-posedness of the problem and regularity of the solution. Our main theorem is a criterion which roughly states that if the flow remains $C^{1,\alpha}$-regular in shape and the velocity remains Lipschitz-continuous, then the flow remains smooth, i.e., C^∞ in time and space, assuming that the initial data is smooth. Our main focus is on the regularity of the shape of the drop. Indeed, due to the appearance of Taylor cones, which are singularities with Lipschitz-regularity, we expect the $C^{1,\alpha}$-regularity assumption to be optimal. We also quantify the C^∞-regularity via high order energy estimates which, in particular, implies the well-posedness of the problem.

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Contents

Birkhäuser

1. Introduction and the Main Result

1.1. State-of-the-Art

In this paper we study the problem of charged liquid drop from rigorous mathematical point of view. In the model the two effecting forces are the surface tension, which prefers to keep the drop spherical, and the repulsive electrostatic force, which both act on the boundary of the drop. The problem is well-known and goes back to Rayleigh [48] who studied the linear stability of the sphere and showed that the sphere becomes unstable when the total electric charge is above a given threshold. When the total electric charge is above this Rayleigh threshold, the drop begins to elongate and may eventually form a conical singularity at the tip with a certain opening angle. Such singularities are called *Taylor cones* due to the work by Taylor [52] and the numerical and experimental evidence suggest that the charged drop typically forms such a singularity [24,45,52,61]. In this paper our goal is to study the well-posedness of the problem and the regularity of the solution. We refer to [46] and [30] for an introduction to the topic.

The static problem of charged liquid drop can be seen as a nonlocal isoperimetric problem and it has been studied from the point of view of Calculus of Variations in recent years [28,29,37,46]. The main issue is that the associated minimization problem, formulated in the framework of Calculus of Variations, is not well-posed, in the sense that the problem does not have a minimizer [28,46]. Even more surprising is that the results in [28,46] show that even if the total electric charge is below the Rayleigh threshold, the sphere is not a local minimizer of the associated energy. This means that the electrostatic term is not lower order with respect to the surface tension, which makes the problem mathematically challenging. In order to make the variational problem well-posed one may restrict the problem to convex sets [29] or regularize the functional by adding a curvature term [30] which could lead to the existence of minimizer as the result in [27] suggests.

Here we study this problem from the point of view of fluid-dynamics, which is the framework studied e.g. in [24], where the authors derive the PDE system in the irrotational case (see also [5]). Indeed, as it is observed in [24] the problem is by nature evolutionary, where the drop deforms as a function of time given by the Euler equations with the surface of the drop being the free boundary, which law of motions is coupled with the system which we give in (1.3) below. The problem can thus be seen as the Euler equations for incompressible fluids with free boundary with an additional term given by the electric field. The Euler equations with free boundary without the electric field has been studied rather extensively in recent years. We give only a brief overview on this challenging problem below and refer to [12,44] for more detailed introduction to the topic. Regarding the problem with electric field we mention the recent works by Yang [59,60] and Wang-Yang [55], where the authors study the case of the water-wave problem. We also mention the work [20], where the authors study the Stokes flow associated with the charged liquid drop near the sphere and show that under smallness assumption the flow is well defined.

We stress that in our case it is crucial to include the surface tension in the model since otherwise the problem might be ill-posed. For the problem without the electric field one may study the Euler equations also without the surface tension, when one assumes the so called Rayleigh-Taylor sign condition [17], which one should not confuse with the Rayleigh threshold mentioned above. For the water-wave problem the well-posedness is proven by Wu [56,57] and the general case is due to Lindblad [40], see also [3,38]. Concerning the problem with surface tension, which is closer to ours, the short time existence of solution in the irrotational case for starshaped sets is due to Beyer-Günther [6,7] and the general case is proven by Coutand-Shkoller [11]. We also mention the earlier works concerning the well-posedness of the problem in the planar case [2,34,58]. The works that are closest to ours are Shatah-Zheng [50] and Schweizer [49], where the authors prove regularity estimates for the free boundary Euler equations with surface tension. Our work is also inspired by Masmoudi-Rousset [44], where the authors prove similar estimates for the Euler equations without the surface tension.

As usual with geometric evolution equations, the Euler equations with free boundary may develop singularities in finite time. In [13] the authors construct an example where the equations develop singularities where the drop changes its topology. We stress that in the absence of the electric field, we do not expect the flow to develop conical singularities predicted by Taylor [52], where both the curvature and the velocity become singular. Indeed, Taylor cones are special type of singularities as the evolving sets Ω_t do not change their topology, but only lose their regularity. We also point out that the analysis in [52] does not give a rigorous mathematical proof for the fact that the Taylor cone is a singularity of the associated flow. Indeed, since there is no monotonicity formula for the Euler equations, similar to the one by Huisken [33,43] for the mean curvature flow, there is little hope to have general classification of the singularities at the moment. We refer to [21] and [23] which both study critical points of energy functionals, which are very much similar to ours, with conical or cusp-like singularities. Then again, as we interpret the Taylor cone as a singularity of the flow given by the Euler equations (1.3), it is not clear why the singularity is a critical point of the potential energy.

1.2. Statement of the Main Theorem

We study the motion of an incompressible charged drop in vacuum in $\mathbb{R}^3$ and denote the fluid domain by Ω_t. We assume that we have an initial smooth and compact set $\Omega_0 \subset \mathbb{R}^3$ and a smooth initial velocity field $v_0 : \Omega_0 \to \mathbb{R}^3$ which evolve to a smooth family of sets and vector fields $(\Omega_t, v(\cdot,t))_{t\in[0,T)}$. The total energy is given by

$$J_t(\Omega_t, v) = \frac{1}{2}\int_{\Omega_t} |v(x,t)|^2\, dx + \mathcal{H}^2(\partial\Omega_t) + \frac{Q}{\mathrm{Cap}(\Omega_t)}, \tag{1.1}$$

where $Q > 0$ and $\mathrm{Cap}(\Omega)$ is the electrostatic capacity given by

$$\mathrm{Cap}(\Omega) := \inf\Big\{ \int_{\mathbb{R}^3} \frac{1}{2}|\nabla u|^2\, dx : \, u(x) \geq 1 \text{ for all } x \in \Omega\,, u \in \dot{H}^1(\mathbb{R}^3)\Big\},$$

and by $\mathcal{H}^2(\partial\Omega)$ we denote the two dimensional Hausdorff measure of the set $\partial\Omega$. Define the norm $\|u\|_{\dot{H}^1(\mathbb{R}^3)} = \|\nabla u\|_{L^2(\mathbb{R}^3)} + \|u\|_{L^6(\mathbb{R}^3)}$. We denote the capacitary potential as $U_\Omega \in \dot{H}^1(\mathbb{R}^3)$ which is the function for which $\mathrm{Cap}(\Omega) = \int_{\mathbb{R}^3} \frac{1}{2}|\nabla U_\Omega|^2\, dx$. This is equivalent to say that $U_\Omega \in \dot{H}^1(\mathbb{R}^3)$ satisfies

$$\begin{cases} -\Delta U_\Omega = 0 & \text{in } \mathbb{R}^3 \setminus \overline{\Omega} \\ U_\Omega = 1 & \text{on } \overline{\Omega}. \end{cases} \tag{1.2}$$

We denote the mean curvature of $\Sigma_t = \partial\Omega_t$ by H_{Σ_t}, which for us is the sum of the principal curvatures given by orientation via the outer normal ν_{Σ_t}. With this convention convex sets have positive mean curvature. As usual we denote the material derivative of a vector field F by

$$\mathcal{D}_t F = \partial_t F + (v\cdot\nabla)F.$$

The equations of motion are given by the Euler equations with free boundary (for the derivation see [24])

$$\begin{cases} \mathcal{D}_t v + \nabla p = 0 & \text{in } \Omega_t \\ \operatorname{div} v = 0 & \text{in } \Omega_t \\ v_n = V_t & \text{on } \Sigma_t = \partial\Omega_t \\ p = H_{\Sigma_t} - \frac{Q}{2C_t^2}|\nabla U_{\Omega_t}|^2 & \text{on } \Sigma_t, \end{cases} \tag{1.3}$$

where $C_t = \mathrm{Cap}(\Omega_t)$, V_t is the normal velocity, $v_n = v\cdot\nu$ and p is the pressure. We say that the system (1.3) has a smooth solution in time-interval $(0,T)$ with initial data (Ω_0, v_0), if there is a family of C^∞-diffeomorphisms $(\Phi_t)_{t\in[0,T)}$, which depend smoothly on t, such that $\Phi_0 = id$ and $\Phi_t(\Omega_0) = \Omega_t$, the functions $v(t,\Phi(t,x))$ and $p(t,\Phi(t,x))$ are smooth and the equations hold in the classical sense. Moreover we require that $v(t,\Phi(t,\cdot)) \to v_0$ as $t \to 0$, where $v_0 : \Omega_0 \to \mathbb{R}^3$ is the initial velocity field. When the total electric charge is zero, i.e. $Q = 0$, the system reduces to the more familiar Euler equations with free boundary with surface tension. We stress that formally it may seem that the term given by the electric field in the pressure is of lower order than the curvature. However, even for Lipschitz domains this naive intuition fails as we will observe in the beginning of Sect. 3.

The characteristic property of the solution of (1.3) is the conservation of the energy (1.1), i.e.,

$$\frac{d}{dt} J(\Omega_t, v) = 0,$$

which follows from straightforward calculation. Therefore one could guess, and we will prove this in our main theorem, that assuming that the flow given by the system (1.3) does not develop singularities, then it preserves the regularity of the initial data (Ω_0, v_0) at least in sense of certain Sobolev norm. In particular, we point out that, unlike the mean curvature flow [43], the flow given by the system (1.3) is not smoothing.

We parametrize the moving boundary $\Sigma_t = \partial\Omega_t$ by using a fixed reference surface Γ which we assume to be smooth and compact. We use the height function parametrization which means that for every t we associate the function $h(\cdot, t) : \Gamma \to \mathbb{R}$ with the moving boundary Σ_t as

$$\Sigma_t = \{x + h(x,t)\nu_\Gamma(x) : x \in \Gamma\}.$$

We assume that Γ satisfies the interior and exterior ball condition with radius $\eta > 0$ and note that η is not necessarily small. For example, from application point of view a relevant case is when the initial set is star-shaped in which case it is natural to choose the reference manifold to be a sphere in which case η is its radius. It is clear that the height-function parametrization is well defined as long as

$$\sup_{t\in[0,T)} \|h(\cdot,t)\|_{L^\infty(\Gamma)} < \eta.$$

Therefore we define the quantity

$$\sigma_T := \eta - \sup_{t\in[0,T)} \|h(\cdot,t)\|_{L^\infty(\Gamma)} \tag{1.4}$$

and the above condition reads as $\sigma_T > 0$.

As in [44,49,50] we note that we do not consider the existence in this paper. Instead, as in [49] we assume that the following qualitative short time existence result holds.

Throughout the paper we assume that for every smooth initial set and smooth initial velocity field the system (1.3) *yhas a smooth solution which exists a short interval of time.*

Since we will prove a priori estimates, we expect the existence to follow from an argument in the spirit of [51].

In this paper we are interested in finding a priori estimates which guarantee that the system (1.3) does not develop singularities. To this aim we fix a small $\alpha > 0$ and define

$$\Lambda_T := \sup_{t\in[0,T)} \left(\|h(\cdot,t)\|_{C^{1,\alpha}(\Gamma)} + \|\nabla v(\cdot,t)\|_{L^\infty(\Omega_t)} + \|v_n(\cdot,t)\|_{H^2(\Sigma_t)} \right). \tag{1.5}$$

We note that α can be any positive number. We could also replace the $C^{1,\alpha}$-norm by the $C^{1,\mathrm{Dini}}$-norm, but we choose to work with Hölder norms as the problem is already technically involved. Our goal is to show that if the quantity Λ_T is bounded then the flow can be extended beyond time T and is smooth if the initial data is smooth. We will prove this in a quantitative way and define an energy quantity of order $l \geq 1$ as

$$\hat{E}_l(t) := \sum_{k=0}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|v(\cdot,t)\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)}, \tag{1.6}$$

where $\lfloor \frac{3}{2}(l+1)\rfloor$ denotes the integer part of $\frac{3}{2}(l+1)$. We define the Hilbert space for half-integers $H^{\frac{3}{2}k}(\Omega_t)$ via extension in Sect. 2. In the last term we use a Hilbert space of integer order since it simplifies the calculations. The fact that the boundedness of $\tilde{E}_l(t)$ for every l implies the smoothness of the flow will be clear from the results in Sect. 8. Indeed, we first show that the bound on $\hat{E}_l(t)$ implies a bound for the pressure p. By the a priori estimate we know that the fluid domain remains $C^{1,\alpha}$-regular. We use this and estimates for harmonic functions to conclude that the bound on the pressure implies bound on the curvature (see Lemma 5.2), which then gives the regularity of the fluid domain Ω_t.

Our main result reads as follows. Recall that we assume that the reference surface $\Gamma \subset \mathbb{R}^3$ satisfies the interior and exterior ball condition with radius η.

Main Theorem. *Assume that Ω_0 is a smooth initial set which boundary satisfies $\partial\Omega_0 = \{x + h_0(x)\nu_\Gamma(x) : x \in \Gamma\}$ with $\|h_0\|_{L^\infty(\Gamma)} < \eta$ and let $v_0 \in C^\infty(\Omega_0; \mathbb{R}^3)$ be the initial velocity field. Assume that the system (1.3) has a smooth solution in time-interval $[0,T)$ and the parametrization satisfies*

$$\Lambda_T \leq M \quad \text{and} \quad \sigma_T \geq \frac{1}{M} \tag{1.7}$$

for some $M > 0$, where σ_T is defined in (1.4) and Λ_T in (1.5). Then for every $l \in \mathbb{N}$ there is a constant C_l, which depends on $M, l, \hat{E}_l(0)$, and on T if $T > 1$, such that the flow satisfies

$$\sup_{0<t<T} \hat{E}_l(t) \leq C_l,$$

where $\hat{E}_l(t)$ is defined in (1.6). In particular, the system (1.3) does not develop singularity at time T, but remains quantitatively smooth.

Moreover, there are $T_0 > 0$ and M, which depend on σ_0, i.e., σ_t at $t = 0$, $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$, $\|v\|_{H^3(\Omega_0)}$ and the $C^{1,\alpha}$-norm of h_0, such that the a priori estimates (1.7) hold for M up to time $\hat{T} = \min\{T, T_0\}$.

Let us make a few comments on the Main Theorem. First, from the point of view of the shape of the drop, the result says that if the parametrization of the flow remains $C^{1,\alpha}$-regular then the flow does not develop singularities. We expect this to be optimal in the sense that, we cannot relax the $C^{1,\alpha}$-regularity to Lipschitz regularity as the flow may create conical singularities as discussed before.

From the point of view of the velocity, the assumption on Lipschitz regularity of v, which is stronger than the boundedness of the curl v, is in the spirit of the Beale-Kato-Majda criterion and thus natural in the theory of the Euler equations [41]. Indeed, in the case when the drop does not chance its shape, i.e., $\Omega_t = \Omega_0$ the condition (1.7) reduces to

$$\sup_{t\in[0,T)} \|\nabla v(\cdot, t)\|_{L^\infty(\Omega_0)} < \infty,$$

which guarantees that the equations do not develop singularities by standard results for the Euler-equations [41]. Whether one may remove this condition is beyond our reach at the moment as the gradient level estimates are a fundamental problem in the theory of the Euler equations without the free boundary. The condition on the H^2-integrability of the normal component of the velocity v on the other hand is related to the fact that the boundary Σ_t is moving. We do not expect this to be optimal but again this problem is too involved for us to solve at the moment. Our main contribution to the problem is to find the optimal sufficient condition for the shape of the drop which guarantee that the flow is well-defined and provide the regularity estimates of all order l. For a drop without surface tension similar type of estimate is proven by Ginsberg [26] with an a priori assumption on the uniform curvature bound.

Finally the last statement of the Main Theorem says that the first statement is not empty, i.e., that the a priori estimates stay bounded up to time T_0, which depends on the initial data by requiring that $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$ and $\|v_0\|_{H^3(\Omega_0)}$ are bounded. We also note that since the regularity estimates in Main Theorem are quantitative, the result can be applied for non-smooth initial data by standard approximation. We note that all quantities in the paper depend of course on the chosen reference surface Γ even if it is not explicitly mentioned.

1.3. Overview of the Proof and the Structure of the Paper

As the paper is long we give a brief overview of the proof of the Main Theorem and of the structure of the paper. The proof is based on energy estimates and to that aim we define the energy functional of order $l \geq 1$ as

$$\mathcal{E}_l(t) = \frac{1}{2}\int_{\Omega_t} |\mathcal{D}_t^{l+1} v|^2\, dx + \frac{1}{2}\int_{\Sigma_t} |\nabla_\tau(\mathcal{D}_t^l v\cdot\nu)|^2\, d\mathcal{H}^2 - \frac{Q}{2C_t^2}\int_{\Omega_t^c} |\nabla(\partial_t^{l+1} U_{\Omega_t})|^2\, dx + \int_{\Omega_t} |\nabla^{\lfloor \frac{1}{2}(3l+1)\rfloor}(\operatorname{curl} v)|^2\, dx,$$

which is similar to the quantity in [49] defined on graphs. Here v is the velocity, $\mathcal{D}_t^l v$ is the material derivative of order l and $\lfloor \frac{1}{2}(3\,l+1)\rfloor$ denotes the largest integer smaller than $\frac{1}{2}(3l+1)$. Note that we need an additional term involving the time derivative of the capacitary potential U_{Ω_t}, as it appears as a high order term in the linearization of the pressure (see Lemma 4.7). This additional term causes problems as it is not immediately clear why the energy is positive or even bounded from below. We also define the associated energy quantity, where we include the spatial regularity

$$E_l(t) = \sum_{k=0}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}(1+1)\rfloor}(\Omega_t)} + \|\mathcal{D}_t^l v\cdot\nu\|^2_{H^1(\Sigma_t)} + 1.$$

Note that this quantity takes into account the natural scaling of the system (1.3), where time scales of order $\frac{3}{2}$ with respect to space as observed e.g. in [50].

We prove high order energy estimates by first showing that if Λ_T, σ_T satisfy (1.7) then for all $t < T$ it holds

$$\frac{d}{dt}\mathcal{E}_l(t) \leq C_l E_l(t) \tag{1.8}$$

for $l \geq 2$. The novelty of (1.8) is that the RHS has linear and not polynomial dependence on $E_l(t)$, which is crucial in order to show that the flow remains smooth as long as (1.7) holds. This makes the proof technically challenging as we need to estimate all nonlinear error terms in $\frac{d}{dt}\mathcal{E}_l(t)$ in an optimal way. We complete the argument by proving

$$E_l(t) \leq C_l(\tilde{C}_l + \mathcal{E}_l(t)) \tag{1.9}$$

which holds for $l \geq 1$. The inequalities (1.8) and (1.9) then imply the energy estimates for $l \geq 2$ and the quantitative C^∞-regularity of the flow.

We prove (1.8) and (1.9) by an induction argument over l, where the constants depend on $\sup_{t<T} E_{l-1}(t)$ which is bounded by the previous step. Therefore the first challenge is to start the argument and to bound $E_1(t)$. The issue is that (1.8) does not hold for $l = 1$. Instead, we show a weaker estimate

$$\frac{d}{dt}\mathcal{E}_1(t) \leq C_1(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t), \tag{1.10}$$

which we expect to be sharp. Therefore in order to start the induction argument we use an ad-hoc argument to show

$$\int_0^T \|p\|^2_{H^2(\Omega_t)}\, dt \leq C. \tag{1.11}$$

The inequalities (1.9), (1.10) and (1.11) then imply the first order energy estimate.

We show (1.11) by studying the function

$$\Phi(t) = -\int_{\Sigma_t} p\,\Delta_{\Sigma_t} v_n\, d\mathcal{H}^2,$$

where p is the pressure and Δ_{Σ_t} the Laplace-Beltrami operator, and prove that it holds

$$\frac{d}{dt}\Phi(t) \leq -\frac{1}{3}\|p\|^2_{H^2(\Omega_t)} + \text{lower order terms}.$$

We show that the a priori estimates (1.7) imply that Φ is bounded and thus we obtain (1.11) by integrating the above inequality over $(0, T)$. We point out that the low order energy estimate is the most challenging part of the proof as we have to work with domains with low regularity. We need rather deep results from differential geometry, boundary regularity for harmonic functions and elliptic regularity in order to overcome this problem. Let us finally outline the structure of the paper.

In Sect. 2 we introduce our notation. Due to the presence of the surface tension, the problem is geometrically involved and we need notation and tools from differential geometry to overcome these issues. We also define the function spaces that we need which include the Hilbert spaces in the domain $H^k(\Omega)$ and on the boundary $H^k(\Sigma)$ for half-integers $k = 0, \frac{1}{2}, 1, \dots$. We also recall functional inequalities such as interpolation inequality and Kato-Ponce inequality.

In Sect. 3 we prove div-curl type estimates in order to transform the high order energy estimates into regularity for the shape and the velocity. We first recall the result from [10] and prove its lower order version in Theorem 3.6. In Theorem 3.9 we prove sharp boundary regularity estimates for harmonic functions with Dirichlet boundary data by using methods from [22]. We believe that these two results are of independent interest.

In Sect. 4 we derive commutation formulas as in [50] and formula for the material derivatives of the pressure on the moving boundary. These formulas include four different error terms which we bound in Sect. 5. All the error terms have different structure and therefore we need to treat them one by one, which makes the Sect. 5 long. The further difficulty is due to the fact that the time and space derivatives have different scaling.

The core of the proof of the Main Theorem is in the next three sections. In Sect. 6 we prove (1.11), in Sect. 7 we prove (1.8) and (1.10), and in Sect. 8 we prove (1.9). The short final section then contains the proof of the Main Theorem.

2. Notation and Preliminary Results

In this section we introduce our notation and recall some basic results on function spaces and geometric inequalities. Many of these results are well-known for experts but we include them since they might be difficult to find, while some results we did not find at all in the existing literature. Throughout the paper C denotes a large constant, which value may change from line to line.

We first introduce notation related to Riemannian geometry. As an introduction to the topic we refer to [39]. We will always deal with compact hypersurfaces $\Sigma \subset \mathbb{R}^3$, which then can be seen as boundaries of sets Ω, i.e., $\partial\Omega = \Sigma$. We denote its outer unit normal by ν_Ω and denote it sometimes by ν_Σ or merely ν when its meaning is obvious from the context. We use the outward orientation and denote the second fundamental form by B_Σ and the mean curvature by H_Σ, which is defined as the sum of the principal curvatures. Again we write simply B and H when the meaning is clear from the context. We note that we use the convention in our notation that $\Sigma = \partial\Omega$ denotes a generic surface, $\Sigma_t = \partial\Omega_t$ denotes the evolving surface given by the equations (1.3) and $\Gamma = \partial G$ is our reference surface which we introduce later. We note that the constants in the paper will depend on the chosen reference surface. We take this for granted and do not mention it in the statements.

Since Σ is embedded in $\mathbb{R}^3$ it has natural metric g induced by the Euclidian metric. Then (Σ, g) is a Riemannian manifold and we denote the inner product on each tangent space $X, Y \in T_x\Sigma$ by $\langle X, Y\rangle$, which we may write in local coordinates as

$$\langle X, Y\rangle = g(X, Y) = g_{ij}X^iY^j.$$

We extend the inner product in a natural way for tensors. We denote smooth vector fields on Σ by $\mathscr{T}(\Sigma)$ and by a slight abuse of notation we denote smooth kth order tensor fields on Σ by $\mathscr{T}^k(\Sigma)$. We write X^i for vectors and Z_i for covectors in local coordinates.

We denote the Riemannian connection on Σ by $\bar{\nabla}$ and recall that for a function $u \in C^\infty(\Sigma)$ the covariant derivative $\bar{\nabla}u$ is a 1-tensor field defined for $X \in \mathscr{T}(\Sigma)$ as

$$\bar{\nabla}u(X) = \bar{\nabla}_X u = Xu,$$

i.e., the derivative of u in the direction of X. The covariant derivative of a smooth k-tensor field $F \in \mathscr{T}^k(\Sigma)$, denoted by $\bar{\nabla}F$, is a $(k+1)$-tensor field and we have the following recursive formula for $Y_1, \ldots, Y_k, X \in \mathscr{T}(\Sigma)$

$$\bar{\nabla} F(Y_1, \ldots, Y_k, X) = (\bar{\nabla}_X F)(Y_1, \ldots, Y_k),$$

where

$$(\bar{\nabla}_X F)(Y_1, \ldots, Y_k) = XF(Y_1, \ldots, Y_k) - \sum_{i=1}^k F(Y_1, \ldots, \bar{\nabla}_X Y_i, \ldots, Y_k).$$

Here $\bar{\nabla}_X Y$ is the covariant derivative of Y in the direction of X (see [39]) and since $\bar{\nabla}$ is the Riemannian connection it holds $\bar{\nabla}_X Y = \bar{\nabla}_Y X + [X, Y]$ for every $X, Y \in \mathscr{T}(\Sigma)$. We denote the kth order covariant derivative of a function u on Σ by $\bar{\nabla}^k u \in \mathscr{T}^k(\Sigma)$. The notation $\bar{\nabla}_{i_1} \cdots \bar{\nabla}_{i_k} u$ means a coefficient of $\bar{\nabla}^k u$ in local coordinates. We may raise the index of $\bar{\nabla}_i u$ by using the inverse of the metric tensor g^{ij} as $\bar{\nabla}^i u = g^{ij} \bar{\nabla}_j u$. We denote the divergence of a vector field $X \in \mathscr{T}(\Sigma)$ by $\operatorname{div}_\Sigma X$ and the Laplace-Beltrami operator for a function $u : \Sigma \to \mathbb{R}$ by $\Delta_\Sigma u$. We recall that by the divergence theorem

$$\int_\Sigma \operatorname{div}_\Sigma X \, d\mathcal{H}^2 = 0.$$

We will first fix our reference surface which we denote by Γ which is a boundary of a smooth, compact set G, i.e., $\Gamma = \partial G$. Since G is smooth it satisfies the interior and exterior ball condition with radius η, and we denote the tubular neighborhood of Γ by $\mathcal{N}_\eta(\Gamma)$ which is defined as

$$\mathcal{N}_\eta(\Gamma) = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, \Gamma) < \eta\}.$$

Then the map $\Psi : \Gamma \times (-\eta, \eta) \to \mathcal{N}_\eta(\Gamma)$ defined as $\Psi(x, s) = x + s\nu_\Gamma(x)$ is a diffeomorphism. We say that a hypersurface Σ, or a domain Ω with $\partial\Omega = \Sigma$, is $C^{1,\alpha}(\Gamma)$-regular for some small $\alpha > 0$, when it can be written as

$$\Sigma = \{x + h(x)\nu_\Gamma(x) : x \in \Gamma\},$$

for a $C^{1,\alpha}(\Gamma)$-regular function $h : \Gamma \to \mathbb{R}$ with $\|h\|_{L^\infty} < \eta$. In particular, all $C^{1,\alpha}(\Gamma)$-regular sets are diffeomorphic. We say that a set Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular if the height-function satisfies $\|h\|_{C^{1,\alpha}(\Gamma)} \leq C$ and $\|h\|_{L^\infty} \leq c\eta$ for constants C and $c < 1$. Finally we say that Σ is uniformly C^1-regular if $\|h\|_{C^1(\Gamma)} \leq C$.

Let us next fix our notation in the ambient space $\mathbb{R}^3$. We denote the kth order differential of a vector field $F : \mathbb{R}^3 \to \mathbb{R}^m$ by $\nabla^k F$, the divergence of $F : \mathbb{R}^3 \to \mathbb{R}^3$ by $\operatorname{div} F$ and the Laplace operator in $\mathbb{R}^3$ by Δ. The notation $(\nabla F)^T$ stands for the transpose of ∇F. When we restrict $F : \mathbb{R}^3 \to \mathbb{R}^3$ on Σ, we define its normal and tangential part as

$$F_n := F \cdot \nu_\Sigma \qquad \text{and} \qquad F_\tau = F - F_n \, \nu_\Sigma.$$

We use the notation $x \cdot y$ for the inner product of two vectors in $\mathbb{R}^n$.

Since Σ is a smooth hypersurface we may extend every function and vector field defined on Σ to $\mathbb{R}^3$. We may thus define a tangential differential of a vector field $F : \Sigma \to \mathbb{R}^m$ by

$$\nabla_\tau F = \nabla F - (\nabla F \nu_\Sigma) \otimes \nu_\Sigma$$

where we have extended F to $\mathbb{R}^3$. We may then extend the definition of $\operatorname{div}_\Sigma$ to fields $F : \Sigma \to \mathbb{R}^3$ by $\operatorname{div}_\Sigma F = \operatorname{Tr}(\nabla_\tau F)$ and the divergence theorem generalizes to

$$\int_\Sigma \operatorname{div}_\Sigma F \, d\mathcal{H}^2 = \int_\Sigma H_\Sigma (F \cdot \nu_\Sigma) \, d\mathcal{H}^2.$$

We note that the tangential gradient of $u \in C^\infty(\Sigma)$ is equivalent to its covariant derivative in the sense that for every vector field $X \in \mathscr{T}(\Sigma)$ we find a vector field $\tilde{X} : \Sigma \to \mathbb{R}^3$ which satisfies $\tilde{X} \cdot \nu_\Sigma = 0$ and

$$\bar{\nabla}_X u = \nabla_\tau u \cdot \tilde{X}.$$

Let us comment briefly on the notation related to the equations (1.3). We denote the derivative with respect to time by $\partial_t F$ and the material derivative as

$$\mathcal{D}_t F := \partial_t F + (v \cdot \nabla) F.$$

The material derivative does not commute with the spatial derivative and we denote the commutation

$$[\mathcal{D}_t, \nabla]u = \mathcal{D}_t \nabla u - \nabla \mathcal{D}_t u.$$

We denote by U_Ω the capacitary potential defined in (1.2) and denote $U_t = U_{\Omega_t}$, $H_t = H_{\Sigma_t}$ etc... when the meaning is clear from the context. To shorten further the notation we denote

$$Q(t) := \frac{Q}{(\mathrm{Cap}(\Omega_t))^2}. \tag{2.1}$$

We may thus write the pressure in (1.3) as

$$p = H_t - \frac{Q(t)}{2}|\nabla U_t|^2.$$

Let us next fix the notation for the function spaces. We define the Sobolev space $W^{l,p}(\Sigma)$ in a standard way for $p \in [1,\infty]$, see e.g. [4], denote the Hilbert space $H^l(\Sigma) = W^{l,2}(\Sigma)$ and define the associated norm for $u \in W^{l,p}(\Sigma)$ as

$$\|u\|^p_{W^{l,p}(\Sigma)} = \sum_{k=0}^{l} \int_\Sigma |\bar\nabla^k u|^p \, d\mathcal{H}^2$$

and for $p = \infty$

$$\|u\|_{W^{l,\infty}(\Sigma)} = \sum_{k=0}^{l} \sup_{x\in\Sigma} |\bar\nabla^k u|.$$

We often denote $\|u\|_{C^0(\Sigma)} = \|u\|_{L^\infty(\Sigma)} = \sup_{x\in\Sigma}|u(x)|$ for continuous function $u : \Sigma \to \mathbb{R}$ and $\|u\|_{C^m(\Sigma)} = \|u\|_{W^{m,\infty}(\Sigma)}$. We define the Hölder norm of a continuous function $u : \Sigma \to \mathbb{R}$ by

$$\|u\|_{C^\alpha(\Sigma)} = \|u\|_{L^\infty(\Sigma)} + \sup_{\substack{x\neq y\\ x,y\in\Sigma}} \frac{|u(y)-u(x)|}{|y-x|^\alpha}.$$

We define the Hölder norm for a tensor field $F \in \mathscr{T}^k(\Sigma)$ as in [36]

$$\|F\|_{C^\alpha(\Sigma)} = \sup\{\|F(X_1,\dots,X_k)\|_{C^\alpha(\Sigma)} : X_i \in \mathscr{T}(\Sigma) \ \text{ with } \ \|X_i\|_{C^1(\Sigma)} \le 1\}.$$

Finally we define the $H^{-1}(\Sigma)$-norm by duality, i.e.,

$$\|u\|_{H^{-1}(\Sigma)} := \sup\Big\{ \int_\Sigma u\, g\, d\mathcal{H}^2 : \|g\|_{H^1(\Sigma)} \le 1\Big\}.$$

For functions defined in the domain $u : \Omega \to \mathbb{R}$ we define the Sobolev space $W^{l,p}(\Omega)$ as functions which have kth order weak derivative in Ω and the corresponding norm is bounded

$$\|u\|^p_{W^{l,p}(\Omega)} := \sum_{k=0}^{l} \int_\Omega |\nabla^k u|^p \, dx < \infty.$$

As before we denote the Hilbert space as $H^l(\Omega) = W^{l,2}(\Omega)$ and define $H^{-1}(\Omega)$ by duality. Finally given an index vector $\alpha = (\alpha)_{i=1}^k \in \mathbb{N}^k$ we define its norm by

$$|\alpha| = \sum_{i=1}^{k} \alpha_i.$$

Throughout the paper we use the notation $S \star T$ from [32, 42] to denote a tensor formed by contraction on some indexes of tensors S and T, using the coefficients of the metric tensor g_{ij} if S and T are defined on the boundary Σ. We also use the convention that $\bar\nabla^k u \star \bar\nabla^l v$ denotes contraction of some indexes of tensors $\bar\nabla^i u$ and $\bar\nabla^j v$ for any $i \le k$ and $j \le l$. In other words, we include also the lower order covariant derivatives.

Following the notation from [54], we first introduce the real interpolation method and then the interpolation spaces. Let X and Y be Banach spaces endowed respectively with the norm $\|\cdot\|_X$ and $\|\cdot\|_Y$. The couple (X,Y) is said to be an interpolation couple if both X and Y are embedded in a Hausdorff topological vector space V. In this case we have that $X \cap Y$ endowed with the norm $\|v\|_{X\cap Y} = \max\{\|v\|_X, \|v\|_Y\}$ is a Banach space. Moreover, we also have that $X+Y = \{z = x+y,\, x \in X,\, y \in Y\}$ endowed with the norm

$$\|z\|_{X+Y} = \inf_{x\in X,\, y\in Y}\{\|x\|_X + \|y\|_Y,\, z = x+y\}$$

is a Banach space and it is immediate to check that

$$X \cap Y \subset X, Y \subset X+Y.$$

For $z \in X+Y$ and $t>0$ we introduce the K functional

$$K(t,z,X,Y) = \inf_{x\in X,\, y\in Y}\{\|x\|_X + t\|y\|_Y,\, x+y=z\}.$$

For $\theta \in (0,1)$, $p \in [1,\infty)$ and $z \in X+Y$ we let

$$\|z\|_{\theta,p}^p = \int_0^\infty \left(\frac{K(t,z,X,Y)}{t^\theta}\right)^p \frac{dt}{t}$$

and define

$$(X,Y)_{\theta,p} = \{z \in X+Y : \|z\|_{\theta,p} < \infty\}.$$

We note that $(X,Y)_{\theta,p}$ is a Banach space.

Finally, we recall that if (X_1,X_2) and (Y_1,Y_2) are interpolation couples and $\mathcal{F}: X_1+X_2 \to Y_1+Y_2$ is a linear operator which is bounded $X_i \to Y_i$ by M_i. Then for $\theta \in (0,1)$ and $p \in [1,\infty)$ the operator

$$\mathcal{F} : (X_1,X_2)_{\theta,p} \to (Y_1,Y_2)_{\theta,p} \tag{2.2}$$

is bounded and we may estimate its norm by $M_1^{1-\theta}M_2^\theta$.

2.1. Half-Integer Sobolev Spaces

Before giving the definition of half-integer Sobolev space in a domain, we exploit the extension properties of Sobolev functions. Throughout the paper we assume that the boundary $\Sigma = \partial\Omega$ is uniformly $C^{1,\alpha}(\Gamma)$-regular and thus it is H^1 extension domain, i.e., there is a linear operator $T : H^1(\Omega) \to H^1(\mathbb{R}^3)$ such that

$$\|T(u)\|_{H^1(\mathbb{R}^3)} \le C\|u\|_{H^1(\Omega)}.$$

We refer to [9] for the study of Sobolev spaces under Lipschitz-regularity and the references therein.

We need more regularity for the boundary for higher order Sobolev extension $m \ge 2$, although we do not need the optimal condition. Instead, we assume the following for the second fundamental form

$$\|B_\Sigma\|_{L^4(\Sigma)} \le C_m \quad \text{if } m=2, \qquad \|B_\Sigma\|_{L^\infty(\Sigma)} + \|B_\Sigma\|_{H^{m-2}(\Sigma)} \le C_m \quad \text{if } m>2, \tag{H$_m$}$$

which guarantees that we may extend a given function $u \in H^m(\Omega)$ to the whole space. Note that for $m \ge 4$ the condition (H$_m$) is implied by $\|B_\Sigma\|_{H^{m-2}(\Sigma)} \le C_m$ by the Sobolev-embedding, which agrees with the assumption e.g. in [10]. In the following we do not specify that a given quantity depends on the constant C_m, but take it for granted when we refer to the condition (H$_m$).

Even if there are many results for extensions of Sobolev functions in the literature, the condition (H$_m$) is too weak to apply them. To this aim we need the following result.

Proposition 2.1. *Let $m \in \mathbb{N}$, with $m \ge 2$, and let Ω be a smooth domain which is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies (H$_m$). Then there is an extension operator $T : H^m(\Omega) \to H_0^m(\mathbb{R}^3)$ such that*

$$\|T(u)\|_{H^m(\mathbb{R}^3)} \le C\|u\|_{H^m(\Omega)}.$$

Proof. Let $x_0 \in \partial\Omega$. There exist $\delta > 0$ and a diffeomorphism $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\Psi^{-1}(\Omega \cap B_\delta(x_0)) = B_1^+ = B_1 \cap \mathbb{R}^3_+$. We note that the $C^{1,\alpha}$ regularity of $\partial\Omega$ and (H_m) imply that we may choose the diffeomorphism such that it satisfies $\|\Psi\|_{C^{1,\alpha}(\mathbb{R}^3)} \le C$ and

$$\|\nabla^2\Psi\|_{L^4(\mathbb{R}^3)} \le C \quad \text{if } m = 2, \qquad \|\Psi\|_{H^m(\mathbb{R}^3)} \le C \quad \text{if } m > 2. \tag{2.3}$$

For $u : \Omega \to \mathbb{R}$ smooth we let $u' = u \circ \Psi$. We may extend u' to a function $T(u') \in H^m(B_1)$ such that

$$\|T(u')\|_{H^m(B_1)} \le C\|u'\|_{H^m(B_1^+)}.$$

The construction of $T(u') \in H^m(B_1)$ is classical but we recall it for the reader's convenience. We define

$$T(u')(x', x_n) = \begin{cases} u'(x', x_n) & x_n \ge 0 \\ \sum_{j=1}^{m+1} \lambda_j u'(x', -j^{-1}x_n) & x_n < 0 \end{cases}$$

where $\lambda = (\lambda_1, \dots, \lambda_{m+1})$ solve the system

$$\begin{cases} \sum_j \lambda_j = 1 \\ \sum_j (-j)^{-1}\lambda_j = 1 \\ \vdots \\ \sum_j (-j)^{-m}\lambda_j = 1. \end{cases}$$

This system, known as Vandermonde system, has a unique solution, hence Tu' is well defined. Finally we define the extension operator as

$$T(u) := T(u') \circ \Psi^{-1}.$$

Let us show that T is a bounded operator.

It is straightforward to check that

$$\|Tu'\|_{H^m(B_1)} \le C\|u'\|_{H^m(B_1^+)}.$$

Let us then show that

$$\|u'\|_{H^m(B_1^+)} \le C\|u\|_{H^m(\Omega)}. \tag{2.4}$$

We first note that

$$\nabla u' = \nabla u \star \nabla\Psi$$

and for $m \ge 2$

$$\nabla^m u' = \sum_{|\alpha| \le m-1} \nabla^{1+\alpha_1}\Psi \star \cdots \star \nabla^{1+\alpha_m}\Psi \star \nabla^{1+\alpha_{m+1}} u.$$

For $m = 2$ we have then by Hölder's inequality, by (2.3) and by the Sobolev embedding

$$\|\nabla^2 u'\|_{L^2(B_1)} \le C\|\Psi\|_{C^1(\mathbb{R}^3)}\|\nabla^2 u\|_{L^2(\Omega)} + \|\nabla^2\Psi\|_{L^4(\mathbb{R}^3)}\|\nabla u\|_{L^4(\Omega)} \le C\|u\|_{H^2(\Omega)}.$$

To treat the case $m \ge 3$ we first observe that by Sobolev embedding it holds

$$\|\nabla^{m-2}u\|_{L^\infty(B_1^+)} \le \|u\|_{H^m(\Omega)} \quad \text{and} \quad \|\nabla^{m-2}\Psi\|_{L^\infty(B_1^+)} \le \|\Psi\|_{H^m(\mathbb{R}^3)}.$$

Hence for $m \ge 3$ we have by Hölder's inequality, by (2.3) and by the Sobolev embedding

$$\begin{aligned} \|\nabla^m u'\|_{L^2(B_1)} &\le C\big((1 + \|\Psi\|^m_{H^m(\mathbb{R}^3)})\|u\|_{H^m(\Omega)} + \|\nabla^2\Psi\|_{L^4(\mathbb{R}^3)}\|\nabla^{m-1}u\|_{L^4(\Omega)} \\ &\quad + \|\nabla^{m-1}\Psi\|_{L^4(\mathbb{R}^3)}\|\nabla^2 u\|_{L^4(\Omega)}\big) \\ &\le C(1 + \|\Psi\|^m_{H^m(\mathbb{R}^3)})\|u\|_{H^m(\Omega)} \\ &\le C\|u\|_{H^m(\Omega)}. \end{aligned}$$

Thus we have (2.4).

Similarly we show that

$$\|T(u') \circ \Psi^{-1}\|_{H^m(\Omega \cap B_\delta(x_0))} \le C\|Tu'\|_{H^m(B_1)}.$$

The claim then follows from standard covering argument. □

Throughout the paper we will refer to the operator T as the canonical extension operator or simply as the extension operator. We define the half-integer Sobolev space in the domain Ω using the canonical extension operator.

Definition 2.2. We say that a function $u \in L^2(\Omega)$ is in $H^{\frac{1}{2}}(\Omega)$ if

$$\|u\|^2_{H^{\frac{1}{2}}(\Omega)} := \|u\|^2_{L^2(\Omega)} + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|Tu(x) - Tu(y)|^2}{|x-y|^3}\, dxdy < \infty.$$

For $k \geq 1$ we say that $u \in H^{k+\frac{1}{2}}(\Omega)$ if $u \in H^k(\Omega)$ and

$$\|u\|_{H^{k+\frac{1}{2}}(\Omega)} := \|u\|_{H^k(\Omega)} + \|T(\nabla^k u)\|_{H^{\frac{1}{2}}(\Omega)} < \infty.$$

Finally we define the $H^{-\frac{1}{2}}(\Omega)$-norm by duality.

We define $H^{\frac{1}{2}}_\star(\Omega)$ as the space of functions via interpolation such that $u \in H^{\frac{1}{2}}_\star(\Omega)$ if $T(u) \in (L^2(\mathbb{R}^3), H^1(\mathbb{R}^3))_{\frac{1}{2},2}$ and endow it with the norm

$$\|u\|^2_{H^{\frac{1}{2}}_\star(\Omega)} := \|u\|^2_{L^2(\Omega)} + \int_0^\infty \left(\frac{K(t, T(u), L^2(\mathbb{R}^3), H^1(\mathbb{R}^3))}{t^{1/2}}\right)^2 \frac{dt}{t}. \tag{2.5}$$

This gives an equivalent definition for the half-integer Sobolev space.

Proposition 2.3. *Let $m \in \mathbb{N}$, with $m \geq 2$, and let Ω be a smooth domain which is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies (H_m). The norms in Definition 2.2 and (2.5) are equivalent, i.e.,*

$$\|u\|_{H^{\frac{1}{2}}(\Omega)} \simeq \|u\|_{H^{\frac{1}{2}}_\star(\Omega)}.$$

We do not give the details of the proof, but only refer to [9] and mention that it follows from the fact that $H^{\frac{1}{2}}(\mathbb{R}^3) = (L^2(\mathbb{R}^3), H^1(\mathbb{R}^3))_{\frac{1}{2},2}$, see [54].

2.2. Half-Integer Sobolev Spaces on a Surfaces

We begin by defining the space $H^{\frac{1}{2}}(\Sigma)$. Again there are many ways to do this. We choose the definition via harmonic extension.

Definition 2.4. Let $\Sigma = \partial\Omega$ be uniformly $C^{1,\alpha}(\Gamma)$-regular. We say that $u \in H^{\frac{1}{2}}(\Sigma)$ if $u \in L^2(\Sigma)$ and

$$\|u\|_{H^{\frac{1}{2}}(\Sigma)} = \|u\|_{L^2(\Sigma)} + \inf\{\|\nabla v\|_{L^2(\Omega)} : v - u \in H^1_0(\Omega)\} < \infty.$$

We define the space $H^{-\frac{1}{2}}(\Sigma)$ and its norm by duality.

By standard theory the $C^{1,\alpha}(\Gamma)$-regularity of Σ ensures that Definition 2.4 is equivalent to the definition via Gagliardo seminorm

$$\|u\|^2_{H^{\frac{1}{2}}(\Sigma)} \simeq \|u\|^2_{L^2(\Sigma)} + \int_\Sigma\int_\Sigma \frac{|u(x)-u(y)|^2}{|x-y|^3}\, d\mathcal{H}^2_x d\mathcal{H}^2_y.$$

Moreover, this norm is also equivalent to the norm obtained via interpolation. Indeed, let us define the interpolation space (see beginning of Sect. 2)

$$H^{\frac{1}{2}}_\star(\Sigma) = (L^2(\Sigma), H^1(\Sigma))_{\frac{1}{2},2}.$$

Let us show that

$$\|u\|_{H^{\frac{1}{2}}(\Sigma)} \simeq \|u\|_{H^{\frac{1}{2}}_\star(\Sigma)}. \tag{2.6}$$

Due to the non-local nature of the problem we give the proof of (2.6) in detail.

We fix a small $\delta > 0$ and cover Σ with finitely many balls of radius δ centered at $x_i \in \Sigma$, i.e.,

$$\Sigma \subset \bigcup_{i=1}^{N} B_\delta(x_i).$$

Since Σ is $C^{1,\alpha}(\Gamma)$, there are $C^{1,\alpha}$-regular functions ϕ_i such that $\Sigma \cap B_{2\delta}(x_i)$ is contained in the graph of ϕ_i for every $i = 1,\dots,N$, when δ is small enough. Let $\{\eta_i\}_{i=1,\dots,N}$ be a partition of unity subordinated to the open covering $B_\delta(x_i)$. Then it holds

$$\|u\|_{H^{\frac{1}{2}}(\Sigma)} \le \sum_{i=1}^{N} \|\eta_i u\|_{H^{\frac{1}{2}}(\Sigma)}.$$

Let us fix $i = 1,\dots,N$ and by rotating and translating the coordinates we may assume that $x_i = 0$ and $\Sigma \cap B_{2\delta} \subset \{(x', \phi_i(x')) : x' \in \mathbb{R}^2\}$ with $\phi_i(0) = 0$ and $\nabla \phi_i(0) = 0$. Denote $u_i = \eta_i u$ and $v_i(x') = u_i(x', \phi_i(x'))$. Note that then $v_i : \mathbb{R}^2 \to \mathbb{R}$, $\operatorname{supp} v_i \subset B_{\delta'} \subset \mathbb{R}^2$ and $\operatorname{supp} u_i \subset B_\delta \subset \mathbb{R}^3$ with $\delta/2 \le \delta' \le \delta$. Therefore we deduce by the $C^{1,\alpha}$-regularity of ϕ_i that

$$\begin{aligned}
&\int_\Sigma \int_\Sigma \frac{|u_i(x) - u_i(y)|^2}{|x-y|^3}\, d\mathcal{H}^2_x\, d\mathcal{H}^2_y \\
&= \int_{\Sigma\cap B_{2\delta}} \int_{\Sigma\cap B_{2\delta}} \frac{|u_i(x) - u_i(y)|^2}{|x-y|^3}\, d\mathcal{H}_x\, d\mathcal{H}_y + 2\int_{\Sigma\setminus B_{2\delta}} \int_{\Sigma\cap B_{2\delta}} \frac{|u_i(x) - u_i(y)|^2}{|x-y|^3}\, d\mathcal{H}_x\, d\mathcal{H}_y \\
&\le C \int_{B_{2\delta'}} \int_{B_{2\delta'}} \frac{|u_i(x', \phi_i(x')) - u_i(y', \phi_i(y'))|^2}{(|x'-y'|^2 + (\phi_i(x') - \phi_i(y')^2)^{3/2}}\, dx'\, dy' + C\|u\|^2_{L^2(\Sigma)} \\
&\le C\|v_i\|^2_{H^{\frac{1}{2}}(\mathbb{R}^2)} + C\|u\|^2_{L^2(\Sigma)} \\
&\le C\|v_i\|^2_{H^{\frac{1}{2}}_\star(\mathbb{R}^2)} + C\|u\|^2_{L^2(\Sigma)}.
\end{aligned}$$

This implies $\|u_i\|_{H^{\frac{1}{2}}(\Sigma)} \le C\|v_i\|_{H^{\frac{1}{2}}_\star(\mathbb{R}^2))} + C\|u\|_{L^2(\Sigma)}$. Let us denote by $L^2_0(B_{2\delta'})$ and $H^1_0(B_{2\delta'})$ for functions $f \in L^2(\mathbb{R}^2)$, and respectively $f \in H^1(\mathbb{R}^2)$, with $\operatorname{supp} f \subset B_{2\delta'}$. Denote also $\Psi : B_{2\delta'} \to \Psi(B_{2\delta'}) \subset \mathbb{R}^3$, $\Psi(x') = (x', \phi_i(x'))$. We may estimate

$$\begin{aligned}
&K(t, v_i, L^2(\mathbb{R}^2), H^1(\mathbb{R}^2)) \\
&\quad = \inf_{f+g=v_i} \{\|f\|_{L^2(\mathbb{R}^2)} + t\|g\|_{H^1(\mathbb{R}^2)},\ f \in L^2(\mathbb{R}^2),\ g \in H^1(\mathbb{R}^2)\} \\
&\quad \le \inf_{f+g=v_i} \{\|f\|_{L^2(\mathbb{R}^2)} + t\|g\|_{H^1(\mathbb{R}^2)},\ f \in L^2_0(B_{2\delta'}),\ g \in H^1_0(B_{2\delta'})\} \\
&\quad \le C \inf_{f+g=v_i} \{\|f\circ\Psi^{-1}\|_{L^2(\Sigma\cap B_{2\delta})} + t\|g\circ\Psi^{-1}\|_{H^1(\Sigma\cap B_{2\delta})},\ f \in L^2_0(B_{2\delta'}),\ g \in H^1_0(B_{2\delta'})\} \\
&\quad \le C \inf_{\tilde f+\tilde g=u_i} \{\|\tilde f\|_{L^2(\Sigma\cap B_{2\delta})} + t\|\tilde g\|_{H^1(\Sigma\cap B_{2\delta})},\ \tilde f \in L^2_0(\Sigma\cap B_{2\delta}),\ \tilde g \in H^1_0(\Sigma\cap B_{2\delta})\} \\
&\quad = C\, K(t, u_i, L^2_0(\Sigma\cap B_{2\delta}), H^1_0(\Sigma\cap B_{2\delta})).
\end{aligned}$$

Since $\operatorname{supp} u_i \subset \Sigma \cap B_\delta$, it is easy to see that

$$K(t, u_i, L^2_0(\Sigma\cap B_{2\delta}), H^1_0(\Sigma\cap B_{2\delta})) \le C\, K(t, u_i, L^2(\Sigma), H^1(\Sigma)).$$

Therefore, we deduce

$$\|v_i\|_{H^{\frac{1}{2}}_\star(\mathbb{R}^2)} \le C\|u_i\|_{H^{\frac{1}{2}}_\star(\Sigma)} \le C\|u\|_{H^{\frac{1}{2}}_\star(\Sigma)}.$$

Repeating the argument for every $i = 1,\dots,N$ yields

$$\|u\|_{H^{\frac{1}{2}}(\Sigma)} \le C\|u\|_{H^{\frac{1}{2}}_\star(\Sigma)}.$$

The opposite inequality can be proved in a similar way.

