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Results in Mathematics

How to Define Sine and Cosine as Functions over Reals Rigorously and with Minimal Prerequisites

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Abstract. Sine and cosine as real functions on the real axis can be defined in several ways. However, the standard way used in undergraduate courses in Calculus is the unit circle definition: shortly, for a given real number t, the $x-$ and $y-$coordinates of the point P_t of the unit circle at the *relative* arc length t from the point $(1, 0)$, are called $\cos t$ and $\sin t$, respectively. The heart of the matter is that the notion of arc length is either postponed after the exposition of integral calculus, or, when it is given through the notion of polygonal path, such notion seems never used to prove the existence of the point P_t of the unit circle. In this paper we show, through a new proof of the existence of P_t, how the definition of sine and cosine can be formalized using only a minimal knowledge of classical Euclidean Geometry and properties of real numbers, avoiding to use the notions of area, limit, derivative, series, integral and complex number.

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1. Motivation

In principle, in any mathematical treatment, for the sake of consistency, all assertions should involve terms whose meaning should be clear to readers and therefore should have been rigorously defined before. However, frequently it happens that definitions are missing. Sometimes this is due to the fact that

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they are standard, no confusion can arise, and full details would make the expositions much longer: for instance, one cannot expect to read the whole Dedekind's construction of the system of real numbers before the theory of Calculus 1, and authors may prefer to introduce real numbers through a system of axioms, or through an intuitive geometric representation. In other cases expositions do not require exactly the definitions, but only some of their consequences. For instance, treatises in PDEs, Calculus of Variations or Harmonic Analysis cannot contain notions of Calculus 2 and must take for granted at least the basics of Real and Functional Analysis. In the case of undergraduate texts about Calculus, since rigorous treatment cannot be done with a complete background, authors must use intuitive approaches for some basic notions.

In this paper we deal with the notions of sine and cosine. In Geometry, such notions are given for acute angles building a right triangle: the sine of an acute angle is the ratio of the length of the opposite side to the length of the hypotenuse, while the cosine is the ratio of the length of the adjacent side to the length of the hypotenuse.

In Calculus treatises, sine and cosine are considered as *functions* over the set of real numbers. The theory requires, for a given real number t, to provide a meaning to $\sin t$ and $\cos t$. The well known *unit circle definition*, which has been introduced by De Morgan (see e.g. the old 1894 booklet [15, p.4] by Macfarlane about the definition of trigonometric functions, which contains some historical details), is the most popular, and it is adopted in many many old and recent books: an obviously incomplete list is Adams [1, p.94], Apostol [2, p.102], Edwards-Penney [6, Appendix C p.A-15], Johnson-Kiokemeister-Wolk [11, p.301], Krantz [13, p.22], Peng [18, p.211], Silverman [23, n.17], Stein [25, p.42], Stewart [26, p.A26], Strang [28, p.29], Swokowski-Cole [29, p.342], Weir [30, p.50], see also the scheme after the index, at the end of the book Larson-Hostetler-Edwards [14]. It sounds as follows: for a given real number t, the $x-$ and $y-$coordinates of the point P_t of the unit circle (i.e. the circle centered in the origin having radius 1, whose equation is $x^2 + y^2 = 1$) at the relative arc length t from the point $(1, 0)$ (i.e. $|t|$ is the length of the arc of the circle obtained when the point $(1, 0)$ moves counterclockwise, $-|t|$ is the relative length when the point $(1, 0)$ moves in the clockwise direction), are called $\cos t$ and $\sin t$, respectively. The heart of the matter is that the notion of arc length is left to the intuition of the reader (namely, it is not rigorously defined), or it is postponed after the exposition of integral calculus, or, even if it is given through the notion of polygonal path, it is never used to prove the existence of the point P_t of the unit circle. Sometimes in literature (see Hunt [8, p.46]) one can find even sentences like *Since the radian measure of an angle* (in our terms: "since t is the relative measure of an arc $\widehat{AP_t}$") *is a real number, any real number may be considered to be the radian measure of an angle* (in our terms: "then for any real number t there exists a point P_t such that the relative measure of the arc $\widehat{AP_t}$ is t"). This way the invertibility seems automatic just because there exists the map from angles to real numbers.

Some authors try to overcome the problem changing the definition in various ways. Readers can find many of them in books and in the web, given as definitions or characterizations. Here there are some of them (in the sequel we will deal mostly with the sine function only, because the cosine function can be easily defined in terms of the sine). The *power series* definition

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} t^{2n+1} ,$$

which goes back to Newton (see Dörrie [4, p.60]), can be found e.g. in Apostol [2, p.436], Bartle-Sherbert [3, Examples 9.4.14 (a) p.286], Dörrie [4, p.62], Hewitt-Stromberg [7, p.50], Stewart-Tall [27, p.59]. The *Euler's formula* definition

$$\sin t = \frac{e^{it} - e^{-it}}{2i} ,$$

where e is the Napier number and i the imaginary unit in the complex number system, is e.g. in the widely known monograph by Walter Rudin [21, p.182]. We highlight that at the end of the section where the main properties of the sine and cosine are proved, Rudin relates this definition to the *usual* definition, referring to the unit circle definition. The *continued fractions* characterization

$$\sin t = \cfrac{t}{1 + \cfrac{t^2}{2 \cdot 3 - t^2 + \cfrac{2 \cdot 3t^2}{4 \cdot 5 - t^2 + \cfrac{4 \cdot 5t^2}{6 \cdot 7 - t^2 + \cdots}}}} ,$$

which could be, in principle, adopted as equivalent definition, has been proved in Khovanskii [12, p.170] (see also Olds [17, p.138]). The *differential equation* definition comes after the proof of the unicity of the initial value problem

$$\begin{cases} y'' + y = 0 \\ y(0) = 0 \\ y'(0) = 1 \end{cases}$$

see e.g. Bartle-Sherbert [3, Def. 8.4.5 p.263] or Peng [18, p.215]. The *infinite product* definition

$$\sin t = t \prod_{n=1}^{\infty} \left(1 - \frac{t^2}{\pi^2 n^2}\right)$$

is e.g. in the short paper by Jha [10]. One more way to introduce sine and cosine is in Apostol [2, Note p.254], where the treatment starts from the arcsin function defined through an integral.

