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SPECTRA AND BALANCE OF SIGNED GRAPHS OBTAINED FROM VERTEX SPLITTINGS

Callum Huntington

Department of Mathematics and Applications R. Caccioppoli
University of Naples Federico II, Italy

e-mail: callum.huntington@unina.it

Abstract

This paper introduces new constructions of nonregular cospectral signed graphs via two operations: the neighbours splitting (NS) join and the non-neighbours splitting (NNS) join. We compute the adjacency and Laplacian characteristic polynomials for each join, allowing spectral analysis for arbitrary signed graphs and explicit eigenvalue calculations for co-regular signed graphs. A second approach employs pseudo-potential functions to define more robust switching-stable versions of these joins, preserving spectral properties under switching equivalence. As an application of these techniques, we construct infinite families of nonisomorphic signed graphs exhibiting cospectrality for both adjacency and Laplacian matrices. We also characterise balancedness conditions for each of the constructions.

Keywords: signed graph, graph operation, adjacency, Laplacian, cospectral, switching equivalence.

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1. Introduction

A signed graph $\Sigma = (G, \sigma)$ is a pair consisting of an ordinary graph $G = (V_G, E_G)$ on n vertices, called the underlying graph, together with a map $\sigma\colon E \to \{+1, -1\}$, referred to as the signature. The vertex sets V_Σ and V_G are identical and may be used interchangeably. However, the same does not generally hold for the edge sets E_Σ and E_G, since the edge set of a signed graph is composed of subsets of positive and negative edges. We think of an ordinary graph G as the all-positive signed graph $\Sigma = (G, +)$ whose signature assigns the value $+1$ to all of its edges. All graphs considered in this paper shall be simple.

The adjacency matrix of a signed graph $\Sigma = (G, \sigma)$ on n vertices is a square matrix of order n that is denoted by $A_\Sigma = (a_{ij})_{n\times n}$ and defined by

$$a_{ij} = \begin{cases} \sigma(v_i v_j), & \text{if } v_i v_j \in E, \\ 0, & \text{otherwise.} \end{cases}$$

For a graph G with n vertices, let $d_i = d_G(v_i)$ be the degree of vertex v_i, and let D_G denote the degree matrix of G, the $n \times n$ diagonal matrix with entries $d_1, d_2, \ldots, d_n$. The Laplacian matrix L_Σ of a signed graph $\Sigma = (G, \sigma)$ is defined by $L_\Sigma = D_G - A_\Sigma$.

Let M be an $n \times n$ matrix. Its characteristic polynomial, $\det(\lambda I_n - M)$, will be denoted by $P(M; \lambda)$, with I_n representing the $n \times n$ identity matrix. For a signed graph Σ, we define $P(M_\Sigma; \lambda)$ to be the M-characteristic polynomial of Σ for $M \in \{A, L\}$. The roots of the M-characteristic polynomial of Σ are identified to be the M-eigenvalues of Σ and the collection of these M-eigenvalues, including multiplicities, is called the M-spectrum of Σ. By the characteristic polynomial, the eigenvalues, and the spectrum of a signed graph (without any reference to the matrix), we mean the A-characteristic polynomial, the A-eigenvalues, and the A-spectrum, respectively.

In an ordinary graph, the degree of a vertex is simply the number of edges that are incident to that vertex. In a signed graph, the net degree $d^{\pm}_\Sigma(v)$ of a vertex $v \in V_\Sigma$ is defined as $d^{\pm}_\Sigma(v) = d^{+}_\Sigma(v) - d^{-}_\Sigma(v)$, where $d^{+}_\Sigma(v)$ and $d^{-}_\Sigma(v)$ denote respectively the number of positive and the number of negative edges incident to v in Σ. The notion of co-regular signed graphs was proposed in 2015 by Hameed et al. [7]. There, a signed graph $\Sigma = (G, \sigma)$ is said to be co-regular if its underlying graph G is r-regular for some positive integer r, and Σ is net regular with net degree k. The corresponding co-regularity pair is then (r, k).

