

Regularity for a strongly degenerate equation with explicit u -dependence

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Abstract

We consider local weak solutions of widely degenerate elliptic PDEs of the type

$$\operatorname{div} \left(a(x)(|Du| - 1)_+^{p-1} \frac{Du}{|Du|} \right) = b(x, u) \quad \text{in } \Omega, \quad (0.1)$$

where $2 \leq p < \infty$, Ω is an open subset of $\mathbb{R}^n$, $n > 2$, and $(\cdot)_+$ stands for the positive part. We establish a higher differentiability result for the composition of the gradient with a suitable function that vanishes in the unit ball for the gradient, under suitable assumptions on the datum $b(x, u)$ and the coefficient $a(x)$. The novelty here with respect to previous papers on the subject is that the right hand side explicitly depends on the solution u .

Keywords: - Widely degenerate problems, explicit u -dependence, higher differentiability

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1 Introduction

In this paper, we are interested in the regularity properties of the local weak solutions to strongly degenerate elliptic equation of the form

$$\operatorname{div} \left(a(x)(|Du| - 1)_+^{p-1} \frac{Du}{|Du|} \right) = b(x, u) \quad \text{in } \Omega, \quad (1.1)$$

where $2 \leq p < \infty$, Ω is an open subset of $\mathbb{R}^n$, $n > 2$, and $(\cdot)_+$ stands for the positive part. When dealing with equations as in (1.1), one has to face two different major difficulties: the principal part of the equation is widely degenerate and the right hand side explicitly depends on the solution itself. Actually, the operator on the left hand side of (1.1) is uniformly elliptic only outside the unit ball centered at the origin, where it behaves asymptotically as the classical p -Laplace operator. One of the main motivations for the study of equations of this type comes from the optimal transport problems with congestion effects, and when $a(x) = 1$ and $b(x, u) = f(x)$ such connection has been exhaustively studied in [10, 11, 12, 13].

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The study of such an equation fits into the wider class of the asymptotically regular problems that have been extensively studied starting from the pioneering paper by Chipot and Evans [16], concerning the homogeneous, autonomous quadratic growth case. Later on, still for the homogeneous, autonomous case, the Lipschitz continuity of weak solutions has been established for the superquadratic growth [26], and in the subquadratic growth case [31]. Since then, many contributions to the regularity theory of weak solutions of widely degenerate equations have been established. Among the others we quote the results in [7, 8, 15, 17, 18, 23, 25, 28, 34]. It is worth to point out that no more than Lipschitz regularity can be expected for the solutions of widely degenerate problems. Indeed, every 1-Lipschitz function is a solution of the homogeneous elliptic equation

$$\operatorname{div} \left((|Du| - 1)_+^{p-1} \frac{Du}{|Du|} \right) = 0.$$

However, many higher differentiability results are available for a suitable function of the gradient that vanishes outside the unit ball. This is a common aspect of nonlinear elliptic problems with growth rate $p \geq 2$ for which the higher differentiability is proven for a nonlinear function of the gradient that takes into account the growth and the structure of the equation.

Indeed, already for the p -Poisson equation, the higher differentiability is established for the function

$$V_p(Du) = |Du|^{\frac{p-2}{2}} Du$$

as can be seen already in the pioneering paper by Uhlenbeck [37]. In case of widely degenerate problems, this phenomenon persists and higher differentiability results hold true for the function

$$H_{p/2}(Du) = (|Du| - 1)_+^{p-1} \frac{Du}{|Du|}$$

(see [9, 11, 12, 19, 20]), that, as one may expect, does not provide any information about the second-order regularity of the solutions in the set where the equation becomes degenerate.

Differently, from the above mentioned papers our datum depends on the solution itself. Even in the simplest setting of the Laplacian, the dependence on u constitutes a difficulty by itself and this has been clear since the pioneering paper by Brezis and Nirenberg [14], where the aim was to prove the existence of a positive solution to the elliptic equation

$$-\Delta u = u^p + f(x, u) \quad \text{on } \Omega.$$

Already in [14], a critical exponent was identified, for the growth of the function $f(x, u)$ with respect to the function u . Later on, the problem has been studied in many different directions, replacing the Laplace with more general non-linear operator and trying to find suitable conditions on the right hand side, to establish existence, regularity and the symmetry of solutions, (see for example [4, 5, 21, 22, 30, 24, 33, 35, 36]).

In this work we combine both the difficulties mentioned above with the aim of investigating the higher differentiability of solutions. Actually, the $W^{2,2}$ regularity for solutions of equations whose right hand side depends on the solution itself has been widely investigated, since, under suitable sign assumption on f , yields a comparison principle for the solutions. The first paper that deals with widely degenerate equation with right hand side depending on the solution is [29], where the authors were able to prove a higher differentiability result for $H_{p/2}(Du)$, under quite more restrictive conditions on the right hand side. In fact, with the respect to the result

in [29], not only we deal with weaker assumptions on $b(x, u)$ but also allow for discontinuous coefficients, assuming on the function $a(x)$ only a $W^{1,n}$ regularity.

To state and describe our results precisely let us specify the assumptions on our data.

The coefficient $a(x)$ is assumed to satisfy

$$\begin{aligned} \text{(i)} \quad & a(x) \in W_{loc}^{1,n}(\Omega), \\ \text{(ii)} \quad & \exists m, M > 0 : m < a(x) < M \quad \text{a.e. in } \Omega. \end{aligned} \quad (1.2)$$

For the datum $b(x, u)$ we assume that there exists an exponent q such that

$$1 \leq q \leq p^* \left(\frac{p-2}{p} \right) + 1, \quad (1.3)$$

and non-negative functions $k(x)$ and $h(x)$ such that

$$\begin{aligned} \text{(i)} \quad & |b(x, z) - b(y, z)| \leq (k(x) + k(y))|x - y||z|^q, \\ \text{(ii)} \quad & |b(x, z) - b(x, t)| \leq h(x)|z - t|(|z|^{q-1} + |t|^{q-1}), \end{aligned} \quad (1.4)$$

for a.e. $x, y \in \Omega$ and every $s, t \in \mathbb{R}$. As usual, p^* denotes the Sobolev conjugate exponent of p , i.e.

$$p^* = \begin{cases} \frac{np}{n-p} & \text{if } p < n, \\ \text{any finite } r & \text{if } p \geq n. \end{cases}$$

For the functions appearing in (1.4), we shall assume that $k(x) \in L_{loc}^s(\Omega)$, and $h(x) \in L_{loc}^\gamma(\Omega)$, where γ and s are such that

$$\frac{1}{s} = 1 - \frac{q}{p^*} - \frac{n-2}{np}, \quad (1.5)$$

$$\frac{1}{\gamma} = 1 - \frac{q-1}{p^*} - \frac{2(n-1)}{np}. \quad (1.6)$$

Note that by (1.3) both s and γ are greater than 1. As it is natural, the values of s and γ depend on q , more precisely when q goes to 0 we obtain a datum $b(x, u) = b(x)$ that does not depend on the solution u , while when q goes to its critical value we obtain specific ranges for the values of s and γ ,

$$\begin{cases} \frac{np}{np-n+2} \leq s_q \leq \frac{np}{p+2}, \\ \frac{np}{np-n-p+2} \leq \gamma_q \leq \frac{np}{2}. \end{cases} \quad (1.7)$$

As an example of the datum $b(x, u)$ that satisfies (1.4) we can consider

$$b(x, u) = f(x)g(u) \text{ where } f(x) \in W^{1,s}(\Omega) \text{ and } |g(u)| \leq C|u|^q.$$

Moreover, for simplicity, we shall assume that $b(x, 0) = 0$ which yields, through the second assumption in (1.4), that

$$|b(x, u)| \leq h(x)|u|^q. \quad (1.8)$$

Our main result is the following

Theorem 1.1. *Let $u \in W_{loc}^{1,p}(\Omega)$ be a weak solution of the equation (1.1), under assumptions (1.2), (1.3), (1.4), (1.5) and (1.6). Then $H_{p/2}(Du) \in W_{loc}^{1,2}(\Omega)$ and the following estimate holds*

$$\begin{aligned} \int_{B_{R/2}} |D(H_{p/2}(Du))|^2 dx &\leq C \int_{B_R} (1 + |Du|^p) dx \\ &+ C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left[\left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} + \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \right], \end{aligned}$$

for every $B_R \subset B_{R_0} \Subset \Omega$ and a constant $C = C(n, p, q, s, \gamma, m, M, R)$.

Let us briefly summarize the proof of the previous Theorem, describing its main points. First of all, we freeze and regularize the datum $b(x, u)$ and regularize the coefficient $a(x)$, in order to cast the problem into a classical variational framework. This way of arguing leads us to the study of the higher differentiability of the solutions to the problem (3.1) below, in which the right hand side is independent of the solution.

The central part of the proof consists in establishing a suitable *a priori* estimate for the derivatives of the composition between the function $H_{p/2}(\xi)$ and the gradient of the solution of the frozen problem. Next, we introduce a family of regularized problems whose solutions satisfy the assumptions of the *a priori* estimate and we show that this estimate is preserved in passing to the limit. Finally, we come back to the original problem proving that the limit function coincides with the solution of (1.1) outside the ball of radius 1. This allows us to conclude that the solution of (1.1) shares with this limit function the same higher differentiability property.