We also note that we may interpolate between $H^{\frac12}(\Sigma)$ and its dual and obtain by using [9, Theorem 4.1] and by a localization argument that

$$(H^{-\frac12}(\Sigma), H^{\frac12}(\Sigma))_{\frac12,2} = L^2(\Sigma).$$

In order to define higher order half-integer Sobolev spaces on the boundary we use the fact that in our setting the boundary $\partial\Omega = \Sigma$ is given by the parametrization $\Psi_\Sigma : \Gamma \to \Sigma$, $\Psi_\Sigma(x) = x + h(x)\nu_\Gamma(x)$, where Γ is the reference surface. In our case Γ is the boundary of a smooth set G and the map $\Psi : \Gamma \times (-\eta, \eta) \to \mathcal{N}_\eta(\Gamma)$, $\Psi(x, s) = x + s\nu_\Gamma(x)$, is a diffeomorphism. Here $\mathcal{N}_\eta(\Gamma)$ is the tubular neighborhood of Γ. Therefore the projection map $\pi_\Gamma : \mathcal{N}_\eta(\Gamma) \to \Gamma$ is well defined as

$$\pi_\Gamma(y) = x \qquad \text{where } \ y = x + s\nu_\Gamma(x) \ \text{ for some } \ s \in (-\eta, \eta). \tag{2.7}$$

We extend π_Γ to whole $\mathbb{R}^3$ and thus we may extend a given function $u : \Gamma \to \mathbb{R}$ to $\mathbb{R}^3$ as $(u \circ \pi_\Gamma) : \mathbb{R}^3 \to \mathbb{R}$. In particular, the kth order derivative $\nabla^k(u \circ \pi_\Gamma)(x)$ is well defined for all $x \in \Gamma$, and for $x \in \Gamma$ the function $s \mapsto (u \circ \pi_\Gamma)(x + s\nu_\Gamma(x))$ is constant for $|s|$ small. We use this extension to define the half-integer Sobolev norm on the reference surface.

Definition 2.5. For $m \geq 2$ we say that $u \in H^{m-\frac12}(\Gamma)$ if $u \in H^{m-1}(\Gamma)$ and the norm

$$\|u\|_{H^{m-\frac12}(\Gamma)} := \|\nabla^{m-1}(u \circ \pi_\Gamma)\|_{H^{\frac12}(\Gamma)} + \|u\|_{H^{m-1}(\Gamma)}$$

is bounded.

We define the half-integer Sobolev spaces on Σ by mapping a function $u \in C^\infty(\Sigma)$ back to Γ by using the parametrization $\Psi_\Sigma : \Gamma \to \Sigma$, $\Psi_\Sigma(x) = x + h(x)\nu_\Gamma(x)$. Let us fix $m \geq 2$ and recall that

$$\bar\nabla^m(u \circ \Psi_\Sigma) = \sum_{|\alpha| \leq m-1} \bar\nabla^{1+\alpha_1}\Psi_\Sigma \star \cdots \star \bar\nabla^{1+\alpha_k}\Psi_\Sigma \star \bar\nabla^{1+\alpha_{m+1}} u.$$

If Σ satisfies (H_m), then arguing as in the proof of Proposition 2.1 we deduce

$$\|u\|_{H^m(\Sigma)} \simeq \|u \circ \Psi_\Sigma\|_{H^m(\Gamma)}.$$

Based on this we define the half-integer Sobolev space of order $m - 1/2$ on Σ in the following way.

Definition 2.6. Let $m \geq 2$ be an integer and assume Σ is $C^{1,\alpha}(\Gamma)$-regular. We say that u is in the space $H^{m-\frac12}(\Sigma)$ if $(u \circ \Psi_\Sigma) \in H^{m-\frac12}(\Gamma)$ and define the norm as

$$\|u\|_{H^{m-\frac12}(\Sigma)} := \|u \circ \Psi_\Sigma\|_{H^{m-\frac12}(\Gamma)},$$

where $\Psi_\Sigma : \Gamma \to \Sigma$ is the parametrization $\Psi_\Sigma(x) = x + h(x)\nu_\Gamma(x)$.

We define the space $H_\star^{m-\frac12}(\Sigma)$ via interpolation as the functions $u \in H^{m-1}(\Sigma)$ such that

$$\|u\|^2_{H_\star^{m-\frac12}(\Sigma)} := \|u\|^2_{H^{m-1}(\Sigma)} + \int_0^\infty \left(\frac{K(t, u, H^{m-1}(\Sigma), H^m(\Sigma))}{t^{1/2}}\right)^2 \frac{dt}{t} < \infty. \tag{2.8}$$

We note that if Σ satisfies the assumption (H_m) for $m \geq 2$ the norm $H^{m-\frac12}(\Sigma)$ in Definition 2.6 is equivalent with the interpolation norm $\|u\|_{H_\star^{m-\frac12}(\Sigma)}$ in (2.8). We state this in the next proposition. The proof is similar to the argument for (2.6) and we omit it.

Proposition 2.7. *Let $m \geq 2$ and assume that Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the assumption (H_m). Then it holds*

$$H^{m-\frac12}(\Sigma) = H_\star^{m-\frac12}(\Sigma) \qquad \text{and} \qquad \|u\|_{H^{m-\frac12}(\Sigma)} \simeq \|u\|_{H_\star^{m-\frac12}(\Sigma)}.$$

2.3. Geometric Preliminaries

We begin by recalling basic results from differential geometry. We define the Riemann curvature tensor $R \in \mathscr{T}^4(\Sigma)$ [39,43] via interchange of covariant derivatives of a vector field Y^i and a covector field Z_i as

$$\begin{aligned}\bar{\nabla}_i \bar{\nabla}_j Y^s - \bar{\nabla}_j \bar{\nabla}_i Y^s &= R_{ijkl} g^{ks} Y^l,\\ \bar{\nabla}_i \bar{\nabla}_j Z_k - \bar{\nabla}_j \bar{\nabla}_i Z_k &= R_{ijkl} g^{ls} Z_s,\end{aligned} \tag{2.9}$$

where we have used the Einstein summation convention. We may write the Riemann tensor in local coordinates by using the second fundamental form B as

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk}. \tag{2.10}$$

We will also need the Simon's identity which reads as

$$\Delta_\Sigma B_{ij} = \bar{\nabla}_i \bar{\nabla}_j H + H B_{il} g^{ls} B_{sj} - |B|^2 B_{ij}. \tag{2.11}$$

Let us recall that the interpolation inequality holds for smooth compact n-dimensional hypersurface $\Sigma \subset \mathbb{R}^{n+1}$, see e.g. [4],

$$\|\bar{\nabla}^k u\|_{L^p(\Sigma)} \leq C_\Sigma \|u\|_{W^{l,r}(\Sigma)}^\theta \|u\|_{L^q(\Sigma)}^{(1-\theta)}, \tag{2.12}$$

where

$$\frac{1}{p} = \frac{k}{n} + \theta\left(\frac{1}{r} - \frac{l}{n}\right) + \frac{1}{q}(1-\theta).$$

In particular, (2.12) holds on the reference surface $\Gamma \subset \mathbb{R}^3$ and in $\mathbb{R}^n$ for functions with compact support $\operatorname{supp} u \subset B_R$.

In order to have the interpolation inequality for a general surface $\Sigma \subset \mathbb{R}^{n+1}$ with control on the constant C_Σ, we use the result in [42], which states that once the mean curvature H_Σ satisfies the bound $\|H_\Sigma\|_{L^{n+\delta}(\Sigma)} \leq C$, then the above interpolation inequality holds on Σ with uniform bound on the constant. We state this for our purpose, where Σ is 2-dimensional surfaces that is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the bound $\|B_\Sigma\|_{L^4(\Sigma)} \leq C$. The reason for the L^4-curvature bound will be clear from the results in Sect. 6. The following interpolation inequality follows from [42, Proposition 6.5].

Proposition 2.8. *Assume $\Sigma \subset \mathbb{R}^3$ is a compact 2-dimensional hypersurface which is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the bound $\|B_\Sigma\|_{L^4(\Sigma)} \leq M$. Then for integers k, l, $0 \leq k < l$ and numbers $p, r \in [1,\infty)$ and $q \in [1,\infty]$ we have for all tensor fields T that*

$$\|\bar{\nabla}^k T\|_{L^p(\Sigma)} \leq C \|T\|_{W^{l,r}(\Sigma)}^\theta \|T\|_{L^q(\Sigma)}^{(1-\theta)},$$

where p and $\theta \in [0,1]$ are given by

$$\frac{1}{p} = \frac{k}{2} + \theta\left(\frac{1}{r} - \frac{l}{2}\right) + \frac{1}{q}(1-\theta).$$

The constant C depends on M, k, p, l, r, q.

In particular, we have the Sobolev embedding, i.e., for $p \in [1, n)$ it holds $\|u\|_{L^{p^*}(\Sigma)} \leq C\|u\|_{W^{1,p}(\Sigma)}$ with $p^* = \frac{np}{n-p}$, for $p = n$ it holds $\|u\|_{L^q(\Sigma)} \leq C\|u\|_{W^{1,p}(\Sigma)}$ for all $q < \infty$ and for $p > n$ it holds $\|u\|_{C^\alpha(\Sigma)} \leq C\|u\|_{W^{1,p}(\Sigma)}$ for $\alpha = 1 - \frac{n}{p}$.

There is a danger for confusion in terminology when we use interpolation of function spaces and interpolation inequality. We use the term 'interpolation argument', when we interpolate between two function spaces, and 'interpolation inequality' or merely 'interpolation' when we refer to Proposition 2.8.

Let $\Sigma = \partial\Omega \subset \mathbb{R}^3$ be a compact hypersurface in $\mathbb{R}^3$ such that $\Sigma = \partial\Omega$ which is $C^{1,\alpha}(\Gamma)$-regular. Then the Sobolev embedding extends to half-integers, i.e., it holds

$$\|u\|_{L^p(\Sigma)} \leq C\|u\|_{H^{\frac{1}{2}}(\Sigma)}, \qquad \text{for } p \leq 4$$

and

$$\|u\|_{L^p(\Omega)} \le C\|u\|_{H^{\frac{1}{2}}(\Omega)}, \qquad \text{for } p \le 3.$$

We need the above interpolation inequality also for half-integers and for functions defined in Ω. To this aim we need to assume that Σ satisfies the condition (H_m).

Corollary 2.9. *Let $m \in \mathbb{N}$ and $\Sigma \subset \mathbb{R}^3$ is compact 2-dimensional hypersurface which is uniformly $C^{1,\alpha}(\Gamma)$-regular such that $\Sigma = \partial\Omega$ and satisfies the condition (H_m). Then for all half-integers k and l with $k < l \le m$ and for $q \in [1,\infty]$ it holds*

$$\|u\|_{H^k(\Sigma)} \le C\|u\|_{H^l(\Sigma)}^{\theta}\|u\|_{L^q(\Sigma)}^{1-\theta},$$

where $\theta \in [0,1]$ is given by

$$1 = k - \theta(l-1) + \frac{2}{q}(1-\theta).$$

In addition, it holds

$$\|u\|_{H^k(\Omega)} \le C\|u\|_{H^l(\Omega)}^{\theta}\|u\|_{L^q(\Omega)}^{1-\theta},$$

where $\theta \in [0,1]$ is given by

$$\frac{1}{2} = \frac{k}{3} + \theta\left(\frac{1}{2} - \frac{l}{3}\right) + \frac{1}{q}(1-\theta).$$

Moreover, the inequality (2.12) holds on $\Omega \subset \mathbb{R}^3$ with $r = 2$ and integers $k < l \le m$. The constants depends on m, q and on the $C^{1,\alpha}$-norm of the heightfunction.

Proof. We sketch the proof only for the first claim when $k = \tilde{k} - \frac{1}{2}$ for $\tilde{k} \in \mathbb{N}$ and l is an integer. By Proposition 2.7 and by the classical interpolation theory stated in (2.2) we have

$$\|u\|_{H^k(\Sigma)} \le C\|u\|_{H^k_\star(\Sigma)} \le C\|u\|_{H^{\tilde{k}}(\Sigma)}^{\frac{1}{2}}\|u\|_{H^{\tilde{k}-1}(\Sigma)}^{\frac{1}{2}}.$$

Proposition 2.8 yields

$$\|u\|_{H^{\tilde{k}}(\Sigma)} \le C\|u\|_{H^l(E)}^{\theta_1}\|u\|_{L^q(E)}^{1-\theta_1},$$

where θ_1 is given by $1 = \tilde{k} - \theta_1(l-1) + \frac{2}{q}(1-\theta_1)$, and

$$\|u\|_{H^{\tilde{k}-1}(\Sigma)} \le C\|u\|_{H^l(E)}^{\theta_2}\|u\|_{L^q(E)}^{1-\theta_2},$$

where θ_2 is given by $1 = (\tilde{k}-1) - \theta_2(l-1) + \frac{2}{q}(1-\theta_2)$. This implies the claim. The case when l is half-integer follows from the same argument. Finally the second interpolation inequality follows by extending u to whole $\mathbb{R}^3$, where the inequality is well-known, and using Proposition 2.1. □

2.4. Functional and Geometric Inequalities

We begin by recalling the extension of the interpolation inequality (2.12), or the Gagliardo-Nirenberg inequality, in $\mathbb{R}^n$ for fractional Sobolev spaces [8]. We state the result in the setting that we need, where for all $f \in C_0^\infty(B_R)$ it holds

$$\|f\|_{W^{s,p}(B_R)} \le C\|f\|_{W^{s_1,p_1}(B_R)}^{\theta}\|f\|_{L^{p_2}(B_R)}^{1-\theta}, \tag{2.13}$$

for $0 \le s \le s_1$, $p_2 \in (1,\infty)$ and $\theta \in (0,1)$ which satisfy

$$s = \theta s_1 \qquad \text{and} \qquad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}.$$

Next we recall the Kato-Ponce inequality, or the fractional Leibniz rule, in $\mathbb{R}^n$ which is proven e.g. in [31]. We may define the norm $\|f\|_{W^{k,p}(\mathbb{R}^n)}$ for half-integer $k \geq 0$ and $p \in (1,\infty)$ by using Bessel potentials $\langle D\rangle^k$ as

$$\|f\|_{W^{k,p}(\mathbb{R}^n)} = \|\langle D\rangle^k f\|_{L^p(\mathbb{R}^n)}.$$

The Kato-Ponce inequality, in the form we are interested in, states that for $f, g \in C_0^\infty(\mathbb{R}^n)$ and for numbers $2 \leq p_1, q_2 < \infty$ and $2 \leq p_2, q_1 \leq \infty$ with

$$\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2} \tag{2.14}$$

it holds

$$\|fg\|_{H^k(\mathbb{R}^n)} \leq C\|f\|_{W^{k,p_1}(\mathbb{R}^n)}\|g\|_{L^{q_1}(\mathbb{R}^n)} + C\|f\|_{L^{p_2}(\mathbb{R}^n)}\|g\|_{W^{k,q_2}(\mathbb{R}^n)}. \tag{2.15}$$

We need the following generalization of the Kato-Ponce inequality both on the boundary Σ and in the domain Ω.

Proposition 2.10. *Let $m \geq 1$ be an integer and assume Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the condition (H_m). Then for all half-integers $k \leq m$ it holds*

$$\|fg\|_{H^k(\Sigma)} \leq C\|f\|_{H^k(\Sigma)}\|g\|_{L^\infty(\Sigma)} + C\|f\|_{L^\infty(\Sigma)}\|g\|_{H^k(\Sigma)}$$

and

$$\|fg\|_{H^k(\Omega)} \leq C\|f\|_{H^k(\Omega)}\|g\|_{L^\infty(\Omega)} + C\|f\|_{L^\infty(\Omega)}\|g\|_{H^k(\Omega)}.$$

Moreover, assume that $\|B\|_{L^4} \leq M$ and let $k \in \mathbb{N}$. Then for $p_1, p_2, q_1, q_2 \in [2,\infty]$ with $p_1, q_2 < \infty$ which satisfies (2.14) it holds

$$\|fg\|_{H^k(\Sigma)} \leq C\|f\|_{W^{k,p_1}(\Sigma)}\|g\|_{L^{q_1}(\Sigma)} + C\|f\|_{L^{p_2}(\Sigma)}\|g\|_{W^{k,q_2}(\Sigma)}.$$

The constants depend on $M, m, k, p_1, p_2, q_1, q_2$ and on the $C^{1,\alpha}$-norm of the heightfunction.

Proof. The second inequality follows immediately from the property of the extension operator given by Proposition 2.1 and by the classical Kato-Ponce inequality (2.15), see e.g. [10]. Also the first inequality follows from a similar localization argument as we used in (2.6).

We prove the third inequality, since we will use the argument also later. First by Leibniz formula we may write

$$\bar{\nabla}^k(fg) = \sum_{i+j=k} \bar{\nabla}^i f \star \bar{\nabla}^j g.$$

The claim thus follows once we prove

$$\sum_{i+j=k} \|\bar{\nabla}^i f \star \bar{\nabla}^j g\|_{L^2(\Sigma)} \leq C\|f\|_{W^{k,p_1}(\Sigma)}\|g\|_{L^{q_1}(\Sigma)} + \|f\|_{L^{p_2}(\Sigma)}\|g\|_{W^{k,q_2}(\Sigma)}. \tag{2.16}$$

To this aim we use Hölder's inequality as

$$\sum_{i+j=k} \|\bar{\nabla}^i f \star \bar{\nabla}^j g\|_{L^2(\Sigma)} \leq \sum_{i+j=k} \|\bar{\nabla}^i f\|_{L^4(\Sigma)}\|\bar{\nabla}^j g\|_{L^4(\Sigma)}.$$

By interpolation inequality in Proposition 2.8 we have

$$\|\bar{\nabla}^i f\|_{L^4(\Sigma)} \leq C\|f\|_{W^{k,p_1}(\Sigma)}^{\theta_i}\|f\|_{L^{p_2}(\Sigma)}^{1-\theta_i} \qquad \text{with} \qquad \theta_i = \frac{\frac{i}{2} - \frac{1}{4} + \frac{1}{p_2}}{\frac{k}{2} + \frac{1}{p_2} - \frac{1}{p_1}}$$

and recalling that $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2} = \frac{1}{2}$ we have

$$\|\bar{\nabla}^j g\|_{L^4(\Sigma)} \leq C\|g\|_{W^{k,q_2}(\Sigma)}^{\theta_j}\|g\|_{L^{q_1}(\Sigma)}^{1-\theta_j} \qquad \text{with} \qquad \theta_j = \frac{\frac{j}{2} - \frac{1}{4} + \frac{1}{q_1}}{\frac{k}{2} + \frac{1}{p_2} - \frac{1}{p_1}}.$$

In particular, $i+j=k$ implies $\theta_i+\theta_j=1$. Therefore we have by Young's inequality $a^{\theta_i}b^{\theta_j}\le\theta_i a+\theta_j b\le a+b$ that

$$\sum_{i+j=k}\|\bar\nabla^i f\star\bar\nabla^j g\|_{L^2}\le C\|f\|_{L^{p_2}}\|g\|_{L^{q_1}}\sum_{i+j=k}\|f\|^{\theta_i}_{W^{k,p_1}}\|f\|^{-\theta_i}_{L^{p_2}}\|g\|^{\theta_j}_{W^{k,q_2}}\|g\|^{-\theta_j}_{L^{q_1}}$$
$$\le C\|f\|_{L^{p_2}}\|g\|_{L^{q_1}}\left(\frac{\|f\|_{W^{k,p_1}}}{\|f\|_{L^{p_2}}}+\frac{\|g\|_{W^{k,q_2}}}{\|g\|_{L^{q_1}}}\right)$$

and the claim follows. □

We remark that we do not generalize the last inequality in Proposition 2.10 for half-integers k since we do not define the space $W^{k,p}(\Sigma)$ for $p\neq 2$, when k is not an integer. However, under the assumption of Proposition 2.10, we obtain a weaker version which reads as follows

$$\|fg\|_{H^{\frac12}(\Sigma)}\le C\|f\|_{H^{\frac12}(\Sigma)}\|g\|_{L^\infty(\Sigma)}+\|f\|_{L^p(\Sigma)}\|g\|_{W^{1,q}(\Sigma)},\tag{2.17}$$

for $\frac1p+\frac1q=\frac12$. Again, since the proof is similar to the argument we used in (2.6) we leave it for the reader, but refer to [15, Lemma 4.3] for the proof of the case $p=q=4$. In particular, when g is Lipschitz, we may estimate the product simply by

$$\|fg\|_{H^{\frac12}(\Sigma)}\le C\|f\|_{H^{\frac12}(\Sigma)}\|g\|_{C^1(\Sigma)}.$$

Next we recall (see e.g. [22]) that it holds $\|u\|_{H^{k+2}(\Sigma)}\le C_\Sigma(\|\Delta_\Sigma u\|_{H^k(\Sigma)}+\|u\|_{L^2(\Sigma)})$. However, the constant depends on the curvature of Σ and we need to quantify this dependence.

Proposition 2.11. *Assume that Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies $\|B_\Sigma\|_{L^4}\le M$. For all $\varepsilon>0$ there exists a constant C_ε such that for $k=0,\frac12,1$ it holds*

$$\|u\|_{H^{k+2}(\Sigma)}\le(1+\varepsilon)\|\Delta_\Sigma u\|_{H^k(\Sigma)}+C_\varepsilon\|u\|_{L^2(\Sigma)}.$$

Let m be an integer with $m\ge3$ and assume that Σ satisfies in addition the condition (H_m). Then for every half-integer $2\le k\le m$ it holds

$$\|u\|_{H^k(\Sigma)}\le(1+\varepsilon)\|\Delta_\Sigma u\|_{H^{k-2}(\Sigma)}+C_\varepsilon\|u\|_{L^2(\Sigma)}.$$

The constant C_ε depends on ε, M, m and on the $C^{1,\alpha}$-norm of the heightfunction.

Proof. The case $k=0$ follows from [15, Lemma 4.11] but we give the proof for the reader's convenience. We recall that the Riemann tensor R satisfies by (2.10) $|R|\le C|B|^2$ and deduce by [22, Remark 2.4] (see also [4]) that

$$\|\bar\nabla^2u\|^2_{L^2(\Sigma)}\le\|\Delta_\Sigma u\|^2_{L^2(\Sigma)}+C\int_\Sigma|B|^2|\bar\nabla u|^2\,d\mathcal{H}^2.$$

By Proposition 2.8 there is $\theta\in(0,1)$ such that

$$\int_\Sigma|B|^2|\bar\nabla u|^2\,d\mathcal{H}^2\le\|B\|^2_{L^4}\|\bar\nabla u\|^2_{L^4}\le C\|u\|^{2\theta}_{H^2}\|u\|^{2(1-\theta)}_{L^2}\le\varepsilon\|u\|_{H^2}+C_\varepsilon\|u\|^2_{L^2}.$$

This implies the claim for $k=0$.

For the case $k=1$ we use (2.9) and integration by parts

$$\begin{aligned}\|\bar\nabla\Delta_\Sigma u\|^2_{L^2(\Sigma)}&=\int_\Sigma\bar\nabla^k\bar\nabla_i\bar\nabla^i u\,\bar\nabla_k\bar\nabla^j\bar\nabla_j u\,d\mathcal{H}^2\\&=\int_\Sigma\bar\nabla_i\bar\nabla^k\bar\nabla^i u\,\bar\nabla_k\bar\nabla^j\bar\nabla_j u\,d\mathcal{H}^2+\int_\Sigma(R\star\bar\nabla u\star\bar\nabla^3u)\,d\mathcal{H}^2\\&\ge-\int_\Sigma\bar\nabla^k\bar\nabla^i u\,\bar\nabla_i\bar\nabla_k\bar\nabla^j\bar\nabla_j u\,d\mathcal{H}^2-\|R\star\bar\nabla u\|_{L^2}\,\|\bar\nabla^3u\|_{L^2}.\end{aligned}$$

As before we have by Proposition 2.8 and by $|R|\le C|B|^2$ that

$$\|R\star\bar\nabla u\|_{L^2}\le C\|B\|^2_{L^4}\|\bar\nabla u\|_{L^\infty}\le\varepsilon\|u\|_{H^3}+C_\varepsilon\|u\|_{L^2}.\tag{2.18}$$

We proceed by using (2.9) and by integrating by parts

$$
\begin{aligned}
&-\int_\Sigma \bar\nabla^k\bar\nabla^i u\,\bar\nabla_i\bar\nabla_k\bar\nabla^j\bar\nabla_j u\,d\mathcal{H}^2\\
&\quad\geq -\int_\Sigma \bar\nabla^k\bar\nabla^i u\,\bar\nabla_k\bar\nabla_i\bar\nabla^j\bar\nabla_j u\,d\mathcal{H}^2 + \int_\Sigma \bar\nabla^2 u \star R\star \bar\nabla^2 u\,d\mathcal{H}^2\\
&\quad\geq -\int_\Sigma \bar\nabla^k\bar\nabla^i u\,\bar\nabla_k\bar\nabla^j\bar\nabla_i\bar\nabla_j u\,d\mathcal{H}^2 - \int_\Sigma \bar\nabla^k\bar\nabla^i u\,\bar\nabla_k[\bar\nabla_i\nabla^j - \bar\nabla_j\nabla^i]\bar\nabla_j u\,d\mathcal{H}^2\\
&\qquad - C\|B\|_{L^4}^2\|\bar\nabla^2 u\|_{L^4}^2\\
&\quad\geq \int_\Sigma \bar\nabla^j\bar\nabla^k\bar\nabla^i u\,\bar\nabla_k\bar\nabla_i\bar\nabla_j u\,d\mathcal{H}^2 + \int_\Sigma \bar\nabla_k\bar\nabla^k\bar\nabla^i u\,[\bar\nabla_i\nabla^j - \bar\nabla_j\nabla^i]\bar\nabla_j u\,d\mathcal{H}^2\\
&\qquad - C\|B\|_{L^4}^2\|\bar\nabla^2 u\|_{L^4}^2\\
&\quad\geq \|\bar\nabla^3 u\|_{L^2}^2 - C\|B\|_{L^4}^2\|\bar\nabla^2 u\|_{L^4}^2 - \|R\star\bar\nabla u\|_{L^2}\,\|\bar\nabla^3 u\|_{L^2}.
\end{aligned}
$$

The inequality for $k=1$ then follows from (2.18) and from Proposition 2.8 which yields

$$\|\bar\nabla^2 u\|_{L^4} \leq \varepsilon\|u\|_{H^3} + C_\varepsilon\|u\|_{L^2}.$$

The case $k=1/2$ follows from the previous two estimates and Proposition 2.7 with standard interpolation argument which we briefly sketch here for the reader's convenience. We define a linear operator $\mathcal{F}:\tilde H^k(\Sigma)\to\tilde H^{k+2}(\Sigma)$ such that $\mathcal{F}(g)=u$, where u is the solution of

$$\Delta_\Sigma u = g \qquad \text{on } \Sigma,$$

and $\tilde H^k(\Sigma)=\{f\in H^k(\Sigma): \int_\Sigma f\,d\mathcal{H}^2=0\}$. The operator $\mathcal{F}$ is well-defined and by the previous estimates it satisfies

$$\|\mathcal{F}\|_{\mathcal{L}(L^2,H^2)}\leq C \qquad \text{and} \qquad \|\mathcal{F}\|_{\mathcal{L}(H^1,H^3)}\leq C$$

By the interpolation theory discussed in (2.2) it holds

$$\|\mathcal{F}(g)\|_{H_\star^{\frac52}(\Sigma)}\leq C\|g\|_{H_\star^{\frac12}(\Sigma)}.$$

Proposition 2.7 then yields

$$\|\mathcal{F}(g)\|_{H^{\frac52}(\Sigma)}\leq C\|g\|_{H^{\frac12}(\Sigma)}.$$

We apply this to $\tilde u = u-\bar u$, where $\bar u = ⨍_\Sigma u\,d\mathcal{H}^2$ and the claim follows.

The argument for higher m and k is similar and we merely sketch it. Let k be an integer with $2\leq k\leq m$. Using (2.9) and arguing as above we obtain after long but straightforward calculations that

$$\|\bar\nabla^k u\|_{L^2(\Sigma)}^2 \leq \|\bar\nabla^{k-2}\Delta_\Sigma u\|_{L^2(\Sigma)}^2 + C\sum_{\alpha+\beta\leq k-2}\|\bar\nabla^\alpha R\star\bar\nabla^{1+\beta}u\|_{L^2(\Sigma)}^2.$$

Then by (2.10), (2.16), Proposition 2.8 and by the assumption $\|B\|_{L^\infty},\|B\|_{H^{k-2}}\leq C$ we have

$$
\begin{aligned}
\sum_{\alpha+\beta\leq k-2}\|\bar\nabla^\alpha(R\star\bar\nabla^{1+\beta}u)\|_{L^2(\Sigma)} &\leq C\|B\|_{L^\infty}^2\|u\|_{H^{k-1}} + C\|B\|_{L^\infty}\|B\|_{H^{k-2}}\|\bar\nabla u\|_{L^\infty}\\
&\leq \varepsilon\|u\|_{H^k(\Sigma)} + C_\varepsilon\|u\|_{L^2(\Sigma)}.
\end{aligned}
$$

This yields the claim for integers $2\leq k\leq l$.

If $k\leq m-\frac12$ is an half-integer but not an integer, then we may use the previous argument for integer $l=k+\frac12\leq m$ and deduce

$$\|u\|_{H^l}\leq(1+\varepsilon)\|\Delta_\Sigma u\|_{H^{l-2}(\Sigma)} + C_\varepsilon\|u\|_{L^2(\Sigma)}.$$

The same holds for $l-1$. Hence, the claim follows by Proposition 2.7 and by the same interpolation argument we used above. □

By using the previous proposition and the Simon's identity (2.11) we deduce that we may bound the second fundamental form by the mean curvature.

Proposition 2.12. *Assume that Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular. Then for every $p \in (1,\infty)$ it holds*

$$\|B_\Sigma\|_{L^p(\Sigma)} \leq C(1 + \|H_\Sigma\|_{L^p(\Sigma)}).$$

If in addition $\|B_\Sigma\|_{L^4(\Sigma)} \leq M$, then for $k = \frac{1}{2}, 1, 2$ it holds

$$\|B_\Sigma\|_{H^k(\Sigma)} \leq C(1 + \|H_\Sigma\|_{H^k(\Sigma)}).$$

Finally let $m \geq 3$ be an integer and assume that Σ satisfies in addition the condition (H_m) for m. Then the above estimate holds for all half-integers $k \leq m$. The constants depend on M, p, m and on the $C^{1,\alpha}$-norm of the heightfunction.

Proof. The first claim follows from standard Calderon-Zygmund estimate [25] and we omit it. Let us proof the second claim for $k = \frac{1}{2}$. We recall the geometric fact

$$\Delta_\Sigma x_i = -H_\Sigma \nu_i,$$

where $x_i = x \cdot e_i$ and $\nu_i = \nu_\Sigma \cdot e_i$. Then we have by Proposition 2.11 and (2.17)

$$\begin{aligned}
\|B_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)} &\leq C \sum_{i=1}^{3} (1 + \|\nabla_\Sigma^2 x_i\|_{H^{\frac{1}{2}}(\Sigma)}) \leq \sum_{i=1}^{3} C(1 + \|\Delta_\Sigma x_i\|_{H^{\frac{1}{2}}(\Sigma)}) \\
&= \sum_{i=1}^{3} C(1 + \|H_\Sigma \nu_i\|_{H^{\frac{1}{2}}(\Sigma)}) \\
&\leq C(1 + \|H_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)} + \|H_\Sigma\|_{L^4(\Sigma)} \|\nu_\Sigma\|_{W^{1,4}(\Sigma)}) \leq C(1 + \|H_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)}).
\end{aligned}$$

The argument for $k = 1$ is similar.

In the case $k = 2$ we use the Simon's identity (2.11) to deduce

$$\|\Delta_\Sigma B\|_{L^2(\Sigma)}^2 \leq \|\bar{\nabla}^2 H\|_{L^2(\Sigma)}^2 + C\|B\|_{L^6(\Sigma)}^6.$$

Proposition 2.11 yields $\|B\|_{H^2(\Sigma)} \leq 2\|\Delta_\Sigma B\|_{L^2(\Sigma)} + C$. The claim then follows from interpolation inequality (Proposition 2.8)

$$\|B\|_{L^6(\Sigma)}^6 \leq \|B\|_{H^2(\Sigma)}^{\frac{2}{3}} \|B\|_{L^4(\Sigma)}^{\frac{16}{3}} \leq \varepsilon \|B\|_{H^2(\Sigma)} + C_\varepsilon.$$

Let us then fix $m \geq 3$, assume that Σ satisfies the condition (H_m) for m and let $k \leq m$. We use the Simon's identity (2.11) and Proposition 2.10 to deduce

$$\begin{aligned}
\|\Delta_\Sigma B\|_{H^{k-2}(\Sigma)} &\leq \|H\|_{H^k(\Sigma)} + C\|B \star B \star B\|_{H^{k-2}(\Sigma)} \\
&\leq \|H\|_{H^k(\Sigma)} + C\|B\|_{L^\infty(\Sigma)}^2 \|B\|_{H^{k-2}(\Sigma)} \\
&\leq \|H\|_{H^k(\Sigma)} + \varepsilon\|B\|_{H^k(\Sigma)} + C_\varepsilon,
\end{aligned}$$

where the last inequality follows from $\|B\|_{L^\infty} \leq C$ and from interpolation. The claim then follows from Proposition 2.11. □

Note that by the definition of the space $\|\cdot\|_{H^k(\Sigma)}$ in Definition 2.6 it is not yet clear if it holds

$$\|\nabla_\tau u\|_{H^{k-1}(\Sigma)} \leq C\|u\|_{H^k(\Sigma)}$$

when k is not an integer. We conclude this by section by proving this in the following technical lemma.

Lemma 2.13. *Let m be an integer with $m \geq 3$ and assume that Σ is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the condition (H_m). Then it holds*

$$\|\nabla_\tau u\|_{H^{m-\frac{3}{2}}(\Sigma)} \leq C\|u\|_{H^{m-\frac{1}{2}}(\Sigma)}.$$

Proof. Let us denote $\tilde{u} = u \circ \Psi : \Gamma \to \mathbb{R}$ and in order to simplify the notation denote the extension given by the projection in (2.7) ($\tilde{u} \circ \pi_\Gamma$) simply by $\tilde{u}$. We observe that there is a matrix field $A(x) = A(x, h, \bar{\nabla} h)$ such that

$$(\nabla_\tau u \circ \Psi)(x) = A(x) \nabla \tilde{u}(x) \qquad \text{for } x \in \Gamma.$$

Therefore we have by Definition 2.6 and by Proposition 2.10

$$\begin{aligned}\|\nabla_\tau u\|_{H^{m-\frac{3}{2}}(\Sigma)} &= \|\nabla_\tau u \circ \Psi\|_{H^{m-\frac{3}{2}}(\Gamma)} = \|A\, \nabla \tilde{u}\|_{H^{m-\frac{3}{2}}(\Gamma)} \\ &\le C\|A\|_{L^\infty} \|\nabla \tilde{u}\|_{H^{m-\frac{3}{2}}(\Gamma)} + C\|A\|_{H^{m-\frac{3}{2}}(\Gamma)} \|\nabla \tilde{u}\|_{L^\infty}.\end{aligned}$$

The assumption $\|B\|_{L^\infty(\Sigma)}, \|B\|_{H^{m-2}(\Sigma)} \le C$ implies for the height function $\|h\|_{C^2(\Gamma)} \le C$ and $\|h\|_{H^m(\Gamma)} \le C$ and therefore $\|A\|_{L^\infty(\Gamma)}, \|A\|_{H^{m-1}(\Gamma)} \le C$. Moreover, since $m \ge 3$ the Sobolev embedding yields $\|\nabla \tilde{u}\|_{L^\infty(\Gamma)} \le C\|\nabla \tilde{u}\|_{H^{m-\frac{3}{2}}(\Gamma)}$. Therefore we have

$$\|\nabla_\tau u\|_{H^{m-\frac{3}{2}}(\Sigma)} \le C\|\nabla \tilde{u}\|_{H^{m-\frac{3}{2}}(\Gamma)} \le C\|\tilde{u}\|_{H^{m-\frac{1}{2}}(\Gamma)} = C\|u\|_{H^{m-\frac{1}{2}}(\Sigma)}.$$

□

3. Elliptic Estimates for Vector Fields and Functions

In this section we recall some known and provide some new div-curl type estimates for vector fields in the domain, i.e., $F : \Omega \to \mathbb{R}^3$. We will need estimates where we control the norm $\|F\|_{H^k(\Omega)}$ by the $\operatorname{div} F$, $\operatorname{curl} F$ in Ω and with F_n on the boundary Σ. The main result of the section is Theorem 3.6 where we prove this estimate for $k = 1$ and require the boundary merely to satisfy $\|B_\Sigma\|_{L^4} \le C$. We do not expect the L^4-integrability to be the optimal condition. However, related to this we note that we may construct a cone $\Omega \subset \mathbb{R}^3$ and a harmonic function $u : \Omega \to \mathbb{R}$ with zero Neumann boundary data $\partial_\nu u = 0$ arguing as in [21, Section 3], such that u can be written in spherical coordinates as $u(\rho, \theta) = \sqrt{\rho} f(\theta)$ for a smooth functions f. In particular, $u \notin H^2(\Omega \cap B_R)$ and therefore we may deduce that a necessary condition for the curvature is at least $\|B_\Sigma\|_{L^2} \le C$ for Lemma 3.5 and Theorem 3.6 to hold.

We will also prove boundary regularity estimates for harmonic functions in Theorem 3.9, which quantify the boundary regularity of the harmonic functions with respect to the regularity of the boundary. We note that in Theorem 3.9 it is crucial to assume that the boundary is uniformly $C^{1,\alpha}(\Gamma)$-regular. Indeed, the statement does not hold for Lipschitz domains.

3.1. Regularity Estimates for Vector Fields

We begin this section by recalling the following result which is essentially from [10] (see also [53]). Recall that we define

$$\operatorname{curl} F = \nabla F - (\nabla F)^T.$$

Throughout the section we assume that Ω is connected, but its boundary $\Sigma = \partial\Omega$ may have many components.

Theorem 3.1. *Let $l \ge 2$ be an integer and let Ω be a domain such that $\Sigma = \partial\Omega$ is uniformly $C^{1,\alpha}(\Gamma)$-regular and $\|B_\Sigma\|_{H^{\frac{3}{2}l-1}(\Sigma)} \le M$. Then there exists a constant C, which depends on M, l and on the $C^{1,\alpha}$-norm of the heightfunction, such that for all smooth vector fields $F : \Omega \to \mathbb{R}^3$ and every half-integers $1 \le k \le \frac{3}{2}l$ it holds*

$$\|F\|_{H^k(\Omega)} \le C(\|F_n\|_{H^{k-\frac{1}{2}}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}}).$$

Moreover, for $k = \lfloor \frac{3}{2}(l+1) \rfloor$ it holds

$$\|F\|_{H^k(\Omega)} \le C(\|\nabla_\tau F_n\|_{H^{k-\frac{3}{2}}(\Sigma)} + (1 + \|B_\Sigma\|_{H^{\frac{3}{2}l}})\|F\|_{L^\infty} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}}).$$

Proof. We first note that the assumption $\|B\|_{H^{\frac{3}{2}l-1}(\Sigma)} \leq M$ implies that Σ satisfies the condition (H_m) for $m = \lfloor \frac{3}{2}l + 1 \rfloor \geq 4$. We use [10, Theorem 1.3] to deduce

$$\|F\|_{H^k(\Omega)} \leq C(\|\nabla_\tau F \cdot \nu\|_{H^{k-\frac{3}{2}}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)})$$

for all $k \leq \lfloor \frac{3}{2}(l+1) \rfloor$. We write $\nabla_\tau F \cdot \nu = \nabla_\tau F_n + F \star B$ and use Proposition 2.10 to obtain

$$\|\nabla_\tau F \cdot \nu\|_{H^{k-\frac{3}{2}}(\Sigma)} \leq \|\nabla_\tau F_n\|_{H^{k-\frac{3}{2}}(\Sigma)} + C\|F\|_{L^\infty}\|B\|_{H^{k-\frac{3}{2}}(\Sigma)} + C\|B\|_{L^\infty}\|F\|_{H^{k-\frac{3}{2}}(\Sigma)}.$$

Interpolation inequality yields

$$\|F\|_{H^{k-\frac{3}{2}}(\Sigma)} \leq \|F\|_{H^{k-1}(\Omega)} \leq \varepsilon\|F\|_{H^k(\Omega)} + C_\varepsilon\|F\|_{L^\infty(\Omega)}.$$

Thus we have the second inequality. The first one follows from the fact that for $k \leq \frac{3}{2}l$ it holds $\|B\|_{H^{k-\frac{3}{2}}(\Sigma)} \leq M$ by the assumption. □

We combine Proposition 2.11 and Theorem 3.1 and obtain the following inequality which is suitable to our purpose.

Proposition 3.2. *Let l and Ω be as in Theorem 3.1. Then for all smooth vector fields $F : \Omega \to \mathbb{R}^3$ and every half-integer $\frac{5}{2} \leq k \leq \frac{3}{2}l$ it holds*

$$\|F\|_{H^k(\Omega)} \leq C(\|\Delta_\Sigma F_n\|_{H^{k-\frac{5}{2}}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}).$$

Moreover, for $k = \lfloor \frac{3}{2}(l+1) \rfloor$ it holds

$$\|F\|_{H^k(\Omega)} \leq C(\|\Delta_\Sigma F_n\|_{H^{k-\frac{5}{2}}(\Sigma)} + (1 + \|B\|_{H^{\frac{3}{2}l}})\|F\|_{L^\infty} + \|\operatorname{div} F\|_{H^{k-1}(\Omega)} + \|\operatorname{curl} F\|_{H^{k-1}(\Omega)}).$$

Proof. Recall that $\|B\|_{H^{\frac{3}{2}l-1}(\Sigma)} \leq M$ implies that Σ satisfies the condition (H_m) for $m = \lfloor \frac{3}{2}l + 1 \rfloor \geq 4$. The first inequality then follows from Theorem 3.1 and Proposition 2.11.

Let us then prove the last inequality. We have by Proposition 2.11 that

$$\|\nabla_\tau F_n\|_{H^{k-\frac{3}{2}}(\Sigma)} \leq C(\|\Delta_\Sigma \nabla_\tau F_n\|_{H^{k-\frac{7}{2}}(\Sigma)} + \|F_n\|_{L^2(\Sigma)}).$$

We use the commutation formula (2.9) for the tangential gradient of $u : \Sigma \to \mathbb{R}$ and obtain

$$\Delta_\Sigma(\nabla_\tau u) = \nabla_\tau(\Delta_\Sigma u) + (B \star B) \star \nabla_\tau u.$$

Therefore we have by Lemma 2.13

$$\begin{aligned}\|\Delta_\Sigma \nabla_\tau F_n\|_{H^{k-\frac{7}{2}}(\Sigma)} &\leq C\left(\|\nabla_\tau(\Delta_\Sigma F_n)\|_{H^{k-\frac{7}{2}}(\Sigma)} + \|(B \star B) \star \nabla_\tau F_n\|_{H^{k-\frac{7}{2}}(\Sigma)}\right)\\ &\leq C(\|\Delta_\Sigma F_n\|_{H^{k-\frac{5}{2}}(\Sigma)} + \|(B \star B) \star \nabla_\tau F_n\|_{H^{k-\frac{7}{2}}(\Sigma)}).\end{aligned}$$

Recall that $k = \lfloor \frac{3}{2}(l+1) \rfloor$. We have by Proposition 2.10, by Lemma 2.13 and by the assumption $\|B\|_{H^{\frac{3}{2}l-1}(\Sigma)} \leq C$ that

$$\begin{aligned}\|(B \star B) \star \nabla_\tau F_n\|_{H^{k-\frac{7}{2}}(\Sigma)} &\leq C\left(\|B\|_{L^\infty}^2\|\nabla_\tau F_n\|_{H^{k-\frac{7}{2}}} + \|B\|_{L^\infty}\|B\|_{H^{k-\frac{7}{2}}}\|\nabla_\tau F_n\|_{L^\infty}\right)\\ &\leq C(\|F_n\|_{H^{k-\frac{5}{2}}(\Sigma)} + \|\nabla_\tau F_n\|_{L^\infty(\Sigma)}).\end{aligned}$$

Finally we have by the Sobolev embedding and by Corollary 2.9

$$\|\nabla_\tau F_n\|_{L^\infty(\Sigma)} + \|F_n\|_{H^{k-\frac{5}{2}}(\Sigma)} \leq \varepsilon\|\nabla_\tau F_n\|_{H^{k-\frac{3}{2}}(\Sigma)} + \varepsilon\|F\|_{H^k(\Omega)} + C_\varepsilon\|F\|_{L^\infty}.$$

The second inequality then follows from Theorem 3.1 and combining the above inequalities. □

Proposition 3.2 provides the inequality we need when we have the bound $\|B_\Sigma\|_{H^{\frac{3}{2}l-1}(\Sigma)} \leq C$ for $l \geq 2$. When $l = 1$ the above bound reduces to $\|B_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)}$, which is the bound that we are able to prove in Sect. 6, but is not enough to apply the results from [10,53]. Note that by the Sobolev embedding this implies $\|B_\Sigma\|_{L^4(\Sigma)} \leq C$. We need to work more in order to prove the first inequality in Theorem 3.1 under the assumption $\|B_\Sigma\|_{L^4(\Sigma)} \leq C$.

We begin by recalling the following Reilly's type identity for vector fields. First, if $\psi : \Omega \to \mathbb{R}^3$ is a smooth divergence free vector field such that $\psi \cdot \nu = 0$ on Σ then it holds

$$\|\nabla\psi\|^2_{L^2(\Omega)} = \frac{1}{2}\|\mathrm{curl}\,\psi\|^2_{L^2(\Omega)} - \int_\Sigma \langle B_\Sigma\,\psi, \psi\rangle\, d\mathcal{H}^2. \tag{3.1}$$

Second, if $u : \Omega \to \mathbb{R}$ is a smooth function then it holds

$$\begin{aligned}\|\nabla^2 u\|^2_{L^2(\Omega)} =& \|\Delta u\|^2_{L^2(\Omega)} - 2\int_\Sigma \Delta_\Sigma u\, \partial_\nu u\, d\mathcal{H}^2 \\ &- \int_\Sigma \langle B_\Sigma \bar\nabla u, \bar\nabla u\rangle\, d\mathcal{H}^2 - \int_\Sigma H_\Sigma (\partial_\nu u)^2\, d\mathcal{H}^2.\end{aligned} \tag{3.2}$$

We give the calculations for (3.1) and (3.2) for the reader's convenience. First, for a generic smooth vector field $F : \Omega \to \mathbb{R}^3$ it holds

$$\begin{aligned}\int_\Omega |\,\mathrm{div}\, F|^2 + \frac{1}{2}|\mathrm{curl}\, F|^2\, dx &= \sum_{i,j=1}^3 \int_\Omega (\partial_i F_j)^2\, dx + \sum_{i,j=1}^3 \int_\Omega (\partial_i F_i \partial_j F_j - \partial_i F_j \partial_j F_i)\, dx \\ &= \|\nabla F\|^2_{L^2(\Omega)} + \sum_{i,j=1}^3 \int_\Omega (\partial_i F_i \partial_j F_j - \partial_i F_j \partial_j F_i)\, dx.\end{aligned}$$

By using divergence theorem twice we obtain

$$\begin{aligned}\int_\Omega \partial_i F_i \partial_j F_j\, dx &= -\int_\Omega F_i \partial_i \partial_j F_j\, dx + \int_\Sigma \partial_j F_j\, F_i \nu_i,\ d\mathcal{H}^2 \\ &= \int_\Omega \partial_j F_i \partial_i F_j\, dx + \int_\Sigma \partial_j F_j\, F_i \nu_i,\ d\mathcal{H}^2 - \int_\Sigma F_i \partial_i F_j \nu_j\ d\mathcal{H}^2.\end{aligned}$$

Combining the two above equalities yield

$$\begin{aligned}\|\nabla F\|^2_{L^2(\Omega)} &= \|\,\mathrm{div}\, F\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mathrm{curl}\, F\|^2_{L^2(\Omega)} \\ &\quad + \int_\Sigma (\nabla F\, F) \cdot \nu\ d\mathcal{H}^2 - \int_\Sigma \mathrm{div}\, F\, F_n\, d\mathcal{H}^2.\end{aligned} \tag{3.3}$$

Assume now that ψ is a divergence free vector field such that $\psi \cdot \nu = 0$ on Σ. Since ψ is a tangent field on Σ we have $\nabla_\psi(\psi \cdot \nu) = 0$ and thus $\nabla\psi\, \psi \cdot \nu = -\langle \bar\nabla_\psi \nu, \psi\rangle = -\langle B_\Sigma \psi, \psi\rangle$. Therefore the equality (3.1) follows from (3.3) and from $F_n = \psi \cdot \nu = 0$.