However, all such definitions/characterizations require the knowledge of the notions of area, limit, derivative, series, integral or complex number.

If one looks for a definition requiring a minimal background of Real Analysis, which is worth especially at the beginning of a course in Calculus at undergraduate level, the choice for introducing the sine and cosine as functions over reals is restricted to the unit circle definition and *almost always* it forces authors to use an intuitive approach. However, since about 60 years some authors wanted to overcome the problem of an exposition without rigour. For instance, Apostol [2, p.95] states four properties which allow an axiomatically introduction of the trigonometric functions and their statements do not require any background (except the notions of function and real number). However, in order to show the existence of functions satisfying such properties, Apostol builds a geometric method, and unfortunately in [2, p.103], for a given $x \in]0, 2\pi[$, the existence of a point on the unit circle such that the area of a sector equals $x/2$ is again based on intuition (Apostol wrote that it is not used the notion of arc length, because it has been postponed). We note that a *rigorous* argument to show that the same properties determine uniquely the functions sine and cosine appears in Ilyin-Poznyak [9, p.140], but there the long proof uses the notions of limit, continuity and series. A result of the same type (i.e. properties which can be stated without knowledge of deep Calculus), with a much simpler proof, is in the short article [20] by Robison; however, the proof of the uniqueness requires once again continuity and the notion of limit. In an article published in 1966, hence still about 60 years ago, in [5] Eberlein highlights the problem to avoid an intuitive approach, but his solution makes use of complex valued functions and differentials. More recently the problem of a rigorous exposition is explained with a great detail in the Calculus book by Michael Spivak [24, p.288], where in the end trigonometric functions are introduced as follows: at first the number π is defined through

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}dx\,,$$

then, setting

$$A(x) = \frac{x\sqrt{1-x^2}}{2} + \int_{x}^{1} \sqrt{1-t^2}dt\,,$$

for $0 \le x \le \pi$ the number $\cos x$ is the unique solution in $[-1, 1]$ of the equation $A(\cos x) = x/2$, and finally $\sin x = \sqrt{1-\cos^2 x}$. This is a precious formalization (because e.g. in Larson-Hostetler-Edwards [14, p. 1050] there is the same method without formalization), however it requires integrals. The problem to use a minimal background is raised in a 1993 note by Richman [19], where in the final paragraph "recap" he asserts that defining arc length as the supremum of the lengths of polygonal approximations makes necessary a limiting process.

The goal of this paper is to show, through a new proof of the existence of P_t, how the definition of the sine and cosine can be given in the usual way and, at the same time, formalized using only a minimal knowledge of classical Euclidean Geometry and properties of real numbers, avoiding to use the notions of area, limit, derivative, series, integral and complex number.

In spite of this paper, our opinion is that, from the teaching point of view, the intuition approach for the introduction of sine and cosine remains the best one, because for instance the prerequisites of Euclidean Geometry needed for the definition cannot be part of a course in Calculus. However, we believe that this paper throws some light on the motivation which is behind the choice made by several authors when they make expositions of Real Analysis at the undergraduate level.

The plan of the paper is the following. In Section 2 we give the whole set of prerequisites necessary for the proof that the definition of sine and cosine is consistent: readers can check that it contains only really basic notions and that there are no "ε/δ arguments". In Section 3 we prove our main result, which is the existence of a point in the unit circle whose coordinates will be the cosine and the sine. At the end of Section 3 readers can see the definition as widely known from undergraduate Real Analysis. Finally, we close the paper with a short Appendix, where we highlight that a similar question does not exist for the notion of power – used in our proofs – in the real number system, because in such case the definitions can be given using only basics of the real number system.

2. Prerequisites and Notation

2.1. Basics from the Real Number System

We will consider the set $\mathbb{R}$ of the real numbers as totally ordered set, i.e. as set endowed with the standard order denoted by $\leq$. As usual, for $x, y \in \mathbb{R}$, we write $y \geq x$ if $x \leq y$ and we write $x < y$ or $y > x$ if $x \leq y$ and $x \neq y$. Notation about intervals will be with square brackets: if $a, b \in \mathbb{R}$, $a < b$, we set

$$\begin{aligned}
[a, b] &= \{x \in \mathbb{R} : a \leq x \leq b\} \\
]a, b[&= \{x \in \mathbb{R} : a < x < b\} \\
]a, b] &= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b[&= \{x \in \mathbb{R} : a \leq x < b\} .
\end{aligned}$$

The following theorem is at the heart of the real number system (see e.g. [21, Theorem 1.11 p. 5 and Theorem 1.19 p.8]), because it is conceptually equivalent to the completeness property.

Theorem 2.1. *The set $\mathbb{R}$, endowed with its standard order, has the least-upper-bound property, namely, any nonempty subset E of $\mathbb{R}$ which is bounded above admits a unique supremum denoted by $\sup E$, defined as the minimum of the set of the upper bounds of E. Equivalently, any nonempty subset F of $\mathbb{R}$ which is bounded below admits a unique infimum denoted by $\inf F$, defined as the maximum of the set of the lower bounds of F.*

As usual, if a nonempty subset E of $\mathbb{R}$ is not bounded above, we set $\sup E = +\infty$, so that the supremum exists for every nonempty subset of $\mathbb{R}$. Analogously, if a nonempty subset F of $\mathbb{R}$ is not bounded below, we set $\inf F = -\infty$.