2 Preliminaries

2.1 Notation and essential definitions

In this paper we shall denote by C or c a general positive constant that may vary on different occasions. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm we use on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for the vectors $\xi, \eta \in \mathbb{R}^k$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

In what follows, $B_r(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < r\}$ will denote the n -dimensional open ball centered at x_0 with radius r . We shall sometimes omit the dependence on the center when all balls occurring in a proof are concentric. Unless otherwise stated, different balls in the same context will have the same center.

For further needs, we now define the auxiliary function $H_\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$H_\gamma(\xi) := \begin{cases} (|\xi| - 1)_+^\gamma \frac{\xi}{|\xi|} & \text{if } \xi \neq 0, \\ 0 & \text{if } \xi = 0, \end{cases} \quad (2.1)$$

where $\gamma > 0$ is a parameter. We conclude this first part of the preliminaries by recalling the following definition.

Definition 2.1. *A function $u \in W_{loc}^{1,p}(\Omega)$ is a local weak solution of equation (1.1) if and only if, for any test function $\varphi \in W_0^{1,p}(\Omega)$, the following integral identity holds:*

$$\int_{\Omega} \langle a(x) H_{p-1}(Du), D\varphi \rangle dx = \int_{\Omega} b(x, u) \varphi dx.$$

2.2 Algebraic inequalities

In this section, we gather some relevant algebraic inequalities that will be needed later on. We recall the following estimate, whose proof can be found in [32, Chapter 12].

Lemma 2.2. *Let $p \in [2, \infty)$ and $k \in \mathbb{N}$. Then, for every $\xi, \eta \in \mathbb{R}^k$, the following inequality*

$$|\xi - \eta|^p \leq C \left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2$$

holds for a constant $C \equiv C(p) > 0$.

Combining [1, Lemma 2.2] with [26, Formula (2.4)], we obtain the following

Lemma 2.3. *Let $1 < p < \infty$. There exists a constant $c \equiv c(n, p) > 0$ such that*

$$c^{-1}(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}} \leq \frac{\left| |\xi|^{\frac{p-2}{2}} \xi - |\eta|^{\frac{p-2}{2}} \eta \right|^2}{|\xi - \eta|^2} \leq c(|\xi|^2 + |\eta|^2)^{\frac{p-2}{2}}$$

for every $\xi, \eta \in \mathbb{R}^n$ with $\xi \neq \eta$.

For the function $H_{p-1}(\xi)$ defined at (2.1) with $\gamma = p - 1$, we record the following estimates, which can be obtained by suitably modifying the proofs of [12, Lemma 4.1] (for the case $p \geq 2$), [2, Lemma 2.5] (for the case $1 < p < 2$) and [7, Lemma 2.8].

Lemma 2.4. *Let $p \in (1, \infty)$. Then, there exists a constant $c \equiv c(n, p) > 0$ such that*

$$\langle H_{p-1}(\xi) - H_{p-1}(\eta), \xi - \eta \rangle \geq c |H_{\frac{p}{2}}(\xi) - H_{\frac{p}{2}}(\eta)|^2, \quad (2.2)$$

for every $\xi, \eta \in \mathbb{R}^n$. Moreover, if $2 \leq p < \infty$ we have

$$|H_{p-1}(\xi) - H_{p-1}(\eta)| \leq (p-1) \left(|H_{p/2}(\xi)|^{\frac{p-2}{p}} + |H_{p/2}(\eta)|^{\frac{p-2}{p}} \right) |H_{p/2}(\xi) - H_{p/2}(\eta)|.$$

Lemma 2.5. *For every $(\alpha, \varepsilon) \in \mathbb{R}^+ \times \mathbb{R}^+$ with $\alpha < \varepsilon$, there exist two positive constants $\beta_1(\alpha, \varepsilon)$ and $\beta_2(\alpha, \varepsilon, n)$ such that*

$$\beta_1 |H_\varepsilon(\xi) - H_\varepsilon(\eta)| \leq c \frac{|H_\alpha(\xi) - H_\alpha(\eta)|}{((|\xi| - 1)_+^\varepsilon + (|\eta| - 1)_+^\varepsilon)^{\frac{\alpha - \varepsilon}{\varepsilon}}} \leq \beta_2 |H_\varepsilon(\xi) - H_\varepsilon(\eta)| \quad (2.3)$$

for every $\xi, \eta \in \mathbb{R}^n$.

We conclude by recalling a well-known iteration Lemma, whose proof can be found in [27, Lemma 6.1].

Lemma 2.6. *Let $Z(t)$ be a bounded non-negative function in the interval $[\rho, R]$. Assume that for $\rho \leq s < t \leq R$ we have*

$$Z(s) \leq [A(t-s)^{-\alpha} + B(t-s)^{-\beta} + C] + \theta Z(t)$$

with $A, B, C \geq 0$, $\alpha, \beta > 0$ and $0 \leq \theta < 1$. Then,

$$Z(\rho) \leq c(\alpha, \theta) [A(R-\rho)^{-\alpha} + B(R-\rho)^{-\beta} + C].$$

2.3 Difference quotients

We recall here the definition and some elementary properties of the difference quotients that will be useful in the following (see, for example, [27]).

Definition 2.7. For every vector-valued function $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$ the finite difference operator in the direction x_j is defined by

$$\tau_{j,h}F(x) = F(x + he_j) - F(x),$$

where $h \in \mathbb{R}$, e_j is the unit vector in the direction x_j and $j \in \{1, \dots, n\}$.

The difference quotient of F with respect to x_j is defined for $h \in \mathbb{R} \setminus \{0\}$ by

$$\Delta_{j,h}F(x) = \frac{\tau_{j,h}F(x)}{h}.$$

When no confusion can arise, we shall omit the index j and simply write τ_h or Δ_h instead of $\tau_{j,h}$ or $\Delta_{j,h}$, respectively.

Proposition 2.8. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \in W^{1,q}(\Omega)$, with $q \geq 1$. Moreover, let $G : \Omega \rightarrow \mathbb{R}$ be a measurable function and consider the set

$$\Omega_{|h|} := \{x \in \Omega : \text{dist}(x, \partial\Omega) > |h|\}.$$

Then:

- (i) $\Delta_h F \in W^{1,q}(\Omega_{|h|})$ and $\partial_i(\Delta_h F) = \Delta_h(\partial_i F)$ for every $i \in \{1, \dots, n\}$.
- (ii) If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then

$$\int_{\Omega} F \Delta_h G \, dx = - \int_{\Omega} G \Delta_{-h} F \, dx.$$

- (iii) We have

$$\Delta_h(FG)(x) = F(x + he_j)\Delta_h G(x) + G(x)\Delta_h F(x).$$

The next result about the finite difference operator is a kind of integral version of the Lagrange Theorem and its proof can be found in [27, Lemma 8.1].

Lemma 2.9. If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < q < +\infty$ and $F \in L^q(B_R, \mathbb{R}^k)$ is such that $DF \in L^q(B_R, \mathbb{R}^{k \times n})$, then

$$\int_{B_\rho} |\tau_h F(x)|^q \, dx \leq c^q(n) |h|^q \int_{B_R} |DF(x)|^q \, dx.$$

Moreover

$$\int_{B_\rho} |F(x + he_j)|^q \, dx \leq \int_{B_R} |F(x)|^q \, dx.$$

Finally, we recall the following fundamental result, whose proof can be found in [27, Lemma 8.2].

Lemma 2.10. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^k$, $F \in L^q(B_R, \mathbb{R}^k)$ with $1 < q < +\infty$. Suppose that there exist $\rho \in (0, R)$ and a constant $M > 0$ such that

$$\sum_{j=1}^n \int_{B_\rho} |\tau_{j,h} F(x)|^q dx \leq M^q |h|^q$$

for every $h \in \mathbb{R}$ with $|h| < \frac{R-\rho}{2}$. Then $F \in W^{1,q}(B_\rho, \mathbb{R}^k)$. Moreover

$$\|DF\|_{L^q(B_\rho)} \leq M$$

and

$$\Delta_{j,h} F \rightarrow \partial_j F \quad \text{in } L^q_{loc}(B_R, \mathbb{R}^k) \quad \text{as } h \rightarrow 0,$$

for each $j \in \{1, \dots, n\}$.