To obtain (3.2) we apply (3.3) for $F = \nabla u$ and deduce

$$\|\nabla^2 u\|^2_{L^2(\Omega)} = \|\Delta u\|^2_{L^2(\Omega)} + \int_\Sigma ((\nabla^2 u\, \nabla u) \cdot \nu - \Delta u\, \partial_\nu u)\, d\mathcal{H}^2.$$

We write $\nabla u = \nabla_\tau u + \partial_\nu u\, \nu$ and observe

$$\begin{aligned}(\nabla^2 u\, \nabla u) \cdot \nu &= (\nabla^2 u\, \nabla_\tau u) \cdot \nu + (\nabla^2 u\, \nu) \cdot \nu\, \partial_\nu u \\ &= \nabla_\tau(\partial_\nu u) \cdot \nabla_\tau u - \langle B_\Sigma\, \bar\nabla u, \bar\nabla u\rangle + (\nabla^2 u\, \nu) \cdot \nu\, \partial_\nu u.\end{aligned}$$

Again by divergence theorem

$$\int_\Sigma \nabla_\tau(\partial_\nu u) \cdot \nabla_\tau u\, d\mathcal{H}^2 = -\int_\Sigma \Delta_\Sigma u\, \partial_\nu u\, d\mathcal{H}^2.$$

The equality (3.2) then follows from

$$\Delta_\Sigma u = \Delta u - (\nabla^2 u\,\nu)\cdot\nu - H_\Sigma\,\partial_\nu u. \tag{3.4}$$

We remark that it is crucial in Theorem 3.1 that the boundary term on the RHS has only the normal component of the vector field. The next lemma is a generalization of [35] and it essentially states that we may control the vector field on the boundary by its normal or its tangential component.

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^3$ with $\Sigma = \partial\Omega$ be uniformly $C^{1,\alpha}(\Gamma)$-regular. Then for all vector fields $F \in \dot H^1(\Omega;\mathbb{R}^3)$ it holds*

$$\|F\|^2_{L^2(\Sigma)} \le C\left(\|F_n\|^2_{L^2(\Sigma)} + \|F\|^2_{L^2(\Omega)} + \|\operatorname{div} F\|^2_{L^2(\Omega)} + \|\operatorname{curl} F\|^2_{L^2(\Omega)}\right)$$

and

$$\|F\|^2_{L^2(\Sigma)} \le C\left(\|F_\tau\|^2_{L^2(\Sigma)} + \|F\|^2_{L^2(\Omega)} + \|\operatorname{div} F\|^2_{L^2(\Omega)} + \|\operatorname{curl} F\|^2_{L^2(\Omega)}\right),$$

where $F_n = F\cdot\nu$ and $F_\tau = F - F_n\,\nu$. Here $F \in \dot H^1(\Omega;\mathbb{R}^3)$ means that $\|F\|_{\dot H^1(\Omega)} = \|\nabla F\|_{L^2(\Omega)} + \|F\|_{L^6(\Omega)} < \infty$. Note that Ω may be unbounded, but its boundary is compact.

Proof. We only consider the case when Ω is bounded. Let us first assume that Ω is uniformly star shaped with respect to the origin, i.e., we have that there exists a constant $c_0 > 0$ such that $x\cdot\nu_\Omega \ge c_0$ for all $x \in \Sigma$. We claim that the following identity holds

$$\operatorname{div}\left(|F|^2 x - 2(F\cdot x)F\right) = |F|^2 - 2\operatorname{curl} F(F\cdot x) - 2\operatorname{div} F\,(F\cdot x). \tag{3.5}$$

Indeed, this follows from the following straightforward computation, where we denote the Dirac delta by δ_i^j,

$$\begin{aligned}\operatorname{div}(|F|^2x - 2(F\cdot x)F) &= \sum_{i,j=1}^3 \partial_i\left(F_j^2 x_i - 2x_jF_jF_i\right)\\ &= 3|F|^2 + \sum_{i,j=1}^3 2x_i\partial_iF_jF_j - 2\delta_i^jF_jF_i - 2x_j\partial_iF_jF_i - 2x_jF_j\partial_iF_i\\ &= |F|^2 - 2(F\cdot x)\operatorname{div} F - 2\sum_{i,j=1}^3(\partial_jF_i - \partial_iF_j)F_jx_i.\end{aligned}$$

Thus we integrate (3.5) to find

$$\int_\Sigma \left((x\cdot\nu)|F|^2 - 2F_n(F\cdot x)\right)d\mathcal{H}^2 = \int_\Omega |F|^2 - 2\operatorname{curl} F(F\cdot x) - 2\operatorname{div} F(F\cdot x)\,dx.$$

Note that $|F|^2 = |F_\tau|^2 + F_n^2$ and $(F\cdot x) = (x\cdot\nu)F_n + (F_\tau\cdot x)$. Therefore we have the equality

$$\begin{aligned}&\int_\Sigma \left(-(x\cdot\nu)F_n^2 + (x\cdot\nu)|F_\tau|^2 - 2F_n(F_\tau\cdot x)\right)d\mathcal{H}^2\\ &\qquad = \int_\Omega |F|^2 + 2\operatorname{curl} F(F\cdot x) - 2\operatorname{div} F(F\cdot x)\,dx.\end{aligned}$$

We use the fact that $x\cdot\nu \ge c_0$ on Σ and obtain the first claim by re-organizing the terms in above and estimating $|F_n(F_\tau\cdot x)| \le \varepsilon|F_\tau|^2 + C_eF_n^2$

$$\begin{aligned}c_0\int_\Sigma |F_\tau|^2\,d\mathcal{H}^2 &\le \int_\Sigma(\varepsilon|F_\tau|^2 + C_\varepsilon|F_n|^2)\,d\mathcal{H}^2\\ &\quad + C(\|F\|^2_{L^2(\Omega)} + \|\operatorname{curl} F\|^2_{L^2(\Omega)} + \|\operatorname{div} F\|^2_{L^2(\Omega)}).\end{aligned}$$

This yields the first inequality. The second follows from similar argument.

To prove the general case, i.e. when Ω is not starshaped, we use a localization argument which is similar to [1]. □

Remark 3.4. We observe that the proof gives us a slightly stronger estimate. Indeed, we may improve the second inequality in Lemma 3.3 as

$$\|F\|_{L^2(\Sigma)}^2 \le C\left(\|F_\tau\|_{L^2(\Sigma)}^2 + \|F\|_{L^2(\Omega)}^2 + \|F \operatorname{div} F\|_{L^1(\Omega)} + \|F \operatorname{curl} F\|_{L^1(\Omega)})\right).$$

In order to estimate $\|\nabla F\|_{H^1(\Omega)}$ we first consider the case when F is curl-free, i.e., $F = \nabla u$. Since we define the norm $\|\partial_\nu u\|_{H^{\frac{1}{2}}(\Sigma)}$ via the harmonic extension, we prove the next lemma using standard results from harmonic analysis instead of localizing and flattening the boundary.

Lemma 3.5. *Assume that Ω, with $\Sigma = \partial\Omega$, is uniformly $C^{1,\alpha}(\Gamma)$-regular and $\|B_\Sigma\|_{L^4} \le M$ and $u : \Omega \to \mathbb{R}$ is a smooth function. There exists a constant C, depending on M and the $C^{1,\alpha}$-norm of the heightfunction, such that it holds*

$$\|u\|_{H^2(\Omega)} \le C\left(\|\partial_\nu u\|_{H^{\frac{1}{2}}(\Sigma)} + \|u\|_{L^2(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right).$$

The reverse also holds $\|\partial_\nu u\|_{H^{\frac{1}{2}}(\Sigma)} \le C\|u\|_{H^2(\Omega)}$.

Proof. We have by (3.2) and (3.4) that

$$\|\nabla^2 u\|_{L^2(\Omega)}^2 \le \|\Delta u\|_{L^2(\Omega)}^2 + 2\int_\Sigma ((\nabla^2 u\, \nu)\cdot \nu - \Delta u)\, \partial_\nu u \, d\mathcal{H}^2 + C\int_\Sigma |B_\Sigma|\, |\nabla u|^2 \, d\mathcal{H}^2.$$

To estimate the last terms, we use the interpolation inequality (Proposition 2.8), Lemma 3.3 and the assumption $\|B\|_{L^4} \le C$ and have for some $\theta \in (0,1)$

$$\begin{aligned}\int_\Sigma |B_\Sigma|\, |\nabla u|^2 \, d\mathcal{H}^2 &\le C\|B\|_{L^4(\Sigma)} \|\nabla u\|_{L^{\frac{8}{3}}(\Sigma)}^2 \le \|\nabla u\|_{H^{\frac{1}{2}}(\Sigma)}^{2\theta} \|\nabla u\|_{L^2(\Sigma)}^{2(1-\theta)} \\ &\le \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon \|\nabla u\|_{L^2(\Sigma)}^2 \\ &\le \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon(\|\partial_\nu u\|_{L^2(\Sigma)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2) \\ &\le \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon(\|\partial_\nu u\|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.6}$$

Let us then show that

$$\int_\Sigma ((\nabla^2 u\, \nu)\cdot \nu - \Delta u)\, \partial_\nu u \, d\mathcal{H}^2 \le \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon(\|\partial_\nu u\|_{H^{\frac{1}{2}}(\Sigma)}^2 + \|u\|_{L^2(\Omega)}^2 + \|\Delta u\|_{L^2(\Omega)}^2). \tag{3.7}$$

To this aim we denote the harmonic extension of ν by $\tilde{\nu}$ and denote $f = \Delta u$. Then we have

$$\begin{aligned}\int_\Sigma (\nabla^2 u\, \nu)\cdot \nu\, \partial_\nu u \, d\mathcal{H}^2 &= \int_\Sigma \big(\partial_\nu(\nabla u \cdot \tilde{\nu}) - (\partial_\nu \tilde{\nu} \cdot \nabla u)\big)\, (\nabla u \cdot \tilde{\nu})\, d\mathcal{H}^2 \\ &\le \int_\Sigma \partial_\nu(\nabla u \cdot \tilde{\nu})\, (\nabla u \cdot \tilde{\nu})\, d\mathcal{H}^2 + C\|\partial_\nu \tilde{\nu}\|_{L^4(\Sigma)} \|\nabla u\|_{L^{\frac{8}{3}}(\Sigma)}^2.\end{aligned}$$

We argue as in (3.6) and obtain

$$\|\nabla u\|_{L^{\frac{8}{3}}(\Sigma)}^2 \le \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon(\|\partial_\nu u\|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2).$$

Next we use the result from [19] for harmonic functions $\varphi : \Omega \to \mathbb{R}$ in $C^{1,\alpha}$-domains which states that

$$\|\partial_\nu \varphi\|_{L^p(\Sigma)} \le C_p \|\nabla_\tau \varphi\|_{L^p(\Sigma)} \qquad \text{for } p \in (1,\infty).$$

We use this for $\tilde{\nu}$ component-wise, use the fact that on Σ it holds $\tilde{\nu} = \nu$ and obtain

$$\|\nabla \tilde{\nu}\|_{L^4(\Sigma)} \le C\|\nabla_\tau \tilde{\nu}\|_{L^4(\Sigma)} \le \|B\|_{L^4(\Sigma)} \le C. \tag{3.8}$$

Therefore we have

$$\begin{aligned}\int_\Sigma (\nabla^2 u\, \nu\cdot)\nu\, \partial_\nu u \, d\mathcal{H}^2 \le &\int_\Sigma \partial_\nu(\nabla u \cdot \tilde{\nu})\, (\nabla u \cdot \tilde{\nu})\, d\mathcal{H}^2 \\ &+ \varepsilon\|u\|_{H^2(\Omega)}^2 + C_\varepsilon(\|\partial_\nu u\|_{L^2(\Sigma)}^2 + \|u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2).\end{aligned} \tag{3.9}$$

Let us denote $\tilde u = (\nabla u \cdot \tilde \nu)$ for short and let v be the harmonic extension of $\tilde u$ to Ω. Let us show that v is close to $\tilde u$, i.e., we show that

$$\|\nabla(\tilde u - v)\|_{L^2(\Omega)} \le \varepsilon \|\nabla^2 u\|_{L^2(\Omega)} + C_\varepsilon(\|f\|_{L^2(\Omega)} + \|\tilde u - v\|_{L^2(\Omega)}). \tag{3.10}$$

To this aim we calculate (recall that $f = \Delta u$)

$$\Delta \tilde u = \nabla f \cdot \tilde \nu + 2\nabla^2 u : \nabla \tilde \nu. \tag{3.11}$$

This implies by integration by parts

$$\begin{aligned}
\|\nabla(\tilde u - v)\|_{L^2(\Omega)}^2 &= -\int_\Omega \Delta \tilde u (\tilde u - v)\, dx = \int_\Omega (\nabla f \cdot \tilde \nu + 2\nabla^2 u : \nabla \tilde\nu)(\tilde u - v)\, dx\\
&= \int_\Omega f \operatorname{div}((\tilde u - v)\, \tilde \nu) - 2(\nabla^2 u : \nabla \tilde \nu)(\tilde u - v)\, dx\\
&\le C\|\nabla(\tilde u - v)\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} + C(\|\nabla^2 u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)})\|\nabla \tilde \nu\|_{L^4(\Omega)}\|(\tilde u - v)\|_{L^4(\Omega)}.
\end{aligned}$$

By standard estimates from harmonic analysis [16] and by (3.8) it holds

$$\|\nabla \tilde \nu\|_{L^4(\Omega)} \le C\|\nabla \tilde \nu\|_{L^4(\Sigma)} \le C. \tag{3.12}$$

On the other hand we have by Hölder's inequality and by Sobolev embedding (recall that $\tilde u - v = 0$ on Σ)

$$\|(\tilde u - v)\|_{L^4(\Omega)} \le \|(\tilde u - v)\|_{L^6(\Omega)}^{\frac12}\|\tilde u - v\|_{L^2(\Omega)}^{\frac12} \le C\|\nabla(\tilde u - v)\|_{L^2(\Omega)}^{\frac12}\|\tilde u - v\|_{L^2(\Omega)}^{\frac12}.$$

Therefore by combining the previous inequalities we obtain (3.10) by Young's inequality.

We proceed by using (3.11) and by integrating by parts

$$\begin{aligned}
\int_\Sigma \partial_\nu \tilde u\, \tilde u\, d\mathcal{H}^2 &= \int_\Sigma \partial_\nu \tilde u\, v\, d\mathcal{H}^2 = \int_\Omega (\nabla \tilde u \cdot \nabla v + \Delta \tilde u\, v)\, dx\\
&\le 2\|\nabla v\|_{L^2(\Omega)}^2 + 2\|\nabla(\tilde u - v)\|_{L^2(\Omega)}^2 + \int_\Omega (\nabla f \cdot \tilde \nu + 2\nabla^2 u : \nabla \tilde \nu)\, v\, dx\\
&= 2\|\nabla v\|_{L^2(\Omega)}^2 + 2\|\nabla(\tilde u - v)\|_{L^2(\Omega)}^2 + \int_\Sigma f\, v\, (\tilde \nu \cdot \nu)\, d\mathcal{H}^2\\
&\quad + \int_\Omega -f \operatorname{div}(v\, \tilde \nu) + 2(\nabla^2 u : \nabla \tilde \nu) v\, dx.
\end{aligned}$$

Recall that $\tilde \nu = \nu$ and $v = \tilde u$ on Σ. Therefore we obtain by the above inequality, by $\|v\|_{L^4(\Omega)} \le C\|v\|_{H^1(\Omega)}$, (3.10) and (3.12) that

$$\begin{aligned}
\int_\Sigma \partial_\nu \tilde u\, \tilde u - f\, \tilde u\, d\mathcal{H}^2 &\le \varepsilon \|\nabla^2 u\|_{L^2(\Omega)}\\
&\quad + C_\varepsilon(\|\nabla \nu\|_{L^4(\Omega)}\|v\|_{L^4(\Omega)} + \|v\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|\tilde u - v\|_{L^2(\Omega)}^2)\\
&\le \varepsilon \|\nabla^2 u\|_{L^2(\Omega)} + C_\varepsilon(\|v\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|\tilde u - v\|_{L^2(\Omega)}^2).
\end{aligned}$$

The inequality (3.7) then follows from the above and (3.9) together with

$$\|v\|_{H^1(\Omega)} = \|\partial_\nu u\|_{H^{\frac12}(\Sigma)}, \qquad \|\tilde u - v\|_{L^2(\Omega)}^2 \le \varepsilon \|\nabla^2 u\|_{L^2(\Omega)} + C_\varepsilon \|u\|_{L^2(\Omega)}^2 + \|v\|_{L^2(\Omega)}^2,$$

and by recalling that $\tilde u = (\nabla u \cdot \tilde \nu) = \partial_\nu u$ on Σ and $f = \Delta u$. This yields the first claim. The second inequality follows from reversing the previous calculations. □

We state our lower order version of Theorem 3.1.

Theorem 3.6. *Assume that Ω, with $\Sigma = \partial\Omega$, is uniformly $C^{1,\alpha}(\Gamma)$-regular and $\|B_\Sigma\|_{L^4(\Sigma)} \le M$. There exists a constant C, depending on M and the $C^{1,\alpha}$-norm of the heightfunction, such that for all vector fields $F \in H^1(\Omega;\mathbb{R}^3)$ it holds*

$$\|F\|_{H^1(\Omega)} \le M(\|F_n\|_{H^{\frac12}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{L^2(\Omega)} + \|\operatorname{curl} F\|_{L^2(\Omega)}).$$

Proof. By approximation argument we may assume that F and Ω are smooth. We use the Helmholtz-Hodge decomposition and write $F = \nabla\phi + \psi$ where ϕ is the unique solution of the Neumann problem

$$\begin{cases} \Delta\phi = \operatorname{div} F & x \in \Omega \\ \partial_\nu \phi = F_n & x \in \Sigma \end{cases}$$

with zero average and ψ solves

$$\begin{cases} \operatorname{curl}\psi = \operatorname{curl} F & x \in \Omega \\ \operatorname{div}\psi = 0 & x \in \Omega \\ \psi \cdot \nu = 0 & x \in \Sigma. \end{cases}$$

We also note that $\nabla\phi$ and ψ are orthogonal in $L^2(\Omega)$ and thus

$$\int_\Omega |\nabla\phi|^2 + |\psi|^2 dx = \int_\Omega |F|^2 dx.$$

For ϕ we have by Lemma 3.5 that

$$\begin{aligned} \|\phi\|_{H^2(\Omega)} &\le C(\|\partial_\nu\phi\|_{H^{\frac12}(\Sigma)} + \|\nabla\phi\|_{L^2(\Omega)} + \|\Delta\phi\|_{L^2(\Omega)}) \\ &\le C(\|F_n\|_{H^{\frac12}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{L^2(\Omega)}). \end{aligned} \tag{3.13}$$

For ψ we have by (3.1)

$$\|\nabla\psi\|^2_{L^2(\Omega)}\, dx = \frac12 \|\operatorname{curl}\psi\|^2_{L^2(\Omega)} - \int_\Sigma \langle B_\Sigma\, \psi, \psi\rangle \, d\mathcal{H}^2.$$

We use the assumption $\|B_\Sigma\|_{L^4} \le C$, Hölder's inequality and interpolation inequality to deduce

$$\begin{aligned} -\int_\Sigma \langle B_\Sigma\, \psi, \psi\rangle \, d\mathcal{H}^2 &\le \|B_\Sigma\|_{L^4(\Sigma)} \|\psi\|^2_{L^{\frac83}(\Sigma)} \\ &\le \varepsilon \|\psi\|^2_{H^{\frac12}(\Sigma)} + C_\varepsilon \|\psi\|^2_{L^2(\Omega)} \le \varepsilon \|\nabla\psi\|^2_{L^2(\Omega)} + C_\varepsilon \|F\|^2_{L^2(\Omega)}. \end{aligned}$$

Thus we have

$$\|\nabla\psi\|_{L^2(\Omega)} \le C(\|\operatorname{curl} F\|_{L^2(\Omega)} + \|F\|_{L^2(\Omega)})$$

This together with (3.13) yields the claim. □

We proceed by using Theorem 3.6 to control the higher order norms $\|F\|_{H^2(\Omega)}$ and $\|F\|_{H^3(\Omega)}$. We do not need the sharp dependence on the curvature for these estimates and we state the result in a form that is suitable for us. We also treat the case $\|F\|_{H^{\frac32}(\Omega)}$, but only for curl-free vector fields. In this case we need the 'sharp' curvature dependence but this time we have non-optimal dependence on the divergence.

Lemma 3.7. *Assume that Ω, with $\Sigma = \partial\Omega$, is uniformly $C^{1,\alpha}(\Gamma)$-regular and $\|B_\Sigma\|_{H^{\frac12}(\Sigma)} \le M$. Then for all vector fields $F \in H^3(\Omega; \mathbb{R}^3)$ it holds*

$$\|F\|_{H^3(\Omega)} \le C\big(\|\Delta_\Sigma F_n\|_{H^{\frac12}(\Sigma)} + (1 + \|H_\Sigma\|_{H^2(\Sigma)})\|F\|_{L^\infty(\Omega)} + \|\operatorname{div} F\|_{H^2(\Omega)} + \|\operatorname{curl} F\|_{H^2(\Omega)}\big)$$

and

$$\|F\|_{H^2(\Omega)} \le C\big(\|\Delta_\Sigma F_n\|_{H^{\frac12}(\Sigma)} + \|F\|_{L^\infty(\Omega)} + \|\operatorname{div} F\|_{H^1(\Omega)} + \|\operatorname{curl} F\|_{H^1(\Omega)}\big)$$

for some constant C, depending on M and the $C^{1,\alpha}$-norm of the heightfunction. Moreover, if $F = \nabla u$ then it holds

$$\|\nabla u\|_{H^{\frac32}(\Omega)} \le C(\|\partial_\nu u\|_{H^1(\Sigma)} + \|u\|_{L^2(\Omega)} + \|\Delta u\|_{H^1(\Omega)})$$

and for $k = \frac12, 1$ it holds

$$\|u\|_{H^k(\Sigma)} \le C\|u\|_{H^{k+\frac12}(\Omega)}.$$

Proof. By approximation argument we may assume that F and Ω are smooth. Note also that by Sobolev embedding $\|B_\Sigma\|_{L^4(\Sigma)} \le \|B_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)} \le M$.

Let $\tilde{\nu}$ be the harmonic extension of the normal field ν to Ω. Let us define the vector fields $\tau_i = e_i - (\tilde{\nu} \cdot e_i)\tilde{\nu}$ for $i = 1,2,3$, where $\{e_i\}_i$ is a coordinate basis of $\mathbb{R}^3$. For i,j we define a vector field $F_{ij} : \Omega \to \mathbb{R}^3$ as $(F_{ij})_k = \nabla^2 F_k \tau_i \cdot \tau_j$. We apply Theorem 3.6 for F_{ij} and obtain

$$\|\nabla F_{ij}\|_{L^2(\Omega)} \le C(\|F_{ij}\cdot \nu\|_{H^{\frac{1}{2}}(\Sigma)} + \|F_{ij}\|_{L^2(\Omega)} + \|\operatorname{div} F_{ij}\|_{L^2(\Omega)} + \|\operatorname{curl} F_{ij}\|_{L^2(\Omega)}).$$

Recall that (3.12) implies $\|\nabla \tilde{\nu}\|_{L^4(\Omega)} \le C$. Moreover by maximum principle it holds $\|\tilde{\nu}\|_{L^\infty(\Omega)} \le C$. Therefore

$$\begin{aligned}\|\operatorname{div} F_{ij}\|_{L^2(\Omega)} &\le C\|\operatorname{div} F\|_{H^2(\Omega)} + C\|\nabla^2 F\|_{L^4(\Omega)}\|\nabla \nu\|_{L^4(\Omega)}\\ &\le C\|\operatorname{div} F\|_{H^2(\Omega)} + C\|\nabla^2 F\|_{L^4(\Omega)}.\end{aligned}$$

By interpolation we have $\|\nabla^2 F\|_{L^4(\Omega)} \le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon\|F\|_{L^2(\Omega)}$ and thus

$$\|F_{ij}\|_{L^2(\Omega)} + \|\operatorname{div} F_{ij}\|_{L^2(\Omega)} \le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon(\|\operatorname{div} F\|_{H^2(\Omega)} + \|F\|_{L^2(\Omega)}).$$

By a similar argument

$$\|\operatorname{curl} F_{ij}\|_{L^2(\Omega)} \le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon(\|\operatorname{curl} F\|_{H^2(\Omega)} + \|F\|_{L^2(\Omega)})$$

and

$$\|\nabla F_{ij}\|^2_{L^2(\Omega)} \ge \sum_{k,l=1}^{3} \|\nabla^2 \nabla_l F_k\, \tau_i \cdot \tau_j\|^2_{L^2(\Omega)} - \varepsilon\|F\|^2_{H^3(\Omega)} - C_\varepsilon\|F\|^2_{L^2(\Omega)}.$$

Let us fix a point $x \in \Omega$ and estimate the norm

$$\sum_{i,j,k,l=1}^{3} |\nabla^2 \nabla_l F_k(x)\, \tau_i \cdot \tau_j|^2.$$

First we observe that the above quantity does not depend on the choice of the coordinates in $\mathbb{R}^3$. Let us choose the coordinates such that $\tilde{\nu}(x) \cdot e_i = 0$ for $i = 1,2$. Then we have

$$\sum_{i,j,k,l=1}^{3} |\nabla^2 \nabla_l F_k(x)\, \tau_i \cdot \tau_j|^2 \ge \sum_{k,l=1}^{3}\sum_{i,j=1}^{2} |\nabla_i \nabla_j \nabla_l F_k(x)|^2.$$

By a simple combinatorial argument we deduce

$$\begin{aligned}&\sum_{i,j,k,l=1}^{3} |\nabla_i \nabla_j \nabla_l F_k(x)|^2\\ &\le C\sum_{k,l=1}^{3}\sum_{i,j=1}^{2} |\nabla_i \nabla_j \nabla_l F_k(x)|^2 + C|\nabla^2 \operatorname{div} F(x)|^2 + C|\nabla^2 \operatorname{curl} F(x)|^2.\end{aligned}$$

By applying the above argument for every x we have

$$\sum_{i,j,k,l=1}^{3} \|\nabla F_{ij}\|^2_{L^2(\Omega)} \ge c_0\|\nabla^3 F\|^2_{L^2(\Omega)} - C(\|\operatorname{div} F\|^2_{H^2(\Omega)} + \|\operatorname{curl} F\|^2_{H^2(\Omega)} + \|F\|^2_{L^2(\Omega)}).$$

Combing all the previous estimates we obtain

$$\|\nabla^3 F\|_{L^2(\Omega)} \le \sum_{i,j=1}^{3} C(\|F_{ij}\cdot \nu\|_{H^{\frac{1}{2}}(\Sigma)} + \|F\|_{L^2(\Omega)} + \|\operatorname{div} F\|_{H^2(\Omega)} + \|\operatorname{curl} F\|_{H^2(\Omega)}).$$

The first inequality follows once we show

$$\sum_{i,j=1}^{3} \|F_{ij}\cdot\nu\|_{H^{\frac12}(\Sigma)} \le C\|\Delta_\Sigma F_n\|_{H^{\frac12}(\Sigma)} + \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon(1+\|H_\Sigma\|_{H^2})\|F\|_{L^\infty(\Sigma)}. \tag{3.14}$$

To this aim we first note that on Σ it holds $\tau_i = e_i - (\nu\cdot e_i)\nu$ and therefore τ_i is tangential on Σ. Thus we have

$$(\nabla^2 F_n)\tau_i\cdot\tau_j = F_{ij}\cdot\nu + \nabla_\tau F \star B_\Sigma + F\star\nabla_\tau B_\Sigma. \tag{3.15}$$

We use Proposition 2.10 and get

$$\|\nabla_\tau F\star B_\Sigma\|_{H^{\frac12}(\Sigma)} \le C\|\nabla F\|_{H^{\frac12}(\Sigma)}\|B_\Sigma\|_{L^\infty(\Sigma)} + C\|\nabla_\tau F\|_{L^\infty(\Sigma)}\|B_\Sigma\|_{H^{\frac12}(\Sigma)}.$$

Recall that $\|B_\Sigma\|_{H^{\frac12}(\Sigma)} \le C$. By interpolation we have

$$\|\nabla_\tau F\|_{L^\infty(\Sigma)} \le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon\|F\|_{L^\infty}.$$

Moreover, by Sobolev embedding, Proposition 2.8 and by Proposition 2.12 we have for $\theta < \frac12$

$$\|B_\Sigma\|_{L^\infty(\Sigma)} \le C\|B_\Sigma\|_{W^{1,\frac73}(\Sigma)} \le C\|B_\Sigma\|_{H^2(\Sigma)}^{\theta}\|B_\Sigma\|_{L^4(\Sigma)}^{1-\theta} \le C(1+\|H_\Sigma\|_{H^2(\Sigma)}^{\theta}).$$

On the other hand, Corollary 2.9 implies

$$\|\nabla F\|_{H^{\frac12}(\Sigma)} \le C\|\nabla F\|_{H^1(\Omega)} \le C\|F\|_{H^3(\Omega)}^{\frac12}\|\nabla F\|_{L^2(\Omega)}^{\frac12}.$$

Therefore we have by the above estimates and by Young's inequality

$$\begin{aligned}\|\nabla_\tau F\star B_\Sigma\|_{H^{\frac12}(\Sigma)} &\le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon\|F\|_{L^\infty} + C_\varepsilon(1+\|H_\Sigma\|_{H^2(\Sigma)}^{\theta})\|\nabla F\|_{L^2(\Omega)}\\ &\le \varepsilon\|F\|_{H^3(\Omega)} + C_\varepsilon(1+\|H_\Sigma\|_{H^2(\Sigma)})\|F\|_{L^\infty(\Omega)},\end{aligned}$$

where the last inequality follows from interpolation.

Let us then bound the last term in (3.15). We have by Proposition 2.10

$$\begin{aligned}\|F\star\nabla_\tau B_\Sigma\|_{H^{\frac12}(\Sigma)} &\le \|F\star\nabla_\tau B_\Sigma\|_{H^1(\Sigma)}\\ &\le C\|F\|_{L^\infty(\Sigma)}\|\bar\nabla B_\Sigma\|_{H^1(\Sigma)} + C\|F\|_{W^{1,3}(\Sigma)}\|\bar\nabla B_\Sigma\|_{L^6(\Sigma)}.\end{aligned}$$

Proposition 2.12 yields $\|B_\Sigma\|_{H^2(\Sigma)} \le C(1+\|H_\Sigma\|_{H^2(\Sigma)})$. Interpolation implies

$$\|\bar\nabla B_\Sigma\|_{L^6(\Sigma)} \le C\|B_\Sigma\|_{H^2(\Sigma)}^{\theta}\|B_\Sigma\|_{L^2(\Sigma)}^{1-\theta}$$

and

$$\|F\|_{W^{1,3}(\Sigma)} \le \|F\|_{H^3(\Omega)}^{1-\theta}\|F\|_{L^\infty(\Omega)}^{\theta}$$

for some $\theta\in(0,1)$. Therefore we have by (3.15)

$$\|F_{ij}\cdot\nu\|_{H^{\frac12}(\Sigma)} \le \|\bar\nabla^2 F_n\|_{H^{\frac12}(\Sigma)} + \varepsilon\|F\|_{H^3(E)} + C_\varepsilon(1+\|H_\Sigma\|_{H^2(\Sigma)})\|F\|_{L^\infty(\Omega)}.$$

The inequality (3.14) then follows from Proposition 2.11 as

$$\|\bar\nabla^2 F_n\|_{H^{\frac12}(\Sigma)} \le 2\|\Delta_\Sigma F_n\|_{H^{\frac12}(\Sigma)} + C_\varepsilon\|F_n\|_{L^2(\Sigma)}.$$

The second inequality follows from a similar argument.

Let us next prove the last part of the statement, i.e. the inequalities when $F=\nabla u$. Let u be a solution of the Neumann boundary problem

$$\begin{cases}\Delta u = f & \text{in } \Omega\\ \partial_\nu u = g & \text{on } \Sigma,\end{cases}$$

where $\int_\Sigma g = \int_\Omega f$ and $\int_\Omega u\,dx = 0$. First, clearly $\|u\|_{H^{\frac12}(\Sigma)} \le C\|u\|_{H^1(\Omega)}$. By the equation and by divergence theorem

$$\int_\Omega |\nabla u|^2\,dx + \int_\Omega u\,f\,dx = \int_\Sigma u\,\partial_\nu u\,d\mathcal{H}^2 \le \|u\|_{H^{\frac12}(\Sigma)}\|g\|_{H^{-\frac12}(\Sigma)} \le C\|u\|_{H^1(\Omega)}\|g\|_{H^{-\frac12}(\Sigma)}.$$

Therefore by

$$\left|\int_\Omega u\,f\,dx\right| \le \|u\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)}$$

and by Poincaré inequality $\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)}$ we have

$$\|u\|_{H^1(\Omega)} \le C(\|g\|_{H^{-\frac12}(\Sigma)} + \|f\|_{L^2(\Omega)}).$$

On the other hand Lemma 3.5 implies

$$\begin{aligned}\|u\|_{H^2(\Omega)} &\le C(\|g\|_{H^{\frac12}(\Sigma)} + \|u\|_{L^2} + \|f\|_{L^2(\Omega)})\\ &\le C(\|g\|_{H^{\frac12}(\Sigma)} + \|f\|_{L^2(\Omega)}).\end{aligned}$$

We use the two above inequalities and standard interpolation argument to deduce

$$\|u\|_{H^{3/2}(\Omega)} \le C(\|g\|_{L^2(\Sigma)} + \|f\|_{L^2(\Omega)}). \tag{3.16}$$

We proceed by applying (3.16) for $u_{x_i} = \nabla u \cdot e_i - c_i$, for $i = 1,2,3$, $c_i = \fint_E u_{x_i}$, and obtain

$$\|\nabla u\|_{H^{3/2}(\Omega)} \le C(\|\partial_\nu(\nabla u)\|_{L^2(\Sigma)} + \|f\|_{H^1(\Omega)}). \tag{3.17}$$

In order to treat the first term on the RHS we let $\tilde\nu$ be the harmonic extension of ν to Ω. We write $\nabla u = \nabla_\tau u + (\partial_\nu u)\,\nu$ and have

$$\partial_\nu(\nabla u) = \nabla_\tau(\partial_\nu u) + \partial_\nu(\nabla u\cdot\tilde\nu)\nu + \nabla\tilde\nu \star \nabla u.$$

Recall that we have by maximum principle $\|\tilde\nu\|_{L^\infty} \le C$ and by (3.12) $\|\nabla\tilde\nu\|_{L^4(\Omega)} \le C$. We argue as in (3.6) and obtain

$$\|\nabla\tilde\nu \star \nabla u\|_{L^2(\Sigma)} \le \|\nabla\tilde\nu\|_{L^4(\Sigma)}\,\|\nabla u\|_{L^4(\Sigma)} \le C\|u\|_{H^2(\Omega)}.$$

We use Remark 3.4 for $F = \nabla(\nabla u\cdot\tilde\nu)$ and (3.11) and have

$$\begin{aligned}\|\partial_\nu(\nabla u\cdot\tilde\nu)\|_{L^2(\Sigma)}^2 &\le C(\|\nabla_\tau(\nabla u\cdot\tilde\nu)\|_{L^2(\Sigma)}^2 + \|\nabla(\nabla u\cdot\tilde\nu)\,\Delta(\nabla u\cdot\tilde\nu)\|_{L^1(\Omega)} + \|\nabla u\cdot\tilde\nu\|_{H^1(\Omega)}^2)\\ &\le C(\|\partial_\nu u\|_{H^1(\Sigma)}^2 + \|u\|_{H^2(\Omega)}^2 + \|f\|_{H^1(\Omega)}^2 + \|\nabla u\cdot\tilde\nu\|_{H^1(\Omega)}^2)\\ &\quad + C(\|\nabla^2 u\star\nabla^2 u\star\nabla\tilde\nu\|_{L^1(\Omega)} + \|\nabla^2 u\star\nabla u\star\nabla\tilde\nu\star\nabla\tilde\nu\|_{L^1(\Omega)}).\end{aligned}$$

First, we obtain by using the previous estimates

$$\|\nabla u\cdot\tilde\nu\|_{H^1(\Omega)}^2 \le C\|u\|_{H^2(\Omega)}^2.$$

We bound the second last term by Hölder's inequality and by the Sobolev embedding

$$\|\nabla^2 u\star\nabla^2 u\star\nabla\tilde\nu\|_{L^1(\Omega)} \le C\|\nabla\tilde\nu\|_{L^4(\Omega)}\|\nabla^2 u\|_{L^{\frac83}(\Omega)}^2 \le \varepsilon\|\nabla u\|_{H^{\frac32}(\Omega)}^2 + C_\varepsilon\|u\|_{H^2(\Omega)}^2.$$

Similarly we estimate the last term

$$\begin{aligned}\|\nabla^2 u\star\nabla u\star\nabla\tilde\nu\star\nabla\tilde\nu\|_{L^1(\Omega)} &\le C\|\nabla\tilde\nu\|_{L^4(\Omega)}^2\|\nabla^2 u\|_{L^{\frac83}(\Omega)}\|\nabla u\|_{L^8(\Omega)}\\ &\le \varepsilon\|\nabla u\|_{H^{\frac32}(\Omega)}^2 + C_\varepsilon\|u\|_{H^2(\Omega)}^2.\end{aligned}$$

Therefore we have

$$\|\partial_\nu(\nabla u)\|_{L^2(\Sigma)} \le \varepsilon\|\nabla u\|_{H^{\frac32}(\Omega)} + C_\varepsilon(\|\partial_\nu u\|_{H^1(\Sigma)} + \|u\|_{H^2(\Omega)} + \|f\|_{H^1(\Omega)}).$$

Recall that we have

$$\|u\|_{H^2(\Omega)} \le C(\|g\|_{H^{\frac12}(\Sigma)} + \|f\|_{L^2(\Omega)}).$$

Therefore the third inequality follows from (3.17).

For the last inequality we recall that while the Trace operator is not bounded $T : H^{\frac{1}{2}}(\mathbb{R}^3) \to L^2(\mathbb{R}^2)$ it is bounded as $T : H^{\frac{3}{2}}(\mathbb{R}^3) \to H^1(\mathbb{R}^2)$. We prove the statement by localization argument similar to the one in the proof of Proposition 2.10 and we only give the sketch of the proof.

We cover Σ with balls of radius δ, $B_\delta(x_i), i = 1, \ldots, N$ such that the set $\Sigma \cap B_{2\delta}(x_i)$ is contained in the graph of ϕ_i and Ω is above the graph. We denote the partition of unity by η_i. Let us fix i and we may assume that $x_i = 0$ and $\phi_i(0) = \nabla\phi_i(0) = 0$. By the regularity assumptions it holds $\|\phi_i\|_{C^{1,\alpha}(\mathbb{R}^2)}, \|\phi_i\|_{W^{2,4}(\mathbb{R}^2)} \leq C$. We define $u_i(x) = \eta_i(x)u_i(x)$ and $v_i(x', x_3) = u_i(x', x_3 + \phi_i(x'))$ for $x_3 \geq 0$ and extend v_i to $\mathbb{R}^3$ by the extension operator. Then we have by the Trace Theorem

$$\|u_i\|_{H^1(\Sigma\cap B_\delta)} \leq C\|v_i\|_{H^1(\mathbb{R}^2)} \leq C\|v_i\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}.$$

Recall that the assumption $\|B_\Sigma\|_{L^4} \leq C$ guarantees that Ω is an H^2-extension domain. Therefore it holds $\|v_i\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \leq C\|u\|_{H^{\frac{3}{2}}(\Omega)}$ and the last inequality follows. □

3.2. Regularity Estimates for Functions

In this subsection we prove regularity estimates for functions $u : \Omega \to \mathbb{R}$ defined as a solution of the Dirichlet problem

$$\begin{cases} \Delta u = f & x \in \Omega \\ u = g & x \in \Sigma \end{cases} \tag{3.18}$$

We first consider the case when $g = 0$ and improve in this case the third inequality in Lemma 3.7. Here we assume that the boundary has the regularity $\|B_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)} \leq C$. Note that by the Sobolev embedding this implies $\|B_\Sigma\|_{L^4(\Sigma)} \leq C$.

Proposition 3.8. *Assume Ω, with $\Sigma = \partial\Omega$, is uniformly $C^{1,\alpha}(\Gamma)$-regular and $\|B_\Sigma\|_{H^{\frac{1}{2}}(\Sigma)} \leq M$. There exists a constant C, depending on M and the $C^{1,\alpha}$-norm of the heightfunction, such that the solution of the problem (3.18) with zero Dircihlet boundary datum, i.e., $g = 0$ satisfies*

$$\|\partial_\nu u\|_{H^1(\Sigma)} + \|\nabla u\|_{H^{\frac{3}{2}}(\Omega)} \leq C\|f\|_{H^{\frac{1}{2}}(\Omega)}.$$

Proof. First we note that since $u = 0$ on Σ then by (3.2) we have

$$\|\nabla^2 u\|^2_{L^2(\Omega)} = \|f\|^2_{L^2(\Omega)} - \int_\Sigma H_\Sigma |\partial_\nu u|^2 \, d\mathcal{H}^2.$$

By (3.6) it holds

$$-\int_\Sigma H_\Sigma |\partial_\nu u|^2 \, d\mathcal{H}^2 \leq \varepsilon\|u\|^2_{H^2(\Omega)} + C_\varepsilon\|\nabla u\|^2_{L^2(\Sigma)}.$$

We apply Lemma 3.3 for $F = \nabla u$ and recall that $u = 0$ on Σ to deduce

$$\begin{aligned} \|\nabla u\|_{L^2(\Sigma)} &\leq C(\|\nabla_\tau u\|_{L^2(\Sigma)} + \|u\|_{H^1(\Omega)} + \|f\|_{L^2(\Omega)}) \\ &\leq \varepsilon\|u\|_{H^2(\Omega)} + C_\varepsilon(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}). \end{aligned}$$

Therefore we have

$$\|\nabla^2 u\|^2_{L^2(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).$$

We bound $\|u\|_{L^2(\Omega)}$ simply by multiplying the equation (3.18) by u and integrating by parts $\|\nabla u\|^2_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}$. Poincaré inequality then implies $\|u\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}$ and we have

$$\|u\|_{H^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{3.19}$$

Let $\tilde{\nu}$ be the harmonic extension of the normal field and let us define $\tau_i = e_i - \langle e_i, \tilde{\nu}\rangle\tilde{\nu}$ as in the proof of Lemma 3.7. Define $u_i = \nabla u \cdot \tau_i$. Observe that $u_i = 0$ on Σ and apply (3.19) to deduce

$$\|\nabla^2 u_i\|_{L^2(\Omega)} \leq C\|\Delta u_i\|_{L^2(\Omega)}. \tag{3.20}$$

We have (recall $\Delta u = f$)

$$\Delta u_i = \nabla f \star \tau_i + \nabla^2 u \star \nabla\tilde{\nu} + \nabla u \star \nabla\tilde{\nu} \star \nabla\tilde{\nu}.$$

Arguing similarly as in the proof of Lemma 3.5 and using (3.19) yields

$$\|\Delta u_i\|_{L^2(\Omega)} \leq \varepsilon\|u\|_{H^3(\Omega)} + C\|f\|_{H^1(\Omega)} \tag{3.21}$$

Let us then treat the LHS of (3.20). We have (recall that $\tau_i = e_i - \langle e_i, \tilde{\nu}\rangle\tilde{\nu}$)

$$\nabla_j\nabla_k u_i = \nabla(\nabla_j\nabla_k u)\cdot\tau_i + \nabla^2 u \star \nabla\tilde{\nu} + \nabla u \star \nabla\tilde{\nu} \star \nabla\tilde{\nu} + \nabla u \star \nabla^2\tilde{\nu}.$$

Therefore arguing as in the proof of Lemma 3.5, we obtain

$$\begin{aligned}\|\nabla^2 u_i\|_{H^2(\Omega)} \geq &\sum_{i,j,k=1}^{3} \|\nabla(\nabla_j\nabla_k u)\cdot\tau_i\|_{L^2(\Omega)} \\ &- \varepsilon\|u\|_{H^3(\Omega)} - C_\varepsilon\|f\|_{H^1(\Omega)} - C\|\nabla u\|_{L^\infty(\Omega)}\|\nabla^2\tilde{\nu}\|_{L^2(\Omega)}.\end{aligned} \tag{3.22}$$

Let us fix a point $x \in \Omega$ and as in the proof of Lemma 3.7 we may assume that $\tilde{\nu}(x)\cdot e_i = 0$ for $i = 1, 2$. Then it is easy to see that

$$\begin{aligned}\sum_{i,j,k=1}^{3} |\nabla(\nabla_j\nabla_k u(x))\cdot\tau_i|^2 &\geq \sum_{i=1}^{2}\sum_{j,k=1}^{3} |\langle\nabla\nabla_j\nabla_k u(x), \tau_i\rangle|^2 \\ &\geq c\sum_{i,j,k=1}^{3} |\nabla_i\nabla_j\nabla_k u(x)|^2 - C|\nabla\Delta u(x)|^2.\end{aligned}$$

This together with (3.22) yields

$$\|u_i\|_{H^2(\Omega)} \geq c\|\nabla^3 u\|_{L^2(\Omega)} - C\|f\|_{H^1(\Omega)} - C\|\nabla u\|_{L^\infty}\|\nabla^2\tilde{\nu}\|_{L^2(\Omega)}. \tag{3.23}$$

We proceed by recalling that $\tilde{\nu}$ is the harmonic extension of ν. We claim that it holds

$$\|\nabla^2\tilde{\nu}\|_{L^2(\Omega)} \leq C. \tag{3.24}$$

Indeed, this follows from already familiar argument and we only give its outline. Define $\tau_i = e_i - \langle e_i, \tilde{\nu}\rangle\tilde{\nu}$ as in the proof of Lemma 3.7 and let $u_{ij} = \langle\nabla\tilde{\nu}\,\tau_i\tau_j\rangle$. Then it holds $u_{ij} = \langle B_\Sigma\tau_i, \tau_j\rangle$ on Σ and therefore by the assumptions it holds $\|u_{ij}\|_{H^{\frac{1}{2}}(\Sigma)} \leq C$. Arguing as in the proof of Lemma 3.5 we deduce

$$\|\nabla u_{ij}\|^2_{L^2(\Omega)} \leq \|u_{ij}\|^2_{H^{\frac{1}{2}}(\Sigma)} + \varepsilon\|\nabla^2\tilde{\nu}\|^2_{L^2(\Omega)} + C_\varepsilon.$$

By applying this to every $i, j, = 1, 2, 3$ and arguing as above we obtain (3.24).

We have by interpolation inequality in Corollary 2.9

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\|\nabla^2 u\|_{L^4(\Omega)} \leq C\|\nabla^3 u\|^{\frac{3}{4}}_{L^2(\Omega)}\|\nabla^2 u\|^{\frac{1}{4}}_{L^2(\Omega)}.$$

Therefore by Young's inequality and by (3.19)

$$\begin{aligned}\|\nabla u\|_{L^\infty(\Omega)}\|\nabla^2\tilde{\nu}\|_{L^2(\Omega)} &\leq \varepsilon\|\nabla^3 u\|_{L^2(\Omega)} + C_\varepsilon\|\nabla^2 u\|_{L^2(\Omega)} \\ &\leq \varepsilon\|\nabla^3 u\|_{L^2(\Omega)} + C_\varepsilon\|f\|_{L^2(\Omega)}.\end{aligned}$$

Hence, (3.20), (3.21) and (3.23) imply

$$\|u\|_{H^3(\Omega)} \leq C\|f\|_{H^1(\Omega)}. \tag{3.25}$$

We set $\mathcal{F}$ to be the linear operator such that it associates f with the unique solution u of the problem (3.18). Then we have by (3.19) and (3.25)

$$\|\mathcal{F}\|_{\mathcal{L}(L^2,H^2)} \le C \qquad \text{and} \qquad \|\mathcal{F}\|_{\mathcal{L}(H^1,H^3)} \le C.$$

Then we have the inequality

$$\|\nabla u\|_{H^{\frac{3}{2}}(\Omega)} \le C\|f\|_{H^{\frac{1}{2}}(\Omega)} \tag{3.26}$$

by standard interpolation theory.