2.2. Absolute Value and Square Root

For $x \in \mathbb{R}$ we write $|x| = x$ if $x \geq 0$, $|x| = -x$ if $x < 0$. The following statement, whose proof is not immediate (see e.g. [21, Theorem 1.21 p.10]), allows the definition of *square root*: for every real number $x \geq 0$ there exists one and only one real nonnegative number whose square equals x. Such number, whose explicit expression will be recalled in the Appendix (along with the definition of power in the real number system), is denoted by $\sqrt{x}$.

We will use, without explicit mentioning, the following properties of the square root. All of them can be obtained by elementary properties of real numbers:

$$\sqrt{x} \geq 0 \quad \forall\, x \geq 0$$

$$x \leq \sqrt{x} \quad \forall\, x \in [0,1]$$

$$\sqrt{x^2} = x \quad \forall\, x \geq 0$$

$$\sqrt{x_1} \leq \sqrt{x_2} \quad \forall\, 0 \leq x_1 \leq x_2$$

2.3. Prerequisites from Classical Euclidean Geometry

If $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ are points in the Euclidean plane endowed with a Cartesian coordinate system, the line segment joining P and Q will be denoted by PQ, and its length by $|PQ|$. We will use that $|PQ| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}$: for a detailed proof which uses the Pythagorean theorem the reader can see Moise [16, p.244].

Next theorem is known as the *triangle inequality property* (here we allow degenerate triangles, therefore in principle the vertices of triangles can be aligned):

Theorem 2.2. *The sum of the lengths of any two sides of a triangle is greater than or equal to the length of the remaining side.*

A *circle* is any set consisting of all points in a plane that are at a given distance from a fixed point, called *centre* of the circle. Any line segment joining any point of the circle and the centre is called *radius* of the circle. A *chord* of a circle is a straight line segment whose both endpoints lie on the circle. A *diameter* of a circle is any chord that passes through the centre of the circle.

Next theorem is also classical (see e.g. Sannia-D'Ovidio [22, Teorema in n. 67 p.75]):

Theorem 2.3. *In a circle, diameters are the longest chords.*

In the following we will denote by $C \subset \mathbb{R}^2$ the unit circle, defined as the set of points in the Euclidean plane that are at distance 1 from the origin $O = (0, 0)$.

The following theorems, written using terms from the Cartesian coordinate system in the plane, deal with the relative position of straight lines and circles, and are part of classical Geometry (see e.g. Sannia-D'Ovidio [22, Teorema 1, Teorema 2, Teorema 3 p.78]):

Theorem 2.4. *If $x_0 \in \mathbb{R}$ is such that $|x_0| > 1$, then $C \cap \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} = \emptyset$.*

Theorem 2.5. $C \cap \{(1, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} = \{(1, 0)\}$.

Theorem 2.6. *If $x_0 \in \mathbb{R}$ is such that $|x_0| < 1$, then $C \cap \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$ consists of exactly two points $P_1 = (x_0, y_1)$ and $P_2 = (x_0, y_2)$, and, assuming e.g. $y_1 < y_2$,*

$$P_1P_2 := \{(x_0, y) \in \mathbb{R}^2 : y_1 < y < y_2 \in \mathbb{R}\} \subset \{P \in \mathbb{R}^2 : |OP| < 1\}.$$

We now record the following straightforward consequences.

Corollary 2.7. *In the assumptions and notation of Theorem 2.6, $y_1 = -\sqrt{1 - x_0^2}$ and $y_2 = \sqrt{1 - x_0^2}$.*

Proof. By Theorem 2.6 we have $y_1 \neq y_2$, and since $P_1, P_2 \in C$ means that $\sqrt{x_0^2 + y_1^2} = \sqrt{x_0^2 + y_2^2} = 1$, then also $Q_1 := (x_0, -y_1)$, $Q_2 := (x_0, -y_2) \in C \cap \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$. Theorem 2.6 entrains that the set $\{P_1, P_2, Q_1, Q_2\}$ consists exactly of two points, and also that $P_1 \neq P_2$, hence we must have $P_1 = Q_1$ or $P_1 = Q_2$ (of course, in principle, P_1 could be different from the other three points, but in such case we apply next argument to P_2). But it cannot be $P_1 = Q_1$, otherwise $y_1 = -y_1$ would mean $y_1 = 0$, from which $P_1 = (x_0, 0) \in C$, which contradicts $|x_0| < 1$. Hence it must be $P_1 = Q_2$, i.e. $y_1 = -y_2$. Therefore

$$C \cap \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\} = \{(x_0, -y_1), (x_0, y_1)\}$$

and since

$$x_0^2 + \left(-\sqrt{1 - x_0^2}\right)^2 = x_0^2 + (1 - x_0^2) = 1$$

$$x_0^2 + \left(\sqrt{1 - x_0^2}\right)^2 = x_0^2 + (1 - x_0^2) = 1,$$

we have

$$\left\{\left(x_0, -\sqrt{1 - x_0^2}\right), \left(x_0, \sqrt{1 - x_0^2}\right)\right\} \subset C \cap \{(x_0, y) \in \mathbb{R}^2 : y \in \mathbb{R}\}$$
$$= \{(x_0, -y_1), (x_0, y_1)\}.$$

Hence

$$\{-y_1, y_1\} = \{y_2, -y_2\} = \left\{-\sqrt{1 - x_0^2}, \sqrt{1 - x_0^2}\right\},$$

and since $y_1 < y_2$ (and square roots are nonnegative), we get the assertion. □

Corollary 2.8. $C = \{(x, -\sqrt{1-x^2}) \in \mathbb{R}^2 : |x| \le 1\} \cup \{(x, \sqrt{1-x^2}) \in \mathbb{R}^2 : |x| \le 1\}$.