For further needs, we record the following

Lemma 2.11. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $p \geq 2$ and $u \in W^{1,p}_{loc}(\Omega)$. Then the implication

$$H_{p/2}(Du) \in W^{1,2}_{loc}(\Omega) \implies |Du| \in L^{\frac{np}{n-2}}_{loc}(\Omega)$$

holds true, together with the estimate

$$\left(\int_{B_R} |Du|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{n}} \leq C \left(\int_{B_R} |DH_{p/2}(Du)|^2 dx + \int_{B_R} (1 + |Du|^p) dx \right).$$

for every $B_R \Subset \Omega$

Proof. By the Sobolev embedding theorem we have that

$$H_{p/2}(Du) \in L^{2^*}_{loc}(B_R)$$

with the estimate

$$\left(\int_{B_R} |H_{p/2}(Du)|^{2^*} dx \right)^{\frac{1}{2^*}} \leq C \left(\int_{B_R} |DH_{p/2}(Du)|^2 dx + \int_{B_R} |H_{p/2}(Du)|^2 dx \right)^{\frac{1}{2}}.$$

By the definition of $H_{p/2}(\xi)$ in (2.1), previous estimate yields

$$\left(\int_{B_R} (|Du| - 1)_+^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{n}} \leq C \left(\int_{B_R} |DH_{p/2}(Du)|^2 dx + \int_{B_R} (|Du| - 1)_+^p dx \right)^{\frac{1}{2}},$$

and from this inequality the conclusion easily follows. $\square$

3 The regularized problem

In this section we establish the higher differentiability of the solutions to an equation similar to that in (1.1) under higher regularity assumptions on the data $a(x)$ and $b(x, u)$ that will be instrumental for the proof of our main result. The main feature of this equation is that we freeze the right hand side $b(x, \cdot)$ calculating it on a fixed function belonging to $W_{loc}^{1,p}(\Omega)$. More precisely, let us fix a function $w \in W_{loc}^{1,p}(\Omega)$ and let $v \in W_{loc}^{1,p}(\Omega)$ be a solution of the following equation

$$\operatorname{div} \left(a(x) (|Dv| - 1)_+^{p-1} \frac{Dv}{|Dv|} \right) = b(x, w) \text{ in } \Omega. \quad (3.1)$$

Our first aim is to prove the following

Theorem 3.1. *Assume that $a(x)$ satisfies (ii) in (1.2) and that for every ball $B_R \Subset \Omega$ there exists $L_{1,R} > 0$ such that*

$$\sup_{\substack{x, y \in B_R \\ x \neq y}} \frac{|a(x) - a(y)|}{|x - y|} \leq L_{1,R}. \quad (3.2)$$

Assume further that for every ball $B_R \Subset \Omega$ there exist constants $L_{2,R}, L_{3,R} > 0$ such that

$$|b(x, s) - b(y, s)| \leq L_{2,R} |x - y| |s|^q, \quad (3.3)$$

$$|b(x, s) - b(x, t)| \leq L_{3,R} |s - t| (|s|^{q-1} + |t|^{q-1}), \quad (3.4)$$

for every $x, y \in B_R$, every $s, t \in \mathbb{R}$ and where q satisfies (1.3). If $v \in W_{loc}^{1,p}(\Omega)$ solves (3.1), then $H_{p/2}(Dv) \in W_{loc}^{1,2}(\Omega)$ with the estimate

$$\int_{B_\rho} |D(H_{p/2}(Dv))|^2 dx \leq C \int_{B_R} (1 + |Dv|^p) dx + C \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}}.$$

for all $B_\rho \subseteq B_{R/2} \subset B_R \Subset \Omega$ and $C = C(m, M, L_1, L_2, L_3, p, \rho, R)$,

Proof. Since v is a weak solution of (3.1), then

$$\int_{\Omega} \langle a(x) H_{p-1}(Dv), D\varphi \rangle dx = \int_{\Omega} b(x, w) \varphi dx \text{ for all } \varphi \in W_0^{1,p}(\Omega). \quad (3.5)$$

Fix a ball $B_R \Subset \Omega$ and let $\zeta \in C_0^\infty(B_{R/2})$ be a cut off function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_{r/2}$ with $r \leq R$ and $|D\zeta| < \frac{C}{R-r}$. In what follows, without loss of generality, we shall suppose that $R < 1$. Choosing $\varphi = \tau_{s,-h}(\zeta^2 \tau_{s,h} v)$ as test function in (3.5) and integrating by parts by means of (ii) in Proposition 2.8, we get

$$\int_{\Omega} \left\langle \tau_{s,h} \left(a(x) H_{p-1}(Dv) \right), \zeta^2 D(\tau_{s,h} v) + 2\zeta D\zeta \tau_{s,h} v \right\rangle dx = \int_{\Omega} \zeta^2 \tau_{s,h} (b(x, w)) \tau_{s,h} v dx.$$

By (iii) in Proposition 2.8 and some elementary calculations, the previous equality can be written as follows

$$0 = \int_{\Omega} \zeta^2 \langle \tau_{s,h}(a(x)) H_{p-1}(Dv), \tau_{s,h}(Dv) \rangle dx$$

$$\begin{aligned}
& + \int_{\Omega} \zeta^2 a(x + he_s) \langle \tau_{s,h}(H_{p-1}(Dv)), \tau_{s,h}(Dv) \rangle dx \\
& + 2 \int_{\Omega} \zeta \langle \tau_{s,h}(a(x)) H_{p-1}(Dv), \tau_{s,h} v D\zeta \rangle dx \\
& + 2 \int_{\Omega} \zeta a(x + he_s) \langle \tau_{s,h}(H_{p-1}(Dv)), \tau_{s,h} v D\zeta \rangle dx \\
& - \int_{\Omega} \zeta^2 \tau_{s,h} v [b(x + h, w(x + h)) - b(x, w(x + h))] dx \\
& - \int_{\Omega} \zeta^2 \tau_{s,h} v [b(x, w(x + h)) - b(x, w(x))] dx \\
& =: I_1 + I_2 + I_3 + I_4 - I_5 - I_6,
\end{aligned}$$

that yields

$$I_2 \leq |I_1| + |I_3| + |I_4| + |I_5| + |I_6|. \quad (3.6)$$

The integral I_2 can be estimated using assumption (ii) in (1.2) and the degenerate ellipticity of the equation given by the first estimate in Lemma 2.4, as follows

$$I_2 = \int_{\Omega} \zeta^2 a(x + he_s) \langle \tau_{s,h}(H_{p-1}(Dv)), \tau_{s,h}(Dv) \rangle dx \geq m \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx. \quad (3.7)$$

By (iii) in Proposition 2.8, we may write

$$\begin{aligned}
H_{p-1}(Dv) \tau_{s,h} Dv &= \tau_{s,h} \left(H_{p-1}(Dv) \cdot Dv \right) - Dv(x + he_s) \tau_{s,h} \left(H_{p-1}(Dv) \right) \\
&= \tau_{s,h} \left((|Dv| - 1)_+^{p-1} |Dv| \right) - Dv(x + he_s) \tau_{s,h} \left(H_{p-1}(Dv) \right) \\
&= \tau_{s,h} \left((|Dv| - 1)_+^p \right) + \tau_{s,h} \left((|Dv| - 1)_+^{p-1} \right) - Dv(x + he_s) \tau_{s,h} \left(H_{p-1}(Dv) \right) \\
&= \tau_{s,h} \left(H_p(|Dv|) \right) + \tau_{s,h} \left(H_{p-1}(|Dv|) \right) - Dv(x + he_s) \tau_{s,h} \left(H_{p-1}(Dv) \right), \quad (3.8)
\end{aligned}$$

where we used the definition of $H_{\gamma}(\xi)$ both with $\gamma = p$ and $\gamma = p - 1$ and the linearity of the finite difference operator.

For the estimate of I_1 we use (3.8), thus getting

$$\begin{aligned}
I_1 &\leq \int_{\Omega} \zeta^2 |\tau_{s,h}(a(x))| |\tau_{s,h}(H_p(|Dv|))| dx + \int_{\Omega} \zeta^2 |\tau_{s,h}(a(x))| |\tau_{s,h}(H_{p-1}(|Dv|))| dx \\
&\quad + \int_{\Omega} \zeta^2 |\tau_{s,h}(a(x))| \left(|Dv(x + he_s)| |\tau_{s,h}(H_{p-1}(Dv))| \right) dx \\
&=: |I_{1,1}| + |I_{1,2}| + |I_{1,3}|. \quad (3.9)
\end{aligned}$$

We now proceed estimating the integrals $|I_{1,j}|$, $j = 1, 2, 3$.

Using Lemma 2.5 with $\alpha = p/2$, $\varepsilon = p$, $\xi = Dv(x + he_s)$ and $\eta = Dv(x)$, we obtain

$$|\tau_{s,h}(H_p(|Dv|))| \leq C \left| \tau_{s,h}(H_{p/2}(|Dv|)) \right| \left[(|Dv(x + he_s)| - 1)_+^{p/2} + (|Dv(x)| - 1)_+^{p/2} \right]. \quad (3.10)$$

Using (3.10) and Young's inequality, we get

$$\begin{aligned}
|I_{1,1}| &\leq C \int_{\Omega} \zeta^2 |\tau_{s,h}(a(x))| |\tau_{s,h} H_{p/2}(|Dv|)| \left[(|Dv(x + he_s)| - 1)_+^{p/2} + (|Dv(x)| - 1)_+^{p/2} \right] dx, \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\
&\quad + C_{\varepsilon} \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 \left((|Dv(x + he_s)| - 1)_+^p + (|Dv(x)| - 1)_+^p \right) dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_{\varepsilon}(L_1) |h|^2 \int_{B_R} (|Dv(x)| - 1)_+^p dx,
\end{aligned} \tag{3.11}$$

where we used assumption (3.2), the properties of ζ , the second estimate in Lemma 2.9 and that

$$|\tau_{s,h} H_{p/2}(|Dv|)| \leq |\tau_{s,h} H_{p/2}(Dv)|. \tag{3.12}$$

To estimate $|I_{1,2}|$, we use the second estimate in Lemma 2.4, assumption (3.2), inequality (3.12) and the properties of ζ as follows

$$\begin{aligned}
|I_{1,2}| &\leq C \int_{\Omega} \zeta^2 |\tau_{s,h}(a(x))| |\tau_{s,h}(H_{p/2}(Dv))| \left(\left| H_{p/2}(Dv(x + he_s)) \right|^{\frac{p-2}{p}} + \left| H_{p/2}(Dv(x)) \right|^{\frac{p-2}{p}} \right) dx \\
&\leq L_1 |h| \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))| \left(\left| H_{p/2}(Dv(x + he_s)) \right|^{\frac{p-2}{p}} + \left| H_{p/2}(Dv(x)) \right|^{\frac{p-2}{p}} \right) dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_{\varepsilon}(L_1) |h|^2 \int_{B_{R/2}} \left(\left| H_{p/2}(Dv(x + he_s)) \right|^{\frac{p-2}{p}} + \left| H_{p/2}(Dv(x)) \right|^{\frac{p-2}{p}} \right)^2 dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_{\varepsilon}(L_1) |h|^2 \int_{B_R} |H_{p/2}(Dv)|^{\frac{2(p-2)}{p}} dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_{\varepsilon}(L_1) |h|^2 \int_{B_R} (|Dv| - 1)_+^{p-2} dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_{\varepsilon}(L_1) |h|^2 \int_{B_R} (1 + |Dv(x)|)^p dx,
\end{aligned} \tag{3.13}$$

where we also used Young's inequality, the definition of $H_{p/2}(Dv)$ and the second estimate in Lemma 2.9.