We need yet to bound $\|\partial_\nu u\|_{H^1(\Sigma)}$. To this aim we extend ∇u to $\mathbb{R}^3$ by T such that

$$\|T(\nabla u)\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \le C\|\nabla u\|_{H^{\frac{3}{2}}(\Omega)}.$$

Let us denote $U = T(\nabla u)$. Let $\tilde{\nu}$ be the Harmonic extension of ν as before, which we may also extend to $\mathbb{R}^3$. We note that we may assume that the extensions have support in B_R. We have by Lemma 3.7

$$\|\nabla u \cdot \nu\|_{H^1(\Sigma)} \le C\|U \cdot \tilde{\nu}\|_{H^{\frac{3}{2}}(\Omega)} \le C\|U \cdot \tilde{\nu}\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}.$$

The Kato-Ponce inequality (2.15) with $p_2 = 8, q_2 = 8/3$ yields

$$\|U \cdot \tilde{\nu}\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \le C\|U\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}\|\tilde{\nu}\|_{L^\infty(\mathbb{R}^3)} + C\|U\|_{L^8(\mathbb{R}^3)}\|\tilde{\nu}\|_{W^{\frac{3}{2},\frac{8}{3}}(\mathbb{R}^3)}.$$

We have $\|\tilde{\nu}\|_{L^\infty(\mathbb{R}^3)} \le C$ and by the Sobolev embedding $\|U\|_{L^8(\mathbb{R}^3)} \le C\|U\|_{H^{\frac{3}{2}}(\mathbb{R}^3)}$. We use (2.13) to deduce that

$$\|\nabla\tilde{\nu}\|_{W^{\frac{1}{2},\frac{8}{3}}(\mathbb{R}^3)} \le C\|\nabla\tilde{\nu}\|_{H^1(\mathbb{R}^3)}^{\frac{1}{2}}\|\nabla\tilde{\nu}\|_{L^4(\mathbb{R}^3)}^{\frac{1}{2}}.$$

By (3.12) we have

$$\|\nabla\tilde{\nu}\|_{L^4(\mathbb{R}^3)} \le \|\tilde{\nabla}\nu\|_{L^4(\Omega)} \le C$$

and by (3.24)

$$\|\nabla\tilde{\nu}\|_{H^1(\mathbb{R}^3)} \le C\|\nabla^2\tilde{\nu}\|_{L^2(\Omega)} \le C.$$

Therefore by combining the previous inequalities we have

$$\|\nabla u \cdot \nu\|_{H^1(\Sigma)} \le C\|U \cdot \tilde{\nu}\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \le C\|U\|_{H^{\frac{3}{2}}(\mathbb{R}^3)} \le C\|\nabla u\|_{H^{\frac{3}{2}}(\Omega)}.$$

The result then follows from (3.26). □

We conclude this section by proving the sharp boundary regularity estimate for the Dirichlet problem. The proof follows the argument in [22, Theoreom 4.1], with the difference that here we have Dirichlet boundary datum, instead of the zero Neumann case.

Theorem 3.9. *Assume Ω, with $\Sigma = \partial\Omega$, is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies (H_m) for $m \ge 2$. Let $u \in \dot{H}^1(\Omega^c)$ be the solution of*

$$\begin{cases} \Delta u = 0 & x \in \Omega^c \\ u = g & x \in \Sigma. \end{cases} \tag{3.27}$$

Then for all integers $0 \le k \le m-1$ it holds

$$\|\nabla^k u\|_{H^{\frac{1}{2}}(\Sigma)} \le C(1 + \|B_\Sigma\|_{H^{k-1}(\Sigma)} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)}) \tag{3.28}$$

for some constant C, depending on m and on the $C^{1,\alpha}$-norm of the heightfunction. Moreover, if g is constant then the above holds for all $k \in \mathbb{N}$.

Proof. **Step 1: Flattening the boundary.** Since Σ is $C^{1,\alpha}(\Gamma)$, for any $x \in \Sigma$ we find $\delta > 0$ such that after rotating and translating the coordinates

$$\Omega^c \cap B_\delta = \{(x', x_3) : \, x_3 > \phi(x')\}$$

with $\phi \in C^{1,\alpha}(B_\delta)$, $\phi(0) = 0$ and $\nabla\phi(0) = 0$. Consider the diffeomorfism $\Psi : \Omega^c \cap B_\delta \to B_\delta^+$ $\Psi(x', x_3) \to (x', x_3 - \phi(x'))$ and let $v := u \circ \Psi^{-1}$ and $w := g \circ \Psi^{-1}$. Let us extend g by its harmonic extension, denote it by $\tilde{g}$, and thus $w = \tilde{g} \circ \Psi^{-1}$ is defined in B_δ^+. By standard calculations we deduce that v is the solution of

$$\begin{cases} \operatorname{div}(A_\phi \nabla v) = 0 & x \in B_\delta^+ \\ v = w & x_3 = 0, \end{cases} \tag{3.29}$$

where A_ϕ is symmetric matrix which can be written as $A_\phi = I + \tilde{A}(\nabla\phi)$ where $\tilde{A}(\nabla\phi(x)) = 0$ if $\nabla\phi(x) = 0$. In particular, by choosing δ small enough A_ϕ is postitive definite. In weak form (3.29) reads as

$$\int_{B_\delta^+} A_\phi \nabla v \cdot \nabla\varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(B_\delta^+)$.

Let k be an integer as in the statement. Let us differentiate the equation (3.29) k times in tangential directions. To this aim let us fix an index vector $\gamma = (\gamma_1, \gamma_2, 0)$ with $\gamma_1 + \gamma_2 = k$, and denote $\bar{v} = \nabla^\gamma v$ and $\bar{w} = \nabla^\gamma w$. Then $\bar{v}$ is the solution of

$$\begin{cases} \operatorname{div}(A_\phi \nabla \bar{v}) = -\sum_{\tilde{\alpha},\beta} \operatorname{div}(\nabla^{\tilde{\alpha}} A_\phi \nabla\nabla^\beta v) & x \in B_\delta^+ \\ \bar{v} = \bar{w} & x_3 = 0. \end{cases} \tag{3.30}$$

with $\tilde{\alpha} = (\tilde{\alpha}_1, \tilde{\alpha}_2, 0)$, $\beta = (\beta_1, \beta_2, 0)$, $|\beta| \le k - 1$ and $|\tilde{\alpha}| + |\beta| \le k$. In the weak form this reads as

$$\int_{B_\delta^+} A_\phi \nabla\bar{v} \cdot \nabla\varphi \, dx = -\sum_{\tilde{\alpha},\beta} \int_{B_\delta^+} (\nabla^{\tilde{\alpha}} A_\phi \nabla\nabla^\beta v) \cdot \nabla\varphi \, dx$$

for all $\varphi \in C_0^\infty(B_\delta^+)$.

Step 2: Choice of the test function that has zero boundary value Let $\zeta \in C_0^\infty(B_\delta^+)$ be a smooth cut-off function such that $\zeta(x) = 1$ for $|x| \le \frac{\delta}{2}$ and $0 \le \zeta \le 1$. We choose a test function $\varphi = (\bar{v} - \bar{w})\zeta^2$, which has zero boundary value. With this choice we have

$$\begin{aligned} &\int_{B_\delta^+} (A_\phi \nabla\bar{v} \cdot \nabla\bar{v})\, \zeta^2 dx = \int_{B_\delta^+} (A_\phi \nabla\bar{v} \cdot \nabla\bar{w})\zeta^2 \, dx + \int_{B_\delta^+} (A_\phi \nabla\bar{v} \cdot \nabla\zeta)\, (\bar{w} - \bar{v})\zeta dx \\ &- \sum_{\tilde{\alpha},\beta} \int_{B_\delta^+} (\nabla^{\tilde{\alpha}} A_\phi \nabla\nabla^\beta v \cdot \nabla(\bar{v} - \bar{w}))\zeta^2 \, dx - 2\sum_{\tilde{\alpha},\beta} \int_{B_\delta^+} (\nabla^{\tilde{\alpha}} A_\phi \nabla\nabla^\beta v \cdot \nabla\zeta)(\bar{v} - \bar{w})\zeta \, dx \\ &= I_1 + I_2 + I_3 + I_4. \end{aligned}$$

By the assumption $\phi \in C^{1,\alpha}$ it holds $\|A_\phi\|_{L^\infty} \le C$. Thus we may bound the first two terms as

$$I_1 + I_2 \le C\|\nabla\bar{v}\|_{L^2(B_\delta^+)}(\|\nabla\bar{w}\|_{L^2(B_\delta^+)} + \|\bar{v} - \bar{w}\|_{L^2(B_\delta^+)}).$$

The term I_3 is more difficult to treat. Note first since A_ϕ is of the form $I + \tilde{A}(\nabla\phi)$ we have a point-wise bound by the Leibniz rule

$$\sum_{\tilde{\alpha},\beta} |\nabla^{\tilde{\alpha}} A_\phi|^2 |\nabla\nabla^\beta v|^2 \le C \sum_{\substack{|\alpha|+|\beta|\le k \\ |\beta|\le k-1}} (1 + |\nabla^{\alpha_1}\nabla\phi|^2 \dots |\nabla^{\alpha_k}\nabla\phi|^2)|\nabla\nabla^\beta v|^2.$$

Hence, we obtain by Hölder's inequality

$$\begin{aligned} I_3 \leq & C\|\nabla(\bar{v}-\bar{w})\|_{L^2(B_\delta^+)} \\ & \cdot \sum_{\substack{|\alpha|+|\beta|\leq k \\ |\beta|\leq k-1}} \Big(1+\|\nabla^{\alpha_1}\nabla\phi\|_{L^{\frac{2k}{\alpha_1}}(B_\delta^+)}\cdots\|\nabla^{\alpha_k}\nabla\phi\|_{L^{\frac{2k}{\alpha_k}}(B_\delta^+)}\Big)\|\nabla\nabla^\beta v\|_{L^{\frac{2k}{|\beta|}}(B_\delta^+)}. \end{aligned}$$

We use interpolation inequality to estimate

$$\|\nabla^{\alpha_i}\nabla\phi\|_{L^{\frac{2k}{\alpha_1}}(B_\delta^+)} \leq \|\nabla\phi\|_{H^k(B_\delta^+)}^{\frac{\alpha_i}{k}}\|\nabla\phi\|_{L^\infty(B_\delta^+)}^{1-\frac{\alpha_i}{k}}.$$

Also by interpolation we have

$$\|\nabla\nabla^\beta v\|_{L^{\frac{2k}{|\beta|}}(B_\delta^+)} \leq \|v\|_{H^{k+1}(B_\delta^+)}^{\frac{|\beta|}{k}}\|\nabla v\|_{L^\infty(B_\delta^+)}^{1-\frac{|\beta|}{k}}$$

and $|\beta| \leq k-1$. Since Σ is $C^{1,\alpha}$-regular, we have by Schauder estimates [25] that $\nabla v \in C^{0,\alpha}(B_\delta^+)$. Note that $\sum_i \frac{\alpha_i}{k} \leq \frac{k-|\beta|}{k}$ and $\frac{|\beta|}{k} < 1$. Therefore by Young's inequality we deduce

$$\begin{aligned} |I_3| \leq & C\|\nabla(\bar{v}-\bar{w})\|_{L^2(B_\delta^+)}(1+\|\nabla\phi\|_{H^k(B_\delta^+)}^{\frac{k-|\beta|}{k}})\|v\|_{H^{k+1}(B_\delta^+)}^{\frac{|\beta|}{k}} \\ \leq & \varepsilon\|\nabla(\bar{v}-\bar{w})\|_{L^2(B_\delta^+)}^2 + \varepsilon\|v\|_{H^{k+1}(B_\delta^+)}^2 + C_\varepsilon(1+\|\nabla\phi\|_{H^k(B_\delta^+)}^2). \end{aligned}$$

We bound the last term I_4 similarly.

Finally we collect the previous estimates, use the ellipticity of the matrix A_ϕ and the definition of $\bar{w}$ and obtain

$$\|\nabla\bar{v}\|_{L^2(B_{\delta/2}^+)}^2 \leq 4\varepsilon\|v\|_{H^{k+1}(B_\delta^+)}^2 + C(1+\|\phi\|_{H^{k+1}(B_\delta^+)}^2 + \|w\|_{H^{k+1}(B_\delta^+)}^2).$$

Summing over all the multi index of the type (γ_1,γ_2) we have the control over the horizontal derivatives. To estimate the vertical derivatives, we use the equation in the strong form as in [22], and obtain

$$\|v\|_{H^{k+1}(B_{\delta/2}^+)}^2 \leq C\varepsilon\|v\|_{H^{k+1}(B_\delta^+)}^2 + C(1+\|\phi\|_{H^{k+1}(B_\delta^+)}^2 + \|w\|_{H^{k+1}(B_\delta^+)}^2). \tag{3.31}$$

Step 3: Going back to the original function. We need to go back to the original function u. The argument is similar to [22] and we merely sketch it. We note that arguing as in [22, Thorem 4.1] we may control

$$\|\phi\|_{H^{k+1}(B_\delta^+)} \leq C(1+\|B_\Sigma\|_{H^{k-1}(\Sigma)})$$

for all $k \in \mathbb{N}$. Recall that $\tilde{g}$ is the harmonic extension of g. Using the assumption that the curvature satisfies the condition (H_m) for m, we may deduce, arguing as in the proof of Proposition 2.1, that for $k \leq m-1$ it holds

$$\|w\|_{H^{k+1}(B_\delta^+)} \leq C\|\tilde{g}\|_{H^{k+1}(\Omega^c\cap B_\delta)} \leq C\|g\|_{H^{k+\frac{1}{2}}(\Sigma)}.$$

Obviously if g is constant the above inequality is trivial.

Fix σ small such that $\cup_{x\in\Sigma}B_\delta(x)$ covers $\mathcal{N}_\delta = \{x \in \Omega^c : d(x,\Omega) \leq \delta\}$ and $\sigma_1 < \sigma_2 < \sigma$. By compactness we may choose a finite family of balls covering $\mathcal{N}_\delta$. Choosing ε small enough we have by (3.31) and by the above inequalities

$$\|u\|_{H^{k+1}(\mathcal{N}_{\sigma_1})}^2 \leq C(\|u\|_{H^{k+1}(\mathcal{N}_{\sigma_2}\setminus\mathcal{N}_{\sigma_1})}^2 + 1 + \|B_\Sigma\|_{H^{k-1}(\Sigma)}^2 + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)}^2).$$

Since u is harmonic, the interior regularity yields

$$\|u\|_{H^{k+1}(\mathcal{N}_{\sigma_2}\setminus\mathcal{N}_{\sigma_1})} \leq C\|u\|_{L^2(\mathcal{N}_{\sigma_2})}.$$

By standard estimates from harmonic analysis [16] it holds for R large

$$\|u\|_{L^2(N_{\sigma_2})} \leq \|u\|_{L^2(\Omega^c\cap B_R)} \leq C(\|u\|_{L^2(\Sigma)} + \|u\|_{L^2(\partial B_R)}) \leq C(1+\|g\|_{L^2(\Sigma)}).$$

Therefore we have

$$\|u\|_{H^{k+1}(\mathcal{N}_{\sigma_1})} \leq C\left(1+\|B_\Sigma\|_{H^{k-1}(\Sigma)} + \|g\|_{H^{k+\frac{1}{2}}(\Sigma)}\right).$$

The claim follows from

$$\|\nabla^k u\|_{H^{\frac{1}{2}}(\Sigma)} \le C\|\nabla^{k+1} u\|^2_{L^2(\mathcal{N}_{\sigma_1})}.$$

□

4. Useful Formulas

In this section we focus on the equations (1.3) and assume that the family of sets $(\Omega_t)_{t\in(0,T)}$ and velocities $v(\cdot,t)$ are solution of (1.3). We derive a general formula for the commutators of the material derivative of high order $\mathcal{D}_t^k$ with spatial derivatives. We apply this to calculate $[\mathcal{D}_t^k,\nabla]v$ and $[\mathcal{D}_t^k,\nabla]p$, which will produce two types of error terms, (4.13) and (4.14), defined in the fluid domain Ω_t. We will also calculate the formula for $\mathcal{D}_t^k p$ on the moving boundary Σ_t in Lemma 4.7, which includes third type of error term defined in (4.26). The precise structures of these error terms are complicated and we only need to trace the order of the derivatives that appear. Therefore we effectively use the notation from [32]

$$\nabla^k f \star \nabla^l g$$

to denote a contraction of some indexes of tensors $\nabla^i f$ and $\nabla^j g$ for $i \le k$ and $j \le l$. Note that we include the lower order derivatives.

We begin by recalling the following formulas from [50]

$$[\mathcal{D}_t,\nabla]f = \mathcal{D}_t\nabla f - \nabla\mathcal{D}_t f = -\nabla v^T\nabla f, \tag{4.1}$$
$$[\mathcal{D}_t,\nabla_\tau]f = -(\nabla_\tau v)^T\nabla_\tau f \tag{4.2}$$

and

$$[\mathcal{D}_t,\Delta_\Sigma]f = \nabla_\tau^2 f\star\nabla v - \nabla_\tau f\cdot\Delta_\Sigma v + B\star\nabla v\star\nabla_\tau f. \tag{4.3}$$

Let us also recall the material derivative of the normal field. We use the shorthand notation $\nu = \nu_{\Sigma_t}$, $B = B_{\Sigma_t}$ and $v_n = v\cdot\nu$. We have by [50]

$$\mathcal{D}_t\nu = -(\nabla_\tau v)^T\nu. \tag{4.4}$$

Since $\nabla_\tau v_n = \nabla_\tau v^T\nu + Bv_\tau$, we may write (4.4) as

$$\mathcal{D}_t\nu = -\nabla_\tau v_n + Bv_\tau. \tag{4.5}$$

We need higher order versions of the commutation formula (4.1), i.e., for

$$[\mathcal{D}_t^l,\nabla^k]f = \mathcal{D}_t^l\nabla^k f - \nabla^k\mathcal{D}_t^l f.$$

Recall the definition of the norm of an index vector $\alpha = (\alpha)_{i=1}^k \in \mathbb{N}^k$

$$|\alpha| = \sum_{i=1}^k \alpha_i$$

and note that we include zero in the set of natural numbers $\mathbb{N}$.

Lemma 4.1. *For $l,k\in\mathbb{N}$ with $l,k\ge 1$ it holds*

$$[\mathcal{D}_t^l,\nabla^k]f = \sum_{\substack{|\alpha|\le k-1\\ |\beta|\le l-1}} \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_l}\mathcal{D}_t^{\beta_l}v\star\nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}}f.$$

Proof. Let us first assume $l=1$ and prove

$$\mathcal{D}_t\nabla^k f - \nabla^k\mathcal{D}_t f = \sum_{|\alpha|\le k-1}\nabla^{1+\alpha_1}v\star\nabla^{1+\alpha_2}f. \tag{4.6}$$

We argue by induction over k and observe immediately that the case $k=1$ follows from (4.1). Assume that (4.6) holds for $k-1$ and note that by (4.1) we have

$$\mathcal{D}_t \nabla^k f = \mathcal{D}_t \nabla(\nabla^{k-1} f) = \nabla \mathcal{D}_t(\nabla^{k-1} f) + \nabla v \star (\nabla^k f).$$

By induction assumption we have

$$\begin{aligned}\nabla \mathcal{D}_t(\nabla^{k-1} f) &= \nabla\big(\nabla^{k-1}\mathcal{D}_t f + \sum_{|\alpha|\le k-2} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} f\big)\\ &= \nabla^k \mathcal{D}_t f + \sum_{|\alpha|\le k-1} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} f.\end{aligned}$$

This yields the claim for $l=1$.

The proof for $l \ge 1$ follows from a similar induction argument. Assume that the claim holds for $l-1$ and note that

$$\mathcal{D}_t^l \nabla^k f = \nabla^k \mathcal{D}_t^l f + \mathcal{D}_t([\mathcal{D}_t^{l-1}, \nabla^k] f) + [\mathcal{D}_t, \nabla^k](\mathcal{D}_t^{l-1} f).$$

By the first claim we have

$$[\mathcal{D}_t, \nabla^k](\mathcal{D}_t^{l-1} f) = \sum_{|\alpha|\le k-1} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} \mathcal{D}_t^{l-1} f.$$

On the other hand, by the induction assumption we have

$$\mathcal{D}_t[\mathcal{D}_t^{l-1}, \nabla^k] f = \mathcal{D}_t \sum_{\substack{|\alpha|\le k-1\\ |\beta|\le l-2}} \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{l-1}}\mathcal{D}_t^{\beta_{l-1}} v \star \nabla^{1+\alpha_l}\mathcal{D}_t^{\beta_l} f.$$

We use the Leibniz rule and the first claim to deduce that

$$\mathcal{D}_t \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v = \nabla^{1+\alpha_1}\mathcal{D}_t^{1+\beta_1} v + \sum_{|\tilde\alpha|\le \alpha_1} \nabla^{1+\tilde\alpha_1} v \star \nabla^{1+\tilde\alpha_2}\mathcal{D}_t^{\beta_1} v.$$

Similar formula holds also for $\mathcal{D}_t \nabla^{1+\alpha_l}\mathcal{D}_t^{\beta_l} f$. Hence, we obtain the claim. □

Let us next prove higher order commutation formulas for (4.2) and a formula for $\mathcal{D}_t^l \nu$ and $\mathcal{D}_t B$. Below $a_\beta(\nu)$ and $a_{\alpha,\beta}(\nu, B)$ denote bounded coefficients which depend on ν and on ν and B respectively.

Lemma 4.2. *For $l \ge 1$ it holds*

$$[\mathcal{D}_t^l, \nabla_\tau] f = \sum_{|\beta|\le l-1} a_\beta(\nu) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \star \nabla_\tau \mathcal{D}_t^{\beta_{l+1}} f \tag{4.7}$$

and

$$[\mathcal{D}_t^l, \nabla_\tau^2] f = \sum_{\substack{|\alpha|\le 1\\ |\beta|\le l-1}} a_{\alpha,\beta}(\nu, B) \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_l}\mathcal{D}_t^{\beta_l} v \star \nabla_\tau^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} f. \tag{4.8}$$

Moreover we have

$$\mathcal{D}_t^l \nu = \sum_{|\beta|\le l-1} a_\beta(\nu)\, \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \tag{4.9}$$

and

$$\mathcal{D}_t^l B = \sum_{\substack{|\alpha|\le 1\\ |\beta|\le l-1}} a_{\alpha,\beta}(\nu, B) \nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_i} v \star \cdots \star \nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} v. \tag{4.10}$$

Proof. Let us first prove (4.9). First, the claim holds for $l = 1$ by (4.4). Let us assume that (4.9) holds for $l - 1$. Then

$$\mathcal{D}_t^l \nu = \mathcal{D}_t \sum_{|\beta| \leq l-2} a_\beta(\nu)\, \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{l-1}} v.$$

By (4.4) it holds $\mathcal{D}_t a_\beta(\nu) = \tilde{a}_\beta(\nu) \nabla v$ and by (4.1) we have

$$\mathcal{D}_t \nabla \mathcal{D}_t^{\beta_i} v = \nabla \mathcal{D}_t^{\beta_i + 1} v + \nabla v \star \nabla \mathcal{D}_t^{\beta_i} v.$$

Thus we deduce

$$\mathcal{D}_t \sum_{|\beta| \leq l-2} a_\beta(\nu)\, \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{l-1}} v = \sum_{|\beta| \leq l-1} \tilde{a}_\beta(\nu)\, \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v$$

which implies (4.9).

Let us next prove (4.7). By (4.2) the claim holds for $l = 1$. Let us assume that the claim holds for $l - 1$. Then

$$\mathcal{D}_t^l \nabla_\tau f = \mathcal{D}_t \nabla_\tau \mathcal{D}_t^{l-1} f + \mathcal{D}_t \sum_{|\beta| \leq l-2} a_\beta(\nu) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{l-1}} v \star \nabla_\tau \mathcal{D}_t^{\beta_l} f.$$

As before we have by (4.4) $\mathcal{D}_t a_\beta(\nu) = \tilde{a}_\beta(\nu) \nabla v$ and by (4.2)

$$\mathcal{D}_t \nabla_\tau \mathcal{D}_t^{\beta_i} f = \nabla_\tau \mathcal{D}_t^{\beta_i + 1} f + a(\nu) \nabla v \star \nabla_\tau \mathcal{D}_t^{\beta_i} f.$$

Therefore we obtain by Leibniz rule

$$\mathcal{D}_t^l \nabla_\tau f = \nabla_\tau \mathcal{D}_t^l f + \sum_{|\beta| \leq l-1} \tilde{a}_\beta(\nu) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \star \nabla_\tau \mathcal{D}_t^{\beta_l} f$$

and (4.7) follows.

We notice next that (4.10) follows from $B = \nabla_\tau \nu$ and by combining (4.7) with (4.9). Finally we obtain (4.8) by first applying (4.7) as

$$\mathcal{D}_t^l \nabla_\tau^2 f = \nabla_\tau (\mathcal{D}_t^l \nabla_\tau f) + \sum_{|\beta| \leq l-1} a_\beta(\nu) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \star \nabla_\tau \mathcal{D}_t^{\beta_{l+1}} \nabla_\tau f.$$

The claim then follows by differentiating (4.7). □

Remark 4.3. By Lemma 4.2 we have in particular that

$$\mathcal{D}_t^l v_n = \sum_{|\beta| \leq l} a_\beta(\nu)\, \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \star D_t^{\beta_{l+1}} v,$$

where $\beta_i \leq l-1$ for $i \leq l$. Moreover, since we may write the Laplace-Beltrami operator as $\Delta_\Sigma f = \mathrm{Tr}(\nabla_\tau^2 f)$ then Lemma 4.2 yields

$$\mathcal{D}_t^l \Delta_\Sigma f = \Delta_\Sigma \mathcal{D}_t^l f + \sum_{\substack{|\alpha| \leq 1 \\ |\beta| \leq l-1}} a_{\alpha,\beta}(\nu, B) \nabla^{1+\alpha_1} \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_l} \mathcal{D}_t^{\beta_l} v \star \nabla_\tau^{1+\alpha_{l+1}} \mathcal{D}_t^{\beta_{l+1}} f.$$

Let us next derive formulas for the divergence and the curl of the vector field $\mathcal{D}_t^l v$. Recall that by (1.3) we have $\operatorname{div} v = 0$ which then implies

$$-\Delta p = \operatorname{div}(\mathcal{D}_t v) = \mathrm{Tr}((\nabla v)^2) = \operatorname{div}\operatorname{div}(v \otimes v). \tag{4.11}$$

For the curl we have $\operatorname{curl}(D_t v) = 0$ and $\omega = \operatorname{curl} v = \nabla v - \nabla v^T$ satisfies (see e.g. [50])

$$\mathcal{D}_t \omega = -\nabla v^T \omega - \omega \nabla v. \tag{4.12}$$

We will derive formulas for $\operatorname{div} \mathcal{D}_t^l v$ and $\operatorname{curl} \mathcal{D}_t^l v$ below by using (4.11), (4.12) and the commutation formula in Lemma 4.1. To this aim we introduce two type of error functions. The first type we denote by R^l_{div}, which stands for any function which can be written in the form

$$R^l_{\mathrm{div}} = \sum_{|\beta| \leq l} a_\beta(\nabla v) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v, \tag{4.13}$$

for $l \geq 0$. We also use the convention that the indexes are ordered as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. The second type of error function is slightly more general and it can be written in the form

$$R^l_{bulk} = \sum_{|\alpha| \leq 1, |\beta| \leq l} a_{\alpha,\beta}(\nabla v) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_l} v \star \nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + \beta_{l+1}} v, \tag{4.14}$$

for $l \geq 0$. Note that R^l_{bulk} has one higher order term compared to R^l_{div}. In particular, all functions of type R^l_{div} are contained in R^l_{bulk}. The reason for introducing these two notations is that we need to estimate them in different norms. We will do this in the next section. Note that using Lemma 4.1 and $-\nabla p = \mathcal{D}_t v$ we deduce that

$$[\mathcal{D}_t^{l+1}, \nabla] p = R^l_{bulk}. \tag{4.15}$$

Lemma 4.4. *Let $l \geq 1$ and denote $\omega = \operatorname{curl} v$. Then it holds*

$$\mathcal{D}_t \nabla^l \omega = \nabla v \star \nabla^l \omega + \sum_{|\alpha| \leq l} \nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} v.$$

Moreover, $\operatorname{curl} \mathcal{D}_t^l v$ and $\operatorname{div} \mathcal{D}_t^l v$ can be written in the form

$$\operatorname{curl} \mathcal{D}_t^l v = R^{l-1}_{\mathrm{div}} \qquad \text{and} \qquad \operatorname{div} \mathcal{D}_t^l v = R^{l-1}_{\mathrm{div}}.$$

We may also write the divergence of $\mathcal{D}_t^{l+1} v$ as

$$\operatorname{div} \mathcal{D}_t^{l+1} v = \operatorname{div} \operatorname{div}(v \otimes \mathcal{D}_t^l v) + \operatorname{div} R^{l-1}_{bulk}.$$

Proof. The first claim is an immediate consequence of Lemma 4.1 and (4.12). The second claim follows from Lemma 4.1 and from $\operatorname{curl}(\mathcal{D}_t v) = 0$. Similarly the third claim follows from Lemma 4.1 and $\operatorname{div} v = 0$.

Let us then prove the last claim. We begin by proving two useful identities. First, we claim that for a vector field F it holds

$$[\mathcal{D}_t, \operatorname{div}] F = -\operatorname{div}(\nabla v F). \tag{4.16}$$

Indeed, since $\operatorname{div} v = 0$ we have

$$\mathcal{D}_t \operatorname{div} F - \operatorname{div} \mathcal{D}_t F = \sum_{i,j=1}^{3} v_i \partial_i (\partial_j F_j) - \partial_i(\partial_j F_i v_j) = -\sum_{i,j=1}^{3} \partial_i v_j \partial_j F_i = -\operatorname{div}(\nabla v F).$$

The second identity follows also from $\operatorname{div} v = 0$ and we may write it

$$\operatorname{div} \operatorname{div}(v \otimes \mathcal{D}_t^l v) = \operatorname{div}(\nabla \mathcal{D}_t^l v \, v). \tag{4.17}$$

Let us prove the last claim in the case $l = 1$. We use (4.1), (4.11), (4.16), (4.17) and the definition of R_{bulk} in (4.14) to deduce

$$\begin{aligned}
\operatorname{div} \mathcal{D}_t^2 v &= \mathcal{D}_t \operatorname{div} \mathcal{D}_t v - [\mathcal{D}_t, \operatorname{div}] \mathcal{D}_t v \\
&= \mathcal{D}_t \operatorname{div}(\nabla v \, v) + \operatorname{div}(\nabla v \, \mathcal{D}_t v) \\
&= \operatorname{div}(\mathcal{D}_t(\nabla v \, v)) - \operatorname{div}(\nabla v \, \nabla v \, v) + \operatorname{div}(R^0_{bulk}) \\
&= \operatorname{div}(\nabla \mathcal{D}_t v \, v)) + \operatorname{div}(R^0_{bulk}) \\
&= \operatorname{div} \operatorname{div}(v \otimes \mathcal{D}_t v)) + \operatorname{div}(R^0_{bulk}).
\end{aligned}$$

Let us assume that the claim holds for $l-1$. We argue as before and obtain by (4.16), (4.17) and by the induction assumption

$$\begin{aligned}\operatorname{div}\mathcal{D}_t^{l+1}v &= \mathcal{D}_t \operatorname{div}\mathcal{D}_t^l v - [\mathcal{D}_t, \operatorname{div}]\mathcal{D}_t^l v\\ &= \mathcal{D}_t \operatorname{div}\left(\nabla\mathcal{D}_t^{l-1}v\, v + R_{\operatorname{div}}^{l-2}\right) + \operatorname{div}(\nabla v\, \mathcal{D}_t^l v).\end{aligned}$$

We use (4.16), (4.1) and the definition of R_{bulk}^{l-1} in (4.14) and obtain

$$\begin{aligned}\mathcal{D}_t \operatorname{div}\left(\nabla\mathcal{D}_t^{l-1}v\, v\right) &= \operatorname{div}\left(\mathcal{D}_t(\nabla\mathcal{D}_t^{l-1}v\, v)\right) + [\mathcal{D}_t, \operatorname{div}](\nabla\mathcal{D}_t^{l-1}v\, v)\\ &= \operatorname{div}(\nabla\mathcal{D}_t^l v\, v) + \operatorname{div}\left(\nabla\mathcal{D}_t^{l-1}v \star \nabla v \star v + \nabla\mathcal{D}_t^{l-1}v \star \mathcal{D}_t v\right)\\ &= \operatorname{div}(\nabla\mathcal{D}_t^l v\, v) + \operatorname{div} R_{bulk}^{l-1}\\ &= \operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^l v) + \operatorname{div} R_{bulk}^{l-1}.\end{aligned}$$

Similarly we have

$$\mathcal{D}_t \operatorname{div} R_{\operatorname{div}}^{l-2} = \operatorname{div} R_{\operatorname{div}}^{l-1} \qquad \text{and} \qquad \operatorname{div}(\nabla v\, \mathcal{D}_t^l v) = \operatorname{div} R_{\operatorname{div}}^{l-1}$$

and the claim follows. □

Let us then turn our focus on the pressure. By (4.11) and (1.3) we have that p is a solution of

$$\begin{cases}-\Delta p = \operatorname{div}\operatorname{div}(v\otimes v), & \text{in } \Omega_t,\\ p = H - \frac{Q(t)}{2}|\nabla U|^2, & \text{on } \Sigma_t,\end{cases}$$

where $Q(t)$ is a real valued function of time defined in (2.1) as

$$Q(t) = \frac{Q}{(\operatorname{Cap}(\Omega_t))^2},$$

$U = U_{\Omega_t}$ is the capacitary potential and $H = H_{\Sigma_t}$ is the mean curvature.

We need to derive the equation for $\mathcal{D}_t^l p$. We obtain the equation for $\mathcal{D}_t^l p$ in the bulk from Lemma 4.4.

Remark 4.5. By Lemma 4.4 and by (4.15) the function $-\Delta\mathcal{D}_t^{l+1}p$ can be written as

$$\begin{aligned}-\Delta\mathcal{D}_t^{l+1}p &= -\operatorname{div}\mathcal{D}_t^{l+1}\nabla p + \operatorname{div}[\mathcal{D}_t^{l+1}, \nabla]p = \operatorname{div}\mathcal{D}_t^{l+2}v + \operatorname{div} R_{bulk}^l\\ &= \operatorname{div}\operatorname{div}(v\otimes\mathcal{D}_t^{l+1}v) + \operatorname{div}(R_{bulk}^l).\end{aligned}$$

To find the formula for $\mathcal{D}_t^l p$ on the boundary Σ_t is more challenging. To that aim we first need to study the capacitary potential U. We introduce an error term which appears when we deal with the capacity term on the boundary, i.e., for $l \geq 0$ we denote by R_U^l as functions on Σ_t, which can be written in the form

$$R_U^l = \sum_{\substack{|\alpha|+|\beta|\leq l+1\\ |\beta|\leq l}} a_{\alpha,\beta}(v)\mathcal{D}_t^{\beta_1}v\star\cdots\star\mathcal{D}_t^{\beta_l}v\star\nabla^{1+\alpha_1}\partial_t^{\alpha_2}U. \tag{4.18}$$

We note that v is defined in Ω_t while U is defined in Ω_t^c, but they are both well-defined on the boundary Σ_t. We have the following formulas for U on Σ_t.

Lemma 4.6. *Let $l \geq 1$. Then on Σ_t it holds*

$$\mathcal{D}_t^l\nabla U = R_U^{l-1}$$

and

$$\begin{aligned}\mathcal{D}_t^{l+1}\nabla U =& \nabla\partial_t^{l+1}U + \nabla^2 U\,\mathcal{D}_t^l v\\ &+ \sum_{\substack{\alpha+|\beta|+\gamma\leq l+1\\ |\beta|\leq l-1, \gamma\leq l}} a_{\alpha,\beta,\gamma}(v)\mathcal{D}_t^{\beta_1}v\star\cdots\star\mathcal{D}_t^{\beta_{l-1}}v\star\nabla^{1+\alpha}\partial_t^{\gamma}U.\end{aligned}$$

Moreover we have the following formula for $\partial_t^{l+1}U$

$$\partial_t^{l+1}U = -\partial_\nu U\, (\mathcal{D}_t^l v \cdot \nu) + R_U^{l-1} \qquad on\ \Sigma_t.$$

Proof. The proof of the first statement is straightforward. Note that

$$\mathcal{D}_t \nabla U = \nabla \partial_t U + \nabla^2 U v$$

and

$$\mathcal{D}_t^2 \nabla U = \nabla \partial_t^2 U + \nabla^2 U \mathcal{D}_t v + \nabla^3 U \star v \star v + \nabla^2 \partial_t U \star v.$$

Thus the first claim holds for $l = 1, 2$ and the second for $l = 1$. The general case $l \geq 1$ follows by an induction argument.

For the third claim we recall that the potential satisfies $U = 1$ on Σ_t. Therefore it holds $\mathcal{D}_t U = 0$ on Σ_t which we write as

$$\partial_t U = -(\nabla U \cdot v).$$

Differentiating this yields

$$\mathcal{D}_t^l \partial_t U = -(\nabla U \cdot \mathcal{D}_t^l v) + \sum_{\substack{i+j=l\\ i\leq l-1}} \mathcal{D}_t^i v \star \mathcal{D}_t^j \nabla U.$$

By the first claim we have $\mathcal{D}_t^j \nabla U = R_U^{j-1}$ and thus by the definition of R_U^{j-1} in (4.18) we may write

$$\mathcal{D}_t^l \partial_t U = -(\nabla U \cdot \mathcal{D}_t^l v) + \sum_{\substack{|\alpha|+|\beta|\leq l\\ |\beta|\leq l-1}} a_{\alpha,\beta}(v) \mathcal{D}_t^{\beta_1} v \star \cdots \star \mathcal{D}_t^{\beta_{l-1}} v \star \nabla^{1+\alpha_1} \partial_t^{\alpha_2} U.$$

It also holds

$$\mathcal{D}_t^l \partial_t U = \partial_t^{l+1} U + \sum_{\substack{|\alpha|+|\beta|\leq l\\ |\beta|\leq l-1}} a_{\alpha,\beta}(v) \mathcal{D}_t^{\beta_1} v \star \cdots \star \mathcal{D}_t^{\beta_{l-1}} v \star \nabla^{1+\alpha_1} \partial_t^{\alpha_2} U.$$

Since U is constant on Σ_t it holds $\nabla U = \partial_\nu U\, \nu$. This implies the third claim. □

We conclude this section by deriving a formula for $\mathcal{D}_t^{l+1} p$. Recall that

$$p = H - \frac{Q(t)}{2} |\nabla U|^2 \qquad \text{on } \Sigma, \tag{4.19}$$

where $Q(t)$ is defined in (2.1). It is well known that (e.g. [15])

$$\mathcal{D}_t H = -\Delta_\Sigma v_n - |B|^2 v_n + \nabla_\tau H \cdot v \tag{4.20}$$

where $v_n = v \cdot \nu$. Using the geometric fact

$$\Delta_\Sigma \nu = -|B|^2 \nu + \nabla_\tau H \tag{4.21}$$

and (4.19) we obtain the formula

$$\mathcal{D}_t p = -\Delta_\Sigma v \cdot \nu - 2B : \nabla_\tau v - Q(t)(\mathcal{D}_t \nabla U \cdot \nabla U) - \frac{Q'(t)}{2} |\nabla U|^2. \tag{4.22}$$

We may write (4.22) in a different form. Indeed we use $\nabla U = \partial_\nu U\, \nu = -|\nabla U| \nu$ and obtain

$$\begin{aligned} -\mathcal{D}_t \nabla U \cdot \nabla U &= -(\nabla \partial_t U \cdot \nabla U) - (\nabla^2 U \nu \cdot v)\, \partial_\nu U \\ &= -(\nabla \partial_t U \cdot \nabla U) + (\nabla^2 U \nu \cdot \nu)\, v_n\, |\nabla U| - (\nabla^2 U \nabla U \cdot v_\tau). \end{aligned}$$

We notice that

$$(\nabla^2 U \nabla U \cdot v_\tau) = \frac{1}{2} (\nabla_\tau |\nabla U|^2 \cdot v). \tag{4.23}$$

Moreover, we recall that U is harmonic in Ω_t^c and constant on Σ_t. Therefore it holds by (3.4)

$$0 = \Delta U = \overbrace{\Delta_\tau U}^{=0} + (\nabla^2 U \nu \cdot \nu) + H \partial_\nu U = (\nabla^2 U \nu \cdot \nu) - H|\nabla U|. \tag{4.24}$$

Thus we have by (4.19), (4.20), (4.23), (4.24) that

$$\mathcal{D}_t p = -\Delta_\Sigma v_n - |B|^2 v_n - Q(t)\left(\partial_\nu U\left(\partial_\nu \partial_t U\right) - H|\nabla U|^2 v_n\right) + \langle \nabla_\tau p, v\rangle - Q'(t)\frac{|\nabla U|^2}{2}. \tag{4.25}$$

In the next lemma we find a suitable expression for $\mathcal{D}_t^l p$ for $l \geq 1$. Again we will have an error term which in this case is defined on the boundary Σ_t and is more complicated than the previous ones. We define the error term R_p^l for $l \geq 1$ as

$$R_p^l = R_I^l + R_{II}^l + R_{III}^l. \tag{4.26}$$

where

$$\begin{aligned} R_I^l &= -(|B|^2 - Q(t)H\,|\nabla U|^2)(\mathcal{D}_t^l v \cdot \nu) + (\nabla_\tau p \cdot \mathcal{D}_t^l v), \\ R_{II}^l &= \sum_{|\alpha|\leq 1,\, |\beta|\leq l-1} a_{\alpha,\beta,\gamma}(B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} v \qquad \text{and} \\ R_{III}^l &= \sum_{\substack{|\alpha|+|\beta|+|\gamma|\leq l+1 \\ |\beta|\leq l-1,\gamma_i\leq l}} a_{\alpha,\beta,\gamma,Q}(v)\mathcal{D}_t^{\beta_1} v \star \cdots \star \mathcal{D}_t^{\beta_{l-1}} v \star \nabla^{1+\alpha_1}\partial_t^{\gamma_1} U \star \nabla^{1+\alpha_1}\partial_t^{\gamma_2} U, \end{aligned} \tag{4.27}$$

where the coefficients $a_{\alpha,\beta,\gamma,Q}(v)$ depend on v and on the derivatives $Q^{(k)}(t)$ for $k \leq l+1$. Above $a_{\alpha,\beta,\gamma}(B)$ means that the coefficient depends on the second fundamental form. For $l = 1$ we need to quantify this dependence in which case R_{II}^1 reads as

$$R_{II}^1 = a_1(\nu, \nabla v) \star B + a_2(\nu, \nabla v) \star \nabla^2 v. \tag{4.28}$$

The reason why R_p^l has three terms is that R_{II}^l contains the error terms arising from the surface tension and R_{III}^l from the capacity. The first term R_I^l is separate merely from notational reasons as it contains the highest order material derivatives.

Lemma 4.7. *For $l \geq 2$ It holds*

$$\mathcal{D}_t^l p = -\Delta_\Sigma(\mathcal{D}_t^{l-1} v \cdot \nu) - Q(t)\,\partial_\nu U\,(\partial_\nu \partial_t^l U) + R_p^{l-1}$$

on Σ_t, where $Q(t)$ is defined in (2.1).

Proof. We first claim that it holds

$$\begin{aligned} \mathcal{D}_t^l p = &-(\Delta_\Sigma \mathcal{D}_t^{l-1} v)\cdot \nu - 2B : \nabla_\tau(\mathcal{D}_t^{l-1} v) \\ &- Q(t)(\nabla \partial_t^l U \cdot \nabla U) - Q(t)(\nabla^2 U \nabla U \cdot \mathcal{D}_t^{l-1} v) + R_{II}^{l-1} + R_{III}^{l-1}. \end{aligned} \tag{4.29}$$

To obtain the claim (4.29) for $l = 2$ we first recall that by (4.3) we have

$$[\mathcal{D}_t, \Delta_\Sigma] v = a_1(\nu, \nabla v) \star B + a_2(\nu, \nabla v) \star \nabla^2 v,$$

and that (4.4) implies $\mathcal{D}_t \nu = -(\nabla_\tau v)^T \nu$ and (4.2) implies $[\mathcal{D}_t, \nabla_\tau] v = a(\nu)\nabla v \star \nabla v$. We use (4.2) and (4.4) to obtain

$$\begin{aligned} \mathcal{D}_t B = \mathcal{D}_t(\nabla_\tau \nu) &= \nabla_\Sigma(\mathcal{D}_t \nu) + [\mathcal{D}_t, \nabla_\tau]\nu \\ &= -\nabla_\tau((\nabla_\tau v)^T \nu) + a_1(\nu, \nabla v) \star B \\ &= a_1(\nu, \nabla v) \star B + a_2(\nu, \nabla v) \star \nabla^2 v. \end{aligned}$$

We differentiate (4.22) and use the above identities and have

$$\mathcal{D}_t^2 p = -(\Delta_\Sigma \mathcal{D}_t v)\cdot \nu - 2B : \nabla_\tau(\mathcal{D}_t v) - \mathcal{D}_t\Big(Q(t)(\mathcal{D}_t \nabla U \cdot \nabla U) + \frac{Q'(t)}{2}|\nabla U|^2\Big) + R_{II}^1.$$

Lemma 4.6 yields

$$\begin{aligned}\mathcal{D}_t(\mathcal{D}_t\nabla U\cdot\nabla U) &= (\mathcal{D}_t^2\nabla U\cdot\nabla U) + (\mathcal{D}_t\nabla U\cdot\mathcal{D}_t\nabla U)\\ &= (\nabla\partial_t^2 U\cdot\nabla U) + (\nabla^2 U\nabla U\cdot\mathcal{D}_t v)\\ &\quad + \sum_{\substack{|\alpha|+|\beta|\le 2,\\ \beta_i\le 1}} a_{\alpha,\beta}(v)\nabla^{1+\alpha_1}\partial_t^{\beta_1}U\star\nabla^{1+\alpha_2}\partial_t^{\beta_2}U.\end{aligned}$$

We may embed the rest of the terms to R^1_{III}. This implies (4.29) for $l=2$.

To obtain the claim (4.29) for $l\ge 2$ we differentiate (4.22) $(l-1)$-times. Since the argument is similar to the case $l=2$, we only highlight the most subtle steps. To identify the error terms we recall that by Lemma 4.2 we have for $i\le l-1$

$$\begin{aligned}\mathcal{D}_t^i\nu &= \sum_{|\beta|\le i-1} a_\beta(\nu)\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_i}v,\\ [\mathcal{D}_t^i,\nabla_\Sigma]v &= \sum_{|\beta|\le i-1} a_\beta(\nu)\nabla\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla\mathcal{D}_t^{\beta_{i+1}}v,\end{aligned}$$

and by Remark 4.3

$$[\mathcal{D}_t^i,\Delta_\Sigma]v = \sum_{\substack{|\alpha|\le 1\\ |\beta|\le i-1}} a_{\alpha,\beta}(B)\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v\star\cdots\star\nabla^{1+\alpha_i}\mathcal{D}_t^{\beta_i}v\star\nabla_\tau^{1+\alpha_{i+1}}\mathcal{D}_t^{\beta_{i+1}}v.$$

In order to treat the capacitary terms we first observe that

$$\mathcal{D}_t^{l-1}\langle\mathcal{D}_t\nabla U,\nabla U\rangle = \langle\mathcal{D}_t^l\nabla U,\nabla U\rangle + \sum_{\substack{i+j\le l\\ i,j\le l-1}}\mathcal{D}_t^i\nabla U\star\mathcal{D}_t^i\nabla U$$

and then use Lemma 4.6 to deduce

$$\begin{aligned}\mathcal{D}_t^{l-1}(\mathcal{D}_t\nabla U\cdot\nabla U) &= (\nabla\partial_t^l U\cdot\nabla U) + (\nabla^2 U\nabla U\cdot\mathcal{D}_t^{l-1}v)\\ &+ \sum_{\substack{|\alpha|+|\beta|+|\gamma|\le l,\\ |\beta|\le l-2,\,\gamma_i\le l-1}} a_{\alpha,\beta,\gamma}(v)\mathcal{D}_t^{\beta_1}v\star\cdots\star\mathcal{D}_t^{\beta_{l-1}}\star\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U\star\nabla^{1+\alpha_2}\partial_t^{\gamma_2}U.\end{aligned}$$

This implies (4.29).

We proceed by calculating and by using (4.21)

$$\begin{aligned}\Delta_\Sigma(\mathcal{D}_t^{l-1}v\cdot\nu) &= (\Delta_\Sigma\mathcal{D}_t^{l-1}v)\cdot\nu + 2B:\nabla_\tau(\mathcal{D}_t^{l-1}v) + (\Delta_\Sigma\nu)\cdot(\mathcal{D}_t^{l-1}v)\\ &= (\Delta_\Sigma\mathcal{D}_t^{l-1}v)\cdot\nu + 2B:\nabla_\tau(\mathcal{D}_t^{l-1}v) - |B|^2(\mathcal{D}_t^{l-1}v\cdot\nu) + (\nabla_\tau H\cdot\mathcal{D}_t^{l-1}v).\end{aligned}$$

Moreover, we recall that $\nabla U=\partial_\nu U\nu$ and that (4.24) implies $(\nabla^2 U\nu\cdot\nu)=H\partial_\nu U$. Therefore we have

$$\begin{aligned}\langle\nabla^2 U\nabla U,\mathcal{D}_t^{l-1}v\rangle &= \langle\nabla^2 U\nu,\nu\rangle\,\partial_\nu U\,(\mathcal{D}_t^{l-1}v\cdot\nu) + \langle\nabla^2 U\nabla U,(\mathcal{D}_t^{l-1}v)_\tau\rangle\\ &= H|\nabla U|^2(\mathcal{D}_t^{l-1}v\cdot\nu) + (\nabla_\tau\frac{|\nabla U|^2}{2}\cdot\mathcal{D}_t^{l-1}v).\end{aligned}$$

By combining the previous identities with (4.29) implies

$$\begin{aligned}\mathcal{D}_t^l p &= -\Delta_\Sigma(\mathcal{D}_t^{l-1}v\cdot\nu) - Q(t)\,\partial_\nu U\,(\partial_\nu\partial_t^l U)\\ &\quad - (|B|^2 - Q(t)\,H|\nabla U|^2)(\mathcal{D}_t^{l-1}v\cdot\nu) + \big(\nabla_\tau(H - Q(t)\frac{|\nabla U|^2}{2})\cdot\mathcal{D}_t^{l-1}v\big) + R_p^{l-1}.\end{aligned}$$

Hence, the claim follows from (4.19). □

5. Estimation of the Error Terms

In the previous section we introduced four error terms $R^l_{\mathrm{div}}, R^l_{bulk}, R^l_U$ and R^l_p, which will appear later in the proof of the Main Theorem. These are nonlinear and characterized by their order $l \geq 0$ and their precise forms can be found in (4.13), (4.14) (4.18) and (4.26) respectively. The first two terms R^l_{div} and R^l_{bulk} are defined in the fluid domain and appear already in the case when the shape of the drop does not change. The term R^l_U is due to the nonlinearity of the capacitary term. The term R^l_p is due to the nonlinear behavior of the pressure on the moving boundary and it is by far the most difficult to treat.