Proof. Any point (x, y) in the set in the right hand side satisfies $x^2 + y^2 = 1$, hence it belongs to C. On the other hand, any point $(x, y) \in C$, by Theorem 2.4, must have $|x| \le 1$. If $x = 1$, then by Theorem 2.5 it must be $y = 0$ and $(1, 0)$ belongs to the set in the right hand side. We argue similarly if $x = -1$. Finally, if $|x| < 1$ we conclude applying Corollary 2.7. □

In the following we will make use of the notation

$$C^- = \left\{\left(x, -\sqrt{1-x^2}\right) \in \mathbb{R}^2 : |x| \le 1\right\}$$

$$C^+ = \left\{\left(x, \sqrt{1-x^2}\right) \in \mathbb{R}^2 : |x| \le 1\right\} .$$

Corollary 2.9. *If $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ are two distinct points of C^+, then $x_P \ne x_Q$.*

Proof. On the contrary, if $x_P = x_Q := x_0$, since P and Q are distinct points of C^+, by Theorem 2.6 and Corollary 2.7, we have at the same time

$$y_P, y_Q \ge 0 , \qquad y_P \ne y_Q , \qquad \{y_P, y_Q\} = \left\{\sqrt{1-x_0^2}, -\sqrt{1-x_0^2}\right\} ,$$

and this is absurd: from the first and third issues

$$\sqrt{1-x_0^2} , \quad -\sqrt{1-x_0^2} \ \ge 0 ,$$

hence $\sqrt{1-x_0^2} = 0$, against $y_P \ne y_Q$. □

2.4. Arc Length

Two distinct points of the unit circle define two *arcs*, however, in the following we will fix the notation for one of them when both points belong to C^+.

Let $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ two distinct points of C^+, $x_Q < x_P$. By the symbol $\widehat{PQ}$ we will denote the arc made by points identified by P when it moves towards Q anticlockwise, namely

$$\widehat{PQ} = \left\{\left(x, \sqrt{1-x^2}\right) \in \mathbb{R}^2 : x_Q \le x \le x_P\right\} \subset C^+ .$$

In the following we will use the notation $A = (1, 0)$, $B = (-1, 0)$, so that $\widehat{AB} = C^+$. A *polygonal path* Π for $\widehat{PQ}$ is a set of the type

$$\Pi_{\widehat{PQ}}(P_0, \cdots, P_n) = P_0P_1 \cup P_1P_2 \cup \cdots \cup P_{n-1}P_n \subset \mathbb{R}^2$$

where $n \ge 1$ is a natural number, $P_0 = P$, $P_n = Q$, $P_i = (x_{P_i}, y_{P_i}) \in C^+$, $i = 0, \ldots, n$, and

$$x_P = x_{P_0} \ge x_{P_1} \ge \cdots \ge x_{P_n} = x_Q .$$

The set of all polygonal paths for $\widehat{PQ}$ will be denoted by $\Pi(\widehat{PQ})$. The *length of the polygonal path* $\Pi_{\widehat{PQ}}(P_0, \cdots, P_n)$ is the number

$$\mathcal{L}(\Pi_{\widehat{PQ}}(P_0, \cdots, P_n)) = |P_0P_1| + |P_1P_2| + \cdots + |P_{n-1}P_n|.$$

Finally, we define the *length of the arc* $\widehat{PQ}$ as

$$\mathcal{L}(\widehat{PQ}) = \sup_{\Pi \in \Pi(\widehat{PQ})} \mathcal{L}(\Pi).$$

We will use the following convention: if $P \in C^+$, then $\mathcal{L}(\widehat{PP}) = 0$.

The first simple property of lengths of arcs is the subadditivity.

Theorem 2.10. *If $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $R = (x_R, y_R)$ are points of C^+ such that $-1 \leq x_R \leq x_Q \leq x_P \leq 1$, then $\mathcal{L}(\widehat{PR}) \leq \mathcal{L}(\widehat{PQ}) + \mathcal{L}(\widehat{QR})$.*

Proof. Let $\Pi_{\widehat{PR}}(P_0, \cdots, P_n)$ be a polygonal path for $\widehat{PR}$. By Theorem 2.2

$$\begin{aligned}
\mathcal{L}(\Pi_{\widehat{PR}}(P_0, \cdots, P_n)) &\leq \mathcal{L}(\Pi_{\widehat{PR}}(P_0, \cdots, P_{j-1}, Q, P_j, \cdots, P_n)) \\
&= |P_0P_1| + \cdots + |P_{j-1}Q| + |QP_j| + \cdots + |P_{n-1}P_n| \\
&= \mathcal{L}(\Pi_{\widehat{PQ}}(P_0, \cdots, P_{j-1}, Q)) + \mathcal{L}(\Pi_{\widehat{QR}}(Q, P_j, \cdots, P_n)) \\
&\leq \mathcal{L}(\widehat{PQ}) + \mathcal{L}(\widehat{QR})
\end{aligned}$$

for some $j \in \{1, \ldots, n\}$. The same estimate holds for the supremum over all polygonal paths for $\widehat{PR}$, hence the assertion follows. □

Next result gives an estimate for the length of arcs contained in the so-called first and second quadrants.

Theorem 2.11. *If $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$ are points of C^+ such that $0 \leq x_Q < x_P \leq 1$, then*

$$\mathcal{L}(\widehat{PQ}) \leq x_P - x_Q + y_Q - y_P.$$

In the case $-1 \leq x_Q < x_P \leq 0$ we have

$$\mathcal{L}(\widehat{PQ}) \leq x_P - x_Q + y_P - y_Q.$$

Proof. Let us consider the case $0 \leq x_Q < x_P \leq 1$. Let $\Pi_{\widehat{PQ}}(P_0, \cdots, P_n) \in \Pi(\widehat{PQ})$.