The sign of a cycle is defined as the product of the signs of its edges. Thus, a cycle is positive precisely when it contains an even number of negative edges. A signed graph is said to be balanced if and only if the product of the edge signs in any closed walk is positive. Hence a signed graph is balanced exactly when all of its cycles, if any exist, are positive.

In signed graph theory, the operation of switching plays a central role. Two signed graphs are switching equivalent if and only if their adjacency matrices satisfy $A'_\Sigma = S^{-1} A_\Sigma S$ for some diagonal matrix S, known as the switching matrix, whose entries are either $+1$ or -1. Switching a signed graph does not alter its spectrum, since matrix conjugation preserves eigenvalues. Hence, switching-equivalent signed graphs share the same characteristic polynomials and eigenvalues. A signed graph is balanced precisely when its signature is switching equivalent to the all-positive signature.

Alternatively, two signed graphs $\Sigma = (G, \sigma)$ and $\Sigma' = (G, \sigma')$ on the same underlying graph are switching equivalent if there exists a function $s \colon V_G \to$

$\{+1, -1\}$ such that for any pair of adjacent vertices v_i and v_j, we have

$$\sigma'(v_i v_j) = s(v_i)\sigma(v_i v_j)s(v_j).$$

A closed walk W of length k in a graph G is an ordered sequence of vertices $v_1 v_2 \cdots v_{k+1}$ such that $v_{k+1} = v_1$ and v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, k$. When G is endowed with a signature σ, it is possible to define the sign of W as $\sigma(W) = \sigma(v_1 v_2)\sigma(v_2 v_3) \cdots \sigma(v_k v_{k+1})$. Given in terms of the signs of closed walks, two signatures σ and σ' on G are switching equivalent if and only if $\sigma(W) = \sigma'(W)$ for every closed walk W in G.

In a more general setting, we say that two signed graphs Σ and Σ' are switching isomorphic if one can find a switching matrix S along with a permutation matrix P such that

$$A_{\Sigma'} = (PS)^{-1} A_{\Sigma} (PS).$$

Indeed, the matrix PS represents a signed permutation matrix, which may also be referred to as a $\{1, 0, -1\}$-monomial matrix.

Two signed graphs Σ and Γ are said to be M-cospectral if they have the same M-spectrum. In this paper, if two or more M-cospectral signed graphs are not switching isomorphic, they are called M-cospectral mates.

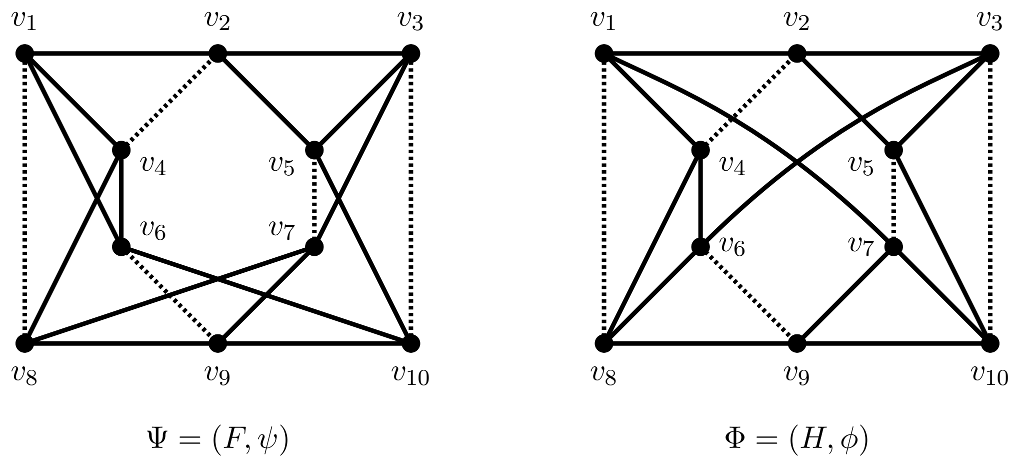


Figure 1. A pair of $(4, 2)$-co-regular nonisomorphic cospectral signed graphs $\Psi = (F, \psi)$ and $\Phi = (H, \phi)$ on ten vertices. Here and throughout this paper solid lines represent positive edges and dashed lines represent negative edges.

