



New variations on the theme of Baer's theorem

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Abstract

Let $\gamma_s(G)$ and $Z_s(G)$ denote the s -th terms of the lower and upper central series of a group G , respectively. A classical theorem by R. Baer states that if $Z_s(G)$ has finite index in G , then $\gamma_{s+1}(G)$ is also finite. In this paper, we prove that if G is a generalized soluble group such that $\gamma_s(G)/(\gamma_s(G) \cap Z_t(G))$ has finite rank r for some s, t , then the rank of $\gamma_{s+t}(G)$ is finite and (r, s, t) -bounded. Moreover, a corresponding result replacing the finite-rank assumption by the condition that $\gamma_s(G)/(\gamma_s(G) \cap Z_t(G))$ is a Chernikov group of bounded size is also obtained. These results extend recent generalizations of the classical Baer's theorem.

Keywords Chernikov group · Rank · Upper and lower central series

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1 Introduction

Let G be a group, and let $\gamma_s(G)$ and $Z_s(G)$ denote the s -th terms of the lower and upper central series of G , respectively. A classical theorem by R. Baer states that if $Z_s(G)$ has finite index n in G , then $\gamma_{s+1}(G)$ is also finite. Moreover, the order of $\gamma_{s+1}(G)$ can be bounded in terms of n and s (see the proof of 14.5.1 in [11]). Throughout the paper, we use the expression “ $(a, b, c, \dots)$ -bounded” to mean that a quantity is finite and bounded by a certain number depending only on the parameters $a, b, c, \dots$. A stronger form of Baer's theorem was obtained by G. Fernández-Alcober and M. Martina Capasso, Liliana Lancellotti, and Pavel Shumyatsky have contributed equally to this work.

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Morigi, who proved that if $\gamma_s(G)/(\gamma_s(G) \cap Z_t(G))$ has finite order n , then $\gamma_{s+t}(G)$ is finite and its order is bounded by a function of n , s and t (see [[3], Theorem B], and the last part of [3] for the quantitative version).

Recall that a group G is said to have *finite (Prüfer or special) rank* if there is a positive integer r such that every finitely generated subgroup of G can be generated by at most r elements, and r is the least integer with this property. A group G is called *generalized radical* if it has an ascending series whose quotients are either locally nilpotent or locally finite. Accordingly, a group G is *locally generalized radical* if every finitely generated subgroup of G is generalized radical. A rank analogue of Baer's result states that if G is a locally generalized radical group and $G/Z_s(G)$ has finite rank r , then $\gamma_{s+1}(G)$ has finite and (r, s) -bounded rank. This result was obtained by N. Yu. Makarenko in [9] when G is finite, and later extended by L. A. Kurdachenko and J. Otal in [7] to the class of locally generalized radical groups. The case $s = 1$ had already been proved in [8].

Along similar lines, A. R. Jamali and S. Zandi proved a rank analogue of the aforementioned result of Fernández-Alcober and Morigi [3], in the case where G is a finite p -group (see [[5], Corollary 3.7]). Our first main result extends this statement to the more general class of locally generalized radical groups.

Theorem 1 *Let G be a locally generalized radical group such that $\gamma_s(G)/(\gamma_s(G) \cap Z_t(G))$ has finite rank r for some s, t . Then the rank of $\gamma_{s+j}(G)/(\gamma_{s+j}(G) \cap Z_{t-j}(G))$ is finite and (j, r, s) -bounded for every j with $0 \leq j \leq t$. In particular, the rank of $\gamma_{s+t}(G)$ is finite and (r, s, t) -bounded.*

Obviously, this is an extension of the result [[7], Theorem A], which can be recovered from Theorem 1 as a particular case where $s = 1$.

A corresponding analogue of Baer's theorem is also known for Chernikov groups, namely that if $G/Z_s(G)$ is a Chernikov group, then $\gamma_{s+1}(G)$ is still a Chernikov group (see for instance [[10] Part 1, Theorem 4.21 and Corollary 2]).