Using the second estimate in Lemma 2.4, Young's inequality, assumption (3.2) and the properties of ζ , we get

$$\begin{aligned}
|I_{1,3}| &\leq \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)| |Dv(x + he_s)| |\tau_{s,h}(H_{p/2}(Dv))| \left| H_{p/2}(Dv(x + he_s)) \right|^{\frac{p-2}{p}} + |H_{p/2}(Dv)|^{\frac{p-2}{p}} dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\
&\quad + C_{\varepsilon} \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 |Dv(x + he_s)|^2 \left| H_{p/2}(Dv(x + he_s)) \right|^{\frac{p-2}{p}} + |H_{p/2}(Dv)|^{\frac{p-2}{p}} dx \\
&\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx
\end{aligned}$$

$$\begin{aligned}
& + C_\varepsilon(L_1)|h|^2 \int_{B_{R/2}} |Dv(x + he_s)|^2 \left| |H_{p/2}(Dv(x + he_s))|^{\frac{p-2}{p}} + |H_{p/2}(Dv)|^{\frac{p-2}{p}} \right|^2 dx \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\
& + C_\varepsilon(L_1)|h|^2 \int_{B_{R/2}} |Dv(x + he_s)|^2 \left(|H_{p/2}(Dv(x + he_s))|^{\frac{2(p-2)}{p}} + |H_{p/2}(Dv(x))|^{\frac{2(p-2)}{p}} \right) dx \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon(L_1)|h|^2 \int_{B_R} |Dv|^p dx,
\end{aligned} \tag{3.14}$$

where in the last line we used the definition of $H_{p/2}(Dv)$ and second estimate in Lemma 2.9. Combining (3.11), (3.13) and (3.14) we obtain the estimate of $|I_1|$, i.e.

$$|I_1| \leq 3\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \int_{B_R} (1 + |Dv|^p) dx. \tag{3.15}$$

We now proceed with the estimate of $|I_3|$. Using (3.2), that $|D\zeta| \leq \frac{C}{R-r}$ and Hölder's inequality, we get

$$\begin{aligned}
|I_3| & \leq 2 \int_{\Omega} \zeta |D\zeta| |\tau_{s,h}(a(x))| (|Dv| - 1)_+^{p-1} |\tau_{s,h} v| dx \\
& \leq \frac{C|h|}{R-r} \int_{B_R} (|Dv| - 1)_+^{p-1} |\tau_{s,h} v| dx \\
& \leq \frac{C|h|^2}{R-r} \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dv| - 1)_+^p dx \right)^{\frac{p-1}{p}} \\
& \leq \frac{C|h|^2}{R-r} \int_{B_R} |Dv|^p dx,
\end{aligned} \tag{3.16}$$

where we also used the first estimate in Lemma 2.9.

For the estimate of $|I_4|$, we use that $|D\zeta| \leq \frac{C}{R-r}$, the second estimate in Lemma 2.4 again, assumption (ii) in (1.2) and Young's inequality, as follows

$$\begin{aligned}
|I_4| & \leq 2 \int_{\Omega} \zeta |D\zeta| |a(x + he_s)| |\tau_{s,h}(H_{p-1}(Dv))| |\tau_{s,h} v| dx \\
& \leq \frac{C(M)}{R-r} \int_{\Omega} \zeta |\tau_{s,h}(H_{p/2}(Dv))| |H_{p/2}(Dv(x + he_s))^{\frac{p-2}{p}} + H_{p/2}(Dv(x))^{\frac{p-2}{p}}| |\tau_{s,h} v| dx \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx \\
& + \frac{C_\varepsilon(M)}{R-r} \int_{B_{R/2}} \left| H_{p/2}(Dv(x + he_s))^{\frac{p-2}{p}} + H_{p/2}(Dv(x))^{\frac{p-2}{p}} \right|^2 |\tau_{s,h} v|^2 dx \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx \\
& + \frac{C_\varepsilon(M)}{R-r} \left(\int_{B_{R/2}} |\tau_{s,h} v|^p dx \right)^{\frac{2}{p}} \left(\int_{B_{R/2}} \left| H_{p/2}(Dv(x + he_s))^{\frac{p-2}{p}} + H_{p/2}(Dv(x))^{\frac{p-2}{p}} \right|^{\frac{2p}{p-2}} dx \right)^{\frac{p-2}{p}} \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{C_\varepsilon(M)}{R-r} |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{2}{p}} \left(\int_{B_R} |H_{p/2}(Dv(x))|^2 dx \right)^{\frac{p-2}{p}} \\
& = \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx \\
& + \frac{C_\varepsilon(M)}{R-r} |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{2}{p}} \left(\int_{B_R} |H_{p/2}(Dv(x))|^2 dx \right)^{\frac{p-2}{p}} \\
& \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx + \frac{C_\varepsilon(M)}{R-r} |h|^2 \int_{B_R} (1 + |Dv|^p) dx,
\end{aligned} \tag{3.17}$$

where we also used Hölder's inequality, Lemma 2.9 and the definition of $H_{p/2}(Dv)$. We now proceed to estimate $|I_5|$, arguing as follows

$$\begin{aligned}
|I_5| & \leq \int_{\Omega} \zeta^2 |\tau_{s,h}v| \left| (b(x+h, w(x+h)) - b(x, w(x+h))) \right| dx \\
& \leq L_2 |h| \int_{B_{R/2}} |\tau_{s,h}v| |w(x+h)|^q dx \\
& \leq L_2 |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} |w|^{\frac{qp}{p-1}} dx \right)^{\frac{p-1}{p}},
\end{aligned}$$

where we used assumption (3.3), Hölder's inequality and Lemma 2.9. Note that the use of Hölder's inequality is legitimate since

$$\frac{qp}{p-1} < p^* \iff q < \frac{p^*(p-1)}{p}$$

which is true by assumption (1.3). Therefore, we can estimate $|I_5|$ further by the use of Sobolev embedding theorem thus getting

$$\begin{aligned}
|I_5| & \leq C(L_2, R) |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} |w|^{p^*} dx \right)^{\frac{q}{p^*}} \\
& \leq C(L_2, R) |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{q}{p}} \\
& \leq C(L_2, R) |h|^2 \left(\int_{B_R} (1 + |Dv|^p) dx \right) + C |h|^2 \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{q}{p-1}},
\end{aligned} \tag{3.18}$$

where we used Young's inequality in the last line. To estimate $|I_6|$ we use assumptions (3.4), (1.3) and Hölder's inequality, thus getting

$$\begin{aligned}
|I_6| & \leq \int_{\Omega} \zeta^2 |\tau_{s,h}v| |b(x, w(x+h)) - b(x, w(x))| dx \\
& \leq L_3 \int_{B_{R/2}} |\tau_{s,h}v| |\tau_{s,h}w| \left(|w(x)|^{q-1} + |w(x+h)|^{q-1} \right) dx \\
& \leq C(L_3) |h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} |Dw|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} |w|^{\frac{(q-1)p}{p-2}} dx \right)^{\frac{p-2}{p}}
\end{aligned}$$

$$\begin{aligned}
&\leq C(L_3)|h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{1}{p}} \left(\int_{B_R} |w|^{p^*} dx \right)^{\frac{q-1}{p^*}} \\
&\leq C(L_3)|h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{q-1}{p}} \\
&\leq C(L_3)|h|^2 \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{q}{p}} \\
&\leq C(L_3)|h|^2 \left(\int_{B_R} (1 + |Dv|^p) dx \right) + C|h|^2 \left(\int_{B_R} (|Dw|^p + |w|^p) dx \right)^{\frac{q}{p-1}}, \tag{3.19}
\end{aligned}$$

where we used again the first estimate in Lemma 2.9, Sobolev embedding theorem and Young's inequality. Also here, we note that

$$\frac{(q-1)p}{p-2} < p^* \iff q-1 < \frac{p^*(p-2)}{p} \iff q < 1 + \frac{p^*(p-2)}{p}$$

which is precisely assumption (1.3). At this point we insert estimates (3.7), (3.15), (3.16), (3.17), (3.18) and (3.19) in (3.6), to get

$$\begin{aligned}
m \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx &\leq 4\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\
&+ |h|^2 \frac{C_\varepsilon}{R-r} \int_{B_R} (1 + |Dv|^p) dx + C|h|^2 \left(\int_{B_R} |Dw|^p dx + |w|^p dx \right)^{\frac{q}{p-1}}.
\end{aligned}$$