It is known (see [5, p. 398]) that all regular unsigned graphs on less than ten vertices are determined by their spectrum, that is, any other unsigned graph that is cospectral is also isomorphic. Furthermore, there are four pairs of cospectral regular unsigned graphs on ten vertices, as shown by van Dam and Haemers [6].

By identifying a perfect matching — that is, a subset of the edge set in which every vertex is incident with exactly one edge — that is present in both unsigned graphs forming such a pair, and turning precisely those edges negative, we obtain a nontrivial pair of co-regular, nonisomorphic, cospectral signed graphs.

Recently, several papers have focused on constructions of cospectral signed graphs that are not switching isomorphic [1, 2, 14]. An example of a pair of nonisomorphic cospectral signed graphs is given in Figure 1. There it can be seen that the graphs are nonisomorphic, since in Ψ there exist edges, such as the edge between v_1 and v_4, that lie in three triangles, whereas no edge of Φ has this property.

Typically, a well-designed generalisation of a product or join operation on signed graphs should aim to be switching stable, meaning that replacing the signatures of the components with switching-equivalent ones does not change the switching equivalence class of the product. This is particularly useful in spectral computations due to the fact that switching-equivalent signed graphs are cospectral.

In [3], the authors introduce the concept of pseudo-potential functions on signed graphs and provide a solution to an open problem regarding the switching stability of the lexicographic product of signed graphs. Following the approach of the work done in [8] and [12], we begin by directly extending two join operations to the setting of signed graphs. Subsequently, we apply a result from [15], the pseudo-potential functions of [3], and a method developed in [11] to define distinct and more robust constructions that avoid the stability and balance issues present in the simpler versions.

For any results or concepts concerning graph spectra or signed graphs used in this paper which have not been clearly stated the reader is directed to [5] and [16].

The remainder of this paper is structured as follows. In Section 2, we collect some preliminary facts and results. In Section 3, we define the signed NS and NNS joins, derive their adjacency and Laplacian characteristic polynomials for two arbitrary signed graphs, compute their spectra for two co-regular signed graphs, and demonstrate a method of constructing cospectral mates. In Section 4, we use pseudo-potentials to provide alternative definitions for the signatures of the NS and NNS joins that are stable under switchings of their components. In Section 5, we offer open problems which are beyond the scope of this paper.

2. Preliminaries

We denote the identity matrix by I, the all-one matrix by J, and the all-zero matrix by O. A column vector of size n with all entries equal to one is written as

$\mathbf{1}_n$. For an unsigned graph G, its complement $\overline{G}$ is the graph on the same vertex set in which two distinct vertices are adjacent if and only if they are not adjacent in G.

Remark 2.1. Let $\Sigma = (G, \sigma)$ be an (r, k)-co-regular signed graph on n vertices. We shall denote its adjacency eigenvalues by setting $\lambda_1(\Sigma) = k$ and ordering the remainder so that $\lambda_2(\Sigma) \geq \lambda_3(\Sigma) \geq \cdots \geq \lambda_n(\Sigma)$. For the underlying graph G, the adjacency eigenvalues can be written as $\lambda_1(G) = r \geq \lambda_2(G) \geq \lambda_3(G) \geq \cdots \geq \lambda_n(G)$.

Remark 2.2. Let $\Sigma = (G, \sigma)$ be an (r, k)-co-regular signed graph on n vertices. We shall denote its Laplacian eigenvalues by setting $\mu_1(\Sigma) = r - k$ and ordering the remainder so that $\mu_2(\Sigma) \leq \mu_3(\Sigma) \leq \cdots \leq \mu_n(\Sigma)$. For the underlying graph G, the Laplacian eigenvalues can be written as $\mu_1(G) = 0 \leq \mu_2(G) \leq \mu_3(G) \leq \cdots \leq \mu_n(G)$.

The graph invariant known as the coronal was introduced in 2011 by McLeman and McNicholas [13]. In the following, we adapt their definition directly to a matrix.