Recall that a group G is *Chernikov* if it contains a radicable abelian normal subgroup R (the finite residual) which is the direct product of a finite number $m(G)$ of groups of Prüfer type, such that the factor group G/R is finite, of order $n(G)$ say. In general a group G is called *radicable* if the equation $x^n = a$ has a solution in G for every positive integer n and every $a \in G$. By a deep result obtained independently by Shunkov [12] and Kegel and Wehrfritz [6], Chernikov groups are precisely the locally finite groups satisfying the minimal condition on subgroups, that is, any non-empty set of subgroups possesses a minimal subgroup. Furthermore, if G is a Chernikov group with $m = m(G)$ and $n = n(G)$, the pair (m, n) is usually called the *size* of G . Notice also that both the rank of G and the number of primes involved in G are bounded by $m(G) + n(G)$. Of course, a Chernikov group G is finite if and only if $m(G) = 0$, and it is a radicable abelian group if and only if $n(G) = 1$. It is well-known that the class of Chernikov groups is closed with respect to subgroups, homomorphic images and extensions. In particular, if G is a Chernikov group of size (m, n) , then each quotient of G is a Chernikov group of size at most (m, n) .

Finally, we establish the following theorem.

Theorem 2 *Let m, n be non-negative integers, with $n \neq 0$. Let G be a group such that $\gamma_s(G)/(\gamma_s(G) \cap Z_t(G))$ is a Chernikov group of size (m, n) for some s, t . Then $\gamma_{s+j}(G)/(\gamma_{s+j}(G) \cap Z_{t-j}(G))$ is a Chernikov group of size bounded in terms of j, m, n and s for every j with $0 \leq j \leq t$. In particular, $\gamma_{s+t}(G)$ is a Chernikov group of size bounded in terms of m, n, s and t .*

2 Proofs

In the present paper we make no attempts to write down explicit bounds for the rank and for the Chernikov size of $\gamma_{s+t}(G)$ in Theorem 1 and Theorem 2, respectively. Furthermore, we use without explicit references the facts that if $r(G) = r$, then every subgroup and every quotient of G has rank at most r and that if G has a normal subgroup N such that $r(N) = r_1$ and $r(G/N) = r_2$, then $r(G) \leq r_1 + r_2$.

The following two results of [8] are essential to our proof of Theorem 1.

Theorem 3 ([8], Theorem 1.1) *Let G be a finite group such that $G/Z(G)$ has rank r . Then the rank of G' is r -bounded.*

Theorem 4 ([8], Theorem 1.2) *Let G be a locally generalized radical group such that $G/Z(G)$ has finite rank r . Then the rank of G' is finite and r -bounded.*

The following results will be useful later on. Recall that if G is a group, then $\gamma_\infty(G)$ denotes the last term of the lower central series of G .

Lemma 5 ([1], Lemma 2.4) *Let G be a finite metanilpotent group, and let p be a prime. If P is a Sylow p -subgroup of $\gamma_\infty(G)$ and H is a Hall p' -subgroup of G , then $P = [P, H]$.*

Before starting the proofs, we recall a simple fact that will be used several times. If G is a group, N a normal subgroup of G and $\overline{G} = G/N$, then $\gamma_s(\overline{G})/(\gamma_s(\overline{G}) \cap Z(\overline{G}))$ is a homomorphic image of the section $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$. Consequently, the property that this section has finite rank r is inherited by quotients. The same holds when such section is a Chernikov group of bounded size.

Therefore, as an immediate consequence of [[5], Proposition 3.6], we obtain the following corollary.

Corollary 6 *Let G be a finite nilpotent group and let $s \geq 1$ be an integer such that $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ has rank r . Then $\gamma_{s+1}(G)$ has (r, s) -bounded rank.*

The next lemma shows that the hypothesis that G is nilpotent in Corollary 6 is superfluous.

Lemma 7 *Let G be a finite group and let $s \geq 1$ be an integer such that $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ has rank r . Then $\gamma_{s+1}(G)$ has (r, s) -bounded rank.*

Proof Since $\gamma_s(G)Z(G)/Z(G)$ has rank r , Theorem 3 yields that $(\gamma_s(G)Z(G))' = \gamma_s(G)'$ has r -bounded rank. Thus, passing to the quotient $G/\gamma_s(G)'$, we may assume that $\gamma_s(G)$ is abelian. In particular, G is metanilpotent.

Let P be a Sylow p -subgroup of $\gamma_\infty(G)$, where p is a prime, and let H be a Hall p' -subgroup of G . By Lemma 5, we have $P = [P, H]$. Moreover, since P is normal in G , it follows that $P = C_P(H) \times [P, H]$ (see [[4], Theorem 2.3]), hence $C_P(H)$ is trivial. On the other hand, $P \cap Z(G) \leq C_P(H)$, and therefore P embeds into $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$. Consequently, P has rank at most r . Since this holds for every Sylow subgroup of $\gamma_\infty(G)$, we deduce that $\gamma_\infty(G)$ itself has rank at most r .