Set

$$R_i = (x_{P_i}, y_{P_{i-1}}), \quad i = 1, \ldots, n.$$

By Theorem 2.2 we have

$$\mathcal{L}(\Pi_{\widehat{PQ}}(P_0, \cdots, P_n)) = |P_0P_1| + \cdots + |P_{i-1}P_i| + \cdots + |P_{n-1}P_n|$$

$$\leq (|P_0R_1| + |R_1P_1|) + \cdots + (|P_{i-1}R_i| + |R_iP_i|) + \cdots + (|P_{n-1}R_n| + |R_nP_n|)$$

$$= (x_{P_0} - x_{P_1} + y_{P_1} - y_{P_0}) + \cdots + (x_{P_{i-1}} - x_{P_i} + y_{P_i} - y_{P_{i-1}})$$

$$+\cdots+(x_{P_{n-1}}-x_{P_n}+y_{P_n}-y_{P_{n-1}})$$
$$=x_{P_0}-x_{P_n}+y_{P_n}-y_{P_0}=x_P-x_Q+y_Q-y_P\,.$$

The second part of the statement follows arguing exactly in the same way, setting $R_i=(x_{P_{i-1}},y_{P_i}),\quad i=1,\dots,n.$ □

Theorem 2.12. *If $P=(x_P,y_P)$, $Q=(x_Q,y_Q)$ are two distinct points of C^+, then $\mathcal{L}(\widehat{PQ})\le\mathcal{L}(C^+)\le 4$.*

Proof. Let $\Pi_{\widehat{PQ}}(P_0,\cdots,P_n)$ be a polygonal path for $\widehat{PQ}$. Setting $T=(0,1)$, $P_{-1}=A=(1,0)$, $P_{n+1}=B=(-1,0)$, by Theorem 2.2

$$\mathcal{L}(\Pi_{\widehat{PQ}}(P_0,\cdots,P_n))\le\mathcal{L}(\Pi_{\widehat{AB}}(A,P_0,\cdots,P_{j-1},T,P_j,\cdots,P_n,B))\le\mathcal{L}(\widehat{AB}) \qquad (1)$$

for some $j\in\{0,\dots,n+1\}$. The same estimate holds for the supremum over all polygonal paths for $\widehat{PQ}$, hence

$$\mathcal{L}(\widehat{PQ})\le\mathcal{L}(\widehat{AB})=\mathcal{L}(C^+)\,.$$

On the other hand, by Theorem 2.10

$$\mathcal{L}(\widehat{AB})\le\mathcal{L}(\widehat{AT})+\mathcal{L}(\widehat{TB})$$

and by Theorem 2.11

$$\mathcal{L}(C^+)=\mathcal{L}(\widehat{AB})\le(x_A-x_T+y_T-y_A)+(x_T-x_B+y_T-y_B)$$
$$=x_A+y_T-x_B+y_T=4\,,$$

hence the assertion follows. □

As a byproduct of Theorem 2.12, we have that $\mathcal{L}(C^+)$ is a (finite supremum and therefore a) real number. Therefore next definition makes sense.

Definition 2.13. $\pi=\mathcal{L}(C^+)$.

Remark 2.14. From Theorem 2.12 we have immediately the estimate $\pi\le 4$. For our goals we will need this estimate, and it will be not necessary to prove any finer inequality. From the logic point of view, the notions of sine and cosine as functions over reals is independent of the exact value of π, which is linked to the use of the choice of the measure of lengths.

Remark 2.15. In Moise [16, p.321] the reader can find the same definition of arc length as ours. In Theorem 1 the author proves the finiteness of the supremum by arguments which use purely geometric statements for triangles, vs the more direct triangle inequality used in our Theorem 2.11. Moreover, the upper bound for the length of the whole circle found in Moise's book is the perimeter of the square circumscribing the circle, which is the same our estimate in the previous Remark 2.14. We observe also that our Theorem 2.11 is equally short and elementary, but our more analytic treatment allows to get the existence result in our next Theorem 3.1, missing in Moise's book. Finally, we observe that in Sections 21.3 and 21.4 of the Moise's book, and in particular in [16, Theorem 1 p.329], readers can find the *additivity* of arc lengths, proved using the ε/δ argument of [16, Theorem 1 p.326]. But this whole part of Moise's book is not necessary for our goals: we need just the *subadditivity* proved through our very short Theorem 2.10.

Corollary 2.16. *If $P \in C^+$, then $0 \leq \mathcal{L}(\widehat{AP}) \leq \pi$.*

Proof. The statement is trivial if $P = B$. If not, every polygonal path defining $\mathcal{L}(\widehat{AP})$ can be extended making the union with the segment PB: this way we obtain particular polygonal paths defining $\mathcal{L}(\widehat{AB}) = \mathcal{L}(C^+) = \pi$. □

In order to prove the main result of this paper in the most interesting case, we need another version of Theorem 2.11, namely, we will need rougher but analytic estimates of $\mathcal{L}(\widehat{PQ})$.

Theorem 2.17. *If $Q = (x_Q, y_Q)$, $0 \leq x_Q < 1$, then*

$$\mathcal{L}(\widehat{AQ}) \leq 1 - x_Q + \sqrt{1 - x_Q^2} \leq 2\sqrt{1 - x_Q^2}\,.$$

If $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $0 \leq x_Q < x_P < 1$, then

$$\mathcal{L}(\widehat{PQ}) \leq \frac{x_P - x_Q}{1 - x_P}\,.$$

If $-1 < x_Q < x_P \leq 0$ we have

$$\mathcal{L}(\widehat{PQ}) \leq \frac{x_P - x_Q}{1 + x_Q}\,.$$

If $P = (x_P, y_P)$, $-1 < x_P \leq 0$, then

$$\mathcal{L}(\widehat{PB}) \leq x_P + 1 + \sqrt{1 - x_P^2} \leq 2\sqrt{1 - x_P^2}\,.$$

Proof. The first and the last case follow directly from Theorem 2.11, using that $x^2 \leq x \leq \sqrt{x}$ for all $x \in [0, 1]$.