Definition 2.3 [13, Definition 1]**.** Let M be an $n \times n$ matrix. The coronal $\chi_M(\lambda)$ of M is defined to be the sum of the entries of the inverse of the characteristic matrix of M. This can be calculated as

$$\chi_M(\lambda) = \mathbf{1}_n^T(\lambda I_n - M)^{-1}\mathbf{1}_n.$$

We also have the following useful result which can be applied, for instance, when considering the adjacency matrix of a signed graph that is net regular with net degree k.

Proposition 2.4 [13, Proposition 6]**.** *Let M be an $n \times n$ matrix with all row sums equal to k. Then*

$$\chi_M(\lambda) = \frac{n}{\lambda - k}.$$

The following lemma is known as Schur's complement formula and appears in many textbooks, see [5].

Lemma 2.5. *Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix such that A and D are square matrices. It follows that*

(a) *if A is an invertible matrix,* $\det(M) = \det(A)\det(D - CA^{-1}B)$*,*

(b) *if D is an invertible matrix,* $\det(M) = \det(D)\det(A - BD^{-1}C)$*.*

This formula can be further extended to simplify the computation of determinants of block matrices with various structures. By introducing a scalar component to a result presented by Hamud and Berman, see [8, Lemma 2.6], we have the following.

Proposition 2.6. *Let M be a block matrix*

$$M = \begin{pmatrix} A & B & \alpha J_{n_1 \times n_2} \\ B & C & O_{n_1 \times n_2} \\ \alpha J_{n_2 \times n_1} & O_{n_2 \times n_1} & D \end{pmatrix}$$

where A, B, C are square matrices of order n_1, D is a square matrix of order n_2, and α is any real number. Then for the determinant of the characteristic matrix of M we have

$$\det(\lambda I_{2n_1+n_2} - M) = \det(\lambda I_{n_2} - D) \det \begin{pmatrix} \lambda I_{n_1} - A - \alpha^2 \chi_D(\lambda) J_{n_1} & -B \\ -B & \lambda I_{n_1} - C \end{pmatrix}.$$

Proof. The characteristic matrix of M is

$$\lambda I_{2n_1+n_2} - M = \begin{pmatrix} \lambda I_{n_1} - A & -B & -\alpha J_{n_1 \times n_2} \\ -B & \lambda I_{n_1} - C & O_{n_1 \times n_2} \\ -\alpha J_{n_2 \times n_1} & O_{n_2 \times n_1} & \lambda I_{n_2} - D \end{pmatrix}.$$

Suppose that λ is not an eigenvalue of D. Then $\lambda I_{n_2} - D$ is invertible and by Schur's complement formula we obtain the determinant of the characteristic matrix of M as

$$\begin{aligned} \det(\lambda I_{2n_1+n_2} - M) &= \det(\lambda I_{n_2} - D) \det \left(\begin{pmatrix} \lambda I_{n_1} - A & -B \\ -B & \lambda I_{n_1} - C \end{pmatrix} \right. \\ &\quad \left. - \begin{pmatrix} -\alpha J_{n_1 \times n_2} \\ O_{n_1 \times n_2} \end{pmatrix} (\lambda I_{n_2} - D)^{-1} \begin{pmatrix} -\alpha J_{n_2 \times n_1} & O_{n_2 \times n_1} \end{pmatrix} \right) \\ &= \det(\lambda I_{n_2} - D) \det \left(\begin{pmatrix} \lambda I_{n_1} - A & -B \\ -B & \lambda I_{n_1} - C \end{pmatrix} \right. \\ &\quad \left. - \alpha^2 \begin{pmatrix} \mathbf{1}_{n_1} \mathbf{1}_{n_2}^T \\ O_{n_1 \times n_2} \end{pmatrix} (\lambda I_{n_2} - D)^{-1} \begin{pmatrix} \mathbf{1}_{n_2} \mathbf{1}_{n_1}^T & O_{n_2 \times n_1} \end{pmatrix} \right) \\ &= \det(\lambda I_{n_2} - D) \det \begin{pmatrix} \lambda I_{n_1} - A - \alpha^2 \chi_D(\lambda) J_{n_1} & -B \\ -B & \lambda I_{n_1} - C \end{pmatrix}. \end{aligned}$$

Since both sides are polynomials in λ that coincide for all λ that are not eigenvalues of D, and since the set of eigenvalues of D is finite, the two polynomials must coincide identically. ∎

The adjugate of a matrix A, defined as the transpose of its cofactor matrix, is denoted by $\operatorname{adj}(A)$, and satisfies the identity $A \operatorname{adj}(A) = \det(A)I$. Consequently, when A is invertible, we have $\operatorname{adj}(A) = \det(A)A^{-1}$. This final result is a special case of the matrix determinant lemma and can be found in textbooks such as [10].