Finally, applying Corollary 6 to the nilpotent quotient $G/\gamma_\infty(G)$, we conclude that $\gamma_{s+1}(G)/\gamma_\infty(G)$ has (r, s) -bounded rank. Hence, $\gamma_{s+1}(G)$ has (r, s) -bounded rank, as required. $\square$

The key step in the proof of Theorem 1 is in the following proposition. It is a rank analogue of [[3], Proposition 2.3]. Note that quotients of a generalized radical group are generalized radical as well. Consequently, the class of locally generalized radical group is also closed under taking quotient.

Proposition 8 *Let G be a locally generalized radical group and let $s \geq 1$ be an integer such that $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ has finite rank r . Then the rank of $\gamma_{s+1}(G)$ is finite and (r, s) -bounded.*

Proof Since the group $\gamma_s(G)Z(G)/Z(G)$ has rank r , Theorem 4 implies that $(\gamma_s(G)Z(G))' = \gamma_s(G)'$ has r -bounded rank. Therefore passing to the quotient $G/\gamma_s(G)'$ we may assume that $\gamma_s(G)$ is abelian.

Suppose first that G is finitely generated. In this case G is residually finite (see [[10] Part 2, Theorem 9.51]). Observe that

$$\gamma_{s+1}(G)/(\gamma_{s+1}(G) \cap Z(G))$$

naturally embeds into $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$. Therefore the quotient $\gamma_{s+1}(G)/(\gamma_{s+1}(G) \cap Z(G))$ has finite rank at most r . So it is sufficient to show that $\gamma_{s+1}(G) \cap Z(G)$ has finite (r, s) -bounded rank. Let E be a finitely generated subgroup of $\gamma_{s+1}(G) \cap Z(G)$. Since E is abelian,

$$r(E) = \max_{p \in \mathbb{P}} r_p(E).$$

Fix a prime number p . Note that E^p is normal (even central) in G and put $\overline{G} = G/E^p$. We have that $\overline{E} = E/E^p$ is a finite p -elementary abelian subgroup of the residually finite group $\overline{G}$. Therefore there exists a normal subgroup $\overline{N}$ of $\overline{G}$ such that $\overline{G}/\overline{N}$ is finite and $\overline{E} \cap \overline{N} = \{1\}$. It follows that $\overline{E}$ is isomorphic to a subgroup of $\gamma_{s+1}(\overline{G}/\overline{N})$.

Applying Lemma 7 to the finite group $\overline{G}/\overline{N}$, we obtain that the rank of $\gamma_{s+1}(\overline{G}/\overline{N})$ is finite and (r, s) -bounded, and thus also E/E^p has finite (r, s) -bounded rank. Since this holds for every prime p , it follows that $\gamma_{s+1}(G) \cap Z(G)$ has finite (r, s) -bounded rank and this proves the claim when G is finitely generated.

In other words, there exists an (r, s) -bounded number, say R_0 , such that $r(\gamma_{s+1}(K)) \leq R_0$ whenever a group K satisfies the hypothesis of the theorem and is finitely generated. Suppose now that our group G is not necessarily finitely generated. If $r(\gamma_{s+1}(G)) \geq R_0 + 1$, we can choose $y_1, \dots, y_{R_0+1} \in \gamma_{s+1}(G)$ such that the subgroup $\langle y_1, \dots, y_{R_0+1} \rangle$ cannot be generated by R_0 elements. We can also choose a

finitely generated subgroup K in G such that $y_1, \dots, y_{R_0+1} \in \gamma_{s+1}(K)$. This yields a contradiction since we know that $r(\gamma_{s+1}(K)) \leq R_0$. The proof is now complete. $\square$

We are now ready to prove Theorem 1.

Proof of Theorem 1 We proceed by induction on t . Clearly, the statement holds if $t = 0$. Assume that $t > 0$, and consider the group $\bar{G} = G/Z_{t-1}(G)$. Since $\bar{G}$ is still a locally generalized radical group, applying Proposition 8 to $\bar{G}$, it follows that $\gamma_{s+1}(\bar{G})$ has finite and (r, s) -bounded rank. Now, by inductive hypothesis, we obtain that $\gamma_{s+j}(G)/(\gamma_{s+j}(G) \cap Z_{t-j}(G))$ has (j, r, s) -bounded rank, for every j with $0 \leq j \leq t$. $\square$

The following proposition provides an analogue of Proposition 8 for the class of Chernikov groups of bounded size.