In this section our goal is to estimate these error terms by the energy quantity of order $l \in \mathbb{N}$ which we define as

$$E_l(t) := \sum_{k=0}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}(1+1) \rfloor}(\Omega_t)} + \|\mathcal{D}_t^l v \cdot \nu\|^2_{H^1(\Sigma_t)} + 1. \tag{5.1}$$

The most difficult is to estimate the lowest order terms $R^1_{div}, \dots$, i.e., the case $l = 1$, and we treat it separately. The difficulty of the case $l = 1$ makes the arguments in this section long and cumbersome.

As we explained in the introduction, the proof of the Main Theorem is by induction argument, where we assume that we have the bound $E_{l-1}(t) \leq C$ and then use this to bound $E_l(t)$. We begin this section by proving that the bound $E_{l-1}(t) \leq C$ for $l \geq 2$ implies

$$\|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \leq M(C).$$

This will guarantee that every step improves the regularity of the flow. Perhaps the most challenging part is to start the argument and we show in Sect. 6 that the a priori bounds (1.7) imply the following estimate on the pressure

$$\|p\|_{H^1(\Omega_t)} \leq C$$

for all $t \leq T$. We will show that this implies the following curvature bound

$$\|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C,$$

which in particular implies the bound $\|B\|_{L^4(\Sigma_t)} \leq C$.

We notice that the above curvature bounds ensure that Σ_t satisfies the condition (H_m) for $m = \lfloor \frac{3}{2}l \rfloor + 1$ when $l \geq 2$ and $m = 2$ for $l = 1$. This means that the results from Sect. 2 such as Proposition 2.1, Proposition 2.7, Corollary 2.9, Proposition 2.10, Proposition 2.11 and Proposition 2.12 hold for all $k \leq m$. We take this for granted in the calculations throughout this section without further mention in order to make the presentation less heavy.

We begin by estimating the capacitary potential U by the pressure via the identity (4.19). We note that in the next lemma the a priori $C^{1,\alpha}$-bound for the boundary $\Sigma_t = \partial\Omega_t$ is crucial.

Lemma 5.1. *Let $l \geq 1$ and assume that Σ_t is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies the condition $\|B\|_{L^4(\Sigma_t)} \leq M$ when $l = 1$ and $\|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \leq M$ when $l \geq 2$. Let U be the capacitary potential defined in (1.2). There exists a constant C, depending on M, l and on the $C^{1,\alpha}$-norm of the heightfunction, such that*

$$\|\nabla^{1+k} U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1 + \|p\|_{H^k(\Sigma_t)}) \qquad \text{on } \Sigma_t$$

for all integers $k \leq \frac{3}{2}l + \frac{1}{2}$.

Proof. Let us note that the assumptions on the curvature imply that Σ_t satisfies the condition (H_m) for $m = \lfloor \frac{3}{2}l \rfloor + 1$. In particular, the condition $k \leq \frac{3}{2}l + \frac{1}{2}$ implies $k \leq m$.

Let us prove the claim by induction over k and consider first the case $k = 0$. This follows immediately from Theorem 3.9 and from $\|B\|_{L^4(\Sigma_t)} \leq C$ as

$$\|\nabla U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1 + \|B\|_{L^2(\Sigma_t)}) \leq C.$$

Let us then fix k and assume that the claim holds for $k-1$. Since U is constant on Σ_t, Theorem 3.9 implies

$$\|\nabla^{1+k}U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1+\|B\|_{H^k(\Sigma_t)}).$$

By Proposition 2.12 we have

$$\|B\|_{H^k(\Sigma_t)} \leq C(1+\|H\|_{H^k(\Sigma_t)}) \leq C(1+\|p\|_{H^k(\Sigma_t)} + \||\nabla U|^2\|_{H^k(\Sigma_t)}).$$

Proposition 2.10 yields

$$\||\nabla U|^2\|_{H^k(\Sigma_t)} \leq C\|\nabla U\|_{L^\infty(\Sigma_t)}\|\nabla U\|_{H^k(\Sigma_t)}.$$

Since Ω_t is uniformly $C^{1,\alpha}$-regular we have by Schauder estimates $\|U\|_{C^{1,\alpha}(\Sigma_t)} \leq C$. By combining the previous inequalities we obtain

$$\|\nabla^{1+k}U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1+\|p\|_{H^k(\Sigma_t)} + \|\nabla U\|_{H^k(\Sigma_t)}) \tag{5.2}$$

We claim next that under the assumptions of the lemma, it holds for every smooth function $u : \Omega_t \to \mathbb{R}$ and for all $k \leq m$

$$\|\nabla u\|_{H^k(\Sigma_t)} \leq C(\|u\|_{L^2(\Sigma_t)} + \|\nabla^{1+k}u\|_{L^2(\Sigma_t)}). \tag{5.3}$$

Indeed, for $k=1$ we have $\|\nabla u\|_{H^1(\Sigma_t)} \leq \|u\|_{L^2(\Sigma_t)} + \|\nabla^2 u\|_{L^2(\Sigma_t)}$, while for $k=2$ the assumption $\|B\|_{L^4} \leq C$ and the Sobolev embedding imply

$$\begin{aligned}\|\nabla u\|_{H^2(\Sigma_t)} &\leq \|u\|_{L^2(\Sigma_t)} + \|\nabla^3 u\|_{L^2(\Sigma_t)} + \|B \star \nabla^2 u\|_{L^2(\Sigma_t)} \\ &\leq \|u\|_{L^2(\Sigma_t)} + \|\nabla^3 u\|_{L^2(\Sigma_t)} + \|B\|_{L^4(\Sigma_t)}\|\nabla^2 u\|_{L^4(\Sigma_t)} \\ &\leq C(\|u\|_{L^2(\Sigma_t)} + \|\nabla^3 u\|_{L^2(\Sigma_t)}).\end{aligned}$$

The case $k \geq 3$ follows from the same argument. We will take (5.3) for granted from now on.

We obtain by (5.2) and (5.3) that

$$\|\nabla^{1+k}U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1+\|p\|_{H^k(\Sigma_t)} + \|\nabla^{1+k}U\|_{L^2(\Sigma_t)}).$$

We deduce by Lemma 3.3, by interpolation and by the induction assumption (that the claim holds for $k-1$)

$$\begin{aligned}\|\nabla^{1+k}U\|_{L^2(\Sigma_t)} &\leq C(1+\|\nabla^k U\|_{H^1(\Sigma_t)}) \leq \varepsilon\|\nabla^k U\|_{H^{\frac{3}{2}}(\Sigma_t)} + C_\varepsilon(1+\|\nabla^k U\|_{L^2(\Sigma_t)}) \\ &\leq \varepsilon\|\nabla^{1+k}U\|_{H^{\frac{1}{2}}(\Sigma_t)} + C_\varepsilon(1+\|p\|_{H^{k-1}(\Sigma_t)}).\end{aligned}$$

Thus by choosing ε small enough we obtain the claim. □

From Lemma 5.1 we deduce that an estimate on the pressure implies bound on the curvature. The statement follows from the proof of Lemma 5.1 and we leave the proof for the reader.

Lemma 5.2. *Let $l \geq 1$ and assume that Σ_t is uniformly $C^{1,\alpha}(\Gamma)$-regular and for $l=1$ it holds $\|p\|_{H^1(\Omega_t)} \leq M$ and for $l \geq 2$ it holds $E_{l-1}(t) \leq M$. In the case $l=1$ we have*

$$\|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C$$

and $\|B\|_{L^4(\Sigma_t)} \leq C$. In the case $l \geq 2$ we have

$$\|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \leq C.$$

Moreover for $l \geq 1$ we have

$$\|B\|_{H^k(\Sigma_t)} \leq M(1+\|p\|_{H^k(\Sigma_t)})$$

for integers $k \leq \frac{3}{2}l + \frac{1}{2}$. The constants depend on M, l and on the $C^{1,\alpha}$-norm of the heightfunction.

From now on we will assume that, in addition to (1.7), we have for $l = 1$ the estimate $\|p\|_{H^1(\Omega_t)} \le C$ and for $l \ge 2$ $E_{l-1}(t) \le C$. By Lemma 5.2 these imply curvature bounds that we mentioned at the beginning of the section.

We begin to estimate the error terms and we begin with R^l_{div} defined in (4.13).

Lemma 5.3. *Consider R^l_{div} defined in (4.13). Assume that (1.7) holds and $\|p\|_{H^1(\Omega_t)} \le M$. Then we have*

$$\|R^1_{\mathrm{div}}\|^2_{H^{\frac12}(\Omega_t)} \le C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t)$$

for $C = C(M)$.
Let $l \ge 2$ and assume also that $E_{l-1}(t) \le M$. Then there exists $C = C(M, l)$, such that

$$\|R^l_{\mathrm{div}}\|^2_{H^{\frac12}(\Omega_t)} \le CE_l(t) \tag{5.4}$$

and for integers $1 \le k \le l$ and every $\varepsilon > 0$ it holds

$$\|R^{l-k}_{\mathrm{div}}\|^2_{H^{\frac32 k-1}(\Omega_t)} \le \varepsilon E_l(t) + C_\varepsilon \tag{5.5}$$

for some $C_\varepsilon = C_\varepsilon(M, l, \varepsilon)$.

Proof. For $l = 1$ we have by the definition of R^1_{div} (4.13) that

$$R^1_{\mathrm{div}} = a(\nabla v) \star \nabla \mathcal{D}_t v,$$

where a is smooth. Note that in this case $E_1(t)$, defined in (5.1), reads as

$$E_1(t) = \|\mathcal{D}^2_t v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t v\|^2_{H^{\frac32}(\Omega_t)} + \|v\|^2_{H^3(\Omega_t)} + \|\mathcal{D}_t v \cdot \nu\|^2_{H^1(\Sigma_t)} + 1.$$

Since $\|B\|_{L^4} \le C$, we may extend ∇v and $\nabla \mathcal{D}_t v$ to $\mathbb{R}^3$, denote the extensions F_v and G_{v_t} respectively, such that the extensions satisfy

$$\|F_v\|_{L^\infty(\mathbb{R}^3)} \le C\|\nabla v\|_{L^\infty(\Omega_t)}, \qquad \|F_v\|_{H^2(\mathbb{R}^3)} \le C\|\nabla v\|_{H^2(\Omega_t)}$$

and

$$\|G_{v_t}\|_{L^2(\mathbb{R}^3)} \le C\|\nabla \mathcal{D}_t v\|_{L^2(\Omega_t)}, \qquad \|G_{v_t}\|_{H^{\frac12}(\mathbb{R}^3)} \le C\|\mathcal{D}_t v\|_{H^{\frac32}(\Omega_t)}.$$

Moreover, since Ω_t is bounded we may assume that $F_v, G_{v_t} \in C^\infty_0(B_R)$.

We use the Kato-Ponce inequality (2.15) in $\mathbb{R}^3$ with $p_1 = 2, q_1 = \infty$, $p_2 = \frac{12}{5}$ and $q_2 = 12$ to deduce

$$\begin{aligned}\|R^1_{\mathrm{div}}\|_{H^{\frac12}(\Omega_t)} &\le C\|\nabla \mathcal{D}_t v \star a(\nabla v)\|_{H^{\frac12}(\Omega_t)} \le C\|G_{v_t} \star a(F_v)\|_{H^{\frac12}(\mathbb{R}^3)} \\ &\le C\|G_{v_t}\|_{H^{\frac12}(\mathbb{R}^3)}\|F_v\|_{L^\infty(\mathbb{R}^3)} + C\|G_{v_t}\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|F_v\|_{W^{\frac12,12}(\mathbb{R}^3)}.\end{aligned}$$

Since $\|F_v\|_{L^\infty} \le C$ we have

$$\|G_{v_t}\|_{H^{\frac12}(\mathbb{R}^3)}\|F_v\|_{L^\infty(\mathbb{R}^3)} \le C\|\mathcal{D}_t v\|_{H^{\frac32}(\Omega_t)} \le CE_1(t)^{\frac12}.$$

We have by using the Sobolev embedding $\|u\|_{L^p(B_R)} \le C\|u\|_{H^s(B_R)} = C\|u\|_{W^{s,2}(B_R)}$, for $p = \frac{6}{3-2s}$ and $s = \frac14$, and by the general Gagliardo-Nirenberg inequality (2.13) that

$$\|G_{v_t}\|_{L^{\frac{12}{5}}(\mathbb{R}^3)} \le C\|G_{v_t}\|_{H^{\frac14}(\mathbb{R}^3)} \le C\|G_{v_t}\|^{\frac12}_{H^{\frac12}(\mathbb{R}^3)}\|G_{v_t}\|^{\frac12}_{L^2(\mathbb{R}^3)}.$$

By (2.13) we also have

$$\|F_v\|_{W^{\frac12,12}(\mathbb{R}^3)} \le C\|F_v\|^{\frac12}_{W^{1,6}(\mathbb{R}^3)}\|F_v\|^{\frac12}_{L^\infty(\mathbb{R}^3)} \le C\|F_v\|^{\frac12}_{H^2(\mathbb{R}^3)}\|F_v\|^{\frac12}_{L^\infty(\mathbb{R}^3)}.$$

Therefore by $\|F_v\|_{L^\infty(\mathbb{R}^3)} \le C$, $\|F_v\|_{H^2(\mathbb{R}^3)} \le C\|\nabla v\|_{H^2(\Omega_t)} \le C\|v\|_{H^3(\Omega_t)}$ and

$$\|G_{v_t}\|_{L^2(\mathbb{R}^3)} \le C\|\nabla \mathcal{D}_t v\|_{L^2(\Omega_t)} = C\|p\|_{H^2(\Omega_t)}$$

we have

$$\|G_{v_t}\|_{L^{\frac{12}{5}}(\mathbb{R}^3)}\|F_v\|_{W^{\frac12,12}(\mathbb{R}^3)} \le C\|p\|^{\frac12}_{H^2(\Omega_t)}\|\mathcal{D}_t v\|^{\frac12}_{H^{\frac32}(\Omega_t)}\|F_v\|^{\frac12}_{H^2(\mathbb{R}^3)} \le C\|p\|^{\frac12}_{H^2(\Omega_t)}E_1(t)^{\frac12}.$$

This implies the first inequality.

Let $l \geq 2$. In order to estimate the product (4.13) we use Proposition 2.10 to deduce

$$\|R^l_{\mathrm{div}}\|_{H^{\frac{1}{2}}(\Omega_t)} \leq \sum_{|\beta|\leq l} \|\nabla \mathcal{D}_t^{\beta_1} v\|_{H^{\frac{1}{2}}(\Omega_t)} \prod_{i=2}^{l} \|\nabla \mathcal{D}_t^{\beta_i} v\|_{L^\infty(\Omega_t)},$$

where we use the convention that $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. By Recalling the definition of $E_{l-1}(t)$ in (5.1), by the assumption $E_{l-1}(t) \leq C$ and by Sobolev embedding it holds for $\beta_i \leq l-2$

$$\|\nabla \mathcal{D}_t^{\beta_i} v\|^2_{L^\infty(\Omega_t)} \leq C\|\mathcal{D}_t^{\beta_i} v\|^2_{H^3(\Omega_t)} \leq \sum_{k=0}^{l-1} \|\mathcal{D}_t^{l-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \leq CE_{l-1}(t) \leq C.$$

For future purpose we also note that by the same argument we have

$$\|\nabla^{1+\alpha} \mathcal{D}_t^{\beta} v\|^2_{L^\infty(\Omega_t)} \leq CE_{l-1}(t) \quad \text{for } \alpha + \beta \leq l-2. \tag{5.6}$$

Moreover, by the same argument it holds

$$\|\nabla \mathcal{D}_t^{l-1} v\|^2_{L^\infty(\Omega_t)} \leq CE_l(t).$$

We also have

$$\|\nabla \mathcal{D}_t^{l-1} v\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq C\|\mathcal{D}_t^{l-1} v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \leq CE_{l-1}(t) \leq C.$$

Recall that by the definition of R^l_{div} above, the norm of the index is $|\beta| \leq l$. Therefore since $l \geq 2$ it holds $\beta_i \leq l-1$ for $i \geq 2$ and $\beta_i \leq l-2$ for $i \geq 3$. Thus we conclude by the above estimates that

$$\|R^l_{\mathrm{div}}\|_{H^{\frac{1}{2}}(\Omega_t)} \leq C(1 + \|\nabla \mathcal{D}_t^{l} v\|_{H^{\frac{1}{2}}(\Omega_t)} + \|\nabla \mathcal{D}_t^{l-1} v\|_{H^{\frac{1}{2}}(\Omega_t)} \|\mathcal{D}_t^{l-1} v\|_{L^\infty(\Omega_t)}) \leq CE_l(t)^{\frac{1}{2}},$$

which implies (5.4).

The proof of (5.5) follows from similar argument and we merely sketch it. For $k = l$ the statement is trivial. For $1 \leq k \leq l-1$ we recall that

$$R^{l-k}_{\mathrm{div}} = \sum_{|\beta|\leq l-k} a_\beta(\nabla v) \nabla \mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla \mathcal{D}_t^{\beta_{l-k}} v.$$

First, if $k = 1$ then by applying the previous estimate for $l-1$ we obtain

$$\|R^{l-1}_{\mathrm{div}}\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1 + \|p\|^2_{H^2(\Omega_t)}) E_{l-1}(t).$$

But now the condition $E_{l-1}(t) \leq C$ yields

$$\|p\|^2_{H^2(\Omega_t)} \leq \|\mathcal{D}_t v\|^2_{H^1(\Omega_t)} \leq CE_{l-1}(t) \leq C.$$

This implies the inequality for $k = 1$.

Assume $2 \leq k \leq l-1$. We apply Proposition 2.10 to bound

$$\|R^{l-k}_{\mathrm{div}}\|_{H^{\frac{3}{2}k-1}(\Sigma_t)} \leq \sum_{|\beta|\leq l-k} \|\nabla \mathcal{D}_t^{\beta_1} v\|_{L^\infty(\Sigma_t)} \cdots \|\nabla \mathcal{D}_t^{\beta_{l-k-1}} v\|_{L^\infty(\Sigma_t)} \|\nabla \mathcal{D}_t^{\beta_{l-k}} v\|_{H^{\frac{3}{2}k-1}(\Sigma_t)}.$$

Since $k \geq 2$ then $\beta_i \leq l-2$ for all i. Therefore by (5.6) we have $\|\nabla \mathcal{D}_t^{\beta_i} v\|_{L^\infty(\Omega_t)} \leq C$ for all i. Moreover since $\beta_i \leq l-k$ it holds by the Trace Theorem and by interpolation

$$\begin{aligned}
\|\nabla \mathcal{D}_t^{\beta_i} v\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} &\leq C\|\mathcal{D}_t^{\beta_i} v\|^2_{H^{\frac{3}{2}k+1}(\Omega_t)} \\
&\leq \varepsilon\|\mathcal{D}_t^{\beta_i} v\|^2_{H^{\frac{3}{2}(k+1)}(\Omega_t)} + C_\varepsilon\|\mathcal{D}_t^{\beta_i} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \\
&\leq \varepsilon E_l(t) + C_\varepsilon E_{l-1}(t) \leq \varepsilon E_l(t) + C_\varepsilon.
\end{aligned}$$

Hence, we have (5.5). □

We proceed to bound the L^2-norm of R^l_{bulk}, which is defined in (4.14), in the fluid domain Ω_t. Formally R^l_{bulk} is of order 1/2 higher than R^l_{div}, and therefore this bound is of the same order than the previous lemma.

Lemma 5.4. *Consider R^l_{bulk} defined in (4.14). Assume that (1.7) holds and $\|p\|_{H^1(\Omega_t)} \leq M$. There exists $C = C(M)$ such that*

$$\|R^1_{bulk}\|^2_{L^2(\Omega_t)} \leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t).$$

Let $l \geq 2$ and assume also that $E_{l-1}(t) \leq M$. There exists $C = C(M, l)$ such that

$$\|R^l_{bulk}\|^2_{L^2(\Omega_t)} \leq C E_l(t)$$

and for integers $1 \leq k \leq l-1$ and for $\varepsilon > 0$ it holds

$$\|R^{l-k}_{bulk}\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \leq \varepsilon E_l(t) + C_\varepsilon$$

for some constant $C_\varepsilon = C_\varepsilon(M, l, \varepsilon)$.

Proof. By (4.14) and the uniform bound on ∇v given by (1.7) we have a pointwise estimate

$$|R^l_{bulk}| \leq C \sum_{|\alpha|\leq 1, |\beta|\leq l} |\nabla \mathcal{D}_t^{\beta_1} v| \cdots |\nabla \mathcal{D}_t^{\beta_l} v|\, |\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2+\beta_{l+1}} v|.$$

Let us first consider the case $l = 1$. Then we have by the above inequality, by $\mathcal{D}_t v = -\nabla p$ and by ignoring the terms of the form $|\nabla v|$, as they are uniformly bounded, and obtain a pointwise bound

$$|R^1_{bulk}| \leq C \left(1 + |\mathcal{D}_t^2 v| + |\nabla \mathcal{D}_t v||\nabla p|\right).$$

Therefore we have by Hölder's inequality and by the Sobolev embedding

$$\begin{aligned}
\|R^1_{bulk}\|^2_{L^2(\Omega_t)} &\leq C(\|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{L^6(\Omega_t)}\|\nabla \mathcal{D}_t v\|^2_{L^3(\Omega_t)}) \\
&\leq C(1 + \|p\|^2_{H^2(\Omega_t)})(1 + \|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} + \|\mathcal{D}_t v\|^2_{H^{3/2}(\Omega_t)}) \\
&\leq C(1 + \|p\|^2_{H^2(\Omega_t)})(1 + E_1(t)).
\end{aligned}$$

This implies the claim for $l = 1$.

Let us then treat the case $l \geq 2$. Let us assume that the first l indexes are ordered as $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_l$. As before we ignore all the terms in above which are uniformly bounded by the a priori assumption and by the assumption $E_{l-1}(t) \leq C$. Recall first that by (5.6) it holds

$$\|\nabla \mathcal{D}_t^{\beta_i} v\|_{L^\infty} \leq C \quad \text{when} \quad \beta_i \leq l-2.$$

Recall that it holds $|\beta| \leq l$. Therefore, if $\beta_{l+1} \geq l-1$ then $\beta_1 \leq 1$ and $\beta_i = 0$ for $i \geq 2$. When $\beta_{l+1} = l-2$ then the only possible other none-zero indexes are when $\beta_1 = 2$ or when $\beta_1 = \beta_2 = 1$. Finally when $l \geq 3$ and $\beta_{l+1} \leq l-3$ then $\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_{l+1}} v$ is itself uniformly bounded and the only nontrivial terms are given by the indexes $\beta_1 = l-1$ and $\beta_2 = 1$. Hence, we have a pointwise bound which we write by relabeling the indexes as

$$\begin{aligned}
|R^l_{bulk}| \leq\, & C \sum_{|\alpha|\leq 1} |\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l} v| + C \sum_{|\alpha|\leq 1, \beta_i \leq l-1} |\nabla \mathcal{D}_t^{\beta_1} v||\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_2} v| \\
&+ C \sum_{|\alpha|\leq 1} (|\nabla \mathcal{D}_t^2 v| + |\nabla \mathcal{D}_t v|^2)|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-2} v| + C(|\nabla \mathcal{D}_t^l v| + |\nabla \mathcal{D}_t^{l-1} v||\nabla \mathcal{D}_t v|)|\nabla \mathcal{D}_t v|
\end{aligned}$$

Then by Hölder's inequality we deduce

$$\begin{aligned}
\|R^l_{bulk}\|^2_{L^2(\Omega_t)} \leq\, & C \sum_{|\alpha|\leq 1} \|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l} v\|^2_{L^2} + C \sum_{|\alpha|\leq 1, \beta_i \leq l-1} \|\nabla \mathcal{D}_t^{\beta_1} v\|^2_{L^3}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_2} v\|^2_{L^6} \\
&+ C \sum_{|\alpha|\leq 1} (\|\nabla \mathcal{D}_t^2 v\|^2_{L^3} + \|\nabla \mathcal{D}_t v\|^4_{L^6})\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+l-2} v\|^2_{L^6} \\
&+ C(\|\nabla \mathcal{D}_t^l v\|^2_{L^3} + \|\nabla \mathcal{D}_t^{l-1} v\|^2_{L^6}\|\nabla \mathcal{D}_t v\|^2_{L^6})\|\mathcal{D}_t v\|^2_{L^6}.
\end{aligned} \tag{5.7}$$

We bound the first term on RHS of (5.7) for $\alpha_1 + \alpha_2 \leq 1$

$$\|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + l} v\|_{L^2(\Omega_t)}^2 \leq \|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 + \|\mathcal{D}_t^l v\|_{H^{3/2}(\Omega_t)}^2 \leq E_l(t).$$

We claim that the next term with the L^3-norm, $\|\nabla \mathcal{D}_t^{\beta_1} v\|_{L^3}$, is bounded. Indeed, we use the Sobolev embedding, the induction assumption and the fact that $\beta_1 \leq l-1$ and have

$$\|\nabla \mathcal{D}_t^{\beta_1} v\|_{L^3(\Omega_t)}^2 \leq \|\mathcal{D}_t^{\beta_1} v\|_{H^{\frac{3}{2}}(\Omega_t)}^2 \leq \sum_{k=0}^{l} \|\mathcal{D}_t^{l-k} v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 \leq E_{l-1}(t) \leq C.$$

We bound the coupling term with $\alpha_1 + \alpha_2 \leq 1$ and $\beta_2 \leq l-1$ by

$$\|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + \beta_2} v\|_{L^6(\Omega_t)}^2 \leq C \|\mathcal{D}_t^{\alpha_2 + \beta_2} v\|_{H^{1+\alpha_1}(\Omega_t)}^2 \leq C \sum_{k=0}^{l} \|\mathcal{D}_t^{l+1-k} v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 \leq C E_l(t).$$

We proceed to the next row in (5.7) and for $\alpha_1 + \alpha_2 \leq 1$ we have

$$\|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + l - 2} v\|_{L^6(\Omega_t)}^2 \leq \|\mathcal{D}_t^{\alpha_2 + l - 2} v\|_{H^{1+\alpha_1}(\Omega_t)}^2 \leq C \sum_{k=0}^{l} \|\mathcal{D}_t^{l-k} v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 \leq C E_{l-1}(t) \leq C.$$

We also have

$$\|\nabla \mathcal{D}_t^2 v\|_{L^3(\Omega_t)}^2 \leq \|\mathcal{D}_t^2 v\|_{H^{3/2}(\Omega_t)}^2 \leq C E_2(t) \leq C E_l(t)$$

since $l \geq 2$. Moreover, by interpolation it holds

$$\|\nabla \mathcal{D}_t v\|_{L^6} \leq C \|\nabla \mathcal{D}_t v\|_{H^2}^{\frac{1}{2}} \|\nabla \mathcal{D}_t v\|_{L^2}^{\frac{1}{2}} \leq C E_2(t)^{\frac{1}{4}} E_1(t)^{\frac{1}{4}} \leq C E_l(t)^{\frac{1}{4}},$$

when $l \geq 2$. Hence, the second row in (5.7) is bounded by $E_l(t)$.

We are left with the last row in (5.7). We bound the first term by

$$\|\nabla \mathcal{D}_t^l v\|_{L^3(\Omega_t)}^2 \leq \|\nabla \mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Omega_t)}^2 \leq \|\mathcal{D}_t^l v\|_{H^{\frac{3}{2}}(\Omega_t)}^2 \leq E_l(t)$$

and the last by $\|\mathcal{D}_t v\|_{L^6(\Omega_t)}^2 \leq \|\mathcal{D}_t v\|_{H^1(\Omega_t)}^2 \leq C E_1(t) \leq C$. Finally we treat the two remaining terms by the same argument. Indeed for $\beta \leq l-1$ we have by interpolation as before

$$\|\nabla \mathcal{D}_t^{\beta} v\|_{L^6(\Omega_t)} \leq C \|\nabla \mathcal{D}_t^{\beta} v\|_{H^2(\Omega_t)}^{\frac{1}{2}} \|\nabla \mathcal{D}_t^{\beta} v\|_{L^2(\Omega_t)}^{\frac{1}{2}} \leq C E_l(t)^{\frac{1}{4}} E_{l-1}(t)^{\frac{1}{4}} \leq C E_l(t)^{\frac{1}{4}}.$$

Hence, we have

$$\|R_{bulk}^l\|_{L^2(\Omega_t)}^2 \leq C E_l(t).$$

We are left with the last inequality. For $k = 1$ the claim follows by applying the previous inequality with $l-1$. Let us then assume $l \geq 3$ and $2 \leq k \leq l-1$. By definition of R_{bulk}^{l-k} in (4.14) it holds $|\beta| \leq l-k \leq l-2$. Therefore (5.6) implies

$$\|\nabla \mathcal{D}_t^{\beta_i} v\|_{L^\infty(\Omega_t)} \leq \|\mathcal{D}_t^{\beta_i} v\|_{H^3(\Omega_t)} \leq C$$

for all i. Therefore by Proposition 2.10 and by relabeling the indexes we have

$$\begin{aligned} \|R_{bulk}^{l-k}\|_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \leq C \sum_{\substack{|\beta| \leq l-k \\ |\alpha| \leq 1}} & \|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + \beta_2} v\|_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \\ & + \|\nabla \mathcal{D}_t^{\beta_1} v\|_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + \beta_2} v\|_{L^\infty(\Omega_t)}. \end{aligned}$$

Since $\beta_2 \leq l-k$, we may estimate the first term on RHS as

$$\|\nabla^{\alpha_1} \mathcal{D}_t^{\alpha_2 + \beta_2} v\|_{H^{\frac{3}{2}(k-1)}(\Omega_t)}^2 \leq C \sum_{i=0}^{l-1} \|\mathcal{D}_t^{l-i} v\|_{H^{\frac{3}{2}i}(\Omega_t)}^2 \leq C E_{l-1}(t) \leq C.$$

We estimate the second similarly by using $\beta_1 \le l-k$

$$\|\nabla \mathcal{D}_t^{\beta_1} v\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \le \|\mathcal{D}_t^{\beta_1} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \le C E_{l-1}(t) \le C.$$

Finally we bound the last term by the Sobolev embedding, by $\beta_2 \le l-k$, $\alpha_1+\alpha_2 \le 1$ and by interpolation

$$\begin{aligned}\|\nabla^{\alpha_1}\mathcal{D}_t^{\alpha_2+\beta_2} v\|^2_{L^\infty(\Omega_t)} &\le C\|\mathcal{D}_t^{\alpha_2+\beta_2} v\|^2_{H^{2+\alpha_1}(\Omega_t)}\\ &\le \varepsilon \|\mathcal{D}_t^{\alpha_2+\beta_2} v\|^2_{H^{\frac{3}{2}(2+\alpha_1)}(\Omega_t)} + C_\varepsilon \|\mathcal{D}_t^{\alpha_2+\beta_2} v\|^2_{L^2(\Omega_t)}\\ &\le \varepsilon E_l(t) + C_\varepsilon E_{l-1}(t) \le \varepsilon E_l(t) + C_\varepsilon.\end{aligned}$$

Thus we have

$$\|R^{l-k}_{bulk}\|^2_{H^{\frac{3}{2}(k-1)}(\Omega_t)} \le \varepsilon E_l(t) + C_\varepsilon.$$

□

The two previous error bounds in Lemma 5.3 and Lemma 5.4 are similar in the sense that they only involve the material derivatives of the velocity field. We proceed to the error terms which involve the time derivatives of the capacity potential. Note that $\partial_t^k U$ for all k is a harmonic function in Ω_t^c but not constant on Σ_t. We will use again Theorem 3.9 together with Lemma 4.6 which gives the formula for $\partial_t^k U$ on the boundary Σ_t.

We first prove a generic bound which will be useful when we bound the pressure.

Lemma 5.5. *Let $l \ge 2$ and assume that (1.7) and the condition $E_{l-1}(t) \le M$ hold. Let $\alpha, \beta \ge 0$ be integers. When $\alpha + \beta \le l$, it holds*

$$\|\nabla^{1+\alpha}\partial_t^\beta U\|^2_{H^{\frac{\alpha}{2}+\frac{1}{2}}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon, \tag{5.8}$$

for $C_\varepsilon = C_\varepsilon(M, l, \varepsilon)$. On the other hand, when $\alpha+\beta \le l+1$ and $\beta \le l$ then

$$\|\nabla^{1+\alpha}\partial_t^\beta U\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le C E_l(t) \tag{5.9}$$

for $C = C_\varepsilon(M, l)$.

Proof. Instead of (5.8) we prove in fact a slightly stronger result, namely

$$\begin{aligned}&\|\nabla^{1+\alpha}\partial_t^\beta U\|^2_{H^{\frac{\alpha}{2}+\frac{1}{2}}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon \quad \text{when } \alpha \text{ is even and}\\ &\|\nabla^{1+\alpha}\partial_t^\beta U\|^2_{H^{\frac{\alpha}{2}+1}(\Sigma_t)} \le C E_l(t) \quad \text{when } \alpha \text{ is odd.}\end{aligned} \tag{5.10}$$

The inequality (5.8) then follows from (5.10) by interpolation.

We prove (5.10) by induction over β and consider first the case $\beta = 0$. Note that then $\alpha \le l$. Let us first consider the case when α is even. Then by Lemma 5.1 and by interpolation we have

$$\begin{aligned}\|\nabla^{1+\alpha} U\|_{H^{\frac{\alpha}{2}+\frac{1}{2}}(\Sigma_t)} &\le C(1+\|p\|_{H^{\frac{3}{2}\alpha}(\Sigma_t)}) \le \varepsilon\|p\|_{H^{\frac{3}{2}\alpha+\frac{1}{2}}(\Sigma_t)} + C_\varepsilon\|p\|_{L^2(\Sigma_t)}\\ &\le \varepsilon\|p\|_{H^{\frac{3}{2}\alpha+\frac{1}{2}}(\Sigma_t)} + C_\varepsilon.\end{aligned}$$

We use Lemma 3.7, $-\nabla p = \mathcal{D}_t v$, $\alpha \le l$ and the definition of $E_l(t)$ to estimate

$$\|p\|^2_{H^{\frac{3}{2}\alpha+\frac{1}{2}}(\Sigma_t)} \le C\|p\|^2_{H^{\frac{3}{2}\alpha+1}(\Omega_t)} \le C(1+\|\mathcal{D}_t v\|^2_{H^{\frac{3l}{2}}(\Omega_t)}) \le C E_l(t).$$

This implies (5.10) when α is even. When α is odd we have again by Lemma 5.1, Lemma 3.7 and by the definition of E_l that

$$\|\nabla^{1+\alpha} U\|^2_{H^{\frac{\alpha}{2}+1}(\Sigma_t)} \le C(1+\|p\|^2_{H^{\frac{3\alpha}{2}+\frac{1}{2}}(\Sigma_t)}) \le C(1+\|p\|^2_{H^{\frac{3l}{2}+1}(\Omega_t)}) \le C E_l(t).$$

Let us assume that (5.10) holds for $\beta \le k-1$ for $2 \le k \le l$. In particular, this implies that (5.8) holds for $\beta \le k-1$. Let us consider only the case when α is even, since the argument for α odd is similar. We first observe that since $\alpha \le l-k \le l-1$ then by the Trace Theorem

$$\|p\|^2_{H^{\frac{3}{2}\alpha}(\Sigma_t)} \le C(1+\|\nabla p\|^2_{H^{\frac{3}{2}\alpha}(\Omega_t)}) \le C(1+\|\mathcal{D}_t v\|^2_{H^{\frac{3}{2}(l-1)}(\Omega_t)}) \le CE_{l-1}(t) \le C. \tag{5.11}$$

We have then by Theorem 3.9, Lemma 5.2 and by (5.11) that

$$\begin{aligned}\|\nabla^{1+\alpha}\partial_t^k U\|_{H^{\frac{\alpha}{2}+\frac{1}{2}}(\Sigma_t)} &\le C(1+\|p\|_{H^{\frac{3}{2}\alpha}(\Sigma_t)} + \|\partial_t^k U\|_{H^{\frac{3}{2}\alpha+\frac{3}{2}}(\Sigma_t)})\\ &\le C(1+\|\partial_t^k U\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}).\end{aligned}$$

We use the expression of $\partial_t^k U$ from Lemma 4.6 and Proposition 2.10 to estimate the last term in above as follows

$$\begin{aligned}\|\partial_t^k U\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} \le &\|\nabla U\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}\|\mathcal{D}_t^{k-1}v\|_{L^\infty(\Sigma_t)}\\ &+\|\nabla U\|_{L^\infty(\Sigma_t)}\|\mathcal{D}_t^{k-1}v\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} + \|R_U^{k-2}\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}.\end{aligned}$$

To estimate the first term on RHS let us first assume that $k \le l-1$. Then by (5.6) $\|\mathcal{D}_t^{k-1}v\|_{L^\infty(\Sigma_t)} \le C$ and since $\alpha \le l-1$ we have by (5.8) for $\beta = 0$ that

$$\|\nabla U\|^2_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon.$$

On the other hand, when $k = l$ then $\alpha = 0$. Thus again by (5.8) it holds

$$\|\nabla U\|^2_{H^{\frac{3}{2}}(\Sigma_t)} \le CE_1(t) \le CE_{l-1}(t) \le C.$$

By the Sobolev embedding, by interpolation, by the Trace Theorem and by $\alpha + k \le l$ we have

$$\begin{aligned}\|\mathcal{D}_t^{k-1}v\|^2_{L^\infty(\Sigma_t)} &\le C\|\mathcal{D}_t^{k-1}v\|^2_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}\\ &\le \varepsilon\|\mathcal{D}_t^{k-1}v\|^2_{H^{\frac{3}{2}\alpha+2}(\Sigma_t)} + C_\varepsilon\|\mathcal{D}_t^{k-1}v\|^2_{L^2(\Sigma_t)}\\ &\le C\varepsilon\|\mathcal{D}_t^{k-1}v\|^2_{H^{\frac{3}{2}(\alpha+2)}(\Omega_t)} + C_\varepsilon E_{l-1}(t) \le C\varepsilon E_l(t) + C_\varepsilon.\end{aligned}$$

Hence, we need yet to prove

$$\|R_U^{k-2}\|^2_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon, \tag{5.12}$$

where $k \le l$.

By the definition in (4.18) and by the Kato-Ponce inequality (Proposition 2.10) we may estimate

$$\begin{aligned}\|R_U^{k-2}\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} \le &C\sum_{\substack{|\gamma|+|\beta|\le k-1\\ |\beta|\le k-2}}\Big(\|\mathcal{D}_t^{\beta_1}v\|_{L^\infty}\cdots\|\mathcal{D}_t^{\beta_{k-2}}v\|_{L^\infty}\|\nabla^{1+\gamma_1}\partial_t^{\gamma_2}U\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}\\ &+\|\mathcal{D}_t^{\beta_1}v\|_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}\cdots\|\mathcal{D}_t^{\beta_{k-2}}v\|_{L^\infty}\|\nabla^{1+\gamma_1}\partial_t^{\gamma_2}U\|_{L^\infty(\Sigma_t)}\Big).\end{aligned} \tag{5.13}$$

Since $|\beta| \le k-2 \le l-2$, (5.6) implies $\|\mathcal{D}_t^{\beta_1}v\|_{L^\infty} \le C$ for all i. Note that $\alpha + k \le l$ and $\beta_1 \le k-2$ implies $\alpha + \beta_1 \le l-2$. Therefore we have by the Trace Theorem and by the definition of $E_{l-1}(t)$

$$\|\mathcal{D}_t^{\beta_1}v\|^2_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)} \le C\|\mathcal{D}_t^{\beta_1}v\|^2_{H^{\frac{3}{2}(\alpha+2)}(\Omega_t)} \le CE_{l-1}(t) \le C.$$

We bound the both capacitary terms by the Sobolev embedding and by the induction assumption, which states that (5.8) holds for $\beta \le k-1$. Indeed we have by $\gamma_1 + \alpha \le |\gamma| + \alpha \le k-1+\alpha \le l-1$ and $\gamma_2 \le k-1$ that

$$\begin{aligned}\|\nabla^{1+\gamma_1}\partial_t^{\gamma_2}U\|^2_{L^\infty(\Sigma_t)} &\le C\|\nabla^{1+\gamma_1}\partial_t^{\gamma_2}U\|^2_{H^{\frac{3}{2}(1+\alpha)}(\Sigma_t)}\\ &\le C(1+\|\nabla^{1+(1+\alpha+\gamma_1)}\partial_t^{\gamma_2}U\|^2_{H^{\frac{1}{2}(1+\alpha+\gamma_1)}(\Sigma_t)})\\ &\le \varepsilon E_l(t) + C_\varepsilon.\end{aligned} \tag{5.14}$$

Hence, we have (5.12) when α is even.

Let us then prove (5.9). We notice that Theorem 3.9 and (5.8) imply (5.9) when $2 \leq \alpha \leq l$. On the other Lemma 5.1 implies (5.9) when $\alpha = l + 1$. (Note that the assumption $\alpha \leq \frac{3}{2}l$ is satisfied for all $\alpha \leq l + 1$ since $l \geq 2$.) We need thus to consider the case $\alpha = 1$ and $\beta = l$. For this the argument is similar than before and we only sketch it.

By Theorem 3.9 and by (5.11) we have

$$\begin{aligned}\|\nabla^2 \partial_t^l U\|_{H^{\frac{1}{2}}(\Sigma_t)} &\leq C(1 + \|p\|_{H^1(\Sigma_t)} + \|\partial_t^l U\|_{H^{\frac{5}{2}}(\Sigma_t)}) \\ &\leq C(1 + \|\partial_t^l U\|_{H^{\frac{5}{2}}(\Sigma_t)}).\end{aligned}$$

We use the expression of $\partial_t^l U$ from Lemma 4.6 and Proposition 2.10 to estimate the last term in above as follows

$$\begin{aligned}\|\partial_t^l U\|_{H^{\frac{5}{2}}(\Sigma_t)} \leq\ & \|\nabla U\|_{H^{\frac{5}{2}}(\Sigma_t)} \|\mathcal{D}_t^{l-1} v\|_{L^\infty(\Sigma_t)} \\ & + \|\nabla U\|_{L^\infty(\Sigma_t)} \|\mathcal{D}_t^{l-1} v\|_{H^{\frac{5}{2}}(\Sigma_t)} + \|R_U^{l-2}\|_{H^{\frac{5}{2}}(\Sigma_t)}.\end{aligned}$$

By Lemma 5.1 and Lemma 3.7 we have

$$\|\nabla U\|^2_{H^{\frac{5}{2}}(\Sigma_t)} \leq C(1 + \|p\|^2_{H^2(\Sigma_t)}) \leq C(1 + \|\nabla p\|^2_{H^{\frac{3}{2}}(\Omega_t)}) \leq C E_1(t) \leq C.$$

Recall also that $\|\nabla U\|_{L^\infty} \leq C$. Sobolev embedding and Trace Theorem yield

$$\|\mathcal{D}_t^{l-1} v\|^2_{L^\infty(\Sigma_t)} \leq C \|\mathcal{D}_t^{l-1} v\|^2_{H^2(\Sigma_t)} \leq C \|\mathcal{D}_t^{l-1} v\|^2_{H^3(\Omega_t)} \leq C E_l(t).$$

Therefore we need yet to estimate $\|R_U^{l-2}\|_{H^{\frac{5}{2}}(\Sigma_t)}$.

Arguing as in (5.13) we obtain

$$\begin{aligned}\|R_U^{l-2}\|_{H^{\frac{5}{2}}(\Sigma_t)} \leq \sum_{\substack{|\gamma|+|\beta|\leq l-1 \\ |\beta| \leq l-2}} \Big(& \|\mathcal{D}_t^{\beta_1} v\|_{L^\infty} \cdots \|\mathcal{D}_t^{\beta_{l-2}} v\|_{L^\infty} \|\nabla^{1+\gamma_1} \partial_t^{\gamma_2} U\|_{H^{\frac{5}{2}}(\Sigma_t)} \\ & + \|\mathcal{D}_t^{\beta_1} v\|_{H^{\frac{5}{2}}(\Sigma_t)} \cdots \|\mathcal{D}_t^{\beta_{l-2}} v\|_{L^\infty} \|\nabla^{1+\gamma_1} \partial_t^{\gamma_2} U\|_{L^\infty(\Sigma_t)} \Big).\end{aligned}$$

Arguing as before we deduce $\|\mathcal{D}_t^{\beta_i} v\|_{L^\infty} \leq C$ for all i and

$$\|\mathcal{D}_t^{\beta_1} v\|^2_{H^{\frac{5}{2}}(\Sigma_t)} \leq \|\mathcal{D}_t^{\beta_1} v\|^2_{H^3(\Omega_t)} \leq E_{l-1}(t) \leq C.$$

Finally we use the fact that (5.9) holds for $\beta \leq l - 1$ and $\gamma_1 + \gamma_2 \leq l - 1$ to conclude

$$\begin{aligned}\|\nabla^{1+\gamma_1} \partial_t^{\gamma_2} U\|^2_{L^\infty(\Sigma_t)} &\leq C \|\nabla^{1+\gamma_1} \partial_t^{\gamma_2} U\|^2_{H^{\frac{5}{2}}(\Sigma_t)} \\ &\leq C(1 + \|\nabla^{1+(2+\gamma_1)} \partial_t^{\gamma_2} U\|^2_{H^{\frac{1}{2}}(\Sigma_t)}) \leq C E_l(t).\end{aligned}$$

This concludes the proof. □

We need also the following bound on the capacitary potential and for the error term R_U^l, defined in (4.18), associated with it. In the first statement of the following lemma we need to relax the usual assumption on the quantity (1.5) being bounded to assume that the set Ω_t is uniformly $C^{1,\alpha}(\Gamma)$-regular and that the velocity satisfies $\|v\|_{W^{1,4}(\Sigma_t)} \leq C$. The point is that we need the following estimate when we do not have the Lipschitz bound on v. This does not complicate the proof and will be useful later.

Lemma 5.6. *Consider R_U^l defined in (4.18). Assume that Σ_t is uniformly $C^{1,\alpha}(\Gamma)$-regular and satisfies*

$$\|p\|_{H^1(\Omega_t)} + \|v\|_{W^{1,4}(\Sigma_t)} \leq M.$$

There exists C such that

$$\|\nabla \partial_t^2 U\|^2_{L^2(\Sigma_t)} + \|R_U^1\|^2_{L^2(\Sigma_t)} \leq C(1 + \|p\|^2_{H^2(\Omega_t)}) E_1(t). \tag{5.15}$$

Let $l \geq 2$ and assume that $E_{l-1}(t) \leq M$. Then it holds

$$\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2 + \|R_U^l\|_{L^2(\Sigma_t)}^2 \leq C E_l(t). \tag{5.16}$$

The constants depend on M, l and on the $C^{1,\alpha}$-norm of the heightfunction.

Proof. This time we only prove (5.15) since (5.16) follows from similar argument. Note also that the $C^{1,\alpha}$-bound on Σ_t implies $C^{1,\alpha}$-bound on U. Recall also that $\|p\|_{H^1(\Omega_t)} \leq C$ implies $\|B\|_{L^4(\Sigma_t)} \leq \|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C$ by Lemma 5.2. We begin by noticing that, since $\partial_t^2 U$ is harmonic, it holds by Lemma 3.3

$$\|\nabla \partial_t^2 U\|_{L^2(\Sigma_t)} \leq C(1 + \|\partial_t^2 U\|_{H^1(\Sigma_t)}).$$

Then by Lemma 4.6 we have

$$\|\partial_t^2 U\|_{H^1(\Sigma_t)} \leq C(1 + \|\partial_\nu U(\mathcal{D}_t v \cdot \nu)\|_{H^1(\Sigma_t)} + \|R_U^0\|_{H^1(\Sigma_t)}),$$

where

$$R_U^0 = \sum_{|\alpha| \leq 1} a_\alpha(v) \nabla^{1+\alpha_1} \partial_t^{\alpha_2} U.$$

Since $\|\nabla U\|_{L^\infty(\Sigma_t)} \leq C$, we may estimate by Proposition 2.10, $\|B\|_{L^4} \leq C$ and by the Sobolev embedding

$$\begin{aligned} &\|\partial_t^2 U\|_{H^1(\Sigma_t)} \\ &\quad \leq C + C\|\mathcal{D}_t v \cdot \nu\|_{H^1(\Sigma_t)} + C(\|\nabla^2 U\|_{L^4(\Sigma_t)} + \|B\|_{L^4(\Sigma_t)})\|\mathcal{D}_t v \cdot \nu\|_{L^4(\Sigma_t)} + \|R_U^0\|_{H^1(\Sigma_t)} \\ &\quad \leq C + C(1 + \|\nabla^2 U\|_{L^4(\Sigma_t)})\|\mathcal{D}_t v \cdot \nu\|_{H^1(\Sigma_t)} + \|R_U^0\|_{H^1(\Sigma_t)}. \end{aligned} \tag{5.17}$$

We have by the Sobolev embedding and by Lemma 5.1

$$\|\nabla^2 U\|_{L^4(\Sigma_t)} \leq \|\nabla^2 U\|_{H^{1/2}(\Sigma_t)} \leq C(1 + \|p\|_{H^1(\Sigma_t)}). \tag{5.18}$$

Since $\|\mathcal{D}_t v \cdot \nu\|_{H^1(\Sigma_t)}^2 \leq E_1(t)$, we need yet to show that $\|R_U^0\|_{H^1(\Sigma_t)}^2 \leq C E_1(t)$.