Proposition 2.7. *If A is an $n \times n$ real matrix and α is a real number, then*

$$\det(A + \alpha J_n) = \det(A) + \alpha \mathbf{1}_n^T \operatorname{adj}(A)\mathbf{1}_n.$$

3. The Signed NS and NNS Joins

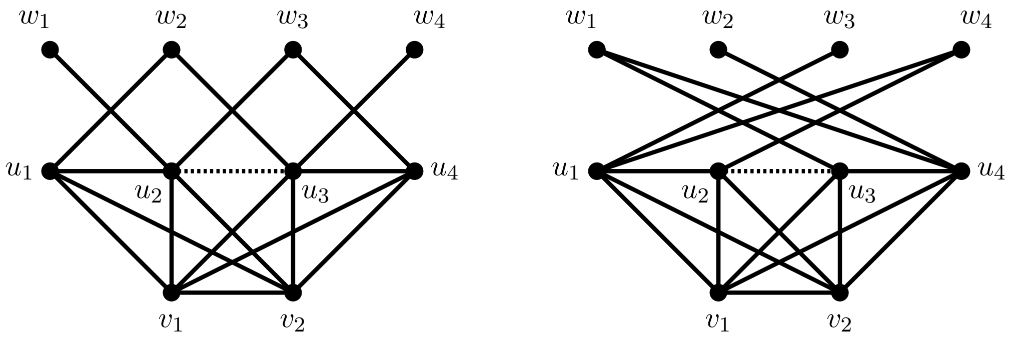
As described by Harary [9], the join of two ordinary graphs G_1 and G_2, denoted $G_1 \vee G_2$, is formed by connecting each vertex of G_1 to every vertex of G_2. In the case of signed graphs, however, there are multiple possible ways to define such a join, since the result depends on the signatures assigned to the newly added edges. We shall mainly focus on the join of two signed graphs Σ_1 and Σ_2 where each newly introduced edge is made to be positive. We will call this procedure the positive join of Σ_1 and Σ_2 and write it as $\Sigma_1 \vee_+ \Sigma_2$.

A variation on the standard join operation was introduced by Lu *et al.* [12] and called the splitting V-vertex join. Throughout the remainder of this paper, we will refer to this operation as the neighbours splitting (NS) join. A further variation, named the non-neighbours splitting (NNS) join, was defined by Hamud and Berman [8]. Both of these were introduced in the setting of ordinary unsigned graphs. A natural way of extending them to two signed graphs Σ_1 and Σ_2 is to follow the same constructions while assigning all of the newly introduced edges to be positive, which is precisely the approach we shall be considering in this section.

Definition 3.1 (Signed NS join, adapted from [12])**.** Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two vertex-disjoint signed graphs of order n_1 and n_2, respectively, and write $V_{\Sigma_1} = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_{\Sigma_2} = \{v_1, v_2, \ldots, v_{n_2}\}$. The signed neighbours splitting (NS) join of Σ_1 and Σ_2, denoted $\Sigma_1 \veebar \Sigma_2$, is the signed graph obtained by forming the positive join $\Sigma_1 \vee_+ \Sigma_2$, introducing the edgeless graph on n_1 vertices, which we shall label $\Sigma_3 = (\overline{K}_{n_1}, \sigma_3)$, where σ_3 is the empty function and $V_{\Sigma_3} = \{w_1, w_2, \ldots, w_{n_1}\}$, and connecting u_i to w_j with a positive edge if and only if u_i is adjacent to u_j in Σ_1.