Choosing $\varepsilon = \frac{m}{8}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side. Using also that $\zeta = 1$ in $B_{r/2}$, dividing both sides of inequality by $|h|^2$ and letting $|h|$ goes to 0, we get

$$\int_{B_{r/2}} |D(H_{p/2}(Dv))|^2 dx \leq C \int_{B_R} (1 + |Dv|^p) dx + C \left(\int_{B_R} (|w|^p + |Dw|^p) dx \right)^{\frac{q}{p-1}}, \tag{3.20}$$

where $r \leq R$ and $C = C(m, M, L_1, L_2, L_3, p, r, R)$, i.e. the conclusion. $\square$

In particular, as a Corollary of Theorem 3.1, we have a first regularity result for the solution u of (1.1) under the same regularity assumptions on $a(x)$ and $b(x, u)$. Indeed, the same proof works in particular if $v = w = u$ and this yields

Corollary 3.2. *Assume that $a(x)$ and $b(x, s)$ satisfy (3.2), (3.3) and (3.4) and that q satisfies (1.3). If $u \in W_{loc}^{1,p}(\Omega)$ solves (1.1), then $H_{p/2}(Du) \in W_{loc}^{1,2}(\Omega)$ holds true with the estimate*

$$\int_{B_\rho} |D(H_{p/2}(Du))|^2 dx \leq C \int_{B_R} (1 + |Du|^p) dx + C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}}.$$

for all $B_\rho \subseteq B_{R/2} \subset B_R \Subset \Omega$ and $C = C(m, M, L_1, L_2, L_3, p, \rho, R)$.

4 A priori estimate

This section is devoted to the proof of an a priori estimate for the solutions to equation (3.1) under the weaker assumptions (1.3) and (1.4), that is the first of the main points in our proof. More precisely, we assume a priori that $H_{p/2}(Dv) \in W_{loc}^{1,2}(\Omega)$ and we estimate the L^2 norm of the derivative of $H_{p/2}(Dv)$ with a constant that depends only on the $W^{1,p}$ norm of w , the L^s norm of k and the L^γ norm of h .

Theorem 4.1. *Let $w \in W_{loc}^{1,p}(\Omega)$ be fixed and $v \in W_{loc}^{1,p}(\Omega)$ be a weak solution of the equation (3.1), under assumptions (1.2), (1.3), (1.4), (1.5) and (1.6). If $H_{p/2}(Dv) \in W_{loc}^{1,2}(\Omega)$ then there exists $R_0 = R_0(\|Da\|_{L^n}, n, p)$ such that the following estimate*

$$\begin{aligned} \int_{B_{R/2}} |D(H_{p/2}(Dv))|^2 dx &\leq C \int_{B_R} (1 + |Dv|^p) dx \\ &+ C \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}} \left[\left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} + \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \right] \end{aligned}$$

holds for every $B_R \subset B_{R_0} \Subset \Omega$ and a constant $C = C(n, p, q, s, \gamma, m, M, R)$.

Proof. Our starting point is inequality (3.6) in Theorem 3.1 that obviously holds true with the same choice of the test function in (3.5).

Here we fix a ball $B_R \Subset \Omega$, radii $R/2 \leq \rho_1 \leq s \leq t \leq \rho_2 \leq R < 1$ and let $\zeta \in C_0^\infty(B_t)$ be a cut off function such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in B_s and $|D\zeta| < \frac{C}{t-s}$. For the estimate of I_1 , we use (3.8) and (3.12), thus getting

$$|I_1| \leq |I_{1,1}| + |I_{1,2}| + |I_{1,3}|. \quad (4.1)$$

where $|I_{1,j}|$, $j = 1, 2, 3$ are defined in (3.9). We now proceed estimating the integrals $|I_{1,j}|$. Arguing as we did in (3.11) we get

$$\begin{aligned} |I_{1,1}| &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\ &+ C_\varepsilon \int_{\Omega} |\tau_{s,h} a(x)|^2 (|Dv(x + he_s)| - 1)_+^p + (|Dv(x)| - 1)_+^p dx \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\ &+ C_\varepsilon \left(\int_{B_t} |\tau_{s,h} a|^n dx \right)^{\frac{2}{n}} \left(\int_{B_t} (|Dv(x + he_s)| - 1)_+^{\frac{pn}{n-2}} + (|Dv(x)| - 1)_+^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} (|Dv(x)| - 1)_+^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{n}}, \end{aligned}$$

where we used in turn, assumption (i) in (1.2), Hölder's inequality, the properties of ζ and Lemma 2.9. The a priori assumption $H_{p/2}(Dv) \in W_{loc}^{1,2}(\Omega)$ allows us to use Lemma 2.11, thus getting

$$|I_{1,1}| \leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx$$

$$+ C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \int_{B_R} |Dv|^p dx. \quad (4.2)$$

Now we proceed with the estimate of $|I_{1,2}|$. Arguing as we did in (3.13), we get

$$\begin{aligned} |I_{1,2}| &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\ &\quad + C_\varepsilon \left(\int_{B_t} |\tau_{s,h}(a(x))|^n dx \right)^{\frac{2}{n}} \left(\int_{B_t} \left(|H_{p/2}(Dv(x + he_s))|^{\frac{p-2}{p}} + |H_{p/2}(Dv(x))|^{\frac{p-2}{p}} \right)^{\frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} |H_{p/2}(Dv(x))|^{\frac{p-2}{p} \frac{2n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} (|Dv| - 1)_+^{\frac{(p-2)n}{n-2}} dx \right)^{\frac{n-2}{n}} \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} \left(1 + (|Dv| - 1)_+^{\frac{pn}{n-2}} \right) dx \right)^{\frac{n-2}{n}} \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \\ &\quad + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + \int_{B_{\rho_2}} (1 + |Dv|^p) dx \right), \end{aligned} \quad (4.3)$$

where we used Lemma 2.4, assumption (1.2), Hölder's inequality and Lemma 2.11. Now we estimate $|I_{1,3}|$, arguing as in (3.14) as follows

$$\begin{aligned} |I_{1,3}| &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 |Dv(x + he_s)|^p dx \\ &\quad + C_\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 \left(|H_{p/2}(Dv(x + he_s))|^2 + |H_{p/2}(Dv(x))^2| \right) dx \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon \left(\int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^n dx \right)^{\frac{2}{n}} \left(\int_{B_t} |Dv(x + he_s)|^{\frac{np}{n-2}} dx \right)^{\frac{n-2}{np}} \\ &\quad + C_\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 \left(|H_{p/2}(Dv(x + he_s))|^2 + |H_{p/2}(Dv(x))^2| \right) dx \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\ &\quad + C_\varepsilon |h|^2 \left(\int_{B_R} |Da(x)|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + \int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \\ &\quad + C_\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} a(x)|^2 \left(|H_{p/2}(Dv(x + he_s))|^2 + |H_{p/2}(Dv(x))^2| \right) dx \\ &\leq \varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx \\ &\quad + C_\varepsilon |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \int_{B_R} |Dv|^p dx, \end{aligned} \quad (4.4)$$

where we used Young's inequality, Hölder's inequality, Lemmas 2.9, 2.11 and in the last line, we argue as in (4.2).

Combining (4.2), (4.3) and (4.4), we obtain

$$\begin{aligned}
|I_1| &\leq 3\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx \\
&\quad + C_{\varepsilon} |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_R} (1 + |Dv|^p) dx \right) \\
&\quad + C_{\varepsilon} |h|^2 \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \left(\int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx \right). \tag{4.5}
\end{aligned}$$

We now proceed with the estimate of $|I_3|$, where I_3 is been defined in (3.6). By the properties of $|D\zeta|$ we have

$$\begin{aligned}
|I_3| &\leq 2 \int_{B_t} |D\zeta| |\tau_{s,h} a(x)| |\tau_{s,h} v| (|Dv| - 1)_+^{p-1} dx \\
&\leq \frac{C}{t-s} \int_{B_t} |\tau_{s,h} a(x)| |\tau_{s,h} v| (|Dv| - 1)_+^{p-1} dx.
\end{aligned}$$

Since

$$\frac{1}{n} + \frac{1}{p} + \frac{(n-2)(p-1)}{np} < 1,$$

we can use Hölder's inequality with exponents $n, p, \frac{np}{(n-2)(p-1)}$ to deduce that

$$\begin{aligned}
|I_3| &\leq \frac{C(R)}{t-s} \left(\int_{B_t} |\tau_{s,h} a|^n dx \right)^{\frac{1}{n}} \left(\int_{B_t} |\tau_{s,h} v|^p dx \right)^{\frac{1}{p}} \left(\int_{B_t} (|Dv| - 1)_+^{\frac{np}{n-2}} dx \right)^{\frac{(n-2)(p-1)}{np}} \\
&\leq \frac{C|h|^2}{t-s} \left(\int_{B_R} |Da|^n dx \right)^{\frac{1}{n}} \left(\int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_{\rho_2}} |D(H_{p/2}(Dv))|^2 dx + \int_{B_{\rho_2}} (1 + |Dv|^p) dx \right)^{\frac{p-1}{p}} \\
&\leq \sigma |h|^2 \int_{B_{\rho_2}} |D(H_{p/2}(Dv))|^2 dx + \sigma |h|^2 \int_{B_R} (1 + |Dv|^p) dx \\
&\quad + \frac{C_{\sigma} |h|^2}{(t-s)^p} \left(\int_{B_R} |Da|^n dx \right)^{\frac{p}{n}} \left(\int_{B_R} |Dv|^p dx \right), \tag{4.6}
\end{aligned}$$

where $\sigma > 0$ will be chosen later, and we also used Lemmas 2.9, 2.11 and Young's inequality. To estimate $|I_5|$, we use assumption (1.5) that allows us to apply Hölder's inequality with exponents $\frac{np}{n-2}, \frac{p^*}{q}$ and s , that is legitimate by virtue of assumption (1.3), where s has been defined in (1.5).