Since $\|v\|_{L^\infty(\Sigma_t)} \leq C\|v\|_{W^{1,4}(\Sigma_t)} \leq C$ we have by the Sobolev embedding

$$\begin{aligned} \|R_U^0\|_{H^1(\Sigma_t)} &\leq C \sum_{|\alpha| \leq 1} \|v\|_{L^\infty} \|\nabla^{1+\alpha_1} \partial_t^{\alpha_2} U\|_{H^1(\Sigma_t)} + C \sum_{|\alpha| \leq 1} \|v\|_{W^{1,4}} \|\nabla^{1+\alpha_1} \partial_t^{\alpha_2} U\|_{L^4(\Sigma_t)} \\ &\leq C(1 + \|\nabla^2 U\|_{H^1(\Sigma_t)} + \|\nabla \partial_t U\|_{H^1(\Sigma_t)}). \end{aligned}$$

Lemma 5.1 and Lemma 3.7 yield

$$\|\nabla^3 U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 \leq C(1 + \|p\|_{H^2(\Sigma_t)}^2) \leq C(1 + \|\nabla p\|_{H^{\frac{3}{2}}(\Omega_t)}^2) \leq C E_1(t). \tag{5.19}$$

We bound $\|\nabla^2 \partial_t U\|_{H^{\frac{1}{2}}(\Sigma_t)}$ with a similar argument and thus we only sketch it. First, we recall that it holds $\partial_t U = -\nabla U \cdot v$ on Σ_t. We use Theorem 3.9, Lemma 5.2, Proposition 2.10 and (5.19) to deduce

$$\begin{aligned} \|\nabla^2 \partial_t U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 &\leq C(1 + \|p\|_{H^1(\Sigma_t)}^2 + \|\nabla U \cdot v\|_{H^{\frac{5}{2}}(\Sigma_t)}^2) \\ &\leq C(1 + \|p\|_{H^1(\Sigma_t)}^2 + \|\nabla^3 U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + \|v\|_{H^3(\Omega_t)}^2) \\ &\leq C E_1(t). \end{aligned} \tag{5.20}$$

We thus deduce by (5.17), (5.18), (5.19) and (5.20) that

$$\|\nabla \partial_t^2 U\|_{L^2(\Sigma_t)}^2 \leq C(1 + \|p\|_{H^2(\Omega_t)}^2) E_1(t). \tag{5.21}$$

We are left with

$$\|R_U^1\|_{L^2(\Sigma_t)}^2 \leq C(1 + \|p\|_{H^2(\Omega_t)}^2) E_1(t). \tag{5.22}$$

To this aim we recall the definition of R_U^1 in (4.18)

$$R_U^1 = \sum_{|\alpha| + \beta \leq 2,\, \beta \leq 1} a_{\alpha,\beta} \mathcal{D}_t^\beta v \star \nabla^{1+\alpha_1} \partial_t^{\alpha_2} U.$$

We use Hölder's inequality as

$$\|R_U^1\|_{L^2(\Sigma_t)} \le C\big(\sum_{|\alpha|\le 2} \|\nabla^{1+\alpha_1}\partial_t^{\alpha_2} U\|_{L^2(\Sigma_t)} + \|\mathcal{D}_t v\|_{L^4(\Sigma_t)} \sum_{|\alpha|\le 1} \|\nabla^{1+\alpha_1}\partial_t^{\alpha_2} U\|_{L^4(\Sigma_t)}\big).$$

We have by (5.19), (5.20) and (5.21)

$$\sum_{|\alpha|\le 2} \|\nabla^{1+\alpha_1}\partial_t^{\alpha_2} U\|_{L^2(\Sigma_t)}^2 \le C(1+\|p\|_{H^2(\Omega_t)}^2)E_1(t).$$

By the Sobolev embedding and $-\nabla p = \mathcal{D}_t v$ it holds

$$\|\mathcal{D}_t v\|_{L^4(\Sigma_t)} \le \|\mathcal{D}_t v\|_{H^1(\Omega_t)} \le C\|p\|_{H^2(\Omega_t)}.$$

Moreover (5.19) and (5.20) imply for $\alpha_1+\alpha_2\le 1$

$$\|\nabla^{1+\alpha_1}\partial_t^{\alpha_2} U\|_{L^4(\Sigma_t)}^2 \le \|\nabla^{1+\alpha_1}\partial_t^{\alpha_2} U\|_{H^{\frac12}(\Sigma_t)}^2 \le CE_1(t).$$

Hence we have (5.22). □

Finally we need to bound the error term R_p^l, defined in (4.26), which is associated with the pressure. This term is the most difficult to treat and it turns out that the lower order case $l=1$ is the most challenging to deal with. Therefore we state it as an own lemma.

Lemma 5.7. *Let R_p^l be as defined in (4.26). Assume that (1.7) holds and $\|p\|_{H^1(\Omega_t)}\le M$. Then it holds*

$$\|R_p^1\|_{H^{\frac12}(\Sigma_t)}^2 \le C(1+\|p\|_{H^2(\Omega_t)}^2)E_1(t).$$

for some constant $C=C(M)$.

Proof. Let us begin by recalling that by the definition of R_p^1 in (4.26), (4.27) (4.28) we may write

$$\begin{aligned} R_p^1 = &- |\nabla_\tau p|^2 - (|B|^2 - Q(t)H|\nabla U|^2)\,\partial_\nu p + a_1(\nu,\nabla v)\star B + a_2(\nu,\nabla v)\star \nabla^2 v \\ &+ \sum_{|\alpha|+|\gamma|\le 2,\, \gamma_i\le 1} a_{\alpha,\gamma,Q}(v)\nabla^{1+\alpha_1}\partial_t^{\gamma_1} U\star \nabla^{1+\alpha_2}\partial_t^{\gamma_2} U. \end{aligned} \tag{5.23}$$

We first recall it holds $\|U\|_{C^{1,\alpha}(\Sigma_t)}\le C$. We may bound the curvature by Sobolev embedding and interpolation as

$$\|B\|_{C^\alpha(\Sigma_t)} \le C(1+\|p\|_{C^\alpha(\Sigma_t)}) \le C(1+\|\bar\nabla p\|_{L^{\frac{11}{5}}(\Sigma_t)}) \le C(1+\|p\|_{H^2(\Sigma_t)}^\theta \|p\|_{L^4(\Sigma_t)}^{1-\theta}),$$

for $\theta<\frac49$. Recall that $\|p\|_{H^1(\Omega_t)}\le C$ implies $\|p\|_{L^4(\Sigma_t)}, \|B\|_{L^4}\le C$. By the Sobolev embedding and by Lemma 3.7 we have

$$\|p\|_{H^2(\Sigma_t)}^2 \le C\|\nabla p\|_{H^1(\Sigma_t)}^2 \le C\|\nabla p\|_{H^{\frac32}(\Omega_t)}^2 \le CE_1(t). \tag{5.24}$$

Therefore we obtain

$$\|B\|_{C^\alpha(\Sigma_t)}^2 \le CE_1(t)^\theta \qquad \text{for } \theta<\frac49. \tag{5.25}$$

We may also bound the curvature simply as

$$\|B\|_{L^\infty(\Sigma_t)} \le \|B\|_{C^\alpha(\Sigma_t)} \le C(1+\|p\|_{C^\alpha(\Sigma_t)}) \le C(1+\|p\|_{H^2(\Omega_t)}). \tag{5.26}$$

Let us bound the first term in (5.23) which is of the highest order. We observe that by interpolation it holds

$$\|\nabla p\|_{L^8(\Sigma_t)} \le C\|\nabla p\|_{H^1(\Sigma_t)}^{\frac12}\|\nabla p\|_{L^4(\Sigma_t)}^{\frac12}.$$

By using this, the Sobolev embedding and (2.17) we have

$$\begin{aligned}\||\nabla_\tau p|^2\|_{H^{1/2}(\Sigma_t)} &\le C\||\nabla p|^2\|_{H^{\frac12}(\Sigma_t)} + C\|\nu\|_{W^{1,4}(\Sigma_t)}\||\nabla p|^2\|_{L^4(\Sigma_t)}\\ &\le C\||\nabla p|^2\|_{H^1(\Omega_t)} + C\|B\|_{L^4(\Sigma_t)}\|\nabla p\|^2_{L^8(\Sigma_t)}\\ &\le C\|\nabla p\|_{L^6(\Omega_t)}\|\nabla^2 p\|_{L^3(\Omega_t)} + C\|\nabla p\|_{L^4(\Sigma_t)}\|\nabla p\|_{H^1(\Sigma_t)}\\ &\le C\|p\|_{H^2(\Omega_t)}\|\nabla p\|_{H^{\frac32}(\Omega_t)}\\ &\le C(1+\|p\|_{H^2(\Omega_t)})E_1(t)^{\frac12}.\end{aligned}$$

This gives bound for the first term.

In order to bound the next term in (5.23) we let $\tilde\nu$ and $\tilde B$ be the harmonic extensions of the normal ν and of the second fundamental form B to Ω_t. By maximum principle $\|\tilde B\|_{L^\infty(\Omega_t)} \le \|B\|_{L^\infty(\Sigma_t)}$, while by standard results from harmonic analysis [16] it holds

$$\|\tilde B\|_{W^{1,3}(\Omega_t)} \le C\|B\|_{W^{1,3}(\Sigma_t)}$$

and by (3.12) $\|\tilde\nu\|_{W^{1,4}(\Omega_t)} \le C$. Then we have

$$\|\nabla p\cdot\tilde\nu\|_{H^1(\Omega_t)} \le C\|p\|_{H^2(\Omega_t)} + C\|\tilde\nu\|_{W^{1,4}(\Omega_t)}\|p\|_{W^{1,4}(\Omega_t)} \le C\|p\|_{H^2(\Omega_t)}. \tag{5.27}$$

We have by (5.25), (5.27) and by the Sobolev embedding

$$\begin{aligned}\||B|^2\,\partial_\nu p\|^2_{H^{1/2}(\Sigma_t)} &\le C(1+\|\nabla(|\tilde B|^2\,(\nabla p\cdot\tilde\nu))\|^2_{L^2(\Omega_t)}\\ &\le C(1+\|B\|^4_{L^\infty(\Sigma_t)}\|\nabla p\cdot\tilde\nu\|^2_{H^1(\Omega_t)} + \|B\|^2_{L^\infty(\Sigma_t)}\|\nabla\tilde B\|^2_{L^3(\Omega_t)}\|\nabla p\|^2_{L^6(\Omega_t)})\\ &\le C(1+\|p\|^2_{H^2(\Omega_t)})E_1(t)^{2\theta} + E_1(t)^\theta\|B\|^2_{W^{1,3}(\Sigma_t)}\|p\|^2_{H^2(\Omega_t)}),\end{aligned}$$

for $\theta<\frac49$. By interpolation in Proposition 2.8, by Lemma 5.2 and (5.24) we have

$$\|B\|_{W^{1,3}(\Sigma_t)} \le C\|B\|^{\frac59}_{H^2(\Sigma_t)}\|B\|^{\frac49}_{L^4(\Sigma_t)} \le C(1+\|p\|^{\frac59}_{H^2(\Sigma_t)}) \le CE_1(t)^{\frac12\cdot\frac59}. \tag{5.28}$$

Therefore, since $\theta<\frac49$, we may bound the second term as

$$\||B|^2\,\partial_\nu p\|^2_{H^{1/2}(\Sigma_t)} \le C(1+\|p\|^2_{H^2(\Omega_t)})E_1(t).$$

By the same argument we also have

$$\|H|\nabla U|^2\partial_\nu p\|^2_{H^{1/2}(\Sigma_t)} \le C(1+\|p\|^2_{H^2(\Omega_t)})E_1(t). \tag{5.29}$$

Indeed, the same calculations as above lead to

$$\|H|\nabla U|^2\partial_\nu p\|^2_{H^{1/2}(\Sigma_t)} \le C(1+\|p\|^2_{H^2(\Omega_t)})E_1(t)^{\frac49}(\|B\|^2_{W^{1,3}(\Sigma_t)} + \||\nabla U|^2\|^2_{W^{1,3}(\Sigma_t)}).$$

Recall that (5.28) yields $\|B\|^2_{W^{1,3}(\Sigma_t)} \le CE_1(t)^{\frac59}$. By Lemma 5.1 we deduce

$$\begin{aligned}\||\nabla U|^2\|_{H^2(\Sigma_t)} &\le C\|\nabla U\|_{L^\infty}\|\nabla U\|_{H^2(\Sigma_t)}\\ &\le C(1+\|\nabla^2 U\|_{H^{\frac12}(\Sigma_t)}) \le C(1+\|p\|_{H^2(\Sigma_t)}) \le CE_1(t)^{\frac12}.\end{aligned}$$

Hence, by interpolation

$$\||\nabla U|^2\|_{W^{1,3}(\Sigma_t)} \le C\||\nabla U|^2\|^{\frac13}_{H^2(\Sigma_t)}\||\nabla U|^2\|^{\frac23}_{L^\infty(\Sigma_t)} \le CE_1(t)^{\frac12\cdot\frac13}$$

and (5.29) follows.

The term $a_1(\nu,\nabla v)\star B$ is easy to bound and leave the details for the reader. Also the term $a_2(\nu,\nabla v)\,\nabla^2 v$ is not difficult and we merely point out that by interpolation

$$\|\nabla^2 v\|_{L^4(\Omega_t)} \le C\|v\|^{1/2}_{H^3(\Omega_t)}\|\nabla v\|^{1/2}_{L^\infty(\Omega_t)} \le C\|v\|^{1/2}_{H^3(\Omega_t)}.$$

Thus we have by (5.26)

$$\begin{aligned}\|a_1(\nu,\nabla v)\,\nabla^2 v\|_{H^{\frac12}(\Sigma_t)} &\le \|a_1(\nu,\nabla v)\,\nabla^2 v\|_{H^1(\Omega_t)}\\ &\le C\|\nabla^3 v\|_{H^3(\Omega_t)} + C\|\nabla^2 v\|_{L^4(\Omega_t)}^2 + C\|B\|_{C^\alpha(\Sigma_t)}\|\nabla^2 v\|_{L^2(\Omega_t)}\\ &\le C(1+\|p\|_{H^2(\Omega_t)})\|v\|_{H^3(\Omega_t)}.\end{aligned}$$

Before we treat the last term in (5.23) we need to show that the coefficients $a_{\alpha,\gamma,Q}$ are bounded. To this aim we need to show that $Q^{(1)} = Q'(t), Q^{(2)} = Q''(t)$, where $Q(t)$ is defined in (2.1), are bounded since $a_{\alpha,\gamma,Q}$ depend smoothly on them. It is clear that it is enough to show that the first and second derivative of $\mathrm{Cap}(\Omega_t)$ are bounded. It is easy to see, and in fact we already calculated, that

$$\frac{d}{dt}\mathrm{Cap}(\Omega_t) = -\frac12\int_{\Sigma_t}|\nabla U|^2 v_n\, d\mathcal{H}^2.$$

This is clearly bounded. We calculate further and obtain

$$\frac{d^2}{dt^2}\mathrm{Cap}(\Omega_t) = -\frac12\int_{\Sigma_t} H_{\Sigma_t}|\nabla U|^2 v_n d\mathcal{H}^2 - \int_{\Sigma_t}(\nabla\partial_t U\cdot\nabla U)v_n + |\nabla U|^2\partial_t(v_n)\, d\mathcal{H}^2. \tag{5.30}$$

The first term on RHS is clearly bounded. For the second term on RHS in (5.30) we note that since U is constant on Σ_t we have $\nabla U = -|\nabla U|\nu$. Therefore

$$|\int_{\Sigma_t}\nabla\partial_t U\cdot\nabla U v_n d\mathcal{H}^2| = |\int_{\Sigma_t}|\nabla U| v_n\nabla\partial_t U\cdot\nu d\mathcal{H}^2| \le \|\nabla U v_n\|_{H^{\frac12}(\Sigma_t)}\|\nabla\partial_t U\cdot\nu\|_{H^{-\frac12}(\Sigma)}.$$

We note that we may use the Kato Ponce inequality (Proposition 2.10) and Lemma 5.1 to deduce

$$\|\nabla U v_n\|_{H^{\frac12}(\Sigma_t)} \le \|\nabla U\|_{H^{\frac12}(\Sigma_t)}\|v_n\|_{L^\infty(\Sigma_t)} + \|v_n\|_{H^{\frac12}(\Sigma_t)}\|\nabla U\|_{L^\infty(\Sigma_t)} \le C.$$

Next we let $\tilde U_t$ the harmonic extension of $\partial_t U$ in Ω_t and note that for any $\phi \in H^{\frac12}(\Sigma_t)$ it holds

$$\begin{aligned}\int_{\Sigma_t}\phi\nabla\partial_t U\cdot\nu\, d\mathcal{H}^2 = \int_{\Omega_t}\mathrm{div}(\phi\nabla\tilde U_t)\, dx &\le \|\nabla\phi\|_{L^2(\Omega_t)}\|\nabla\partial_t\tilde U\|_{L^2(\Omega_t)}\\ &\le \|\phi\|_{H^{\frac12}(\Sigma_t)}\|\partial_t U\|_{H^{\frac12}(\Sigma_t)}.\end{aligned}$$

This and $\partial_t U = -\nabla U\cdot v$ imply

$$\|\nabla\partial_t U\cdot\nu\|_{H^{-\frac12}(\Sigma_t)} \le \|\partial_t U\|_{H^{\frac12}(\Sigma_t)} = \|\partial_\nu U v_n\|_{H^{\frac12}(\Sigma_t)} \le C.$$

Let us estimate the last term in (5.30). We note that

$$\partial_t(v_n) = \mathcal{D}_t v\cdot\nu + a(\nabla v)\star B_{\Sigma_t}.$$

By (4.11) it holds $\mathrm{div}\,\mathcal{D}_t v = -\mathrm{Tr}((\nabla v)^2)$. Therefore we estimate

$$\begin{aligned}|\int_{\Sigma_t}|\nabla U|^2\partial_t v_n\, d\mathcal{H}^2| &\le C + \int_{\Sigma_t}|\nabla U|^2\mathcal{D}_t v\cdot\nu d\mathcal{H}^2\\ &\le C(1+\|\mathrm{div}\,\mathcal{D}_t v\|_{L^2(\Omega_t)} + \|\nabla U\|_{H^{\frac12}(\Sigma_t)}\|\mathcal{D}_t v\|_{L^2(\Omega_t)}) \le C.\end{aligned}$$

Thus we have $|Q^{(2)}(t)| \le C$ and the coefficients $a_{\alpha,\gamma,Q}$ are bounded.

Let us treat the last term in (5.23). We may assume that $\alpha_1+\gamma_1 \ge \alpha_2+\gamma_2$ and assume first that $\alpha_1+\gamma_1 = 2$ (in which case $\alpha_2=\gamma_2=0$). This means that either $\alpha_1 = 2, \gamma_1 = 0$ or $\alpha_1 = 1, \gamma_1 = 1$. Therefore we have by (5.19) and (5.20)

$$\|\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U\|_{H^{\frac12}(\Sigma_t)}^2 \le CE_1(t). \tag{5.31}$$

We extend v to the complement Ω_t^c such that it remains Lipschitz. Recall that U (and $\partial_t U$) is harmonic in Ω_t^c. Since Ω_t is bounded we may choose a large ball such that $\Omega_t \subset B_{R/2}$ and $\|U\|_{H^{\frac{1}{2}}(\Sigma_t)} \simeq \|\nabla U\|_{L^2(B_R\setminus\Omega_t)}$. Then by (5.18), (5.31) and by the Sobolev embedding it holds

$$\begin{aligned}\|a_{\alpha,\gamma}(v)\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U \star \nabla U\|_{H^{\frac{1}{2}}(\Sigma_t)} &\le C\|a_{\alpha,\gamma}(v)\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U \star \nabla U\|_{H^1(B_R\setminus\Omega_t)}\\ &\le C(1+\|\nabla^{2+\alpha_1}\partial_t^{\gamma_1}U\|_{L^2(B_R\setminus\Omega_t)} + \|\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U\|_{L^4(B_R\setminus\Omega_t)}\|\nabla^2 U\|_{L^4(B_R\setminus\Omega_t)})\\ &\le C(1+\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)})(1+\|\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U\|_{H^{\frac{1}{2}}(\Sigma_t)})\\ &\le (1+\|p\|_{H^1(\Sigma_t)})E_1(t)^{\frac{1}{2}}.\end{aligned}$$

We are left with the last term (5.23) in the case when $\alpha_i + \gamma_i \le 1$ for $i = 1,2$. We estimate this by the Kato-Ponce inequality (Proposition 2.10) and the Sobolev embedding as

$$\begin{aligned}\sum_{\alpha_i+\gamma_i\le 1,\, i=1,2} &\|a_{\alpha,\gamma}(v)\nabla^{1+\alpha_1}\partial_t^{\gamma_1}U \star \nabla^{1+\alpha_2}\partial_t^{\gamma_2}U\|_{H^{\frac{1}{2}}(\Sigma_t)}\\ &\le C\sum_{\alpha+\gamma\le 1}\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{L^\infty(\Sigma_t)}\sum_{\alpha+\gamma\le 1}\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{1}{2}}(\Sigma_t)}\\ &\le C\sum_{\alpha+\gamma\le 1}\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{3}{2}}(\Sigma_t)}\sum_{\alpha+\gamma\le 1}\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{1}{2}}(\Sigma_t)}.\end{aligned}$$

We bound the first term in the last row by (5.31)

$$\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|^2_{H^{\frac{3}{2}}(\Sigma_t)} \le CE_1(t).$$

We bound the last term in the last row when $\alpha = 1$ and $\gamma = 0$ by (5.18) as before $\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C(1+\|p\|_{H^1(\Sigma_t)})$. We need yet to prove

$$\|\nabla\partial_t U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C(1+\|p\|_{H^1(\Sigma_t)}) \tag{5.32}$$

to conclude the proof. We use the fact that $\partial_t U$ is harmonic and $\partial_t U = \partial_\nu U\, v_n$ and therefore by Theorem 3.9 we deduce

$$\|\nabla\partial_t U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C(1+\|\partial_\nu U\, v_n\|_{H^{3/2}(\Sigma_t)}).$$

We recall that $\|B\|_{L^4(\Sigma_t)} \le C$ and $\|B\|_{H^1(\Sigma_t)} \le C(1+\|p\|_{H^1(\Sigma_t)})$. Therefore we deduce by by Proposition 2.10, the a priori bound $\|v_n\|_{H^2(\Sigma_t)} \le C$ in (1.7) and by (5.18) that

$$\begin{aligned}\|\partial_\nu U\, v_n\|_{H^{3/2}(\Sigma_t)} &\le C(\|v_n\|_{H^{\frac{3}{2}}(\Sigma_t)} + \|\nabla U\|_{H^{\frac{3}{2}}(\Sigma_t)} + \|\nu\|_{H^{\frac{3}{2}}(\Sigma_t)})\\ &\le C(1+\|v_n\|_{H^2(\Sigma_t)} + \|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} + \|B\|_{H^1(\Sigma_t)})\\ &\le C(1+\|p\|_{H^1(\Sigma_t)}).\end{aligned}$$

Hence, we have (5.32) and the claim follows. □

We conclude this section with the higher order version of Lemma 5.7.

Lemma 5.8. *Let $l \ge 2$ and consider R_p^l defined in (4.26). Assume that (1.7) holds and $E_{l-1}(t) \le M$. There exists $C = C(M,l)$ such that*

$$\|R_p^l\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le CE_l(t) \tag{5.33}$$

and for integers $1 \le k \le l-1$ and $\varepsilon > 0$ it holds

$$\|R_p^{l-k}\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon \tag{5.34}$$

for some constant $C_\varepsilon = C(M, l\varepsilon)$.

Proof. Let us first prove (5.33). We begin by showing

$$\|R_I^l\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le CE_l(t), \tag{5.35}$$

where

$$R_I^l = -(|B|^2 + Q(t)\, H\, |\nabla U|^2)(\mathcal{D}_t^l v \cdot \nu) + \langle \nabla_\tau p, \mathcal{D}_t^l v\rangle,$$

here $Q(t)$ is defined in (2.1). Let us first recall that $E_1(t) \le C$ implies $\|B\|_{H^2(\Sigma_t)} \le C$ and $\|p\|_{H^2(\Sigma_t)} \le C$. By Lemma 5.1 this implies $\|\nabla^3 U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C$. In particular, $\|B\|_{L^\infty} \le C$ and $\|\nabla^2 U\|_{L^\infty} \le C$. Therefore we may bound by Sobolev embedding and by Proposition 2.10

$$\begin{aligned}\|(|B|^2 + Q(t)\, H\, |\nabla U|^2)(\mathcal{D}_t^l v \cdot \nu)\|^2_{H^{\frac{1}{2}}(\Sigma_t)} &\le \|(|B|^2 + Q(t)\, H\, |\nabla U|^2)(\mathcal{D}_t^l v \cdot \nu)\|^2_{H^1(\Sigma_t)}\\ &\le C(1 + \|B\|^2_{W^{1,4}(\Sigma_t)}\|\mathcal{D}_t^l v\cdot\nu\|^2_{L^4(\Sigma_t)} + \|\mathcal{D}_t^l v \cdot \nu\|^2_{H^1(\Sigma_t)})\\ &\le C(1 + \|B\|^2_{H^2(\Sigma_t)})\|\mathcal{D}_t^l v\cdot\nu\|^2_{H^1(\Sigma_t)}\\ &\le CE_l(t).\end{aligned}$$

In order bound $\|\nabla_\tau p \cdot \mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Sigma_t)}$ we observe that by the curvature bound $\|B\|_{L^\infty} \le C$, by $-\nabla p = \mathcal{D}_t v$ and by the Sobolev embeddings $\|u\|_{L^3(\Omega_t)} \le C\|u\|_{H^{\frac{1}{2}}(\Omega_t)}$ and $\|u\|_{L^6(\Omega_t)} \le C\|u\|_{H^1(\Omega_t)}$ we have

$$\begin{aligned}\|\nabla_\tau p\cdot\mathcal{D}_t^l v\|_{H^{\frac{1}{2}}(\Sigma_t)} &\le C\|\mathcal{D}_t v\cdot\mathcal{D}_t^l v\|_{H^1(\Omega_t)}\\ &\le C(1 + \|\nabla\mathcal{D}_t v\cdot\mathcal{D}_t^l v\|_{L^2(\Omega_t)} + \|\mathcal{D}_t v\cdot\nabla\mathcal{D}_t^l v\|_{L^2(\Omega_t)})\\ &\le C(1 + \|\nabla\mathcal{D}_t v\|_{L^3(\Omega_t)}\|\mathcal{D}_t^l v\|_{L^6(\Omega_t)} + \|\mathcal{D}_t v\|_{L^6(\Omega_t)}\|\nabla\mathcal{D}_t^l v\|_{L^3(\Omega_t)})\\ &\le C(1 + \|\mathcal{D}_t v\|_{H^{\frac{3}{2}}(\Omega_t)}\|\mathcal{D}_t^l v\|_{H^1(\Omega_t)} + \|\mathcal{D}_t v\|_{H^1(\Omega_t)}\|\mathcal{D}_t^l v\|_{H^{\frac{3}{2}}(\Omega_t)}).\end{aligned}$$

By definition of $E_l(t)$ it holds

$$\|\mathcal{D}_t v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \le E_1(t) \le E_{l-1}(t) \le C \quad \text{and} \quad \|\mathcal{D}_t^l v\|^2_{H^{\frac{3}{2}}(\Omega_t)} \le E_l(t).$$

Therefore we have $\|\nabla_\tau p\cdot\mathcal{D}_t^l v\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le CE_l(t)$ and (5.35) follows.

Let us next show

$$\|R_{II}^l\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le CE_l(t), \tag{5.36}$$

where

$$R_{II}^l = \sum_{|\alpha|\le 1,\, |\beta|\le l-1} a_{\alpha,\beta}(B)\, \overbrace{\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v \star \cdots \star \nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} v}^{=:R_{\alpha,\beta}(v)}.$$

We first observe that we may ignore the coefficients $a_{\alpha,\beta}(B)$. Indeed, we may extend B to Ω_t, call the extension $\tilde B$, such that $\|\tilde B\|_{H^2(\Omega_t)} \le C$. Then by the above notation

$$\begin{aligned}\|R_{II}^l\|_{H^{\frac{1}{2}}(\Sigma_t)} &\le C\|R_{II}^l\|_{H^1(\Omega_t)} \le C(1 + \sum_{\alpha,\beta}\|\nabla(\nabla a_{\alpha,\beta}(\tilde B)\star R_{\alpha,\beta}(v))\|_{L^2(\Omega_t)}\\ &\le C(1 + \sum_{\alpha,\beta}\|\nabla\tilde B\star R_{\alpha,\beta}(v)\|_{L^2(\Omega_t)} + \|\nabla R_{\alpha,\beta}(v)\|_{L^2(\Omega_t)})\\ &\le C(1 + \sum_{\alpha,\beta}\|\nabla\tilde B\|_{L^4(\Omega_t)}\|R_{\alpha,\beta}(v)\|_{L^4(\Omega_t)} + \|\nabla R_{\alpha,\beta}(v)\|_{L^2(\Omega_t)})\\ &\le C(1 + \sum_{\alpha,\beta}\|\nabla R_{\alpha,\beta}(v)\|_{L^2(\Omega_t)})\\ &\le C\big(1 + \sum_{|\alpha|\le 2,\, |\beta|\le l-1}\|\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1} v\star\cdots\star\nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}} v\|_{L^2(\Omega_t)}\big).\end{aligned} \tag{5.37}$$

By the assumption $E_{l-1}(t) \le C$ and by (5.6) we deduce $\|\nabla^{1+\alpha_i}\mathcal{D}_t^{\beta_i}v\|_{L^\infty} \le C$ for $\alpha_i + \beta_i \le l-2$. We note also that by $|\alpha| \le 2$ and $|\beta| \le l-1$ it follows that $|\alpha|+|\beta| \le l+1$. We ignore all the terms in the last row of (5.37) which indexes satisfy $\alpha_i+\beta_i \le l-2$ as these are uniformly bounded. For the rest of the terms we use Hölder's inequality and relabel the indexes (note that below we assume $\alpha \le 2$ and $\beta \le l-1$)

$$\begin{aligned}\sum_{|\alpha|\le 2,\,|\beta|\le l-1} &\|\nabla^{1+\alpha_1}\mathcal{D}_t^{\beta_1}v \star \cdots \star \nabla^{1+\alpha_{l+1}}\mathcal{D}_t^{\beta_{l+1}}v\|_{L^2(\Omega_t)} \\ &\le C \sum_{\alpha\le 2,\beta\le l-1} \|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^2}^2 + \sum_{\alpha+\beta=l}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6}^2 \cdot \sum_{\alpha+\beta=1}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3}^2 \\ &\quad + \sum_{\alpha+\beta=l-1}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3}^2 \cdot \sum_{\alpha+\beta=2}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6}^2 + \sum_{\alpha+\beta\le l-1}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6}^6.\end{aligned} \tag{5.38}$$

To bound the first term on the RHS of (5.38) we simply note that for $\beta \le l-1$ and $\alpha \le 2$ it holds

$$\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^2(\Omega_t)}^2 \le C\|\mathcal{D}_t^{\beta}v\|_{H^3(\Omega_t)}^2 \le CE_l(t).$$

For the second and the third terms we have first for $\alpha+\beta \le l$ and $\beta \le l-1$ (which include the case $\alpha+\beta=2$ as $l\ge 2$) that

$$\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6(\Omega_t)}^2 \le C\|\mathcal{D}_t^{\beta}v\|_{H^{l-\beta+2}(\Omega_t)}^2 \le C\|\mathcal{D}_t^{l+1-(l+1+\beta)}v\|_{H^{\frac{3}{2}(l+1+\beta)}(\Omega_t)}^2 \le CE_l(t).$$

For $\alpha+\beta \le l-1$ (which includes the case $\alpha+\beta=1$) we deduce

$$\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3(\Omega_t)}^2 \le C\|\mathcal{D}_t^{\beta}v\|_{H^{\frac{1}{2}+(l-\beta)}(\Omega_t)}^2 \le CE_{l-1}(t) \le C. \tag{5.39}$$

For the last term we interpolate in the fluid domain $\Omega_t \subset \mathbb{R}^3$ for $\alpha+\beta \le l-1$ as

$$\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6(\Omega_t)} \le \|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{H^2(\Omega_t)}^{1/3}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3(\Omega_t)}^{2/3} \le CE_l(t)^{1/6}\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3}^{2/3}.$$

By (5.39) we have $\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^3} \le C$ and thus

$$\|\nabla^{1+\alpha}\mathcal{D}_t^{\beta}v\|_{L^6}^6 \le CE_l(t).$$

By combing the previous estimates with (5.37) and (5.38) we obtain

$$\|R_{II}^l\|_{H^{1/2}(\Sigma_t)}^2 \le C(1+\|\nabla R_{II}^l\|_{L^2(\Omega_t)}^2) \le CE_l(t),$$

and (5.36) follows.

We are left to prove

$$\|R_{III}^l\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 \le CE_l(t), \tag{5.40}$$

where

$$R_{III}^l = \sum_{\substack{|\alpha|+|\beta|+|\gamma|\le l+1\\ |\beta|\le l-1,\gamma_i\le l}} a_{\alpha,\beta,\gamma,Q}(v)\mathcal{D}_t^{\beta_1}v \star \cdots \star \mathcal{D}_t^{\beta_{l-1}}v \star \nabla^{1+\alpha_1}\partial_t^{\gamma_1}U \star \nabla^{1+\alpha_1}\partial_t^{\gamma_2}U$$

and the coefficients $a_{\alpha,\beta,\gamma,Q}$ depend on the time derivatives of $Q(t)$ up to order $l+1$. Recall that $Q(t)$ is defined in (2.1)

This time we will not give the argument which proves the boundedness of $Q(t)^{(l+1)} = \frac{d^{l+1}}{dt^{l+1}}Q(t)$ as it simpler than the rest of the proof and is similar to the argument in (5.30).

Recall that by (5.6) it holds $\|\mathcal{D}_t^{\beta_i}v\|_{L^\infty} \le C$ for $\beta_i \le l-2$. On the other hand for $\alpha+\gamma \le l-1$ we have by Lemma 5.5

$$\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{L^\infty(\Sigma_t)}^2 \le C\|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{3}{2}}(\Sigma_t)}^2 \le C(1+\|\nabla^{2+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2) \le CE_{l-1}(t) \le C. \tag{5.41}$$

Similarly we have

$$\|\mathcal{D}_t^{\beta}v\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C \quad \text{for } \beta \le l-1 \quad \text{and} \quad \|\nabla^{1+\alpha}\partial_t^{\gamma}U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C \quad \text{for } \alpha+\gamma \le l. \tag{5.42}$$

Therefore we may ignore all the terms with indexes which satisfy $\beta_i \leq l-2$ and $\alpha_i + \gamma_i \leq l-1$. Recall that we assume $\beta_1 \geq \cdots \geq \beta_{l-1}$. Therefore we may estimate by Proposition 2.10, (5.41) and by (5.42)

$$\begin{aligned}
&\|R^l_{III}\|_{H^{\frac12}(\Sigma_t)} \\
&\quad \leq \sum_{\alpha+\gamma\leq l} C\big(\|\mathcal{D}_t^{l-1}v\|_{L^\infty(\Sigma_t)}\|\nabla^{1+\alpha}\partial_t^\gamma U\|_{H^{\frac12}(\Sigma_t)} + \|\mathcal{D}_t^{l-1}v\|_{H^{\frac12}(\Sigma_t)}\|\nabla^{1+\alpha}\partial_t^\gamma U\|_{L^\infty(\Sigma_t)}\big) \\
&\qquad + C\sum_{\alpha+\gamma\leq l+1, \gamma\leq l} \|\nabla^{1+\alpha}\partial_t^\gamma U\|_{H^{\frac12}(\Sigma_t)} \\
&\quad \leq C\big(\|\mathcal{D}_t^{l-1}v\|_{L^\infty(\Sigma_t)} + \sum_{\alpha+\gamma\leq l}\|\nabla^{1+\alpha}\partial_t^\gamma U\|_{L^\infty(\Sigma_t)} + \sum_{\substack{\alpha+\gamma\leq l+1\\ \gamma\leq l}}\|\nabla^{1+\alpha}\partial_t^\gamma U\|_{H^{\frac12}(\Sigma_t)}\big).
\end{aligned}$$

We estimate the first term in the last row by Sobolev embedding

$$\|\mathcal{D}_t^{l-1}v\|^2_{L^\infty(\Sigma_t)} \leq C\|\mathcal{D}_t^{l-1}v\|^2_{H^3(\Omega_t)} \leq CE_l(t).$$

For the second we use Sobolev embedding and Lemma 5.5 and obtain for $\alpha+\gamma\leq l$

$$\|\nabla^{1+\alpha}\partial_t^\gamma U\|^2_{L^\infty(\Sigma_t)} \leq C(1+\|\nabla^{2+\alpha}\partial_t^\gamma U\|^2_{H^{\frac12}(\Sigma_t)}) \leq CE_l(t).$$

The same argument also implies

$$\|\nabla^{1+\alpha}\partial_t^\gamma U\|^2_{H^{\frac12}(\Sigma_t)} \leq CE_l(t)$$

for $\alpha+\gamma\leq l+1$ and $\gamma\leq l$. Hence we obtain (5.40) and this concludes the proof of (5.33).

Let us then prove (5.34). Let us first treat the first term in the definition of R_p^{l-k} and bound $\|R_I^{l-k}\|_{H^{\frac32 k-1}(\Sigma_t)}$. We first observe that the case $k=1$ follows from (5.33). Let us then assume $k\geq 2$. By the Sobolev embedding it holds $\|u\|_{L^\infty(\Sigma)} \leq C\|u\|_{L^{\frac32 k-1}(\Sigma)}$. We use this and the Kato-Ponce inequality in Proposition 2.10 to deduce that

$$\begin{aligned}
&\|R_I^{l-k}\|_{H^{\frac32 k-1}(\Sigma_t)} \\
&\quad \leq C(1+\|B\|^2_{H^{\frac32 k-1}(\Sigma_t)} + \|\nabla U\|^2_{H^{\frac32 k-1}(\Sigma_t)} + \|\mathcal{D}_t^{l-k}v\|^2_{H^{\frac32 k-1}(\Sigma_t)} + \|\nabla p\|^2_{H^{\frac32 k-1}(\Sigma_t)}).
\end{aligned}$$

Let us show that all the terms on RHS are bounded.

To this aim we first recall that the bound $E_{l-1}(t)\leq C$ implies $\|B\|_{H^{\frac32 l-1}(\Sigma_t)} \leq C$. Since $k\leq l-1$ we deduce $\|B\|_{H^{\frac32 k-1}(\Sigma_t)} \leq C$. Lemma 5.1 implies $\|\nabla U\|^2_{H^{\frac32 k-1}(\Sigma_t)} \leq CE_{l-1}(t) \leq C$. The condition $k\leq l-1$ and the Trace Theorem also yields

$$\|\mathcal{D}_t^{l-k}v\|^2_{H^{\frac32 k-1}(\Sigma_t)} \leq C\|\mathcal{D}_t^{l-k}v\|^2_{H^{\frac32 k}(\Omega_t)} \leq CE_{l-1}(t) \leq C.$$

Similarly we deduce by $-\nabla p = \mathcal{D}_t v$ that $\|\nabla p\|_{H^{\frac32 k-1}(\Sigma_t)} \leq C$. Hence, we have

$$\|R_I^{l-k}\|_{H^{\frac32 k-1}(\Sigma_t)} \leq C.$$

Let us then bound $\|R_{II}^{l-k}\|_{H^{\frac32 k-1}(\Sigma_t)}$. As before, Proposition 2.10 and the Sobolev embedding yield

$$\|R_{II}^{l-k}\|_{H^{\frac32 k-1}(\Sigma_t)} \leq \sum_{\alpha\leq 1, \beta\leq l-k-1} C\left(1+\|a_{\alpha,\beta}(B)\|^2_{H^{\frac32 k-1}(\Sigma_t)} + \|\nabla^{1+\alpha}\mathcal{D}_t^\beta v\|^q_{H^{\frac32 k-1}(\Sigma_t)}\right)$$

for $q\geq 1$. Recall that $\|B\|_{H^{\frac32 k-1}(\Sigma_t)} \leq C$ and $k\geq 2$. Therefore $\|B\|_{L^\infty}\leq C$ and thus $\|a_{\alpha,\beta}(B)\|_{H^{\frac32 k-1}(\Sigma_t)} \leq C$. On the other hand by $\alpha\leq 1$, $\beta\leq l-(k+1)$ and by Lemma 3.7 we deduce

$$\|\nabla^{1+\alpha}\mathcal{D}_t^\beta v\|^2_{H^{\frac32 k-1}(\Sigma_t)} \leq C\|\mathcal{D}_t^\beta v\|^2_{H^{\frac32(k+1)}(\Omega_t)} \leq C\sum_{i=0}^{l-1}\|\mathcal{D}_t^{l-i}v\|^2_{H^{\frac32 i}(\Omega_t)} \leq CE_{l-1}(t) \leq C. \tag{5.43}$$

Hence, we have $\|R_{II}^{l-k}\|_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le C$.

Let us finally treat R_{III}^{l-k}. We note that it holds $\|v\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le CE_{l-1}(t) \le C$ and therefore we may ignore the coefficients $a_{\alpha,\beta,\gamma,Q}(v)$. Then by Proposition 2.10 and by the Sobolev embedding we have

$$\|R_{III}^{l-k}\|_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le C\left(1 + \sum_{\beta \le l-(k+1)} \|\mathcal{D}_t^\beta v\|^q_{H^{\frac{3}{2}k-1}(\Sigma_t)} \cdot \sum_{\substack{|\alpha|+|\gamma|\le l-k+1,\\ |\gamma|\le l-k}} \|\nabla^{1+\alpha_1}\partial_t^{\gamma_1} U\|_{H^{\frac{3}{2}k-1}(\Sigma_t)} \|\nabla^{1+\alpha_2}\partial_t^{\gamma_2} U\|_{H^{\frac{3}{2}k-1}(\Sigma_t)}\right) \tag{5.44}$$

for $q \ge 1$. By (5.43) we have $\|\mathcal{D}_t^\beta v\|_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le C$ for all $\beta \le l-(k+1)$. To bound the last term we may assume that $\alpha_1+\gamma_1 \ge \alpha_1+\gamma_1$. If $\alpha_1+\gamma_1 = l-k+1$ then necessarily $\alpha_2+\gamma_2 \le l-k$. Therefore by Lemma 5.5

$$\|\nabla^{1+\alpha_1}\partial_t^{\gamma_1} U\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le C\left(1 + \|\nabla^{1+(\alpha_1+k-1)}\partial_t^{\gamma_1} U\|^2_{H^{\frac{k-1}{2}+\frac{1}{2}}(\Sigma_t)}\right) \le \varepsilon E_l(t) + C_\varepsilon$$

and

$$\|\nabla^{1+\alpha_2}\partial_t^{\gamma_2} U\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le C\left(1 + \|\nabla^{1+(\alpha_2+k-1)}\partial_t^{\gamma_2} U\|^2_{H^{\frac{k-1}{2}+\frac{1}{2}}(\Sigma_t)}\right) \le CE_{l-1}(t) \le C.$$

Therefore we deduce by (5.44) that

$$\|R_{III}^{l-k}\|^2_{H^{\frac{3}{2}k-1}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon.$$

This concludes the proof of (5.34). □

6. First Regularity Estimates

In this section we prove our first regularity estimates for the solution of (1.3). We assume that the solution satisfies the a priori estimates (1.7), i.e., $\Lambda_T < \infty$ and $\sigma_T > 0$, where Λ_T and σ_T are defined in (1.5) and (1.4). We recall that

$$\Lambda_T := \sup_{t\in(0,T]} \left(\|h(\cdot,t)\|_{C^{1,\alpha}(\Sigma_t)} + \|\nabla v(\cdot,t)\|_{L^\infty(\Omega_t)} + \|v_n(\cdot,t)\|_{H^2(\Sigma_t)}\right).$$

In particular, bound on Λ_T does not imply curvature bounds, and thus we need to be careful as we may not use e.g. the interpolation results from Proposition 2.8. Our goal in this section is to show that the a priori estimates (1.7) imply the following bounds for the pressure

$$\sup_{t\le T} \|p\|_{H^1(\Omega_t)} \le C \qquad \text{and} \qquad \int_0^T \|p\|^2_{H^2(\Omega_t)}\, dt \le C.$$

The first bound above is important as it implies $\|B_\Sigma\|_{L^4(\Sigma_t)} \le C$, which is crucial e.g. for the interpolation inequality in Proposition 2.8 to hold. The second estimate is important for the first order energy estimate which we prove in the next section in Proposition 7.1.

Let us begin by stating regularity estimates that we have by the a priori estimate. First, recall once again that by the uniform $C^{1,\alpha}(\Gamma)$-bound we have for the capacitary potential U that $\|U\|_{C^{1,\alpha}(\bar\Omega_t^c)} \le C$. Let us prove the following estimates for the second fundamental form and for the capcacitary potential.

Lemma 6.1. *Assume that (1.7) holds for $T > 0$. Then for all $t < T$ we have*

$$\|B\|^4_{L^4(\Sigma_t)} \le \varepsilon\|p\|^2_{H^1(\Sigma_t)} + C_\varepsilon$$

for $C = C(M, \varepsilon)$ *and*

$$\|B\|_{H^1(\Sigma_t)} + \|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1 + \|p\|_{H^1(\Sigma_t)})$$

for $C = C(M)$.

Proof. We denote the height-function by $h = h(\cdot, t)$. Then by standard calculations (see e.g. [22,43]) we may write the second fundamental form on Σ as $B = a(h, \bar{\nabla} h)\bar{\nabla}^2 h$. Therefore we may bound

$$\|B\|^4_{L^4(\Sigma_t)} \leq C(1 + \|\bar{\nabla}^2 h\|^4_{L^4(\Gamma)}) \quad \text{and} \quad \|\bar{\nabla}^3 h\|^2_{L^2(\Gamma)} \leq C(1 + \|\bar{\nabla} B\|^2_{L^2(\Sigma_t)} + \|\bar{\nabla}^2 h\|^4_{L^4(\Gamma)}).$$

We use interpolation on Γ as

$$\|\bar{\nabla}^2 h\|_{L^4(\Gamma)} \leq C\|\bar{\nabla}^3 h\|^{\theta}_{L^2(\Gamma)} \|h\|^{1-\theta}_{C^{1,\alpha}(\Gamma)} \leq C\|\bar{\nabla}^3 h\|^{\theta}_{L^2(\Gamma)}$$

for $\theta < 1/2$. This implies by Young's inequality $\|\bar{\nabla}^2 h\|^4_{L^4(\Gamma)} \leq \varepsilon\|\bar{\nabla}^3 h\|^2_{L^2(\Gamma)} + C_\varepsilon$. Thus by choosing ε small we obtain

$$\|\bar{\nabla}^3 h\|^2_{L^2(\Gamma)} \leq C(1 + \|\bar{\nabla} B\|^2_{L^2(\Sigma_t)})$$

and

$$\|B\|^4_{L^4(\Sigma_t)} \leq \varepsilon\|\bar{\nabla}^3 h\|^2_{L^2(\Gamma)} + C_\varepsilon.$$

By the Simon's identity (2.11) we deduce

$$\|\bar{\nabla} B\|^2_{L^2(\Sigma_t)} \leq \|\bar{\nabla} H\|^2_{L^2(\Sigma_t)} + C\|B\|^4_{L^4(\Sigma_t)}.$$

Therefore we have

$$\|B\|^4_{L^4(\Sigma_t)} \leq \varepsilon\|H\|^2_{H^1(\Sigma_t)} + C_\varepsilon \tag{6.1}$$

and

$$\|B\|_{H^1(\Sigma_t)} \leq C(1 + \|H\|_{H^1(\Sigma_t)}). \tag{6.2}$$

Let us consider the capacitary potential U. Let us show that even though we may not use Proposition 2.8, the $C^{1,\alpha}(\Gamma)$-regularity still implies the following weak interpolation inequality

$$\|\nabla U\|_{H^1(\Sigma_t)} \leq \varepsilon\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} + C_\varepsilon\|\nabla U\|_{L^2(\Sigma_t)} \leq \varepsilon\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} + C_\varepsilon. \tag{6.3}$$

In order to prove (6.3) we first observe that the $C^{1,\alpha}(\Gamma)$-regularity of Σ_t implies the following inequalities for $p \in (1, 2)$ and $u, v \in C^\infty(\Sigma_t)$

$$\|u\|_{L^2(\Sigma_t)} \leq \varepsilon\|u\|_{H^{\frac{1}{2}}(\Sigma_t)} + C_\varepsilon\|u\|_{L^p(\Sigma_t)} \quad \text{and} \quad \|\nabla_\tau v\|_{L^p(\Sigma_t)} \leq \delta\|v\|_{H^1(\Sigma_t)} + C_\delta\|v\|_{L^2(\Sigma_t)}.$$

We apply these for $u = \nabla^2 U$ and $v = \nabla U$ and have

$$\|\nabla^2 U\|_{L^2(\Sigma_t)} \leq \varepsilon\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} + C_\varepsilon\|\nabla^2 U\|_{L^p(\Sigma_t)} \quad \text{and}$$
$$\|\nabla_\tau \nabla U\|_{L^p(\Sigma_t)} \leq \delta\|\nabla U\|_{H^1(\Sigma_t)} + C_\delta\|\nabla U\|_{L^2(\Sigma_t)}.$$

Since $\partial_{x_i} U$ is harmonic and Σ_t is $C^{1,\alpha}(\Gamma)$-regular we have by [19] that

$$\|\nabla^2 U\|_{L^p(\Sigma_t)} \leq C(\|\nabla_\tau \nabla U\|_{L^p(\Sigma_t)} + \|\nabla U\|_{L^2(\Omega_t)}).$$

Therefore by first choosing ε small and then δ even smaller we obtain (6.3).

By $p = H - \frac{Q(t)}{2}|\nabla U|^2$, where $Q(t)$ is defined in (2.1), we may estimate

$$\|H\|_{H^1(\Sigma_t)} \leq C(\|p\|_{H^1(\Sigma_t)} + \|\nabla U\|_{H^1(\Sigma_t)}).$$

Then by (6.3) we have

$$\|H\|_{H^1(\Sigma_t)} \leq C_\varepsilon(1 + \|p\|_{H^1(\Sigma_t)}) + \varepsilon\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)}.$$

We use Theorem 3.9 and (6.2) and have

$$\|\nabla^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C(1 + \|B\|_{H^1(\Sigma_t)}) \leq C(1 + \|H\|_{H^1(\Sigma_t)}).$$

Therefore we deduce from the two above inequalities that

$$\|H\|_{H^1(\Sigma_t)} \leq C(1 + \|p\|_{H^1(\Sigma_t)}).$$

The claim follows from this together with (6.1) and (6.2). □

Let us proceed to the following regularity estimate.