The case $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $0 \leq x_Q < x_P < 1$ is consequence of an algebraic estimate. Namely, for any λ, μ such that $|\lambda| \leq 1$, $|\mu| < 1$, using

$$(\lambda - \mu)^2 \geq 0 \;\Leftrightarrow\; -\lambda^2 - \mu^2 \leq -2\lambda\mu\,,$$

we have

$$\begin{aligned}
\sqrt{1-\lambda^2} - \sqrt{1-\mu^2} &= \frac{1}{\sqrt{1-\mu^2}}\left(\sqrt{(1-\lambda^2)(1-\mu^2)} - 1 + \mu^2\right) \\
&= \frac{1}{\sqrt{1-\mu^2}}\left(\sqrt{1-\lambda^2-\mu^2+\lambda^2\mu^2} - 1 + \mu^2\right) \\
&\leq \frac{1}{\sqrt{1-\mu^2}}\left(\sqrt{1-2\lambda\mu+\lambda^2\mu^2} - 1 + \mu^2\right) \\
&= \frac{1}{\sqrt{1-\mu^2}}\left(1 - \lambda\mu - 1 + \mu^2\right)
\end{aligned}$$

$$= \frac{\mu(\mu - \lambda)}{\sqrt{1-\mu^2}}\,.$$

Hence, if $P = (x_P, y_P)$, $Q = (x_Q, y_Q)$, $0 \le x_Q < x_P < 1$, then by Theorem 2.11 and the above estimate applied to $\lambda = x_Q$, $\mu = x_P$

$$\begin{aligned}
\mathcal{L}(\widehat{PQ}) &\le x_P - x_Q + y_Q - y_P = x_P - x_Q + \sqrt{1-x_Q^2} - \sqrt{1-x_P^2} \\
&\le x_P - x_Q + \frac{x_P(x_P - x_Q)}{\sqrt{1-x_P^2}} \\
&= \left(1 + \frac{x_P}{\sqrt{1-x_P^2}}\right)(x_P - x_Q) \le \left(1 + \frac{x_P}{1-x_P^2}\right)(x_P - x_Q) \\
&\le \left(1 + \frac{x_P}{1-x_P}\right)(x_P - x_Q) \\
&= \frac{x_P - x_Q}{1 - x_P}
\end{aligned}$$

Finally, the case $-1 < x_Q < x_P \le 0$ is analogous, or, alternatively, it can be treated using the previous case and the following symmetry argument. Setting $Q' = (-x_Q, y_Q)$, $P' = (-x_P, y_P)$, since $-x_P < -x_Q$,

$$\mathcal{L}(\widehat{Q'P'}) \le \frac{x_{Q'} - x_{P'}}{1 - x_{Q'}} = \frac{-x_Q + x_P}{1 + x_Q}$$

and the theorem is proved. □

3. The Main Result and the Definition of Sine and Cosine as Real Functions

Next result is the core of this paper.

Theorem 3.1. *If $0 \le t \le \pi$, then there exists $P \in C^+$ such that $\mathcal{L}(\widehat{AP}) = t$.*

Proof. If $t = 0$ then we set $P = A$, and if $t = \pi$ then we set $P = B$. Let us assume $0 < t < \pi$. We set

$$E = \{s \in [-1, 1] \,:\, \mathcal{L}(\widehat{AP_s}) > t\}\,,$$

where $P_s = (s, \sqrt{1-s^2}) \in C^+$. Note that $E \ne \emptyset$ because $-1 \in E$: in fact, $P_{-1} = (-1, 0) = B$ and by definition of π we have $\mathcal{L}(\widehat{AP_{-1}}) = \mathcal{L}(\widehat{AB}) = \pi > t$. Moreover, $1 \notin E$ because $P_1 = A$ and $\mathcal{L}(\widehat{AA}) = 0$, and this means that 1 is an upper bound for E. Hence $\sup E \in [-1, 1]$.

We now set $P = (x_P, y_P) := P_{\sup E} = (\sup E, \sqrt{1 - (\sup E)^2}) \in C^+$ and show that $\mathcal{L}(\widehat{AP}) = t$.

On the contrary, let us consider first the case $\mathcal{L}(\widehat{AP}) = t^- \in [0, t[$.

If $\sup E = -1$, then $P = P_{\sup E} = P_{-1} = B$ and therefore $\mathcal{L}(\widehat{AP}) = \mathcal{L}(\widehat{AB}) = \pi > t > t^-$, and this is absurd because $\mathcal{L}(\widehat{AP}) = t^-$.

If $\sup E = x_P \in]-1, 0]$, noticing that

$$-1 < \frac{x_P - \frac{t-t^-}{2}}{1 + \frac{t-t^-}{2}} < x_P\,,$$

it makes sense to consider any $x \in \left]\frac{x_P - \frac{t-t^-}{2}}{1+\frac{t-t^-}{2}}, x_P\right[$ and to set $P^+ = (x_{P^+}, y_{P^+}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorems 2.10 and 2.17 we have

$$\begin{aligned}\mathcal{L}(\widehat{AP^+}) \leq & \mathcal{L}(\widehat{AP}) + \mathcal{L}(\widehat{PP^+}) \leq t^- + \frac{x_P - x_{P^+}}{1 + x_{P^+}} < t^- \\ & + \frac{x_P - \frac{x_P - \frac{t-t^-}{2}}{1+\frac{t-t^-}{2}}}{1 + \frac{x_P - \frac{t-t^-}{2}}{1+\frac{t-t^-}{2}}} = t^- + \frac{\frac{t-t^-}{2} x_P + \frac{t-t^-}{2}}{1 + x_P} = t^- + \frac{t-t^-}{2} < t\end{aligned}$$

from which $x_{P^+} = x \notin E$, and, since also $x_P \notin E$ (because $\mathcal{L}(\widehat{AP}) = t^- < t$), we have that

$$\left]\frac{x_P - \frac{t-t^-}{2}}{1 + \frac{t-t^-}{2}}, x_P\right] \cap E = \emptyset\,,$$

against $\sup E = x_P$.