$$\begin{aligned}
|I_5| &\leq |h| \int_{B_t} |\tau_{s,h} v| |(k(x+h) + k(x))| |w(x+h)|^q dx \\
&\leq C(R) |h| \left(\int_{B_t} |\tau_{s,h} v|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{pn}} \left(\int_{B_{\rho_2}} |w|^{p^*} dx \right)^{\frac{q}{p^*}} \left(\int_{B_{\rho_2}} |k(x)|^s dx \right)^{\frac{1}{s}} \\
&\leq |h|^2 \left(\int_{B_{\rho_2}} |Dv|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{pn}} \left(\int_{B_{\rho_2}} |w|^{p^*} dx \right)^{\frac{q}{p^*}} \left(\int_{B_{\rho_2}} |k(x)|^s dx \right)^{\frac{1}{s}},
\end{aligned}$$

where we used assumption (i) in (1.4) and Lemma 2.9. Using also Lemma 2.11, Young's inequality and Sobolev embedding theorem, we obtain

$$\begin{aligned}
|I_5| &\leq |h|^2 \left(\int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + \int_{B_R} |Dv|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} |w|^{p^*} dx \right)^{\frac{q}{p^*}} \left(\int_{B_{\rho_2}} |k(x)|^s dx \right)^{\frac{1}{s}} \\
&\leq \sigma |h|^2 \int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + \sigma |h|^2 \int_{B_R} |Dv|^p dx \\
&\quad + C_\sigma |h|^2 \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_{\rho_2}} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}}. \tag{4.7}
\end{aligned}$$

For the estimate of $|I_6|$, we use assumption (1.6), and we apply Hölder's inequality with exponents $\frac{np}{n-2}$, p , $\frac{p^*}{q-1}$ and γ , that is legitimate by assumption (1.3) and where γ as been defined in (1.6). More precisely, we get

$$\begin{aligned}
|I_6| &\leq \int_{\Omega} \zeta^2 |\tau_{s,h} v| |\tau_{s,h} w| |w(x+h) + w(x)|^{q-1} |h(x+h)| dx \\
&\leq \left(\int_{B_t} |\tau_{s,h} v|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{np}} \left(\int_{B_t} |\tau_{s,h} w|^p dx \right)^{\frac{1}{p}} \left(\int_{B_{\rho_2}} |w(x)|^{p^*} dx \right)^{\frac{q-1}{p^*}} \left(\int_{B_{\rho_2}} |h(x)|^\gamma dx \right)^{\frac{1}{\gamma}} \\
&\leq |h|^2 \left(\int_{B_{\rho_2}} |Dv|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{pn}} \left(\int_{B_{\rho_2}} |Dw|^p dx \right)^{\frac{1}{p}} \left(\int_{B_{\rho_2}} |w(x)|^{p^*} dx \right)^{\frac{q-1}{p^*}} \left(\int_{B_{\rho_2}} |h(x)|^\gamma dx \right)^{\frac{1}{\gamma}} \\
&\leq |h|^2 \left(\int_{B_{\rho_2}} |Dv|^{\frac{pn}{n-2}} dx \right)^{\frac{n-2}{pn}} \left(\int_{B_{\rho_2}} (|Dw|^p + |w(x)|^p) dx \right)^{\frac{q}{p}} \left(\int_{B_{\rho_2}} |h(x)|^\gamma dx \right)^{\frac{1}{\gamma}} \\
&\leq \sigma |h|^2 \int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + \sigma |h|^2 \int_{B_R} |Dv|^p dx \\
&\quad + C_\sigma |h|^2 \left(\int_{B_R} (|w|^p + |Dw|^p) dx \right)^{\frac{q}{p-1}} \left(\int_{B_{\rho_2}} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}, \tag{4.8}
\end{aligned}$$

where we used the properties of ζ , Lemmas 2.9, 2.11 and Young's inequality. Combining (4.7) and (4.8) we obtain

$$\begin{aligned}
|I_5| + |I_6| &\leq 2\sigma |h|^2 \int_{B_{\rho_2}} |DH_{p/2}(Dv)|^2 dx + 2\sigma |h|^2 \int_{B_R} (1 + |Dv|^p) dx \\
&\quad + C_\sigma |h|^2 \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\
&\quad + C_\sigma |h|^2 \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}. \tag{4.9}
\end{aligned}$$

The estimates of I_2 and I_4 are the same as in the previous Theorem at (3.7) and (3.17) respectively. Next, we insert the estimates (4.5), (3.7), (4.6), (3.17) and (4.9) in (3.6), to get

$$m \int_{\Omega} \zeta^2 |\tau_{s,h}(H_{p/2}(Dv))|^2 dx \leq 4\varepsilon \int_{\Omega} \zeta^2 |\tau_{s,h} H_{p/2}(Dv)|^2 dx$$

$$\begin{aligned}
& + |h|^2 \left(C_\varepsilon \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} + 3\sigma \right) \int_{B_{\rho_2}} |D(H_{p/2}(Dv))|^2 dx \\
& + \frac{C_\sigma |h|^2}{(t-s)^p} \left(\int_{B_R} |Da|^n dx \right)^{\frac{p}{n}} \left(\int_{B_R} |Dv|^p dx \right) \\
& + |h|^2 \left(\frac{C_\varepsilon}{t-s} + 3\sigma + C \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \right) \int_{B_R} (1 + |Dv|^p) dx \\
& + C_\sigma |h|^2 \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\
& + C_\sigma |h|^2 \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}. \tag{4.10}
\end{aligned}$$

Choosing $\varepsilon = \frac{m}{8}$, we can reabsorb the first integral in the right hand side of the previous estimate by the left hand side. Using also that $\zeta = 1$ in B_s , dividing both sides of inequality by $|h|^2$ and letting $|h|$ goes to 0, by the a priori assumption $H_{p/2}(Du) \in W_{loc}^{1,2}(\Omega)$, we get

$$\begin{aligned}
& \frac{m}{2} \int_{B_{\rho_1}} |D(H_{p/2}(Dv))|^2 dx \leq \frac{m}{2} \int_{B_s} |D(H_{p/2}(Dv))|^2 dx \\
& \leq \left(\tilde{C} \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} + 3\sigma \right) \int_{B_{\rho_2}} |D(H_{p/2}(Dv))|^2 dx \\
& + \left(\frac{C}{t-s} + 3\sigma + C \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} \right) \int_{B_R} (1 + |Dv|^p) dx \\
& + \frac{C}{(t-s)^p} \left(\int_{B_R} |Da|^n dx \right)^{\frac{p}{n}} \left(\int_{B_R} |Dv|^p dx \right) \\
& + C_\sigma \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\
& + C_\sigma \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}.
\end{aligned}$$

By the absolute continuity of the integral there exists R_0 such that

$$\left(\int_{B_{R_0}} |Da|^n dx \right)^{\frac{2}{n}} < \frac{1}{4\tilde{C}},$$

hence, choosing $R < R_0$, we have

$$\tilde{C} \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} + 3\sigma \leq \frac{1}{4} + 3\sigma, \tag{4.11}$$

and so, choosing $\sigma = \frac{1}{12}$, we have that

$$\tilde{C} \left(\int_{B_R} |Da|^n dx \right)^{\frac{2}{n}} + 3\sigma \leq \frac{1}{2} \tag{4.12}$$

Moreover, choosing t and s such that

$$\frac{1}{t-s} \cong \frac{1}{\rho_2 - \rho_1},$$

estimate (4.10) can be written as follows

$$\begin{aligned} \frac{m}{2} \int_{B_{\rho_1}} |D(H_{p/2}(Dv))|^2 dx &\leq \frac{1}{2} \int_{B_{\rho_2}} |D(H_{p/2}(Dv))|^2 dx \\ &+ \left(\frac{C}{t-s} + \frac{1}{2} \right) \int_{B_R} (1 + |Dv|^p) dx + \frac{C}{(t-s)^p} \left(\int_{B_R} |Dv|^p \right) \\ &+ C_\sigma \left(\int_{B_R} |w|^p dx + \int_{B_R} |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\ &+ C_\sigma \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}. \end{aligned}$$

At this point we use the iteration Lemma 2.6 with $Z(\rho) = \int_{B_\rho} |D(H_{p/2}(Du))|^2 dx$, that yields

$$\begin{aligned} \int_{B_{R/2}} |D(H_{p/2}(Dv))|^2 dx &\leq C \int_{B_R} (1 + |Dv|^p) dx + C \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\ &+ C \left(\int_{B_R} |w|^p + |Dw|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \end{aligned} \quad (4.13)$$

i.e. the conclusion. $\square$

In particular, as a corollary of Theorem 4.1, we have an a priori estimate for the solution u of (1.1). Indeed, the same proof works in particular if $v = w = u$ and this yields

Corollary 4.2. *Let $u \in W_{loc}^{1,p}(\Omega)$ be a solution of (1.1), under assumptions (1.2), (1.3), (1.4), (1.5) and (1.6). If $H_{p/2}(Du) \in W_{loc}^{1,2}(\Omega)$ then there exist $R_0 = R_0(\|Da\|_{L^n}, n, p)$ such that the following estimate*

$$\begin{aligned} \int_{B_{R/2}} |D(H_{p/2}(Du))|^2 dx &\leq C \int_{B_R} (1 + |Du|^p) dx \\ &+ C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\ &+ C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q(n-p)+p}{n(p-1)}} \left(\int_{B_R} |h(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}, \end{aligned}$$

holds for every $B_R \subset B_{R_0} \Subset \Omega$ and a constant $C = C(n, p, q, s, \gamma, m, M, R)$.