We can see that symbolically for the NS join of $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$, we have the vertex set $V(\Sigma_1 \veebar \Sigma_2) = V_{\Sigma_1} \cup V_{\Sigma_2} \cup V_{\Sigma_3}$, the edge set

$$E(\Sigma_1 \veebar \Sigma_2) = E_{\Sigma_1} \cup E_{\Sigma_2} \cup \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\} \cup \{u_i w_j : u_i u_j \in E_{\Sigma_1}\},$$



$(P_4, \sigma_1) \veebar (P_2, \sigma_2)$ $(P_4, \sigma_1) \barwedge (P_2, \sigma_2)$

Figure 2. The NS and NNS joins $(P_4, \sigma_1) \veebar (P_2, \sigma_2)$ and $(P_4, \sigma_1) \barwedge (P_2, \sigma_2)$.

and the signature

$$\sigma(e) = \begin{cases} \sigma_1(e), & \text{if } e \in E_{\Sigma_1}, \\ \sigma_2(e), & \text{if } e \in E_{\Sigma_2}, \\ +1, & \text{if } e \in \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\} \cup \{u_i w_j : u_i u_j \in E_{\Sigma_1}\}. \end{cases}$$

Definition 3.2 (Signed NNS join, adapted from [8])**.** Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two vertex-disjoint signed graphs of order n_1 and n_2, respectively, and write $V_{\Sigma_1} = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_{\Sigma_2} = \{v_1, v_2, \ldots, v_{n_2}\}$. The signed non-neighbours splitting (NNS) join of Σ_1 and Σ_2, denoted $\Sigma_1 \barwedge \Sigma_2$, is the signed graph obtained by forming the positive join $\Sigma_1 \vee_+ \Sigma_2$, introducing the edgeless graph on n_1 vertices, which we shall label $\Sigma_3 = (\overline{K}_{n_1}, \sigma_3)$, where σ_3 is the empty function and $V_{\Sigma_3} = \{w_1, w_2, \ldots, w_{n_1}\}$, and connecting u_i to w_j with a positive edge if and only if $i \neq j$ and u_i is not adjacent to u_j in Σ_1.

Now we see that for the NNS join of $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$, we have the vertex set $V(\Sigma_1 \barwedge \Sigma_2) = V_{\Sigma_1} \cup V_{\Sigma_2} \cup V_{\Sigma_3}$, the edge set

$$\begin{aligned} E(\Sigma_1 \barwedge \Sigma_2) &= E_{\Sigma_1} \cup E_{\Sigma_2} \cup \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\} \\ &\cup \{u_i w_j : i \neq j \text{ and } u_i u_j \notin E_{\Sigma_1}\}, \end{aligned}$$

and the signature

$$\sigma(e) = \begin{cases} \sigma_1(e), & \text{if } e \in E_{\Sigma_1}, \\ \sigma_2(e), & \text{if } e \in E_{\Sigma_2}, \\ +1, & \text{if } e \in \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\} \\ & \qquad \cup \{u_i w_j : i \neq j \text{ and } u_i u_j \notin E_{\Sigma_1}\}. \end{cases}$$

3.1. Characteristic polynomials of the signed NS and NNS joins

Here we compute the adjacency and Laplacian characteristic polynomials of both $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \bar{\wedge} \Sigma_2$ for two arbitrary signed graphs $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$. These results are then used to construct pairs of nonregular $\{A, L\}$-cospectral mates.

3.1.1. Adjacency characteristic polynomials

Theorem 3.3. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be vertex-disjoint signed graphs on n_1 and n_2 vertices, respectively. Then we have the following.*

(a) *The adjacency characteristic polynomial of the NS join $\Sigma_1 \veebar \Sigma_2$ is*

$$P(A_{\Sigma_1 \veebar \Sigma_2}; \lambda) = \lambda^{n_1} P\big(A_{\Sigma_2}; \lambda\big) \det \Big(\lambda I_{n_1} - N_{\Sigma_1,\lambda}\Big) \Big[1 - \chi_{A_{\Sigma_2}}(\lambda) \chi_{N_{\Sigma_1,\lambda}}(\lambda)\Big],$$

where $N_{\Sigma_1,\lambda} = A_{\Sigma_1} + \frac{1}{\lambda} A_{G_1}^2$.