If $\sup E = x_P \in]0, 1[$, noticing that

$$0 \leq \max\left\{0, x_P - \frac{t-t^-}{2}(1 - x_P)\right\} < x_P\,,$$

it makes sense to consider any $x \in \left]\max\left\{0, x_P - \frac{t-t^-}{2}(1-x_P)\right\}, x_P\right[$ and to set $P^+ = (x_{P^+}, y_{P^+}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorem 2.10 and Theorem 2.17 we have

$$\begin{aligned}\mathcal{L}(\widehat{AP^+}) \leq & \mathcal{L}(\widehat{AP}) + \mathcal{L}(\widehat{PP^+}) \leq t^- + \frac{x_P - x_{P^+}}{1 - x_P} < t^- \\ & + \frac{\frac{t-t^-}{2}(1 - x_P)}{1 - x_P} = t^- + \frac{t-t^-}{2} < t\end{aligned}$$

from which, again, $x_{P^+} = x \notin E$. Since also $x_P \notin E$ (because $\mathcal{L}(\widehat{AP}) = t^- < t$), we have that

$$\left]\max\left\{0, x_P - \frac{t-t^-}{2}(1 - x_P)\right\}, x_P\right] \cap E = \emptyset\,,$$

against $\sup E = x_P$.

If $\sup E = 1$, then $P = P_{\sup E} = P_1 = A$. By Remark 2.14 we have $0 < t - t^- \leq t < \pi \leq 4$, so that

$$0 < \left(\frac{t-t^-}{4}\right)^2 < 1\,,$$

and therefore it makes sense to consider any $x \in \left]\sqrt{1-\left(\frac{t-t^-}{4}\right)^2}, 1\right[$, and to set $P^+ = (x_{P^+}, y_{P^+}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorem 2.17 we have

$$\mathcal{L}(\widehat{AP^+}) \leq 2\sqrt{1-x_{P^+}^2} \leq 2\sqrt{1-\left[1-\left(\frac{t-t^-}{4}\right)^2\right]} = 2\frac{t-t^-}{4} = \frac{t-t^-}{2} < t\,,$$

hence $x_{P^+} = x \notin E$ and, since we already checked that $1 \notin E$, we have

$$\left]\sqrt{1-\left(\frac{t-t^-}{4}\right)^2}, 1\right] \cap E = \emptyset\,,$$

against $\sup E = 1$.

Now let us consider the case $\mathcal{L}(\widehat{AP}) = t^+ > t$. The way to make the argument is similar to that one of $\mathcal{L}(\widehat{AP}) < t$, however, building the whole proof is not really immediate and for the sake of clarity we give here the full details.

If $\sup E = -1$, then $P = P_{\sup E} = P_{-1} = B$, noticing that

$$0 < 1 - \left(\frac{t^+-t}{4}\right)^2 < 1\,,$$

it makes sense to consider any $x \in \left]-1, -\sqrt{1-\left(\frac{t^+-t}{4}\right)^2}\right[$ and to set $P^- = (x_{P^-}, y_{P^-}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorems 2.10 and 2.17 we have

$$\begin{aligned} t^+ &= \mathcal{L}(\widehat{AP}) \leq \mathcal{L}(\widehat{AP^-}) + \mathcal{L}(\widehat{P^-B}) \leq \mathcal{L}(\widehat{AP^-}) + 2\sqrt{1-x_{P^-}^2} \\ &\leq \mathcal{L}(\widehat{AP^-}) + 2\sqrt{1-\left[1-\left(\frac{t^+-t}{4}\right)^2\right]} \\ &= \mathcal{L}(\widehat{AP^-}) + 2\frac{t^+-t}{4} = \mathcal{L}(\widehat{AP^-}) + \frac{t^+}{2} - \frac{t}{2} \end{aligned}$$

from which

$$\mathcal{L}(\widehat{AP^-}) \geq \frac{t^+}{2} + \frac{t}{2} > t$$

and therefore $\sup E = -1 < x_{P^-} \in E$, which is absurd.

If $\sup E = x_P \in]-1, 0[$, noticing that

$$x_P < \min\left\{0, x_P + \frac{t^+-t}{2}(1+x_P)\right\} \leq 0\,,$$

it makes sense to consider any $x \in \left]x_P, \min\left\{0, x_P + \frac{t^+-t}{2}(1+x_P)\right\}\right[$ and to set $P^- = (x_{P^-}, y_{P^-}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorems 2.10 and 2.17 we have

$$\begin{aligned} t^+ &= \mathcal{L}(\widehat{AP}) \leq \mathcal{L}(\widehat{AP^-}) + \mathcal{L}(\widehat{P^-P}) \\ &\leq \mathcal{L}(\widehat{AP^-}) + \frac{x_{P^-} - x_P}{1+x_P} \\ &\leq \mathcal{L}(\widehat{AP^-}) + \frac{\frac{t^+-t}{2}(1+x_P)}{1+x_P} = \mathcal{L}(\widehat{AP^-}) + \frac{t^+-t}{2} \end{aligned}$$

from which

$$\mathcal{L}(\widehat{AP^-}) \geq \frac{t^+}{2} + \frac{t}{2} > t$$

and therefore $\sup E = x_P < x_{P^-} \in E$, which is absurd.