5 Proof of Theorem 1.1

The aim of this section is to conclude the proof of Theorem 1.1 by using a suitable approximation argument. More precisely, let $u \in W_{loc}^{1,p}(\Omega)$ be a local solution of (1.1), fix a ball

$B_R \Subset \Omega$ and assume without loss of generality $R \leq 1$. For $\varepsilon > 0$, let $u_\varepsilon \in W^{1,p}(B_R)$ be the unique solution of the following problem

$$\begin{cases} \operatorname{div}\left(a_\varepsilon(x)H_{p-1}(Du_\varepsilon) + \varepsilon|Du_\varepsilon(x)|^{p-2}Du_\varepsilon\right) = b_\varepsilon(x, u) & \text{in } B_R \\ u_\varepsilon = u & \text{on } \partial B_R \end{cases} \quad (5.1)$$

where

- $a_\varepsilon(x) := a(x) * \rho_\varepsilon$,
- $b_\varepsilon := b * \rho_\varepsilon$, i.e. $b_\varepsilon(x, s) = \int_{B_1} b(x + \varepsilon w, s) \cdot \rho(w)dw$

with $\{\rho_\varepsilon\}_{\varepsilon>0}$ a family of standard compactly supported C^∞ mollifiers.

Observe that the function $b_\varepsilon(x, u)$ satisfies assumption (i) in (1.4), indeed

$$\begin{aligned} |b_\varepsilon(x, s) - b_\varepsilon(y, s)| &= \left| \int_{B_1} b(x + \varepsilon w, s) - b(y + \varepsilon w, s)dw \right| \\ &\leq \int_{B_1} |b(x + \varepsilon w, s) - b(y + \varepsilon w, s)|dw \\ &\leq |x - y||s|^q \left[\int_{B_1} k(x + \varepsilon w)dw + \int_{B_1} k(y + \varepsilon w)dw \right] \\ &= |x - y|(k_\varepsilon(x) + k_\varepsilon(y))|s|^q, \end{aligned} \quad (5.2)$$

where k_ε is the mollifiers of k . Hence, the function $b_\varepsilon(x, u)$ satisfies also (3.3), in fact

$$|b_\varepsilon(x, s) - b_\varepsilon(y, s)| \leq L_{2,\varepsilon}|x - y||s|^q \quad (5.3)$$

where $L_{2,\varepsilon} = 2\|k_\varepsilon\|_{L^\infty(B_R)}$. The same observations can be applied to get assumption (ii) in (1.4), that is

$$\begin{aligned} |b_\varepsilon(x, s) - b_\varepsilon(x, t)| &= \left| \int_{B_1} b(x + \varepsilon w, s) - b(x + \varepsilon w, t)dw \right| \\ &\leq \int_{B_1} |b(x + \varepsilon w, s) - b(x + \varepsilon w, t)|dw \\ &\leq |s - t|(|s|^{q-1} + |t|^{q-1}) \int_{B_1} h(x + \varepsilon w)dw \\ &= |s - t|(|s|^{q-1} + |t|^{q-1})h_\varepsilon(x), \end{aligned} \quad (5.4)$$

where h_ε is the mollifiers of h . Hence, the function $b_\varepsilon(x, u)$ satisfies also (3.4), in fact

$$|b_\varepsilon(x, s) - b_\varepsilon(x, t)| \leq L_{3,\varepsilon}|s - t|(|s|^{q-1} + |t|^{q-1}), \quad (5.5)$$

where $L_{3,\varepsilon} = \|h_\varepsilon\|_{L^\infty(B_R)}$. Since a_ε is Lipschitz continuous and by virtue of (5.3) and (5.5), we can use Theorem 3.1 to derive that $H_{p/2}(Du_\varepsilon) \in W_{loc}^{1,2}(B_R)$. Hence, by virtue of (5.2) and (5.4), we can use Theorem 4.1 with u_ε in place of v and u in place of w to deduce that

$$\int_{B_r} |D(H_{p/2}(Du_\varepsilon))|^2 dx \leq C \int_{B_\rho} (1 + |Du_\varepsilon|^p) dx + C \left(\int_{B_\rho} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_\rho} |k_\varepsilon(x)|^s dx \right)^{\frac{p}{s(p-1)}}$$

$$+ C \left(\int_{B_\rho} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_\rho} |h_\varepsilon(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}}, \quad (5.6)$$

for every $B_\rho \Subset B_R$, $r < \rho$, with a constant C independent of ε . Note that, due to properties of mollifiers, we have

$$0 < m < a_\varepsilon(x) < M \quad \text{a.e. } x \in B_R, \quad (5.7)$$

where m and M are defined in (1.2), and also

$$\|k_\varepsilon(x)\|_{L^s(B_R)} \leq C \|k(x)\|_{L^s(B_R)}, \quad (5.8)$$

$$\|h_\varepsilon(x)\|_{L^\gamma(B_R)} \leq C \|h(x)\|_{L^\gamma(B_R)}. \quad (5.9)$$

Consequently the norm of the functions h_ε and k_ε can be bounded independently of ε . Since u_ε is a weak solution of (5.1), then

$$\int_{B_R} \langle a_\varepsilon(x) H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2} Du_\varepsilon, D\varphi \rangle dx = \int_{B_R} b_\varepsilon(x, u) \varphi dx \quad \text{for all } \varphi \in W_0^{1,p}(B_R). \quad (5.10)$$

Choosing $\varphi = u_\varepsilon - u$ in (5.10), we obtain

$$\begin{aligned} 0 &= \int_{B_R} \langle a_\varepsilon(x) H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2} Du_\varepsilon, Du_\varepsilon - Du \rangle dx - \int_{B_R} b_\varepsilon(x, u) (u_\varepsilon - u) dx \\ &= \int_{B_R} \langle a_\varepsilon(x) H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2} Du_\varepsilon, Du_\varepsilon \rangle dx - \int_{B_R} \langle a_\varepsilon(x) H_{p-1}(Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{p-2} Du_\varepsilon, Du \rangle dx \\ &\quad - \int_{B_R} b_\varepsilon(x, u) (u_\varepsilon - u) dx \\ &=: A - B - C, \end{aligned}$$

that yields

$$A \leq |B| + |C|. \quad (5.11)$$

By (5.7) and Lemma 2.4, we deduce that

$$A \geq m \int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + \varepsilon \int_{B_R} |Du_\varepsilon|^p dx. \quad (5.12)$$

In order to estimate $|B|$, we use again (5.7), Lemma 2.4 and Young's inequality as follows

$$\begin{aligned} B &\leq M \int_{B_R} |H_{p-1}(Du_\varepsilon)| |Du| dx + \varepsilon \int_{B_R} |Du_\varepsilon|^{p-1} |Du| dx \\ &\leq \sigma \int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + C_\sigma \int_{B_R} (1 + |Du|^p) dx + \frac{\varepsilon}{2} \int_{B_R} |Du_\varepsilon|^p dx, \end{aligned} \quad (5.13)$$

where $\sigma > 0$ will be chosen later. Now we estimate $|C|$ using (1.8), Hölder's inequality with exponents $\frac{1}{p^*}$, $\frac{q}{p^*}$ and $\frac{1}{\gamma}$, that is legitimate by virtue of assumption (1.3) and (1.6), Sobolev-Poincaré theorem and Sobolev embedding theorem, as follows

$$|C| \leq \int_{B_R} |u_\varepsilon - u| |b_\varepsilon(x, u)| dx \leq \int_{B_R} |u_\varepsilon - u| h_\varepsilon(x) |u|^q dx$$

$$\begin{aligned}
&\leq \left(\int_{B_R} |u_\varepsilon - u|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{1}{\gamma}} \left(\int_{B_R} |u|^{p^*} dx \right)^{\frac{q}{p^*}} \\
&\leq \left(\int_{B_R} |Du_\varepsilon - Du|^p dx \right)^{\frac{1}{p}} \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{1}{\gamma}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p}} \\
&\leq \sigma \left(\int_{B_R} |Du_\varepsilon|^p + |Du|^p dx \right) + C_\sigma \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p-1}} \\
&\leq \sigma \int_{B_R} (|Du_\varepsilon| - 1)_+^p dx + \sigma \int_{B_R} (1 + |Du|^p) dx \\
&\quad + C_\sigma \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p-1}}, \tag{5.14}
\end{aligned}$$