Lemma 6.2. *Assume that the a priori estimates (1.7) hold for $T > 0$. Then*

$$\sup_{t\in[0,T]} \|p\|^2_{L^2(\Sigma_t)} + \int_0^T \|p\|^2_{H^1(\Sigma_t)}\, dt \leq C(1+T),$$

for $C = C(M)$.

Proof. The idea is to consider the following function

$$\Psi(t) := \int_{\Sigma_t} p\,(\nabla v\, \nu)\cdot \nu + \varepsilon p^2\, d\mathcal{H}^2,$$

where the choice of ε will be clear later. First, we observe that under the a priori estimates (1.7) v is uniformly Lipschitz and therefore Ψ is bounded from below by

$$\Psi(t) \geq -C\|p(\cdot,t)\|_{L^1(\Sigma_t)} + \varepsilon\|p(\cdot,t)\|^2_{L^2(\Sigma_t)} \geq -C_\varepsilon + \frac{\varepsilon}{2}\|p(\cdot,t)\|^2_{L^2(\Sigma_t)}, \tag{6.4}$$

where the last inequality follows from the Young's inequality, i.e., $\|p(\cdot,t)\|_{L^1(\Sigma_t)} \leq \frac{\varepsilon}{2}\|p(\cdot,t)\|^2_{L^2(\Sigma_t)} + C_\varepsilon$ and C_ε is a large constant that depends on ε. By differentiating and using the a priori estimates (1.7) we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\Sigma_t} p\,(\nabla v\,\nu)\cdot\nu\, d\mathcal{H}^2 \\
&\quad = \int_{\Sigma_t} p\,(\nabla v\,\nu)\cdot\nu\,\mathrm{div}_\tau v\, d\mathcal{H}^2 + \int_{\Sigma_t} \mathcal{D}_t p\,(\nabla v\,\nu)\cdot\nu\, d\mathcal{H}^2 + \int_{\Sigma_t} p\,\mathcal{D}_t((\nabla v\,\nu)\cdot\nu)\, d\mathcal{H}^2 \\
&\quad \leq C_\varepsilon + \varepsilon\int_{\Sigma_t} |\mathcal{D}_t p|^2\, d\mathcal{H}^2 + \int_{\Sigma_t} p\,\mathcal{D}_t((\nabla v\,\nu)\cdot\nu)\, d\mathcal{H}^2.
\end{aligned} \tag{6.5}$$

We estimate the second last term in (6.5) by (4.25) and (1.7) and have

$$\int_{\Sigma_t} |\mathcal{D}_t p|^2\, d\mathcal{H}^2 \leq C(1 + \|p\|^2_{H^1(\Sigma_t)} + \|B\|^4_{L^4(\Sigma_t)} + \|\nabla\partial_t U\|^2_{L^2(\Sigma_t)}).$$

Lemma 6.1 yields $\|B\|^4_{L^4(\Sigma_t)} \leq C(1 + \|p\|^2_{H^1(\Sigma_t)})$. On the other hand, by Lemma 3.3 it holds $\|\nabla\partial_t U\|^2_{\Sigma_t} \leq C\|\partial_t U\|^2_{H^1(\Sigma_t)}$. Since $\partial_t U = -\nabla U\cdot v$, we may estimate by Theorem 3.9 and by Lemma 6.1

$$\begin{aligned}
\|\nabla\partial_t U\|^2_{L^2(\Sigma_t)} &\leq C(1 + \|\bar\nabla\partial_t U\|^2_{L^2(\Sigma_t)}) \leq C(1 + \|\bar\nabla(\nabla U\cdot v)\|^2_{L^2(\Sigma_t)}) \leq C(1 + \|\nabla^2 U\|^2_{L^2(\Sigma_t)}) \\
&\leq C(1 + \|\nabla^2 U\|^2_{H^{1/2}(\Sigma_t)}) \leq C(1 + \|p\|^2_{H^1(\Sigma_t)}).
\end{aligned}$$

Therefore we may bound

$$\int_{\Sigma_t} |\mathcal{D}_t p|^2\, d\mathcal{H}^2 \leq C(1 + \|p\|^2_{H^1(\Sigma_t)}). \tag{6.6}$$

Let us treat the last in term in (6.5). First, we have by (4.1), (4.4) and (1.7) that

$$\begin{aligned}
\int_{\Sigma_t} p\,\mathcal{D}_t((\nabla v\,\nu)\cdot\nu)\, d\mathcal{H}^2 &\leq \int_{\Sigma_t} p\,((\nabla\mathcal{D}_t v\,\nu)\cdot\nu)\, d\mathcal{H}^2 + \varepsilon\|p\|^2_{L^2(\Sigma_t)} + C_\varepsilon \\
&= -\int_{\Sigma_t} p\,(\nabla^2 p\,\nu)\cdot\nu)\, d\mathcal{H}^2 + \varepsilon\|p\|^2_{L^2(\Sigma_t)} + C_\varepsilon.
\end{aligned}$$

We use (4.11) to estimate $|\Delta p| \leq C\|\nabla v\|_{L^\infty}^2 \leq C$ and recall that by (3.4) it holds $(\nabla^2 p\,\nu)\cdot\nu = \Delta p - \Delta_\Sigma p - H\partial_\nu p$ to deduce

$$\begin{aligned}-\int_{\Sigma_t} p\,(\nabla^2 p\,\nu)\cdot\nu\,d\mathcal{H}^2 &\leq C + \int_{\Sigma_t} p\,\Delta_\Sigma p\,d\mathcal{H}^2 + \int_{\Sigma_t} Hp\,\partial_\nu p\,d\mathcal{H}^2\\ &\leq C - \int_{\Sigma_t} |\bar\nabla p|^2\,d\mathcal{H}^2 + \int_{\Sigma_t} (\varepsilon|\partial_\nu p|^2 + C_\varepsilon(1+|H|^4))\,d\mathcal{H}^2,\end{aligned}$$

where in the last inequality we have used $p = H - \frac{Q(t)}{2}|\nabla U|^2$, where $Q(t)$ is defined in (2.1). Lemma 3.3 yields $\|\partial_\nu p\|_{L^2(\Sigma_t)} \leq C(1+\|p\|_{H^1(\Sigma_t)})$ while Lemma 6.1 implies $\|H\|_{L^4}^4 \leq \delta\|p\|_{H^1(\Sigma_t)}^2 + C_\delta$. Therefore by first choosing ε small and then δ even smaller, we obtain

$$-\int_{\Sigma_t} p\,(\nabla^2 p\,\nu)\cdot\nu\,d\mathcal{H}^2 \leq -\|\bar\nabla p\|_{L^2(\Sigma_t)}^2 + \varepsilon\|p\|_{H^1(\Sigma_t)}^2 + C_\varepsilon.$$

By direct calculation and by using (6.6) we have

$$\frac{d}{dt}\int_{\Sigma_t} p^2\,d\mathcal{H}^2 \leq C(1+\|p\|_{H^1(\Sigma_t)}^2).$$

Combining the two above inequalities with (6.5) and (6.6) we conclude

$$\frac{d}{dt}\Psi(t) \leq -\frac{1}{2}\|\bar\nabla p\|_{L^2(\Sigma_t)}^2 + \varepsilon\|p\|_{L^2(\Sigma_t)}^2 + C_\varepsilon, \tag{6.7}$$

when ε is small. Finally we use Lemma 6.1 to estimate

$$\|p\|_{L^2(\Sigma_t)}^2 \leq C(1+\|H\|_{L^2(\Sigma_t)}^2) \leq \|B\|_{L^4(\Sigma_t)}^4 + C \leq \varepsilon\|p\|_{H^1(\Sigma_t)}^2 + C_\varepsilon.$$

This yields $\|p\|_{L^2(\Sigma_t)}^2 \leq 2\|\bar\nabla p\|_{L^2(\Sigma_t)}^2 + C_\varepsilon$ when ε is small. Thus we may estimate (6.7)

$$\frac{d}{dt}\Psi(t) \leq -\frac{1}{4}\|\bar\nabla p\|_{L^2(\Sigma_t)}^2 + C_\varepsilon,$$

when ε is small. The conclusion follows by integrating the above over $[0,T]$ and using (6.4). □

We proceed to higher order regularity estimate which is uniform in time.

Proposition 6.3. *Assume that the a priori estimates (1.7) hold for $T > 0$. Then*

$$\sup_{t\in(0,T]} \|\nabla p\|_{L^2(\Omega_t)}^2 \leq e^{C(1+T)}(1+\|\nabla p\|_{L^2(\Omega_0)}^2)$$

for $C = C(M)$.

Proof. We differentiate

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\Omega_t}|\nabla p|^2\,dx &= \frac{1}{2}\int_{\Omega_t}|\nabla p|^2\overbrace{\mathrm{div}(v)}^{=0}\,dx + \int_{\Omega_t}(\mathcal{D}_t\nabla p\cdot\nabla p)\,dx\\ &= \int_{\Omega_t}(\nabla\mathcal{D}_t p\cdot\nabla p)\,dx + \int_{\Omega_t}([\mathcal{D}_t,\nabla]p\cdot\nabla p)\,dx.\end{aligned}$$

By (4.1) and $\|\nabla v\|_{L^\infty} \leq C$ we have a pointwise estimate $|[\mathcal{D}_t,\nabla]p| \leq C|\nabla v||\nabla p| \leq C|\nabla p|$. Therefore we deduce

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}\int_{\Omega_t}|\nabla p|^2\,dx &\leq \int_{\Omega_t}(\nabla\mathcal{D}_t p\cdot\nabla p)\,dx + C\|\nabla p\|_{L^2(\Omega_t)}^2\\ &= \int_{\Omega_t}\mathrm{div}(\mathcal{D}_t p\,\nabla p)\,dx - \int_{\Omega_t}\mathcal{D}_t p\,\Delta p\,dx + C\|\nabla p\|_{L^2(\Omega_t)}^2\\ &= \int_{\Sigma_t}\mathcal{D}_t p\,\partial_\nu p\,dx - \int_{\Omega_t}\mathcal{D}_t p\,\Delta p\,dx + C\|\nabla p\|_{L^2(\Omega_t)}^2\\ &\leq \|\mathcal{D}_t p\|_{L^2(\Sigma_t)}^2 + \|\partial_\nu p\|_{L^2(\Sigma_t)}^2 - \int_{\Omega_t}\mathcal{D}_t p\,\Delta p\,dx + C\|\nabla p\|_{L^2(\Omega_t)}^2.\end{aligned} \tag{6.8}$$

We have by (6.6) that $\|\mathcal{D}_t p\|^2_{L^2(\Sigma_t)} \leq C(1 + \|p\|^2_{H^1(\Sigma_t)})$ and by Lemma 3.3

$$\|\partial_\nu p\|^2_{L^2(\Sigma_t)} \leq C(\|p\|^2_{H^1(\Sigma_t)} + \|\Delta p\|^2_{L^2(\Omega_t)}) \leq C(1 + \|p\|^2_{H^1(\Sigma_t)}).$$

We are left with the second last term in (6.8).

To that aim let $u : \Omega_t \to \mathbb{R}$ be the solution of

$$\begin{cases} -\Delta u = \Delta p \quad \text{in } \ \Omega_t \\ u = 0 \quad \text{on } \ \Sigma_t. \end{cases}$$

Then it holds

$$-\int_{\Omega_t} \mathcal{D}_t p \, \Delta p \, dx = \int_{\Omega_t} \mathcal{D}_t p \, \Delta u \, dx = \int_{\Omega_t} \Delta \mathcal{D}_t p \, u \, dx + \int_{\Sigma_t} \mathcal{D}_t p \, \partial_\nu u \, d\mathcal{H}^2.$$

Since $|\Delta u| = |\Delta p| \leq C$ and $u = 0$ on Σ_t it holds $\|u\|_{H^1(\Omega_t)} \leq C$ and by Lemma 3.3 we deduce $\|\nabla u\|_{L^2(\Sigma_t)} \leq C$. We may bound the last term on the RHS by (6.6)

$$\int_{\Sigma_t} \mathcal{D}_t p \, \partial_\nu u \, d\mathcal{H}^2 \leq \|\mathcal{D}_t p\|^2_{L^2(\Sigma_t)} + \|\partial_\nu u\|^2_{L^2(\Sigma_t)} \leq C(1 + \|p\|^2_{H^1(\Sigma_t)}).$$

We are thus left with the second last term.

We have by (3.2), by Lemma 6.1, by Sobolev embedding and by interpolation that

$$\begin{aligned} \|\nabla^2 u\|^2_{L^2(\Omega_t)} &\leq C + \int_{\Sigma_t} |H_{\Sigma_t}| |\nabla u|^2 \, d\mathcal{H}^2 \leq C + C_\varepsilon \|H_{\Sigma_t}\|^3_{L^3(\Sigma_t)} + \varepsilon \|\nabla u\|^3_{L^3(\Sigma_t)} \\ &\leq C_\varepsilon(1 + \|p\|^2_{L^2(\Sigma_t)}) + \varepsilon \|\nabla u\|^2_{L^4(\Sigma_t)} \|\nabla u\|_{L^2(\Sigma_t)} \\ &\leq C_\varepsilon(1 + \|p\|^2_{L^2(\Sigma_t)}) + C\varepsilon \|\nabla^2 u\|^2_{L^2(\Omega_t)}. \end{aligned}$$

Therefore it holds

$$\|u\|^2_{H^2(\Omega_t)} \leq C(1 + \|p\|^2_{H^1(\Sigma_t)}).$$

By Remark 4.5 we have

$$\Delta \mathcal{D}_t p = \operatorname{div} \operatorname{div}(v \otimes \nabla p) + \operatorname{div}(R^0_{bulk}).$$

Therefore by integrating by parts

$$\begin{aligned} \int_{\Omega_t} \Delta \mathcal{D}_t p \, u \, dx &= \int_{\Omega_t} (v \otimes \nabla p) : \nabla^2 u \, dx + \int_{\Omega_t} R^0_{bulk} \star \nabla u \, dx - \int_{\Sigma_t} (\nabla p \cdot \nu)(\nabla u \cdot v) \, d\mathcal{H}^2 \\ &\leq C(1 + \|p\|^2_{H^1(\Sigma_t)} + \|\nabla p\|^2_{L^2(\Omega_t)}). \end{aligned}$$

We deduce by (6.8) and by the above estimates that

$$\frac{d}{dt} \frac{1}{2} \|\nabla p\|^2_{L^2(\Omega_t)} \leq C(1 + \|p\|^2_{H^1(\Sigma_t)} + \|\nabla p\|^2_{L^2(\Omega_t)}).$$

This implies

$$\frac{d}{dt} \log(1 + \|\nabla p\|^2_{L^2(\Omega_t)}) \leq C(1 + \|p\|^2_{H^1(\Sigma_t)})$$

and the claim follows from Lemma 6.2. □

An important consequence of Proposition 6.3 is that by Lemma 5.2 we have the following bound for the curvature

$$\|B\|_{L^4(\Sigma_t)} + \|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C. \tag{6.9}$$

This means that from now on we may use the general interpolation inequality from Proposition 2.8.

At the end of this section we improve Lemma 6.2. We recall the definition of the energy quantity $E_1(t)$ from (5.1)

$$E_1(t) = \|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} + \|\nabla p\|^2_{H^{\frac{3}{2}}(\Omega_t)} + \|v\|^2_{H^3(\Omega_t)} + \|\partial_\nu p\|^2_{H^1(\Sigma_t)} + 1.$$

In particular, $E_1(0)$ denotes the above quantity at time $t = 0$. It is clear that

$$\|p\|^2_{H^1(\Omega_t)} \leq E_1(t).$$

Lemma 6.4. *Assume that the a priori estimates (1.7) hold for $T > 0$. Then*

$$\int_0^T \|p\|^2_{H^2(\Omega_t)}\, dt \leq C,$$

where the constant C depends on M, T and on $E_1(0)$.

Proof. The proof is similar to Lemma 6.2. This time we differentiate

$$\Phi(t) := -\int_{\Sigma_t} p\Delta_{\Sigma_t} v_n \, d\mathcal{H}^2.$$

Note that by the a priori estimates (1.7) and by Proposition 6.3, Φ is uniformly bounded on $[0, T]$. Note also that by Proposition 6.3 it holds

$$\sup_{t<T} \|p\|^2_{H^1(\Omega_t)} \leq C,$$

where the constant depends on T, Λ_T and on $E_1(0)$.

We calculate as in (6.5) by using (6.6) and $\|v_n\|_{H^2(\Sigma_t)} \leq C$ that

$$\begin{aligned}\frac{d}{dt}\Phi(t) &\leq C + \|\mathcal{D}_t p\|^2_{L^2(\Sigma_t)} - \int_{\Sigma_t} p\,(\mathcal{D}_t \Delta_{\Sigma_t} v_n)\, d\mathcal{H}^2 \\ &\leq C(1 + \|p\|^2_{H^1(\Sigma_t)}) - \int_{\Sigma_t} p\,(\mathcal{D}_t \Delta_{\Sigma_t} v_n)\, d\mathcal{H}^2.\end{aligned}$$

To bound the last term we recall that by (4.3) it holds

$$(\mathcal{D}_t \Delta_{\Sigma_t} v_n) = \Delta_{\Sigma_t}(\mathcal{D}_t v_n) + \nabla^2_\tau v_n \star \nabla v - \nabla_\tau v_n \cdot \Delta_{\Sigma_t} v + B \star \nabla v \star \nabla_\tau v_n.$$

Therefore by $\|\nabla v\|_{L^\infty} + \|v_n\|_{H^2(\Sigma_t)} \leq C$, by $\|\nabla_\tau v_n\|_{L^4(\Sigma_t)} \leq \|v_n\|_{H^2(\Sigma_t)}$ and by (6.9) it holds

$$\begin{aligned}&-\int_{\Sigma_t} p\,(\mathcal{D}_t \Delta_{\Sigma_t} v_n)\, d\mathcal{H}^2 \leq -\int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t v_n)\, d\mathcal{H}^2 + \int_{\Sigma_t} p\,\nabla_\tau v_n \cdot \Delta_{\Sigma_t} v\, d\mathcal{H}^2 \\ &\quad + C(1 + \|p\|_{L^2(\Sigma_t)}\|B\|_{L^4(\Sigma_t)}\|\bar{\nabla} v_n\|_{L^4(\Sigma_t)}) \\ &\leq -\int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t v \cdot \nu)\, d\mathcal{H}^2 - \int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t \nu \cdot v)\, d\mathcal{H}^2 - \int_{\Sigma_t} (\nabla_\tau(p\,\nabla_\tau v_n)\cdot \nabla_\tau v)\, d\mathcal{H}^2 + C.\end{aligned} \tag{6.10}$$

We may write the first term on RHS in (6.10) by the formula (3.2), $\mathcal{D}_t v = -\nabla p$ and by the estimate (6.9)

$$\begin{aligned}-\int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t v \cdot \nu)\, d\mathcal{H}^2 &= \int_{\Sigma_t} \Delta_{\Sigma_t} p\,\partial_\nu p\, d\mathcal{H}^2 \\ &\leq -\frac{1}{2}\int_{\Omega_t} (|\nabla^2 p|^2 - |\Delta p|^2)\, dx + C\int_{\Sigma_t} |B||\nabla p|^2\, d\mathcal{H}^2 \\ &\leq -\frac{1}{2}\int_{\Omega_t} |\nabla^2 p|^2\, dx + C + C_\varepsilon \|B\|_{L^4} + \varepsilon\|\nabla p\|^2_{L^{\frac{8}{3}}(\Sigma_t)} \\ &\leq -\frac{1}{2}\int_{\Omega_t} |\nabla^2 p|^2\, dx + C_\varepsilon + \varepsilon\|\nabla p\|^2_{L^{\frac{8}{3}}(\Sigma_t)}.\end{aligned}$$

By the Sobolev embedding it holds $\|\nabla p\|^2_{L^{\frac{8}{3}}(\Sigma_t)} \leq C\|\nabla p\|^2_{H^{1/2}(\Sigma_t)} \leq C(1 + \|\nabla^2 p\|^2_{L^2(\Omega_t)})$. Therefore by choosing $\varepsilon > 0$ small we deduce

$$-\int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t v \cdot \nu)\, d\mathcal{H}^2 \leq -\frac{1}{3}\int_{\Omega_t} |\nabla^2 p|^2\, dx + C_\varepsilon.$$

We bound the third term on RHS in (6.10) simply by $\|\nabla v\|_{L^\infty} + \|v_n\|_{H^2(\Sigma_t)} \leq C$ as

$$-\int_{\Sigma_t} \nabla_\tau(p\,\nabla_\tau v_n)\cdot\nabla_\tau v\, d\mathcal{H}^2 \leq C(1+\|p\|_{H^1(\Sigma_t)}^2).$$

For the remaining term in (6.10) we recall (4.5) which states $\mathcal{D}_t\nu = -\nabla_\tau v_n + Bv_\tau$. Therefore we obtain by Lemma 6.1 and by $\|\nabla v\|_{L^\infty} + \|v_n\|_{H^2(\Sigma_t)} \leq C$ that

$$\begin{aligned}-\int_{\Sigma_t} p\,\Delta_{\Sigma_t}(\mathcal{D}_t\nu\cdot v)\, d\mathcal{H}^2 &= \int_{\Sigma_t}\langle\bar\nabla p, \bar\nabla(\mathcal{D}_t\nu\cdot v)\rangle\, d\mathcal{H}^2\\ &\leq C(1+\|p\|_{H^1(\Sigma_t)}^2 + \|B\|_{H^1(\Sigma_t)}^2 + \|B\|_{L^4(\Sigma_t)}^4)\\ &\leq C(1+\|p\|_{H^1(\Sigma_t)}^2).\end{aligned}$$

Hence, we have

$$\frac{d}{dt}\Phi(t) \leq -\frac{1}{3}\int_{\Omega_t}|\nabla^2 p|^2\, dx + C(1+\|p\|_{H^1(\Sigma_t)}^2)$$

and the claim follows from Lemma 6.2. □

7. Energy Estimates

As we mentioned before, the fundamental property of the solution of (1.3) is the conservation of the energy (1.1). In this section we define high order energy functions and show that their derivatives are controlled by the quantity (5.1) of the same order. This will be the first step in proving that the high order energy quantities remain bounded along the flow.

We define the energy of order $l \geq 1$ as

$$\begin{aligned}\mathcal{E}_l(t) = \frac{1}{2}\int_{\Omega_t}|\mathcal{D}_t^{l+1}v|^2\, dx + \frac{1}{2}\int_{\Sigma_t}|\nabla_\tau(\mathcal{D}_t^l v\cdot\nu)|^2\, d\mathcal{H}^2\\ - \frac{Q(t)}{2}\int_{\Omega_t^c}|\nabla(\partial_t^{l+1}U)|^2\, dx + \int_{\Omega_t}|\nabla^{\lfloor\frac{1}{2}(3l+1)\rfloor}\omega|^2\, dx,\end{aligned} \tag{7.1}$$

where $\lfloor\frac{1}{2}(3l+1)\rfloor$ is the integer part of $\frac{1}{2}(3l+1)$, ω is the curl of v defined as

$$\omega = \operatorname{curl} v = \nabla v - \nabla v^T$$

and $Q(t)$ is defined in (2.1).

In this section we calculate $\frac{d}{dt}\mathcal{E}_l$ and estimate it in terms of the $E_l(t)$, which we recall is defined in (5.1) as

$$E_l(t) = \sum_{k=0}^{l}\|\mathcal{D}_t^{l+1-k}v\|_{H^{\frac{3}{2}k}(\Omega_t)}^2 + \|v\|_{H^{\lfloor\frac{3}{2}(l+1)\rfloor}(\Omega_t)}^2 + \|\mathcal{D}_t^l v\cdot\nu\|_{H^1(\Sigma_t)}^2 + 1.$$

In particular, it holds $\mathcal{E}_l(t) \leq CE_l(t)$. We state the main results of this section below and prove them later.

Proposition 7.1. *Assume that the a priori estimates (1.7) hold for $T>0$. Then for all $t<T$ it holds*

$$\frac{d}{dt}\mathcal{E}_1(t) \leq C(1+\|p\|_{H^2(\Omega_t)}^2)E_1(t),$$

where the constant depends on M, T and $E_1(0)$, i.e., $E_1(t)$ at time $t=0$.

Proposition 7.2. *Let $l \geq 2$ and assume that (1.7) and $E_{l-1}(t) \leq M$ hold for all $t \in [0,T)$. Then all $t < T$ it holds*

$$\frac{d}{dt}\mathcal{E}_l(t) \leq C E_l(t)$$

where the constant depends on M, l, T and on $\sup_{t<T} E_{l-1}(t)$.

The proof of the both above energy estimates is based on the calculations of the differential of $\mathcal{E}_l(t)$, which we state first for all $l \geq 1$. In the proof of Proposition 7.1 and Proposition 7.2 we then need to estimate the remainder terms by the quantity $E_l(t)$.

We begin by differentiating the first term of $\mathcal{E}_l(t)$ in (7.1) and obtain by $\operatorname{div} v = 0$, by (4.15) and by the definition of $E_l(t)$ that

$$\begin{aligned}
\frac{d}{dt}\frac{1}{2}\int_{\Omega_t} |\mathcal{D}_t^{l+1} v|^2\, dx &= \int_{\Omega_t} (\mathcal{D}_t^{l+2} v \cdot \mathcal{D}_t^{l+1} v)\, dx \\
&= -\int_{\Omega_t} (\nabla \mathcal{D}_t^{l+1} p \cdot \mathcal{D}_t^{l+1} v)\, dx - \int_{\Omega_t} ([\mathcal{D}_t^{l+1}, \nabla] p \cdot \mathcal{D}_t^{l+1} v)\, dx \\
&\leq -\int_{\Omega_t} (\nabla \mathcal{D}_t^{l+1} p \cdot \mathcal{D}_t^{l+1} v)\, dx + \|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 + \|[\mathcal{D}_t^{l+1}, \nabla] p\|_{L^2(\Omega_t)}^2 \\
&= -\int_{\Omega_t} \operatorname{div}(\mathcal{D}_t^{l+1} p\, \mathcal{D}_t^{l+1} v)\, dx + \int_{\Omega_t} \mathcal{D}_t^{l+1} p\, \operatorname{div}(\mathcal{D}_t^{l+1} v)\, dx + E_l(t) + \|R_{bulk}^l\|_{L^2(\Omega_t)}^2 \\
&= -\int_{\Sigma_t} \mathcal{D}_t^{l+1} p\, (\mathcal{D}_t^{l+1} v \cdot \nu)\, d\mathcal{H}^2 + E_l(t) + \|R_{bulk}^l\|_{L^2(\Omega_t)}^2 + \int_{\Omega_t} \mathcal{D}_t^{l+1} p\, \operatorname{div}(\mathcal{D}_t^{l+1} v)\, dx.
\end{aligned} \tag{7.2}$$

Next we differentiate the second term in the energy and obtain by $\|\nabla v\|_{L^\infty} \leq C$ and (4.2)

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_{\Sigma_t} |\nabla_\tau (\mathcal{D}_t^l v \cdot \nu)|^2\, d\mathcal{H}^2 \\
&= \int_{\Sigma_t} (\mathcal{D}_t \nabla_\tau (\mathcal{D}_t^l v \cdot \nu) \cdot \nabla_\tau (\mathcal{D}_t^l v \cdot \nu))\, d\mathcal{H}^2 + \frac{1}{2}\int_{\Sigma_t} |\nabla_\tau (\mathcal{D}_t^l v \cdot \nu)|^2 (\operatorname{div}_\tau v)\, d\mathcal{H}^2 \\
&\leq \int_{\Sigma_t} (\nabla_\tau \mathcal{D}_t (\mathcal{D}_t^l v \cdot \nu) \cdot \nabla_\tau (\mathcal{D}_t^l v \cdot \nu))\, d\mathcal{H}^2 + C\|\mathcal{D}_t^l v \cdot \nu\|_{H^1(\Sigma_t)}^2 \\
&= -\int_{\Sigma_t} (\Delta_{\Sigma_t} (\mathcal{D}_t^l v \cdot \nu))(\mathcal{D}_t (\mathcal{D}_t^l v \cdot \nu))\, d\mathcal{H}^2 + C E_l(t) \\
&= -\int_{\Sigma_t} (\Delta_{\Sigma_t} (\mathcal{D}_t^l v \cdot \nu))(\mathcal{D}_t^{l+1} v \cdot \nu)\, d\mathcal{H}^2 \\
&\qquad + \int_{\Sigma_t} (\nabla_\tau (\mathcal{D}_t^l v \cdot \nu) \cdot \nabla_\tau (\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu)))\, d\mathcal{H}^2 + C E_l(t) \\
&\leq -\int_{\Sigma_t} (\Delta_{\Sigma_t} (\mathcal{D}_t^l v \cdot \nu))(\mathcal{D}_t^{l+1} v \cdot \nu)\, d\mathcal{H}^2 + C E_l(t) + \|\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu\|_{H^1(\Sigma_t)}^2.
\end{aligned} \tag{7.3}$$

We differentiate the third term, use the fact that $\partial_t^{l+1} U$ is harmonic and Lemma 4.6 and have (recall that it holds $\|\nabla \partial_t^{l+1} U\|_{L^2(\Omega_t^c)} \leq C \|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}$)

$$\begin{aligned}
\frac{d}{dt} - \frac{Q(t)}{2}\int_{\Omega_t^c} |\nabla \partial_t^{l+1} U|^2\, dx &= -Q(t)\int_{\Omega_t^c} \langle \nabla \partial_t^{l+2} U, \nabla \partial_t^{l+1} U \rangle\, dx \\
&\quad + \frac{Q(t)}{2}\int_{\Sigma_t} |\nabla \partial_t^{l+1} U|^2 v_n\, d\mathcal{H}^2 - \frac{Q'(t)}{2}\int_{\Omega_t^c} |\nabla \partial_t^{l+1} U|^2\, dx \\
&\leq Q(t)\int_{\Sigma_t} \partial_t^{l+2} U\, (\partial_\nu \partial_t^{l+1} U)\, d\mathcal{H}^2 + C\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2
\end{aligned}$$

$$
\begin{aligned}
&= -Q(t)\int_{\Sigma_t} (\partial_\nu U(\mathcal{D}_t^{l+1} v\cdot\nu) + R_U^l)\,(\partial_\nu \partial_t^{l+1} U)\, d\mathcal{H}^2 + C\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2\\
&\le -Q(t)\int_{\Sigma_t} (\partial_\nu U\,\partial_\nu \partial_t^{l+1} U)(\mathcal{D}_t^{l+1} v\cdot\nu)\, d\mathcal{H}^2 + \|R_U^l\|_{L^2(\Sigma_t)}^2 + C\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2,
\end{aligned}
\tag{7.4}
$$

where R_U^l in the remainder term defined in (4.18) and $Q'(t) = \frac{d}{dt}Q(t)$, where $Q(t)$ is defined in (2.1). Recall that in the proof of Lemma 5.7 we proved that $Q'(t)$ and $Q''(t)$ are bounded, see (5.30).

Finally, we differentiate the fourth term involving the curl. To that aim we denote $\lambda_l := \lfloor \frac{1}{2}(3l+1)\rfloor$. We have by Lemma 4.4 and by (2.16)

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega_t} |\nabla^{\lambda_l}\omega|^2\, dx &= \int_{\Omega_t} \nabla v \star \nabla^{\lambda_l}\omega \star \nabla^{\lambda_l}\omega\, dx + C\|\nabla^{\lambda_l}\omega\|_{L^2(\Omega_t)}^2\\
&\quad + \sum_{|\alpha|\le\lambda_l} \|\nabla^{1+\alpha_1} v \star \nabla^{1+\alpha_2} v\|_{L^2(\Omega_t)}^2\\
&\le C\|\nabla^{\lambda_l}\omega\|_{L^2(\Omega_t)}^2 + C\|\nabla v\|_{L^\infty(\Omega_t)}^2 \|\nabla v\|_{H^{\lambda_l}(\Omega_t)}^2\\
&\le C\|\nabla^{\lambda_l}\omega\|_{L^2(\Omega_t)}^2 + C\|v\|_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)}^2 \le C E_l(t).
\end{aligned}
\tag{7.5}
$$

Let us next show that the highest order terms in (7.2), (7.3) and (7.4) cancel out. Indeed, this follows from Lemma 4.7 which states that

$$
\mathcal{D}_t^{l+1} p = -\Delta_{\Sigma_t}(\mathcal{D}_t^l v\cdot\nu) - Q(t)\,\partial_\nu U\,(\partial_\nu \partial_t^{l+1} U) + R_p^l,
$$

where R_p^l denotes the error term defined in (4.26) and $Q(t)$ is defined in (2.1). Note that we may estimate

$$
\Big|\int_{\Sigma_t} R_p^l(\mathcal{D}_t^{l+1} v\cdot\nu)\, d\mathcal{H}^2\Big| \le \|R_p^l\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + \|\mathcal{D}_t^{l+1} v\cdot\nu\|_{H^{-\frac{1}{2}}(\Sigma_t)}^2.
$$

By divergence theorem and by Lemma 4.4 we have

$$
\|\mathcal{D}_t^{l+1} v\cdot\nu\|_{H^{-\frac{1}{2}}(\Sigma_t)}^2 \le C\|\mathcal{D}_t^{l+1} v\|_{L^2(\Omega_t)}^2 + \|R_{\mathrm{div}}^l\|_{L^2(\Omega_t)}^2 \le CE_l(t) + \|R_{\mathrm{div}}^l\|_{L^2(\Omega_t)}^2.
$$

Therefore we have by (7.2), (7.3), (7.4) and (7.5) that

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}_l(t) &\le CE_l(t) + \|R_{bulk}^l\|_{L^2(\Omega_t)}^2 + \|R_U^l\|_{L^2(\Sigma_t)}^2 + \|R_{\mathrm{div}}^l\|_{L^2(\Omega_t)}^2 + \|R_p^l\|_{H^{\frac{1}{2}}(\Sigma_t)}^2\\
&\quad + C\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2 + \|\mathcal{D}_t^l v\cdot \mathcal{D}_t\nu\|_{H^1(\Sigma_t)}^2 + \int_{\Omega_t} \mathcal{D}_t^{l+1} p\ \mathrm{div}(\mathcal{D}_t^{l+1} v)\, dx.
\end{aligned}
\tag{7.6}
$$

We need thus to bound the remainder terms in (7.6). As we mentioned before, we prove the energy bounds by induction such that we bound $E_l(t)$ when we know that $E_{l-1}(t)$ is already bounded. The difficulty is to bound the first order quantity $E_1(t)$ and we do this separately in Proposition 7.1.

Proof of Proposition 7.1. First we recall that Proposition 6.3 implies

$$
\sup_{t\in(0,T)} \|\nabla p\|_{L^2(\Omega_t)}^2 \le e^{CT}(1 + \|\nabla p\|_{L^2(\Omega_0)}^2) \le C,
$$

where the constant C on the RHS depends on T, Λ_T defined in (1.7) and on $E_1(0)$, which is the energy quantity $E_1(t)$ at time $t = 0$. Then we have the bound (6.9) which we recall is

$$
\|B\|_{L^4(\Sigma_t)} + \|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C
$$

for all $t < T$.

We recall the estimate (7.6) for $l = 1$ and use Lemma 5.3, Lemma 5.4, estimate (5.15) from Lemma 5.6 and Lemma 5.7 to deduce

$$
\frac{d}{dt}\mathcal{E}_1(t) \le C(1 + \|p\|_{H^2(\Omega_t)}^2)E_1(t) + \|\mathcal{D}_t v\cdot\mathcal{D}_t\nu\|_{H^1(\Sigma_t)}^2 + \int_{\Omega_t} \mathcal{D}_t^2 p\ \mathrm{div}(\mathcal{D}_t^2 v)\, dx.
$$

Hence, we need to bound the two last terms. The second last term is easy to treat and we merely claim that it holds

$$\|\mathcal{D}_t v \cdot \mathcal{D}_t \nu\|^2_{H^1(\Sigma_t)} \leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t).$$

Indeed, this follows from $\mathcal{D}_t v = -\nabla p$, $\mathcal{D}_t \nu = -(\nabla_\tau v)^T \nu$ from (4.4) and from Proposition 2.10. We leave the details for the reader. The last term is challenging and we will prove that

$$\int_{\Omega_t} \mathcal{D}_t^2 p \, \operatorname{div}(\mathcal{D}_t^2 v)\, dx \leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t). \tag{7.7}$$

The argument is similar than in the proof of Proposition 6.3. The idea is to use the fact that the term $\operatorname{div}(\mathcal{D}_t^2 v)$ is lower order due to the fact that $\operatorname{div} v = 0$. Indeed, we have by Lemma 4.4 that $\operatorname{div}(\mathcal{D}_t^2 v) = R^1_{\text{div}}$, where R^1_{div} is defined in (4.13) and is lower order than $\mathcal{D}_t^2 v$. To this aim let u be a solution of

$$\begin{cases} -\Delta u = \operatorname{div}(\mathcal{D}_t^2 v), & \text{in } \Omega_t \\ \quad u = 0 & \text{on } \Sigma_t. \end{cases}$$

We have by integration by parts

$$\int_{\Omega_t} \mathcal{D}_t^2 p \, \operatorname{div}(\mathcal{D}_t^2 v)\, dx = -\int_{\Omega_t} \mathcal{D}_t^2 p \, \Delta u \, dx = -\int_{\Omega_t} \Delta \mathcal{D}_t^2 p \, u \, dx - \int_{\Sigma_t} \mathcal{D}_t^2 p \, \partial_\nu u \, d\mathcal{H}^2.$$

We use Remark 4.5 and write

$$-\Delta \mathcal{D}_t^2 p = \operatorname{div}\operatorname{div}(v \otimes \mathcal{D}_t^2 v) + \operatorname{div}(R^1_{bulk}).$$

By integration by parts we deduce

$$\begin{aligned} -\int_{\Omega_t} \Delta \mathcal{D}_t^2 p \, u \, dx = &\int_{\Omega_t} (v \otimes \mathcal{D}_t^2 v) : \nabla^2 u \, dx + \int_{\Omega_t} R^1_{bulk} \star \nabla u \, dx \\ &- \int_{\Sigma_t} (\mathcal{D}_t^2 v \cdot \nu)(\nabla u \cdot v)\, d\mathcal{H}^2. \end{aligned} \tag{7.8}$$

We use divergence theorem, the definition of $E_1(t)$, $\operatorname{div}(\mathcal{D}_t^2 v) = R^1_{\text{div}}$ and Lemma 5.3 for the last term

$$\begin{aligned} -\int_{\Sigma_t} (\mathcal{D}_t^2 v \cdot \nu)(\nabla u \cdot v)\, d\mathcal{H}^2 &= -\int_{\Omega_t} \operatorname{div}\big((\nabla u \cdot v)\mathcal{D}_t^2 v\big) \\ &\leq C E_1(t) + \|u\|^2_{H^2(\Omega_t)} + C\|R^1_{\text{div}}\|^2_{L^2(\Omega_t)} \\ &\leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t) + \|u\|^2_{H^2(\Omega_t)}. \end{aligned}$$

Since $-\Delta u = \operatorname{div}(D_t^2 v)$, we may use the inequality (3.19) and Lemma 5.3 to obtain

$$\|u\|^2_{H^2(\Omega_t)} \leq C\|R^1_{\text{div}}\|^2_{L^2(\Omega_t)} \leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t).$$

Therefore we have by (7.8), by the definition of $E_1(t)$ and by Lemma 5.4 that

$$\int_{\Omega_t} \mathcal{D}_t^2 p \, \operatorname{div}(\mathcal{D}_t^2 v)\, dx \leq C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t) - \int_{\Sigma_t} \mathcal{D}_t^2 p \, \partial_\nu u \, d\mathcal{H}^2. \tag{7.9}$$

We proceed by using Lemma 4.7 to write

$$\mathcal{D}_t^2 p = -\Delta_{\Sigma_t}(\mathcal{D}_t v \cdot \nu) - Q(t)\nabla U \cdot \nabla \partial_t^2 U + R^1_p,$$

where $Q(t)$ is defined in (2.1). We integrate by parts on Σ_t the term

$$-\int_{\Sigma_t} \Delta_{\Sigma_t}(\mathcal{D}_t v \cdot \nu)\partial_\nu u \, d\mathcal{H}^2 = \int_{\Sigma_t} \langle \bar\nabla(\mathcal{D}_t v \cdot \nu), \bar\nabla \partial_\nu u\rangle \, d\mathcal{H}^2$$

and deduce

$$\begin{aligned} -\int_{\Sigma_t} \mathcal{D}_t^2 p \, \partial_\nu u \, d\mathcal{H}^2 \leq &\|\mathcal{D}_t v \cdot \nu\|^2_{H^1(\Sigma_t)} + \|\partial_\nu u\|^2_{H^1(\Sigma_t)} \\ &+ C\|\nabla \partial_t^2 U\|^2_{L^2(\Sigma_t)} + C\|R^1_p\|^2_{L^2(\Sigma_t)}. \end{aligned}$$

Lemma 5.7 and (5.21) imply

$$\|\nabla \partial_t^2 U\|_{L^2(\Sigma_t)}^2 + \|R_p^1\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 \leq C(1 + \|p\|_{H^2(\Omega_t)}^2) E_1(t).$$

We use Proposition 3.8, Lemma 4.4 and Lemma 5.3 to deduce

$$\begin{aligned}\|\partial_\nu u\|_{H^1(\Sigma_t)}^2 &\leq C\| \operatorname{div}(\mathcal{D}_t^2 v)\|_{H^{\frac{1}{2}}(\Omega_t)}^2 \leq C\|R_{\text{div}}\|_{H^{\frac{1}{2}}(\Omega_t)}^2 \\ &\leq C(1 + \|p\|_{H^2(\Omega_t)}^2) E_1(t).\end{aligned}$$

Therefore since $\|\mathcal{D}_t v \cdot \nu\|_{H^1(\Sigma_t)}^2 \leq E_1(t)$ we obtain by combining the previous estimates

$$\int_{\Sigma_t} \mathcal{D}_t^2 p \, \partial_\nu u \, d\mathcal{H}^2 \leq C(1 + \|p\|_{H^2(\Omega_t)}^2) E_1(t).$$

Hence, (7.8) implies (7.7) which concludes the proof. □

By a similar argument we prove the higher order case. We begin again with (7.6) and we need to bound the remainder terms.

Proof of Proposition 7.2. The assumption $E_{l-1}(t) \leq C$ and Lemma 5.2 imply the curvature bound $\|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \leq C$. Thus we may use the estimate (7.6), Lemma 5.3, Lemma 5.4, estimate (5.16) from Lemma 5.6 and Lemma 5.7 to deduce

$$\frac{d}{dt}\mathcal{E}_l(t) \leq C E_l(t) + \|\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu\|_{H^1(\Sigma_t)}^2 + \int_{\Omega_t} \mathcal{D}_t^{l+1} p \, \operatorname{div}(\mathcal{D}_t^{l+1} v) \, dx.$$

Hence, we need to bound the two last terms. Again the second last term is easy to treat and we merely sketch it. We have by (4.4) $\mathcal{D}_t \nu = -(\nabla_\tau v)^T \nu$ and have by Proposition 2.10

$$\|\mathcal{D}_t^l v \cdot \mathcal{D}_t \nu\|_{H^1(\Sigma_t)}^2 \leq C\|\mathcal{D}_t \nu\|_{L^\infty(\Sigma_t)}^2 \|\mathcal{D}_t^l v\|_{H^1(\Sigma_t)}^2 + C\|\mathcal{D}_t \nu\|_{W^{1,4}(\Sigma_t)}^2 \|\mathcal{D}_t^l v\|_{L^4(\Sigma_t)}^2 \leq C E_l(t).$$

We treat the last term similarly as in the previous proof and claim that it holds

$$\int_{\Omega_t} \mathcal{D}_t^{l+1} p \, \operatorname{div}(\mathcal{D}_t^{l+1} v) \, dx \leq C E_l(t). \tag{7.10}$$

Since the argument is almost the same as with (7.7) we only sketch it. We let u be the solution of

$$\begin{cases} -\Delta u = \operatorname{div}(\mathcal{D}_t^{l+1} v) & \text{in } \Omega_t, \\ u = 0 & \text{on } \Sigma_t. \end{cases}$$

Again by integration by parts and by using Remark 4.5, Lemma 5.3 and Lemma 5.4 we obtain the higher order version of (7.9) which reads as

$$\int_{\Omega_t} \mathcal{D}_t^{l+1} p \, \operatorname{div}(\mathcal{D}_t^{l+1} v) \, dx \leq C E_l(t) - \int_{\Sigma_t} \mathcal{D}_t^{l+1} p \, \partial_\nu u \, d\mathcal{H}^2.$$

Lemma 4.7 yields

$$\mathcal{D}_t^{l+1} p = -\Delta_{\Sigma_t}(\mathcal{D}_t^l v \cdot \nu) - Q(t) \nabla U \cdot \nabla \partial_t^{l+1} U + R_p^l,$$

where $Q(t)$ is defined in (2.1). By integration by parts on Σ_t we have

$$\begin{aligned}-\int_{\Sigma_t} \mathcal{D}_t^{l+1} p \, \partial_\nu u \, d\mathcal{H}^2 \leq &\|\mathcal{D}_t^l v \cdot \nu\|_{H^1(\Sigma_t)}^2 + \|\partial_\nu u\|_{H^1(\Sigma_t)}^2 \\ &+ C\|\nabla \partial_t^{l+1} U\|_{L^2(\Sigma_t)}^2 + C\|R_p^l\|_{L^2(\Sigma_t)}^2.\end{aligned}$$

Note that $\|\mathcal{D}_t^l v \cdot \nu\|_{H^1(\Sigma_t)}^2 \leq E_l(t)$. By Lemma 5.6 and Lemma 5.8 we have

$$\|\nabla \partial_t^{l+1} U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + \|R_p^l\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 \leq C E_l(t).$$

Finally by Proposition 3.8, by the curvature bound $\|B\|_{H^2(\Sigma_t)} \leq \|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \leq C$, by Lemma 4.4 and by Lemma 5.3 we have

$$\|\partial_\nu u\|^2_{H^1(\Sigma_t)} \leq C\| \operatorname{div}(\mathcal{D}_t^{l+1} v)\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq C\|R^l_{\mathrm{div}}\|^2_{H^{\frac{1}{2}}(\Omega_t)} \leq CE_l(t).$$

This proves (7.10) and concludes the proof. □

8. Higher Regularity Estimates

Let us recall the definition of the energies $E_l(t)$ and $\mathcal{E}_l(t)$ for $l \geq 1$ in (5.1) and (7.1) respectively. In the previous section we proved energy estimates where we control the derivative of $\mathcal{E}_l(t)$ by $E_l(t)$. In this section we complete the estimate and prove that the energy $\mathcal{E}_l(t)$ in fact controls $E_l(t)$. This, together with the results in the previous section, will give us control for $\mathcal{E}_l(t)$ and implies the regularity of the flow.

Note that the energy $\mathcal{E}_l(t)$ is defined in (7.1) as

$$\begin{aligned}\mathcal{E}_l(t) = \frac{1}{2}\int_{\Omega_t} |\mathcal{D}_t^{l+1} v|^2\, dx + \frac{1}{2}\int_{\Sigma_t} |\nabla_\tau(\mathcal{D}_t^l v\cdot \nu)|^2\, d\mathcal{H}^2 \\ - \frac{Q(t)}{2}\int_{\Omega_t^c} |\nabla(\partial_t^{l+1} U)|^2\, dx + \int_{\Omega_t} |\nabla^{\lfloor \frac{1}{2}(3l+1)\rfloor} \operatorname{curl} v|^2\, dx,\end{aligned}$$

where $Q(t)$ is defined in (2.1). Since $Qt \geq 0$, the energy has one negative term and we define its positive part as

$$\mathcal{E}_l^+(t) := 1 + \frac{1}{2}\|\mathcal{D}_t^{l+1} v\|^2_{L^2(\Omega_t)} + \|\operatorname{curl} v\|^2_{H^{\lfloor \frac{3}{2}l+\frac{1}{2}\rfloor}(\Omega_t)} + \frac{1}{2}\|\mathcal{D}_t^l v\cdot\nu\|^2_{H^1(\Sigma_t)}. \tag{8.1}$$

Then it holds

$$c\,\mathcal{E}_l^+(t) \leq \mathcal{E}_l(t) + \frac{Q(t)}{2}\int_{\Omega_t^c} |\nabla(\partial_t^{l+1} U)|^2\, dx + 1,$$

for $c > 0$.

The first main result of this section states that the energy $\mathcal{E}_1(t)$ controls $E_1(t)$. For later purpose we need this bound when the boundary Σ_t is $C^{1,\alpha}$-regular but the velocity field is only bounded in $W^{1,4}$. This makes the statement slightly heavy.

Proposition 8.1. *Assume that Σ_t is uniformly $C^{1,\alpha}(\Gamma)$-regular and the pressure and the velocity satisfies*

$$\|p\|_{H^1(\Omega_t)} + \|v\|_{W^{1,4}(\Sigma_t)} + \|v\|_{W^{1,4}(\Omega_t)} \leq M.$$

Then there are constants C and C_0 such that

$$E_1(t) \leq C(C_0 + \mathcal{E}_1(t)).$$

The constants depend on σ_t defined in (1.4), the $C^{1,\alpha}$-norm of the heightfunction and on M.