If $\sup E = x_P \in [0,1[$, noticing that

$$x_P < \frac{x_P + \frac{t^+-t}{2}}{1+\frac{t^+-t}{2}} < 1\,,$$

it makes sense to consider any $x \in \left]x_P, \frac{x_P+\frac{t^+-t}{2}}{1+\frac{t^+-t}{2}}\right[$ and to set $P^- = (x_{P^-}, y_{P^-}) = (x, \sqrt{1-x^2}) \in C^+$. By Theorems 2.10 and 2.17 we have

$$\begin{aligned} t^+ = \mathcal{L}(\widehat{AP}) \leq \mathcal{L}(\widehat{AP^-}) + \mathcal{L}(\widehat{P^-P}) &\leq \mathcal{L}(\widehat{AP^-}) + \frac{x_{P^-} - x_P}{1 - x_{P^-}} \\ \leq \mathcal{L}(\widehat{AP^-}) + \frac{\frac{x_P+\frac{t^+-t}{2}}{1+\frac{t^+-t}{2}} - x_P}{1 - \frac{x_P+\frac{t^+-t}{2}}{1+\frac{t^+-t}{2}}} & \\ \leq \mathcal{L}(\widehat{AP^-}) + \frac{\frac{t^+-t}{2} - \frac{t^+-t}{2}x_P}{1-x_P} &= \mathcal{L}(\widehat{AP^-}) + \frac{t^+}{2} - \frac{t}{2} \end{aligned}$$

from which

$$\mathcal{L}(\widehat{AP^-}) \geq \frac{t^+}{2} + \frac{t}{2} > t$$

and therefore $\sup E = x_P < x_{P^-} \in E$, which is absurd.

Finally, if $\sup E = 1$, then $P = P_{\sup E} = P_1 = A$ and therefore $\mathcal{L}(\widehat{AP}) = \mathcal{L}(\widehat{AA}) = 0 < t < t^+$, and this is absurd (because $\mathcal{L}(\widehat{AP}) = t^+$). $\square$

We now have all prerequisites to transfer the previous machinery to the whole circle. With the use of the previous results next step can be quite fast.

If $P = A = (1,0)$ and $Q = (x_Q, y_Q) \in C^-$ are distinct points, then the arc $\widehat{AQ}$ is defined by

$$\widehat{AQ} = C^+ \cup \left\{\left(x, -\sqrt{1-x^2}\right) \in \mathbb{R}^2 \,:\, -1 \leq x \leq x_Q\right\}.$$

The *length of the arc* $\widehat{AQ}$ is defined by

$$\mathcal{L}(\widehat{AQ}) = \pi + \widehat{AP}$$

where $P = -Q = (-x_Q, -y_Q) \in C^+$.

Corollary 3.2. *If* $0 \le t < 2\pi$*, then there exists* $P \in C$ *such that* $\mathcal{L}(\widehat{AP}) = t$*.*

Proof. By Theorem 3.1 we know already the assertion if $0 \le t \le \pi$. If $\pi < t < 2\pi$, again by Theorem 3.1, there exists $Q = (x_Q, y_Q) \in C^+$ such that $\mathcal{L}(\widehat{AQ}) = t - \pi$. Setting $P = -Q = (-x_Q, -y_Q) \in C^-$, we have

$$\mathcal{L}(\widehat{AP}) = \pi + \widehat{AQ} = \pi + (t - \pi) = t\,,$$

and the assertion follows. $\square$

For $t \in [0, 2\pi[$ let $P_t = (x_t, y_t) \in C$ be the point whose existence is ensured by Corollary 3.2.

Definition 3.3. For any $t \in \mathbb{R}$ let $k \in \mathbb{Z}$ be such that $t \in [2k\pi, 2(k+1)\pi[$, so that $t - 2k\pi \in [0, 2\pi[$. We set

$$\sin t = y_{t-2k\pi}\,, \quad \cos t = x_{t-2k\pi}\,.$$

4. Appendix

Even if it is outside of our goals, we recall that the question we studied for the sine and cosine functions is easy in the case of powers and roots, because their definitions need only the notion of least upper bound and do not require the knowledge of limit and derivatives. This is well known, but usually treatises (at least looking at the same references we considered for the sine and cosine functions) do not give a complete and concise scheme. Hence we write the whole set of definitions below. As usual, $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers.

$x^0 = 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$
$x^\alpha = x \cdot x \cdots x$ (α times) $\forall \alpha \in \mathbb{N}, \forall x \in \mathbb{R}$
$x^{m/n} = \sup\{t \ge 0 : t^n \le x^m\}$ $\forall m, n \in \mathbb{N}$, $m/n \notin \mathbb{N}, \forall x \ge 0$
$x^\alpha = \sup\{x^q \in \mathbb{R} : q \in \mathbb{Q}, 0 < q < \alpha\}$ $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha > 0, \forall x \ge 1$
$x^\alpha = 1/[(1/x)^\alpha]$ $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha > 0, \forall 0 < x < 1$
$0^\alpha = 0$ $\forall \alpha \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha > 0$
$x^\alpha = 1/[x^{-\alpha}]$ $\forall \alpha \in \mathbb{Z}$, $\alpha < 0, \forall x \in \mathbb{R} \setminus \{0\}$
$x^\alpha = 1/[x^{-\alpha}]$ $\forall \alpha \in \mathbb{R} \setminus \mathbb{Z}$, $\alpha < 0, \forall x > 0$
$\sqrt{x} = x^{1/2} = \sup\{t \ge 0 : t^2 \le x\}$ $\forall x \ge 0$
and more generally

$$\sqrt[n]{x} = x^{1/n} = \sup\{t \geq 0 \,:\, t^n \leq x\} \; \forall\, n \in \mathbb{N},\; n \text{ even}, \; \forall\, x \geq 0$$
$$\sqrt[n]{x} = \sup\{t \in \mathbb{R} \,:\, t^n \leq x\} \; \forall\, n \in \mathbb{N},\; n > 1,\; n \text{ odd}, \; \forall\, x \in \mathbb{R}$$

The symbols defined above make true the properties $a^b \cdot a^c = a^{b+c}$ and $(a^b)^c = a^{bc}$ if both sides fall in the definitions above. Finally, we remark that definitions recalled above justify why the set of the a's such that a^x has been defined for all $x \in \mathbb{R}$ is the set of the positive numbers.

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