Proposition 9 *Let m, n, s be non-negative integers, with $n, s \neq 0$. Let G be a group such that $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ is a Chernikov group of size (m, n) . Then $\gamma_{s+1}(G)$ is a Chernikov group of size bounded in terms of m, n and s .*

Proof Since $\gamma_s(G)Z(G)/Z(G)$ is a Chernikov group of size (m, n) , it follows from [[2], Lemma 4] that $(\gamma_s(G)Z(G))' = \gamma_s(G)'$ is a Chernikov group of size bounded in terms of m and n . Thus, we may replace G by $G/\gamma_s(G)'$ and assume that $\gamma_s(G)$ is abelian.

Let $a \in \gamma_s(G)$. There exists a positive integer k such that $[a^k, g] = 1$ for every $g \in G$. As $\gamma_s(G)$ is abelian, we have $[a, g]^k = [a^k, g] = 1$ and therefore $\gamma_{s+1}(G)$ is periodic.

Assume first that $\gamma_{s+1}(G)$ is a p -group, for some prime p . The group G obviously satisfies the assumptions of Proposition 8 with $r = m + n$, hence it follows that the rank of $\gamma_{s+1}(G)$ is finite and bounded by a function $u = u(m, n, s)$. It follows that the radicable part R of $\gamma_{s+1}(G)$ is a direct product of at most u Prüfer subgroups. In particular, R is also normal in G . Let $D/(\gamma_s(G) \cap Z(G))$ be the radicable part of $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$; then $\gamma_s(G)/D$ is finite of order n . For convenience, put $Z/R = \gamma_s(G/R) \cap Z(G/R)$ and let $g \in G$. Since $\gamma_s(G)$ is abelian, the map

$$x(\gamma_s(G) \cap Z(G)) \in D/(\gamma_s(G) \cap Z(G)) \rightarrow [x, g] \in [D, g]$$

induces an epimorphism from $D/(\gamma_s(G) \cap Z(G))$ to $[D, g]$. Thus $[D, g]$ is a radicable abelian subgroup of $\gamma_{s+1}(G)$, consequently $[D, g] \leq R$. Therefore $D \leq Z$ and so

$$\frac{\gamma_s(G/R)}{\gamma_s(G/R) \cap Z(G/R)} \simeq \gamma_s(G)/Z$$

is finite of order at most n . Applying [[3], Proposition 2.3] to the quotient G/R , we conclude that $\gamma_{s+1}(G)/R$ has finite order bounded by a function $v = v(n, s)$, and so $\gamma_{s+1}(G)$ is a Chernikov group of size at most (u, v) .

Suppose now that $\gamma_{s+1}(G)$ is not necessarily a p -group. Since the quotient $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ is a Chernikov group of size (m, n) , the set of primes involved in this quotient has cardinality at most $m + n$. Furthermore, since $\gamma_s(G)$

is abelian, for every $g \in G$, the subgroup $[\gamma_s(G), g]$ is a homomorphic image of $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ through the map

$$a \in \gamma_s(G) \rightarrow [a, g] \in [\gamma_s(G), g].$$

Hence every prime occurring in the torsion of $[\gamma_s(G), g]$ also occurs in the quotient $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$. Moreover, since $\gamma_{s+1}(G)$ is abelian, it is generated by the subgroups of the form $[\gamma_s(G), g]$. Hence, also the primes involved in $\gamma_{s+1}(G)$ are contained in the set of the primes involved in $\gamma_s(G)/(\gamma_s(G) \cap Z(G))$ and their number is at most $m + n$. For each prime p , let $O_{p'}(\gamma_{s+1}(G))$ be the subgroup generated by all Sylow q -subgroups of $\gamma_{s+1}(G)$, with $q \neq p$. Therefore we apply the previous case to the quotient $G/O_{p'}(\gamma_{s+1}(G))$ and thus we obtain that each Sylow p -subgroup of $\gamma_{s+1}(G)$ is a Chernikov group of size at most (u, v) , and so $\gamma_{s+1}(G)$ itself is a Chernikov group of size at most $(u(m + n), v^{m+n})$. $\square$

The proof of Theorem 2 follows the same lines as that of Theorem 1, with Proposition 9 taking the place of Proposition 8, and is therefore omitted.

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Declarations

Conflict of interest The authors have no conflict of interest to declare that are relevant to the content of this article.

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