where we used Young's inequality in the fourth line. Inserting (5.12), (5.13) and (5.14) in (5.11) we obtain

$$\begin{aligned}
&m \int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + \varepsilon \int_{B_R} |Du_\varepsilon|^p dx \leq 2\sigma \int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + C_\sigma \int_{B_R} (1 + |Du|^p) dx \\
&\quad + C_\sigma \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p-1}} + \frac{\varepsilon}{2} \int_{B_R} |Du_\varepsilon|^p dx, \tag{5.15}
\end{aligned}$$

where we used that $|H_{p/2}(Du_\varepsilon)|^2 = (|Du_\varepsilon| - 1)_+^p$. Choosing $\sigma = \frac{m}{4}$ we can reabsorb the first integral in the right hand side by the left hand side, and reabsorbing also the last integral in (5.15), we get

$$\begin{aligned}
&\int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + \frac{\varepsilon}{2} \int_{B_R} |Du_\varepsilon|^p dx \leq C \int_{B_R} (1 + |Du|^p) dx \\
&\quad + C \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p-1}}, \tag{5.16}
\end{aligned}$$

and by virtue of (5.9), inequality (5.16) implies that

$$\begin{aligned}
&\int_{B_R} |Du_\varepsilon|^p dx \leq \int_{B_R} (|Du_\varepsilon| - 1)_+^p dx + |B_R| = \int_{B_R} |H_{p/2}(Du_\varepsilon)|^2 dx + |B_R| \\
&\leq C \int_{B_R} (1 + |Du|^p) dx + |B_R| \\
&\quad + C \left(\int_{B_R} h_\varepsilon(x)^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \left(\int_{B_R} |u|^p dx + \int_{B_R} |Du|^p dx \right)^{\frac{q}{p-1}}, \tag{5.17}
\end{aligned}$$

with C independent of ε . Using (5.16) in the right hand side of (5.6), we get

$$\begin{aligned}
&\int_{B_\rho} |DH_{p/2}(Du_\varepsilon)|^2 dx \leq C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |k_\varepsilon(x)|^s dx \right)^{\frac{p}{s(p-1)}} \\
&\quad + C \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \left(\int_{B_R} |h_\varepsilon(x)|^\gamma dx \right)^{\frac{p}{\gamma(p-1)}} \\
&\leq C \|k(x)\|_{L^s}^{\frac{p}{p-1}} \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}}
\end{aligned}$$

$$+ C \|h(x)\|_{L^\gamma}^{\frac{p}{p-1}} \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \quad (5.18)$$

where we used (5.8) and (5.9). By (5.17) and (5.18) we have

$$u_\varepsilon \rightharpoonup v \text{ in } W^{1,p} \quad (5.19)$$

$$H_{p/2}(Du_\varepsilon) \rightharpoonup w \text{ in } W_{loc}^{1,2}(B_R) \quad (5.20)$$

$$H_{p/2}(Du_\varepsilon) \longrightarrow w \text{ in } L_{loc}^2(B_R), \quad (5.21)$$

and therefore also a.e. up to a subsequence. By the continuity of the function $H_{p/2}(\xi)$ and the uniqueness of the weak limit, it holds

$$w = H_{p/2}(Dv).$$

Hence, passing to the limit as $\varepsilon \rightarrow 0$ in (5.18) and applying Fatou Lemma, we get

$$\begin{aligned} \int_{B_\rho} |DH_{p/2}(Dv)|^2 dx &\leq C \|k(x)\|_{L^s}^{\frac{p}{p-1}} \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}} \\ &\quad + C \|h(x)\|_{L^\gamma}^{\frac{p}{p-1}} \left(\int_{B_R} |u|^p + |Du|^p dx \right)^{\frac{q}{p-1}}. \end{aligned} \quad (5.22)$$

Our next aim is to prove that the function $v \in W^{1,p}(\Omega)$ is a solution of (3.1) with u in place of w , i.e. in weak formulation, that the following identity

$$\int_{B_R} \left(a(x)(|Dv| - 1)_+^{p-1} \frac{Dv}{|Dv|} \right) D\varphi \, dx = \int_{B_R} b(x, u) \varphi \, dx, \quad (5.23)$$

holds $\forall \varphi \in W_0^{1,p}(B_R)$.

In order to prove (5.23), we observe that

$$\begin{aligned} \int_{B_R} \langle a(x)H_{p-1}(Dv), D\varphi \rangle dx &= \int_{B_R} a(x) \langle H_{p-1}(Dv), D\varphi \rangle dx \\ &+ \int_{B_R} a(x) \langle H_{p-1}(Du_\varepsilon), D\varphi \rangle dx - \int_{B_R} a(x) \langle H_{p-1}(Du_\varepsilon), D\varphi \rangle dx \\ &= \int_{B_R} a(x) \langle H_{p-1}(Dv), D\varphi \rangle dx + \int_{B_R} b_\varepsilon(x, u) \varphi \, dx \\ &- \int_{B_R} a(x) \langle H_{p-1}(Du_\varepsilon), D\varphi \rangle dx \\ &= \int_{B_R} a(x) \langle H_{p-1}(Dv) - H_{p-1}(Du_\varepsilon), D\varphi \rangle dx \\ &+ \int_{B_R} \left(b_\varepsilon(x, u) - b(x, u) \right) \varphi \, dx + \int_{B_R} b(x, u) \varphi \, dx \\ &=: J_\varepsilon + JJ_\varepsilon + \int_{B_R} b(x, u) \varphi \, dx, \end{aligned} \quad (5.24)$$

where, in the second identity we used that u_ε solves problem (5.1). Our aim is to prove that J_ε and JJ_ε go to 0 as ε goes to 0. Indeed, we have

$$\begin{aligned}
|J_\varepsilon| &\leq M \|D\varphi\|_{L^p} \left(\int_{\text{supp } D\varphi} |H_{p-1}(Dv) - H_{p-1}(Du_\varepsilon)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \\
&\leq C \|D\varphi\|_{L^p} \left(\int_{\text{supp } D\varphi} |H_{p/2}(Dv) - H_{p/2}(Du_\varepsilon)|^{\frac{p}{p-1}} \left((|Dv| - 1)_+ + (|Du_\varepsilon| - 1)_+ \right)^{\frac{p}{p-1} \frac{p-2}{2}} dx \right)^{\frac{p-1}{p}} \\
&\leq C \|D\varphi\|_{L^p} \left(\int_{\text{supp } D\varphi} |H_{p/2}(Dv) - H_{p/2}(Du_\varepsilon)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\text{supp } D\varphi} (|Dv| - 1)_+ + (|Du_\varepsilon| - 1)_+ dx \right)^p \right)^{\frac{p-2}{2p}} \\
&\leq C \|D\varphi\|_{L^p} \left(\int_{\text{supp } D\varphi} |H_{p/2}(Dv) - H_{p/2}(Du_\varepsilon)|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_R} |Du|^p + |Du_\varepsilon|^p dx \right)^{\frac{p-2}{2p}}, \tag{5.25}
\end{aligned}$$

where we used assumption (i) in (1.2), Lemma 2.5 with $\epsilon = p - 1$ and $\alpha = p/2$ and Hölder's inequality twice. Note that since $H_{p/2}(Du_\varepsilon) \rightarrow H_{p/2}(Dv)$ strongly in $L^2_{loc}(B_R)$ and $\int_{B_R} |Du_\varepsilon|^p dx$ is bounded by (5.17), we conclude that $J_\varepsilon \rightarrow 0$.

To estimate JJ_ε , we use that $|b(x, u)| \leq h(x)|u|^q$ as follows

$$\begin{aligned}
\int_{B_R} |b(x, u)| dx &\leq \int_{B_R} h(x) |u|^q dx \\
&\leq \left(\int_{B_R} |u|^{p^*} dx \right)^{\frac{q}{p^*}} \left(\int_{B_R} h(x)^{\frac{p^*}{p^*-q}} dx \right)^{\frac{p^*-q}{p^*}} \\
&\leq C \left(\int_{B_R} |u|^{p^*} dx \right)^{\frac{q}{p^*}} \left(\int_{B_R} h(x)^\gamma dx \right)^{\frac{1}{\gamma}}, \tag{5.26}
\end{aligned}$$

where we used Hölder's inequality and that $\frac{p^*}{p^*-q} < \gamma$. By (5.26) we can conclude that $JJ_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ because $b_\varepsilon(x, u) \rightarrow b(x, u)$ strongly in $L^1(B_R)$. To conclude the proof, we have to prove that $H_{p/2}(Du) = H_{p/2}(Dv)$. Since u and v solve (1.1) and (3.1) respectively, we have

$$\int_{B_R} a(x) \langle H_{p-1}(Du) - H_{p-1}(Dv), D\varphi \rangle dx = 0$$

and, choosing as $\varphi = u - v$, we get

$$0 \leq m \int_{B_R} |H_{p/2}(Du) - H_{p/2}(Dv)|^2 dx \leq \int_{B_R} a(x) \langle H_{p-1}(Du) - H_{p-1}(Dv), Du - Dv \rangle dx = 0, \tag{5.27}$$

where we used the ellipticity of $H_{p/2}(\xi)$ given by Lemma 2.5 and assumption (1.2). Hence (5.27) proves that $H_{p/2}(Du) = H_{p/2}(Dv)$ and consequently, using this information in (5.22), we obtain the desired estimate for the function u .

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