Before the proof we remark that if the a priori estimates (1.7) hold for $T > 0$, then the above assumptions hold for constants C, C_0, which depend on T, Λ_T, σ_T and $E_1(0)$. Indeed, then by Proposition 6.3 it holds

$$\|p\|_{H^1(\Omega_t)} \leq C.$$

Proof. We first recall that by Lemma 5.2 we have

$$\|B\|_{L^4(\Sigma_t)} + \|B\|_{H^{\frac{1}{2}}(\Sigma_t)} \leq C\|p\|_{H^1(\Omega_t)} \leq C.$$

The claim follows once we prove that for any $\varepsilon > 0$ it holds

$$\mathcal{E}_1^+(t) \leq \mathcal{E}_1(t) + \varepsilon E_1(t) + C_\varepsilon \tag{8.2}$$

and

$$E_1(t) \le C\mathcal{E}_1^+(t). \tag{8.3}$$

Let us first prove (8.2). Since $\partial_t^2 U$ is harmonic in Ω_t^c it holds

$$\int_{\Omega_t^c} |\nabla \partial_t^2 U|^2\, dx \le C\|\partial_t^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2.$$

We use interpolation (Corollary 2.9) to deduce

$$\|\partial_t^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)} \le C\|\partial_t^2 U\|_{H^1(\Sigma_t)}^{\frac{1}{2}}\|\partial_t^2 U\|_{L^2(\Sigma_t)}^{\frac{1}{2}}. \tag{8.4}$$

By (5.21) it holds

$$\|\partial_t^2 U\|_{H^1(\Sigma_t)} \le C(1+\|p\|_{H^1(\Sigma_t)})E_1(t)^{\frac{1}{2}}. \tag{8.5}$$

In order to estimate $\|\partial_t^2 U\|_{L^2(\Sigma_t)}$ we use Lemma 4.6 and $\|\nabla U\|_{L^\infty}, \|v\|_{L^\infty} \le C$ and have

$$\begin{aligned}\|\partial_t^2 U\|_{L^2(\Sigma_t)} &\le C\|\mathcal{D}_t v\|_{L^2(\Sigma_t)} + C\sum_{|\alpha|\le 1}\|\nabla^{1+\alpha_1}\partial_t^{\alpha_2}U\|_{L^2(\Sigma_t)}\\ &\le C\|p\|_{H^1(\Sigma_t)} + C(1+\|\nabla^2 U\|_{L^2(\Sigma_t)} + \|\nabla\partial_t U\|_{L^2(\Sigma_t)}).\end{aligned}$$

We have by (5.18) $\|\nabla^2 U\|_{L^2(\Sigma_t)} \le C(1+\|p\|_{H^1(\Sigma_t)})$. We use Lemma 3.3 and $\partial_t U = -\nabla U\cdot v$ to deduce

$$\begin{aligned}\|\nabla\partial_t U\|_{L^2(\Sigma_t)} \le C(1+\|\nabla_\tau\partial_t U\|_{L^2(\Sigma_t)}) &\le C\|v\|_{L^\infty}\|\nabla^2 U\|_{L^2(\Sigma_t)} + C\|\nabla U\|_{L^\infty}\|v\|_{H^1(\Sigma_t)}\\ &\le C(1+\|\nabla^2 U\|_{L^2(\Sigma_t)}) \le C(1+\|p\|_{H^1(\Sigma_t)}).\end{aligned}$$

Therefore by (8.4), (8.5), $\|p\|_{L^4(\Sigma_t)} \le \|p\|_{H^1(\Omega_t)} \le C$ and by interpolation we obtain

$$\begin{aligned}\|\partial_t^2 U\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 &\le C + C\|p\|_{H^1(\Sigma_t)}^2 E_1(t)^{\frac{1}{2}}\\ &\le C + C\|p\|_{H^2(\Sigma_t)}^{\frac{2}{3}}\|p\|_{L^4(\Sigma_t)}^{\frac{4}{3}}E_1(t)^{\frac{1}{2}}\\ &\le \varepsilon(\|p\|_{H^2(\Sigma_t)}^2 + E_1(t)) + C_\varepsilon.\end{aligned}$$

Lemma 3.7 yields $\|p\|_{H^2(\Sigma_t)}^2 \le C\|\nabla p\|_{H^{\frac{3}{2}}(\Omega_t)}^2 \le CE_1(t)$ and (8.2) follows.

Let us then prove (8.3). Recall that it holds

$$E_1(t) \le 2\mathcal{E}_1^+(t) + \|v\|_{H^3(\Omega_t)}^2 + \|\mathcal{D}_t v\|_{H^{\frac{3}{2}}(\Omega_t)}^2. \tag{8.6}$$

By (4.11) we have $-\Delta p = \mathrm{Tr}((\nabla v)^2)$. We use the third inequality in Lemma 3.7 and $\|\bar\nabla\partial_\nu p\|_{L^2(\Sigma_t)}^2 \le 2\mathcal{E}_1^+(t)$ and have by interpolation

$$\begin{aligned}\|\mathcal{D}_t v\|_{H^{\frac{3}{2}}(\Omega_t)}^2 = \|\nabla p\|_{H^{\frac{3}{2}}(\Omega_t)}^2 &\le C(\|\partial_\nu p\|_{H^1(\Sigma_t)}^2 + \|p\|_{L^2(\Omega_t)}^2 + \|\Delta p\|_{H^1(\Omega_t)}^2)\\ &\le C(\|\partial_\nu p\|_{H^1(\Sigma_t)}^2 + \|p\|_{L^2(\Omega_t)}^2 + \|\nabla^2 v\|_{L^4(\Omega_t)}^2\|\nabla v\|_{L^4(\Omega_t)}^2 + 1)\\ &\le C_\varepsilon\mathcal{E}_1^+(t) + \varepsilon\|v\|_{H^3(\Omega_t)}^2.\end{aligned} \tag{8.7}$$

We proceed estimating $\|v\|_{H^3(\Omega_t)}$. By the first inequality in Lemma 3.7, by $\|v\|_{L^\infty(\Omega_t)} \le \|v\|_{W^{1,4}(\Omega_t)} \le C$ and by Lemma 5.2 we have

$$\begin{aligned}\|v\|_{H^3(\Omega_t)}^2 &\le C\big(\|\Delta_{\Sigma_t}v_n\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + (1+\|H_{\Sigma_t}\|_{H^2(\Sigma_t)}^2)\|v\|_{L^\infty}^2 + \|\mathrm{curl}\, v\|_{H^1(\Omega_t)}^2\big)\\ &\le C\big(\|\Delta_{\Sigma_t}v_n\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + \|p\|_{H^2(\Sigma_t)}^2 + \mathcal{E}_1^+(t)\big).\end{aligned}$$

By the fourth inequality in Lemma 3.7 and by (8.7) we have

$$\|p\|_{H^2(\Sigma_t)}^2 \le C(\|p\|_{L^2(\Sigma_t)}^2 + \|\nabla p\|_{H^{\frac{3}{2}}(\Omega_t)}^2) \le C_\varepsilon\mathcal{E}_1^+(t) + \varepsilon\|v\|_{H^3(\Omega_t)}^2.$$

Therefore by choosing ε small enough we deduce

$$\|v\|_{H^3(\Omega_t)}^2 \le C\|\Delta_{\Sigma_t}v_n\|_{H^{\frac{1}{2}}(\Sigma_t)}^2 + C\mathcal{E}_1^+(t) \tag{8.8}$$

In order to control $\|\Delta_{\Sigma_t} v_n\|_{H^{\frac{1}{2}}(\Sigma_t)}$ we use (4.25) which states

$$\mathcal{D}_t p = -\Delta_{\Sigma_t} v_n - Q(t)\nabla U \cdot \nabla \partial_t U + R_p^0, \tag{8.9}$$

where $Q(t)$ is defined in (2.1) and

$$R_p^0 = -(|B|^2 - Q(t)\, H|\nabla U|^2)v_n + (\nabla p \cdot v_\tau) - \frac{Q'(t)}{2}|\nabla U|^2.$$

Therefore we have

$$\|\Delta_{\Sigma_t} v_n\|_{H^{\frac{1}{2}}(\Sigma_t)} \le \|\mathcal{D}_t p\|_{H^{\frac{1}{2}}(\Sigma_t)} + \|\nabla U \cdot \nabla \partial_t U\|_{H^{\frac{1}{2}}(\Sigma_t)} + \|R_p^0\|_{H^{\frac{1}{2}}(\Sigma_t)}.$$

We estimate the first term on RHS as

$$\begin{aligned}\|\mathcal{D}_t p\|^2_{H^{\frac{1}{2}}(\Sigma_t)} &\le C(1 + \|\nabla \mathcal{D}_t p\|^2_{L^2(\Omega_t)})\\ &\le C(1 + \|\mathcal{D}_t \nabla p\|^2_{L^2(\Omega_t)} + \|[\mathcal{D}_t, \nabla]p\|^2_{L^2(\Omega_t)})\\ &\le C(1 + \|\mathcal{D}_t^2 v\|^2_{L^2(\Omega_t)} + \|\nabla v\|^2_{L^4(\Omega_t)}\|\nabla p\|^2_{L^4(\Omega_t)}) \le C\mathcal{E}_1^+(t).\end{aligned}$$

By an already familiar argument we get

$$\|\nabla U \cdot \nabla \partial_t U\|^2_{H^{\frac{1}{2}}(\Sigma_t)} + \|R_p^0\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le \varepsilon E_1(t) + C_\varepsilon.$$

We leave the details for the reader. Combing the previous three inequalities yield

$$\|\Delta_{\Sigma_t} v_n\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le C_\varepsilon \mathcal{E}_1^+(t) + \varepsilon E_1(t).$$

By combining (8.6), (8.7), (8.8) with the above inequality and by choosing ε small enough imply

$$E_1(t) \le C\mathcal{E}_1^+(t)$$

and the claim (8.3) follows. □

Proposition 8.1 implies that the bound on $\operatorname{curl} v$ and $\mathcal{D}_t^2 v$ in the fluid domain and on $\mathcal{D}_t v$ on the boundary imply the bound on v and $\mathcal{D}_t v$ in the domain. In the next lemma we show the converse for the initial set $t = 0$, i.e., the bound on v in the domain and on the mean curvature H_{Σ_0} imply that $E_1(0)$ is bounded.

Lemma 8.2. *Assume that Ω_0 is a smooth set such that $\|h_0\|_{L^\infty(\Sigma)} < \eta$. Then it holds*

$$E_1(0) \le C_0,$$

for a constant C_0 which depends on $\sigma_0 = \eta - \|h_0\|_{L^\infty(\Sigma)}$, $\|v\|_{H^3(\Omega_0)}$ on $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$ and on $\|h_0\|_{C^{1,\alpha}}$.

Proof. The bound $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)} \le C$ and Proposition 2.12 imply $\|B_{\Sigma_0}\|_{H^2(\Sigma_0)} \le C$. Then we obtain by Theorem 3.9 that $\|\nabla^3 U\|_{H^{\frac{1}{2}}(\Sigma_0)} \le C$. Hence, we have

$$\|p\|_{H^2(\Sigma_0)} \le C. \tag{8.10}$$

Let us show that

$$\|\partial_\nu p\|_{H^1(\Sigma_0)} \le C. \tag{8.11}$$

Let $\tilde{\nu}$ be the harmonic extension of the normal field. Note that since

$$\|B\|_{C^\alpha(\Sigma_0)} \le C\|B_{\Sigma_0}\|_{H^2(\Sigma_0)} \le C,$$

then by standard elliptic regularity theory [25] we deduce that $\|\nabla\tilde{\nu}\|_{C^\alpha(\Sigma_0)} \le C$. Then (4.11), $\nabla\Delta p = \nabla^2 v \star \nabla v$ and $\|v\|_{H^3(\Omega_0)} \le C$ imply that

$$\begin{aligned}\|\Delta(\nabla p \cdot \tilde{\nu})\|_{L^2(\Omega_0)} &\le C\|\nabla^2 v \star \nabla v\|_{L^2(\Omega_0)} + C\|\nabla^2 p \star \nabla\tilde{\nu}\|_{L^2(\Omega_0)}\\ &\le C(1 + \|p\|_{H^2(\Omega_0)}).\end{aligned}$$

Lemma 3.5 together with interpolation yields

$$\|p\|_{H^2(\Omega_0)} \le C(1 + \|\partial_\nu p\|_{H^{\frac{1}{2}}(\Sigma_0)}) \le \varepsilon\|\partial_\nu p\|_{H^1(\Sigma_0)} + C_\varepsilon.$$

Therefore by combing the two estimates with Lemma 3.3 we obtain

$$\|\partial_\nu p\|_{H^1(\Sigma_0)} = \|\nabla p \cdot \tilde{\nu}\|_{H^1(\Sigma_0)} \leq C_\varepsilon(1 + \|\partial_\nu(\nabla p \cdot \tilde{\nu})\|_{L^2(\Sigma_0)}) + \varepsilon\|\partial_\nu p\|_{H^1(\Sigma_0)}.$$

Choosing ε small yields

$$\|\partial_\nu p\|_{H^1(\Sigma_0)} \leq C(1 + \|\partial_\nu(\nabla p \cdot \tilde{\nu})\|_{L^2(\Sigma_0)}).$$

Note that by $\|\nabla\tilde{\nu}\|_{C^\alpha(\Sigma_0)} \leq C$, Lemma 3.3 and by (8.10) we have

$$\begin{aligned}\|\partial_\nu(\nabla p \cdot \tilde{\nu})\|_{L^2(\Sigma_0)} &\leq C(\|(\nabla^2 p\, \nu \cdot \nu)\|_{L^2(\Sigma_0)} + C\|\nabla p\|_{L^2(\Sigma_0)})\\ &\leq C\|(\nabla^2 p\, \nu \cdot \nu)\|_{L^2(\Sigma_0)} + C(1 + \|p\|_{H^1(\Sigma_0)})\\ &\leq C(1 + \|(\nabla^2 p\, \nu \cdot \nu)\|_{L^2(\Sigma_0)}).\end{aligned}$$

Therefore since

$$\Delta_{\Sigma_0} p = \Delta p - (\nabla^2 p\, \nu \cdot \nu) - H_{\Sigma_0}\partial_\nu p$$

we obtain by (8.10) and by $\|\partial_\nu p\|_{L^2(\Sigma_t)} \leq C(1 + \|p\|_{H^1(\Sigma_t)}) \leq C$ that

$$\|(\nabla^2 p\, \nu \cdot \nu)\|_{L^2(\Sigma_0)} \leq C(1 + \|p\|_{H^2(\Sigma_0)}) \leq C.$$

Thus we have (8.11) by the three inequalities above.

We estimate $\|\nabla p\|_{H^{\frac{3}{2}}(\Omega_0)}$ similarly. We use (8.11) and Lemma 3.7 to estimate

$$\|\nabla p\|_{H^{\frac{3}{2}}(\Omega_0)} \leq C(\|\partial_\nu p\|_{H^1(\Sigma_0)} + \|p\|_{L^2(\Omega_0)} + \|\Delta p\|_{H^1(\Omega_0)}) \leq C. \tag{8.12}$$

In order to show that $\|\mathcal{D}_t^2 v\|_{L^2(\Omega_0)}$ is bounded we first observe that by (4.1) and by (8.12) we have

$$\|\mathcal{D}_t^2 v\|_{L^2(\Omega_0)} \leq \|\nabla \mathcal{D}_t p\|_{L^2(\Omega_0)} + \|\nabla v \star \nabla p\|_{L^2(\Omega_0)} \leq \|\nabla \mathcal{D}_t p\|_{L^2(\Omega_0)} + C.$$

Recall that we define the $H^{\frac{1}{2}}(\Sigma_0)$-norm using harmonic extension. Then it holds

$$\|\nabla \mathcal{D}_t p\|_{L^2(\Omega_0)} \leq C(\|\mathcal{D}_t p\|_{H^{\frac{1}{2}}(\Sigma_0)} + \|\mathcal{D}_t p\|_{L^2(\Omega_0)} + \|\Delta \mathcal{D}_t p\|_{L^2(\Omega_0)}).$$

Note that it holds $\|\mathcal{D}_t p\|_{L^2(\Omega_0)} \leq C\|\mathcal{D}_t p\|_{H^{\frac{1}{2}}(\Sigma_0)}$. By Remark 4.5 and Lemma 4.4 we have

$$\begin{aligned}\|\Delta \mathcal{D}_t p\|_{L^2(\Omega_0)} &\leq C\|R^1_{\mathrm{div}}\|_{L^2(\Omega_0)} + C\|R^0_{bulk}\|_{H^1(\Omega_0)}\\ &\leq C(1 + \|p\|_{H^2(\Omega_0)} + \|v\|_{H^2(\Omega_0)}) \leq C.\end{aligned}$$

We proceed by using (4.22) to write

$$\mathcal{D}_t p = -\Delta_{\Sigma_0} v \cdot \nu - 2B : \nabla_\tau v - Q(0)(D_t \nabla U \cdot \nabla U) - \frac{Q'(0)}{2}|\nabla U|^2.$$

We only bound the first term on RHS as the others are lower order. By the $C^{1,\alpha}$-regularity of ν we immediately estimate

$$\|\Delta_{\Sigma_0} v \cdot \nu\|_{H^{\frac{1}{2}}(\Sigma_0)} \leq C\|\Delta_{\Sigma_0} v\|_{H^{\frac{1}{2}}(\Sigma_0)} \leq C\|v\|_{H^3(\Omega_0)}.$$

This concludes the proof. □

Let us next prove the higher order version of Proposition 8.1.

Proposition 8.3. *Let $l \geq 2$ and assume that (1.7) and $E_{l-1}(t) \leq M$ hold for all $t \in [0, T)$. Then there are constants C and C_0 such that*

$$E_l(t) \leq C(C_0 + \mathcal{E}_l(t)),$$

where the constants C and C_0 depend on M, l and T.

Proof. We recall that by the definition of $E_l(t)$ in (5.1), $\mathcal{E}_l(t)$ in (7.1) and of $\mathcal{E}_l^+(t)$ it holds

$$c\,\mathcal{E}_l^+(t) \le \mathcal{E}_l(t) + \frac{Q(t)}{2}\int_{\Omega_t^c} |\nabla(\partial_t^{l+1}U)|^2\,dx + 1$$

for $c>0$, $Q(t)$ defined in (2.1), and

$$E_l(t) \le 2\mathcal{E}_l^+(t) + \sum_{k=1}^{l} \|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)}. \tag{8.13}$$

The claim follows once we prove that for any $\varepsilon>0$ it holds

$$\mathcal{E}_l^+(t) \le \mathcal{E}_l(t) + \varepsilon E_l(t) + C_\varepsilon \tag{8.14}$$

and

$$E_l(t) \le C\mathcal{E}_l^+(t). \tag{8.15}$$

In order to prove (8.14) we use the fact that $\partial^{l+1}U$ is harmonic in Ω_t^c, interpolation (Corollary 2.9) and Lemma 5.6 and have

$$\begin{aligned}\int_{\Omega_t^c} |\nabla\partial_t^{l+1}U|^2\,dx &\le C\|\partial_t^{l+1}U\|^2_{H^{\frac12}(\Sigma_t)} \le C\|\partial_t^{l+1}U\|_{H^1(\Sigma_t)}\|\partial_t^{l+1}U\|_{L^2(\Sigma_t)}\\ &\le CE_l(t)^{\frac12}\|\partial_t^{l+1}U\|_{L^2(\Sigma_t)} \le \varepsilon_1 E_l(t) + C_{\varepsilon_1}\|\partial_t^{l+1}U\|^2_{L^2(\Sigma_t)}.\end{aligned}$$

We use Lemma 4.6, $\|\nabla U\|_{L^\infty}\le C$, Lemma 5.6 and the assumption $E_{l-1}(t)\le \tilde{C}$ to deduce

$$\|\partial_t^{l+1}U\|_{L^2(\Sigma_t)} \le \|\nabla U\cdot \mathcal{D}_t^l v\|_{L^2(\Sigma_t)} + \|R_U^{l-1}\|_{L^2(\Sigma_t)} \le C\|\mathcal{D}_t^l v\|_{L^2(\Sigma_t)} + C.$$

By the Trace Theorem, by interpolation (Corollary 2.9) and by the definition of $E_l(t)$ it holds

$$\begin{aligned}\|\mathcal{D}_t^l v\|^2_{L^2(\Sigma_t)} &\le C\|\mathcal{D}_t^l v\|^2_{H^1(\Omega_t)} \le C\|\mathcal{D}_t^l v\|^{\frac43}_{H^{\frac32}(\Omega_t)}\|\mathcal{D}_t^l v\|^{\frac23}_{L^2(\Omega_t)}\\ &\le CE_l(t)^{\frac23}E_{l-1}(t)^{\frac13} \le \varepsilon_2 E_l(t) + C_{\varepsilon_2}.\end{aligned}$$

By choosing first ε_1 and then ε_2 small implies (8.14).

Let us then prove (8.15). By (8.13) we have to bound $\|\mathcal{D}_t^{l+1-k}v\|_{H^{\frac32 k}(\Omega_t)}$ for all $k=1,\dots,l$ and $\|v\|_{H^{\lfloor\frac32(l+1)\rfloor}(\Omega_t)}$. We claim first that it holds

$$\|\mathcal{D}_t^l v\|^2_{H^{\frac32}(\Omega_t)} \le C\mathcal{E}_l^+(t). \tag{8.16}$$

Indeed, by Theorem 3.1, Lemma 4.4 and Lemma 5.3 it holds

$$\begin{aligned}&\|\mathcal{D}_t^l v\|^2_{H^{\frac32}(\Omega_t)}\\ &\le C(\|(\mathcal{D}_t^l v\cdot\nu)\|^2_{H^1(\Sigma_t)} + \|\mathcal{D}_t^l v\|^2_{L^2(\Omega_t)} + \|\operatorname{div}\mathcal{D}_t^l v\|^2_{H^{\frac12}(\Omega_t)} + \|\operatorname{curl}\mathcal{D}_t^l v\|^2_{H^{\frac12}(\Omega_t)})\\ &\le C(\mathcal{E}_l^+(t) + E_{l-1}(t) + \|R^{l-1}_{\mathrm{div}}\|^2_{H^{\frac12}(\Omega_t)})\\ &\le C(\mathcal{E}_l^+(t) + E_{l-1}(t)) \le C\mathcal{E}_l^+(t)\end{aligned}$$

and (8.16) follows.

Next we claim that for $2\le k\le l$ it holds

$$\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac32 k}(\Omega_t)} \le C\|\mathcal{D}_t^{l+3-k}v\|^2_{H^{\frac32 k-3}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon. \tag{8.17}$$

This inequality means that two derivatives in time implies regularity for three derivatives in space. We first use Proposition 3.2, Lemma 4.4 and Lemma 5.3 to deduce

$$\begin{aligned}
&\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \\
&\quad \le C\big(\|\Delta_\Sigma(\mathcal{D}_t^{l+1-k}v\cdot\nu)\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} + \|\mathcal{D}_t^{l+1-k}v\|^2_{L^2(\Omega_t)} \\
&\qquad + \|\operatorname{div}(\mathcal{D}_t^{l+1-k}v)\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)} + \|\operatorname{curl}(\mathcal{D}_t^{l+1-k}v)\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)}\big) \\
&\quad \le C\big(\|\Delta_\Sigma(\mathcal{D}_t^{l+1-k}v\cdot\nu)\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} + E_{l-1}(t) + \|R^{l-k}_{\operatorname{div}}\|^2_{H^{\frac{3}{2}k-1}(\Omega_t)}\big) \\
&\quad \le C\|\Delta_\Sigma(\mathcal{D}_t^{l+1-k}v\cdot\nu)\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} + \varepsilon E_l + C_\varepsilon.
\end{aligned} \tag{8.18}$$

We proceed by using Lemma 4.7 to write

$$\mathcal{D}_t^{l+2-k}p = -\Delta_\Sigma(\mathcal{D}_t^{l+1-k}v\cdot\nu) - Q(t)(\nabla U\cdot\nabla\partial_t^{l+2-k}U) + R_p^{l+1-k}. \tag{8.19}$$

Lemma 5.8 yields

$$\|R_p^{l+1-k}\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} = \|R_p^{l-(k-1)}\|^2_{H^{\frac{3}{2}(k-1)-1}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon. \tag{8.20}$$

Next we claim that

$$\|(\nabla U\cdot\nabla\partial_t^{l+2-k}U)\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon. \tag{8.21}$$

If $k=2$ then we use the fact that by the assumption $\|B\|_{H^2(\Sigma_t)} \le \|B\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le C$ and by Theorem 3.9 the function U is uniformly $C^{2,\alpha}$-regular. Therefore we have by Lemma 5.5

$$\|(\nabla U\cdot\nabla\partial_t^{l}U)\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le C\|\nabla\partial_t^l U\|^2_{H^{\frac{1}{2}}(\Sigma_t)} \le \varepsilon E_l(t) + C_\varepsilon.$$

If $k\ge 3$ then $2 \le \frac{3}{2}k - \frac{5}{2} \le \lfloor\frac{3}{2}l\rfloor - 2$. We have by Proposition 2.10, by the Sobolev embedding, by Lemma 5.1 and by Lemma 5.5 that

$$\begin{aligned}
\|(\nabla U\cdot\nabla\partial_t^{l+2-k}U)\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} &\le C\|\nabla U\|^2_{L^\infty(\Sigma_t)}\|\nabla\partial_t^{l+2-k}U\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} \\
&\quad + \|\nabla U\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)}\|\nabla\partial_t^{l+2-k}U\|^2_{L^\infty(\Sigma_t)} \\
&\le C(1+\|\nabla U\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)})\|\nabla\partial_t^{l+2-k}U\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} \\
&\le C(1+\|p\|^2_{H^{\lfloor\frac{3}{2}l\rfloor-2}(\Sigma_t)})(\varepsilon E_l(t) + C_\varepsilon).
\end{aligned}$$

Hence, (8.21) follows from the Trace Theorem as

$$\|p\|^2_{H^{\lfloor\frac{3}{2}l\rfloor-2}(\Sigma_t)} \le C(1+\|\nabla p\|^2_{H^{\frac{3}{2}l-2}(\Omega_t)}) \le CE_{l-1}(t) \le C.$$

We deduce by (8.18), (8.19), (8.20), (8.21), Lemma 3.7 and (4.15) that

$$\begin{aligned}
&\|\mathcal{D}_t^{l+1-k}v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} \\
&\quad \le C\|\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3}{2}k-\frac{5}{2}}(\Sigma_t)} + \varepsilon E_l(t) + C_\varepsilon \\
&\quad \le C\|\nabla\mathcal{D}_t^{l+2-k}p\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon \\
&\quad \le C\|\mathcal{D}_t^{l+2-k}\nabla p\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + C\|[\mathcal{D}_t^{l+2-k},\nabla]p\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon \\
&\quad \le C\|\mathcal{D}_t^{l+3-k}v\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + C\|R^{l+1-k}_{bulk}\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon.
\end{aligned}$$

Lemma 5.3 implies

$$\|R^{l+1-k}_{bulk}\|^2_{H^{\frac{3}{2}k-3}(\Omega_t)} \le \|R^{l-(k-1)}_{bulk}\|^2_{H^{\frac{3}{2}(k-1)-1}(\Omega_t)} \le \varepsilon E_l(t) + C_\varepsilon$$

and the estimate (8.17) follows.

Let us then prove

$$\|v\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)} \le C\|\mathcal{D}_t^2 v\|^2_{H^{\frac{3}{2}(l-1)}(\Omega_t)} + C\|\mathcal{D}_t v\|^2_{H^{\frac{3}{2}l}(\Omega_t)} + \varepsilon E_l(t) + C_\varepsilon \mathcal{E}_l^+(t). \tag{8.22}$$

We denote $\lambda_l = \lfloor \frac{3}{2}(l+1)\rfloor - 1$ and use the second inequality in Proposition 3.2 and have

$$\|v\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)} \le C(1 + \|\Delta_{\Sigma_t} v_n\|^2_{H^{\lambda_l - \frac{3}{2}}(\Sigma_t)} + \|B\|^2_{H^{\frac{3}{2}l}(\Sigma_t)} + \|\operatorname{curl} v\|^2_{H^{\lambda_l}(\Omega_t)}).$$

By the definition of $\mathcal{E}_l^+(t)$ in (8.1) it holds $\|\operatorname{curl} v\|^2_{H^{\lambda_l}(\Omega_t)} \le \mathcal{E}_l^+(t)$. Lemma 5.2 and Trace Theorem yield

$$\begin{aligned}\|B\|^2_{H^{\frac{3}{2}l}(\Sigma_t)} &\le C\left(1 + \|p\|^2_{H^{\lfloor \frac{3}{2}l + \frac{1}{2}\rfloor}(\Sigma_t)}\right)\\ &\le C\left(1 + \|\nabla p\|^2_{H^{\frac{3}{2}l}(\Omega_t)}) = C(1 + \|\mathcal{D}_t v\|^2_{H^{\frac{3}{2}l}(\Omega_t)}\right).\end{aligned}$$

We treat the term $\|\Delta_{\Sigma_t} v_n\|_{H^{\lambda_l - \frac{3}{2}}(\Sigma_t)}$ by using (8.9) and have

$$\|\Delta_{\Sigma_t} v_n\|_{H^{\lambda_l - \frac{3}{2}}(\Sigma_t)} \le C\|\mathcal{D}_t p\|_{H^{\lambda_l - \frac{3}{2}}(\Sigma_t)} + C\|\nabla U \cdot \nabla \partial_t U\|_{H^{\frac{3}{2}l-1}(\Sigma_t)} + \|R_p^0\|_{H^{\frac{3}{2}l-1}(\Sigma_t)}.$$

By Lemma 5.5 we have

$$\|\nabla \partial_t U\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le \varepsilon E_l + C_\varepsilon$$

and

$$\|\nabla U\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le C E_{l-1}(t) \le C.$$

Therefore we have by Proposition 2.10 and by the Sobolev embedding

$$\|\nabla U \cdot \nabla \partial_t U\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le C\|\nabla U\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)}\|\nabla \partial_t U\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le \varepsilon E_l + C_\varepsilon.$$

Similarly we obtain

$$\|R_p^0\|^2_{H^{\frac{3}{2}l-1}(\Sigma_t)} \le \varepsilon E_l + C_\varepsilon.$$

We leave the details for the reader. Therefore we have by arguing as before

$$\begin{aligned}\|v\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)} &\le C\|\mathcal{D}_t p\|_{H^{\lambda_l - \frac{3}{2}}(\Sigma_t)} + \varepsilon E_l + C_\varepsilon \mathcal{E}_l^+(t)\\ &\le C\|\nabla \mathcal{D}_t p\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \varepsilon E_l + C_\varepsilon \mathcal{E}_l^+(t)\\ &\le C\|\mathcal{D}_t \nabla p\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \|[\mathcal{D}_t, \nabla]p\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \varepsilon E_l + C_\varepsilon \mathcal{E}_l^+(t)\\ &\le C\|\mathcal{D}_t^2 v\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \|\nabla v \star \nabla p\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \varepsilon E_l + C_\varepsilon \mathcal{E}_l^+(t).\end{aligned}$$

Note that $\lambda_l - 2 \le \frac{3}{2}(l-1)$ and $\lambda_l - 1 \le \lfloor \frac{3}{2}l\rfloor$. Thus by the definition of $E_{l-1}(t)$ it holds

$$\|\nabla p\|^2_{H^{\lambda_l - 2}(\Omega_t)} + \|\nabla v\|^2_{H^{\lambda_l - 2}(\Omega_t)} \le \|\mathcal{D}_t v\|^2_{H^{\frac{3}{2}(l-1)}(\Omega_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}l\rfloor}(\Omega_t)} \le C E_{l-1}(t) \le C.$$

Proposition 2.10, the assumption $\|\nabla v\|_{L^\infty(\Omega_t)} \le C$, and the Sobolev embedding then imply

$$\|\nabla v \star \nabla p\|^2_{H^{\lambda_l - 2}(\Omega_t)} \le C E_{l-1}^2(t) \le C$$

and the inequality (8.22) follows.

We deduce by (8.16), (8.22) and by using (8.17) an iterative way that

$$\sum_{k=1}^{l} \|\mathcal{D}_t^{l+1-k} v\|^2_{H^{\frac{3}{2}k}(\Omega_t)} + \|v\|^2_{H^{\lfloor \frac{3}{2}(l+1)\rfloor}(\Omega_t)} \le C_\varepsilon \mathcal{E}_l^+(t) + \varepsilon E_l(t).$$

Thus we obtain (8.15) by using the above inequality and (8.13). □

9. Proof of the Main Theorem

In this short section we collect the results from Sects. 6, 7 and 8 and prove the Main Theorem. The proof is fairly straightforward, and the only delicate part is to show that the a priori estimates (1.7) hold for a short time.

Proof of the Main Theorem. Let us assume that the quantities Λ_T and σ_T, which are defined in (1.5) and (1.4) respectively, satisfy $\Lambda_T \le M$ and $\sigma_T \ge \frac{1}{M}$ for $T > 0$. We show that this implies the bound

$$E_l(t) \le C_l \qquad \text{for all } \; t \le T \tag{9.1}$$

for every positive integer l, where the constant C_l depends on l, T, M and on $E_l(0)$. Here the dependence on T means that if $T < 1$, then the constant C_l may be chosen to be independent of T. The estimate (9.1) is crucial as it quantifies the smoothness of the flow under the assumption that the a priori estimates are bounded.

We obtain first by Lemma 6.4 that

$$\int_0^T \|p\|^2_{H^2(\Omega_t)}\, dt \le \tilde{C}, \tag{9.2}$$

where $\tilde{C}$ depends on T, M and on $E_1(0)$. Proposition 7.1 and Proposition 8.1 in turn imply

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_1(t) &\le C(1 + \|p\|^2_{H^2(\Omega_t)})E_1(t) \\ &\le C(1 + \|p\|^2_{H^2(\Omega_t)})(C_0 + \mathcal{E}_1(t)) \end{aligned} \tag{9.3}$$

for all $t \le T$. In particular, the quantity $C_0 + \mathcal{E}_1(t)$ is positive. Therefore we obtain by integrating over $(0, T)$ and using (9.2) that

$$C_0 + \mathcal{E}_1(t) \le \hat{C}(C_0 + \mathcal{E}_1(0))$$

for all $t \le T$. By using Proposition 8.1 again we have

$$E_1(t) \le C(C_0 + \mathcal{E}_1(t)) \le C\hat{C}(C_0 + \mathcal{E}_1(0)) \le C_1,$$

where the constant C_1 depends on M, T and $E_1(0)$.

We may then use Proposition 7.2 and Proposition 8.3 in an inductive way and deduce that if $E_{l-1}(t) \le C_{l-1}$ for $t \le T$ then it holds

$$\frac{d}{dt}\mathcal{E}_l(t) \le C E_l(t) \le C(C_0 + \mathcal{E}_l(t)).$$

By integrating we deduce

$$C_0 + \mathcal{E}_l(t) \le (C_0 + \mathcal{E}_l(0))e^{CT}$$

and using Proposition 8.3 again we have

$$E_l(t) \le C(C_0 + \mathcal{E}_l(0))e^{CT} \le C_l, \tag{9.4}$$

where the constant C_l depends on l, T, M and on $E_l(0)$. Note that we obtain (9.4) under the assumption $E_{l-1}(t) \le C_{l-1}$ for $t \le T$ and thus an induction argument implies that (9.4) holds for all l for a constant which depends on T, l, M and on $E_l(0)$. Therefore we have (9.1).

Let us then prove the last claim, i.e., that the a priori estimates (1.7) hold for M for a short time

$$T_0 \ge c_0 \tag{9.5}$$

for a positive constant c_0 which depends on $\|H_{\Sigma_0}\|_{H^2(\Sigma)}$, $\|v\|_{H^3(\Omega_0)}$ and on σ_0.

To this aim we define the quantity

$$\lambda_t := \|\nabla p\|^2_{L^2(\Omega_t)} + \|B\|^4_{L^4(\Sigma_t)} + \|\nabla v\|^4_{L^4(\Sigma_t)} + \|\nabla v\|^4_{L^4(\Omega_t)} + 1,$$

where p is the pressure and v the velocity field. Let us also denote by

$$\delta(t) := d_{\mathcal{H}}(\Omega_t, \Omega_0)$$

the Hausdorff distance between the sets Ω_t and Ω_0. The point is that if we would know that it holds $\lambda_t \leq 2\lambda_0$ and $\delta(t) \leq \varepsilon_0$, where λ_0 is the value at time $t = 0$ and ε_0 is a small number, then we have by the curvature bound and by standard argument from regularity theory (e.g. by Allard regularity theory) that Σ_t is uniformly $C^{1,\alpha}(\Gamma)$-regular. We choose the number ε_0 such that it depends also on σ_0 so that $\delta(t) \leq \varepsilon_0$ implies $\sigma_t \geq \frac{\sigma_0}{2}$. Moreover, by Proposition 8.1 we deduce that there are constants C and C_0 such that

$$E_1(t) \leq C(C_0 + \mathcal{E}_1(t)). \tag{9.6}$$

Let us then define $T_0 \in (0, T]$ to be the largest number such that

$$\sup_{t \leq T_0} \lambda_t \leq 2\lambda_0, \quad \sup_{t \leq T_0} \delta(t) \leq \varepsilon_0 \quad \text{and} \quad \sup_{t \leq T_0} \mathcal{E}_1(t) \leq C_0 + \mathcal{E}_1(0),$$

where C_0 is the constant in (9.6). We note that the last condition together with (9.6) implies that

$$E_1(t) \leq C(C_0 + \mathcal{E}_1(t)) \leq C(2C_0 + \mathcal{E}_1(0)) \leq \tilde{C} E_1(0), \tag{9.7}$$

for $t \leq T_0$. It is also easy to see that for Λ_T defined in (1.5) it holds $\Lambda_T^2 \leq C \sup_{t \leq T} E_1(t)$. This means that (9.7) ensures that the a priori estimates (1.7) hold for the time interval $[0, T_0]$. Therefore it is enough to show that $T_0 \geq c_0$. We may assume that $T_0 < \min\{T, 1\}$ since otherwise the claim is trivially true.

If $T_0 < \min\{T, 1\}$ then at least in one of the three conditions in the definition of T_0 we have an equality. Assume that $\lambda_{T_0} = 2\lambda_0$. Note that by (9.7) it holds $E_1(t) \leq \tilde{C} E_1(0)$ for all $t \leq T_0$. We remark that it holds

$$\|B\|^2_{L^\infty(\Sigma_t)} + \|\nabla v\|^2_{L^\infty(\Omega_t)} \leq C E_1(t).$$

Moreover, by using the formula (4.10) in Lemma 4.2 we obtain

$$\|\mathcal{D}_t \nabla p\|_{L^2(\Omega_t)} + \|\mathcal{D}_t B\|_{L^2(\Sigma_t)} + \|\mathcal{D}_t \nabla v\|_{L^2(\Sigma_t)} \leq C E_1(t).$$

We leave the details for the reader. Therefore by a straightforward calculation we deduce that for some $q \geq 1$ it holds

$$\frac{d}{dt} \lambda_t \leq C E_1(t)^q \leq C E_1(0)^q$$

where the last inequality follows from (9.7). By integrating the above over $(0, T_0)$ and using $\lambda_{T_0} = 2\lambda_0$ we obtain

$$\lambda_0 \leq C E_1(0)^q T_0.$$

Since $\lambda_0 \geq 1$ we have $T_0 \geq c_0$, for a constant that depends on $E_1(0)$ and σ_0.

We argue similarly if we have an equality in the third condition in the definition of T_0, i.e., $\mathcal{E}_1(T_0) = C_0 + \mathcal{E}_1(0)$. Indeed, then by the definition of $E_1(t)$ we have that

$$\|p\|^2_{H^2(\Omega_t)} \leq E_1(t).$$

Therefore we obtain by (9.3) and (9.7) that

$$\frac{d}{dt} \mathcal{E}_1(t) \leq C(1 + \|p\|^2_{H^2(\Omega_t)}) E_1(t) \leq C,$$

where the constant C depends on $E_1(0)$ and on σ_0. We integrate the above over $(0, T_0)$ and obtain

$$C_0 = \mathcal{E}_1(T_0) - \mathcal{E}_1(0) \leq C T_0.$$

Thus we have again $T_0 \geq c_0$.

Finally assume that it holds $\delta(T_0) = \varepsilon_0$. By definition the flow gives a diffeomorphism $\Phi_{T_0} : \Sigma_0 \to \Sigma_{T_0}$. We note that the velocity is uniformly bounded by the Sobolev embedding and by (9.7)

$$\|v\|^2_{L^\infty(\Omega_t)} \leq C E_1(t) \leq C E_1(0).$$

Therefore we have by the fundamental Theorem of Calculus that for every $x \in \Sigma_0$ it holds

$$|\Phi_{T_0}(x) - x| \leq \int_0^{T_0} \|v\|_{L^\infty} \, dt \leq C T_0.$$

Since $\sup_{x\in\Sigma_0} |\Phi_{T_0}(x) - x| \geq \delta(T_0) \geq \varepsilon_0$, we again have $T_0 \geq c_0$.

We have thus obtained (9.5) for a constant c_0 which depends on σ_0 and $E_1(0)$. By Lemma 8.2 we deduce that c_0 in fact depends on $\sigma_0, \|v\|_{H^3(\Omega_0)}$, $\|H_{\Sigma_0}\|_{H^2(\Sigma_0)}$ and on $\|h_0\|_{C^{1,\alpha}(\Gamma)}$. This concludes the proof of the second claim. □

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References

[1] Acerbi, E., Fusco, N., Julin, V., Morini, M.: Nonlinear stability results for the modified Mullins–Sekerka and the surface diffusion flow. J. Differ. Geom. **113**(1), 1–53 (2019)
[2] Ambrose, D.M.: Well-posedness of vortex sheets with surface tension. SIAM J. Math. Anal. **35**, 211–244 (2003)
[3] Ambrose, D.M., Masmoudi, N.: The zero surface tension limit of two-dimensional water waves. Commun. Pure Appl. Math. **58**, 1287–1315 (2005)
[4] Aubin, T.: Some Nonlinear Problems in Riemannian Geometry. Springer Monographs in Mathematics, Springer, Berlin (1998)
[5] Basaran, O.A., Scriven, L.E.: Axisymmetric shapes and stability of isolated charged drops. Phys. Fluids A **1**, 795–798 (1989)
[6] Beyer, K., Günther, M.: On the Cauchy problem for a capillary drop. I. Irrotational motion. Math. Methods Appl. Sci. **21**, 1149–1183 (1998)
[7] Beyer, K., Günther, M.: The Jacobi equation for irrotational free boundary flows. Analysis (Munich) **20**, 237–254 (2000)
[8] Brezis, H., Mironescu, P.: Gagliardo–Nirenberg inequalities and non-inequalities: the full story. Ann. Inst. H. Poincaré Anal. Non Linéaire **35**, 1355–1376 (2018)
[9] Chandler-Wilde, S.N., Hewett, D.P., Moiola, A.: Interpolation of Hilbert and Sobolev spaces quantitative estimates and counter examples. Mathematika **61**, 414–443 (2015)
[10] Cheng, C.H.A., Shkoller, S.: Solvability and regularity for an elliptic system prescribing the curl, divergence, and partial trace of a vector field on Sobolev-class domains. J. Math. Fluid Mech. **19**, 375–422 (2017)
[11] Coutand, D., Shkoller, S.: Well-posedness of the free-surface incompressible Euler equations with or without surface tension. J. Am. Math. Soc. **20**, 829–930 (2007)
[12] Coutand, D., Shkoller, S.: Well-posedness in smooth function spaces for the moving-boundary three-dimensional compressible euler equations in physical vacuum. Arch. Rational. Mech. Anal. **206**, 515–616 (2012)
[13] Coutand, D., Shkoller, S.: On the finite-time splash and splat singularities for the 3-D free-surface Euler equations. Commun. Math. Phys. **325**, 143–183 (2014)
[14] Dambrine, M., Lamboley, J.: Stability in shape optimization with second variation. J. Differ. Equ. **267**, 3009–3045 (2019)
[15] Corte, S.D., Diana, A., Mantegazza, C.: Global existence and stability for the modified Mullins–Sekerka and surface diffusion flow. Preprint (2021)
[16] Dahlberg, B.E.J.: On the Poisson integral for Lipschitz and C^1-domains. Studia Math. **66**, 13–24 (1979)
[17] Ebin, D.G.: The equations of motion of a perfect fluid with free boundary are not well posed. Commun. Part. Differ. Equ. **12**, 1175–1201 (1987)

[18] Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence (1998)
[19] Fabes, E.B., Jodeit, M., Jr., Rivière, N.M.: Potential techniques for boundary value problems on C^1-domains. Acta Math. **141**, 165–186 (1978)
[20] Fontelos, M.A., Friedman, A.: Symmetry-breaking bifurcations of charged drops. Arch. Rational Mech. Anal. **172**, 267–294 (2004)
[21] Fusco, N., Julin, V.: On the regularity of critical and minimal sets of a free interface problem. Interfaces Free Bound. **17**, 117–142 (2015)
[22] Fusco, N., Julin, V., Morini, M.: The surface diffusion flow with elasticity in three dimensions. Arch. Rational. Mech. Anal. **237**, 1325–1382 (2020)
[23] Garcia, M.S.V., Varvaruca, E., Weiss, G.S.: Singularities in axisymmetric free boundaries for electrohydrodynamic equations. Arch. Rational. Mech. Anal. **222**, 573–601 (2016)
[24] Garzon, M., Gray, L.J., Sethian, J.A.: Numerical simulations of electrostatically driven jets from nonviscous droplets. Phys. Rev. E **89**, 033011 (2014)
[25] Gilbarg, D., Trudinger, N.S.: Elliptic Partial Differential Equations of Second Order. Springer, Berlin (2001)
[26] Ginsberg, D.: On the breakdown of the solutions to the incompressible Euler equations with free surface boundary. SIAM J. Math. Anal. **53**, 3366–3384 (2021)
[27] Goldman, M., Novaga, M., Röger, M.: Quantitative estimates for bending energies and applications to non-local variational problems. Proc. Roy. Soc. Edinb. Sect. A **150**, 131–169 (2020)
[28] Goldman, M., Novaga, M., Ruffini, B.: Existence and stability for a non-local isoperimetric model of charged liquid drops. Arch. Rat. Mech. Anal. **217**, 1–36 (2015)
[29] Goldman, M., Novaga, M., Ruffini, B.: On minimizers of an isoperimetric problem with long-range interactions under a convexity constraint. Anal. PDE **11**, 1113–1142 (2018)
[30] Goldman, M., Ruffini, B.: Equilibrium shapes of charged liquid droplets and related problems: (mostly) a review. Geom. Flows **2**, 94–104 (2017)
[31] Grafakos, L., Oh, S.: The Kato–Ponce inequality. Commun. Part. Differ. Equ. **39**, 1128–1157 (2014)
[32] Hamilton, R.S.: Three-manifolds with positive Ricci curvature. J. Differ. Geom. **17**(2), 255–306 (1982)
[33] Huisken, G.: Asymptotic behavior for singularities of the mean curvature flow. J. Differ. Geom. **31**, 285–299 (1990)
[34] Iguchi, T.: Well-posedness of the initial value problem for capillary-gravity waves. Funkcial. Ekvac. **44**, 219–241 (2001)
[35] Jerison, D.S., Kenig, C.E.: The Neumann problem on Lipschitz domains. Bull. Am. Math. Soc. **4**, 203–207 (1981)
[36] Julin, V., La Manna, D.A.: Short time existence of classical solution to the fractional mean curvature flow. Ann. Inst. H. Poincaré Anal. Non Linéaire **37**, 983–1016 (2020)
[37] La Manna, D.A.: An isoperimetric problem with a Coulombic repulsion and attractive term. ESAIM COCV (2019). https://doi.org/10.1051/cocv/2018008
[38] Lannes, D.: Well-posedness of the water-waves equations. J. Am. Math. Soc. **18**, 605–654 (2005)
[39] Lee, J.M.: Riemannian Manifolds. An Introduction to Curvature. Graduate Texts in Mathematics, vol. 176. Springer, New York (1997)
[40] Lindblad, H.: Well-posedness for the motion of an incompressible liquid with free surface boundary. Ann. Math. **162**, 109–194 (2005)
[41] Majda, A.J., Bertozzi, A.L.: Vorticity and Incompressible Flow. Cambridge Texts in Applied Mathematics, vol. 27. Cambridge University Press, Cambridge (2002)
[42] Mantegazza, C.: Smooth geometrc evolutions of hypersurfaces. Geom. Funct. Anal. **12**, 138–182 (2002)
[43] Mantegazza, C.: Lecture notes on Mean Curvature Flow Progress in Mathematics, vol. 290. Springer, Basel (2011)
[44] Masmoudi, N., Rousset, F.: Uniform regularity for the Navier–Stokes equation with Navier boundary condition. Arch. Rational Mech. Anal. **203**, 529–575 (2012)
[45] Miksis, M.J.: Shape of a drop in an electric field. Phys. Fluids **24**, 1967–1972 (1981)
[46] Muratov, C.B., Novaga, M.: On well-posedness of variational models of charged drops. Proc. Roy. Soc. Lond. A **472**(2187), 20150808 (2016)
[47] Muratov, C.B., Novaga, M., Ruffini, B.: On equilibrium shape of charged flat drops. Commun. Pure Appl. Math. **71**, 1049–1073 (2018)
[48] Rayleigh, L.: On the equilibrium of liquid conducting masses charged with electricity. Philos. Mag. **14**, 184–186 (1882)
[49] Schweizer, B.: On the three-dimensional Euler equations with a free boundary subject to surface tension. Ann. Inst. H. Poincaré Anal. non linéaire **22**, 753–781 (2005)
[50] Shatah, J., Zeng, C.: Geometry and a priori estimates for free boundary problems of the Euler equation. Commun. Pure App. Math. **61**, 678–744 (2008)
[51] Shatah, J., Zeng, C.: Local well-posedness for fluid interface problems. Arch. Ration. Mech. Anal. **199**, 653–705 (2011)
[52] Taylor, G.I.: Disintegration of water drops in an electric field. Proc. R. Soc. Lond. A **280**, 383–397 (1964)
[53] Taylor, M.E.: Partial Differential Equations III. Applied Mathematical Sciences, vol. 117. Springer, Berlin (2011)
[54] Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators. North-Holland, Amsterdam (1978)
[55] Wang, Z., Yang, J.: Well-posedness of electrohydrodynamic interfacial waves under tangential electric field. SIAM J. Math. Anal. **53**, 2567–2594 (2021)
[56] Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 2-D. Invent. Math. **130**, 39–72 (1997)
[57] Wu, S.: Well-posedness in Sobolev spaces of the full water wave problem in 3-D. J. Am. Math. Soc. **12**, 445–495 (1999)
[58] Yosihara, H.: Capillary-gravity waves for an incompressible ideal fluid. J. Math. Kyoto Univ. **23**, 649–694 (1983)

[59] Yang, J.: Well-posedness of electrohydrodynamic waves under vertical electric field. Z. Angew. Math. Phys. **71**, 171 (2020)
[60] Yang, J.: A priori estimates of the electrohydrodynamic waves with vorticity: vertical electric field. J. Math. Anal. Appl. **498**, 124973 (2021)
[61] Zeleny, J.: Instability of electrified liquid surfaces. Phys. Rev. **10**, 1–6 (1917)
[62] Zhang, P., Zhang, Z.: On the free boundary problem of three-dimensional incompressible Euler equations. Commun. Pure Appl. Math. **61**, 877–940 (2008)

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