(b) *The adjacency characteristic polynomial of the NNS join $\Sigma_1 \bar{\wedge} \Sigma_2$ is*

$$P(A_{\Sigma_1 \bar{\wedge} \Sigma_2}; \lambda) = \lambda^{n_1} P\big(A_{\Sigma_2}; \lambda\big) \det \Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big) \Big[1 - \chi_{A_{\Sigma_2}}(\lambda) \chi_{\overline{N}_{\Sigma_1,\lambda}}(\lambda)\Big],$$

where $\overline{N}_{\Sigma_1,\lambda} = A_{\Sigma_1} + \frac{1}{\lambda} A_{\overline{G}_1}^2$.

Proof. By labelling the vertices of $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \bar{\wedge} \Sigma_2$ in a suitable order, we obtain the adjacency matrices

$$A_{\Sigma_1 \veebar \Sigma_2} = \begin{pmatrix} A_{\Sigma_1} & A_{G_1} & J_{n_1 \times n_2} \\ A_{G_1} & O_{n_1} & O_{n_1 \times n_2} \\ J_{n_2 \times n_1} & O_{n_2 \times n_1} & A_{\Sigma_2} \end{pmatrix},$$
$$A_{\Sigma_1 \bar{\wedge} \Sigma_2} = \begin{pmatrix} A_{\Sigma_1} & A_{\overline{G}_1} & J_{n_1 \times n_2} \\ A_{\overline{G}_1} & O_{n_1} & O_{n_1 \times n_2} \\ J_{n_2 \times n_1} & O_{n_2 \times n_1} & A_{\Sigma_2} \end{pmatrix}.$$

Owing to the resemblance between the corresponding matrices when the graph vertices are ordered appropriately, the two joins admit proofs of a similar structure. For this reason, we shall provide a full proof only for the NNS join.

Hence, by applying Proposition 2.6 with $\alpha = 1$, we get the adjacency characteristic polynomial

$$P(A_{\Sigma_1 \bar{\wedge} \Sigma_2}; \lambda) = \det(\lambda I_{n_2} - A_{\Sigma_2}) \det \begin{pmatrix} \lambda I_{n_1} - A_{\Sigma_1} - \chi_{A_{\Sigma_2}}(\lambda) J_{n_1} & -A_{\overline{G}_1} \\ -A_{\overline{G}_1} & \lambda I_{n_1} \end{pmatrix}.$$

Now we can apply Lemma 2.5 and then make use of Proposition 2.7 to acquire

$$\begin{aligned}P(A_{\Sigma_1 \bar{\wedge} \Sigma_2};\lambda) &= \det\left(\lambda I_{n_2} - A_{\Sigma_2}\right)\det\left(\lambda I_{n_1}\right)\det\left(\lambda I_{n_1} - A_{\Sigma_1} - \chi_{A_{\Sigma_2}}(\lambda)J_{n_1} - A_{\overline{G}_1}(\lambda I_{n_1})^{-1}A_{\overline{G}_1}\right)\\ &= \lambda^{n_1}P\big(A_{\Sigma_2};\lambda\big)\det\Big(\lambda I_{n_1} - A_{\Sigma_1} - \frac{1}{\lambda}A^2_{\overline{G}_1} - \chi_{A_{\Sigma_2}}(\lambda)J_{n_1}\Big)\\ &= \lambda^{n_1}P\big(A_{\Sigma_2};\lambda\big)\det\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda} - \chi_{A_{\Sigma_2}}(\lambda)J_{n_1}\Big)\\ &= \lambda^{n_1}P\big(A_{\Sigma_2};\lambda\big)\Big[\det\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big) - \chi_{A_{\Sigma_2}}(\lambda)\mathbf{1}^T_{n_1}\operatorname{adj}\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big)\mathbf{1}_{n_1}\Big]\\ &= \lambda^{n_1}P\big(A_{\Sigma_2};\lambda\big)\det\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big)\Big[1 - \chi_{A_{\Sigma_2}}(\lambda)\mathbf{1}^T_{n_1}\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big)^{-1}\mathbf{1}_{n_1}\Big].\end{aligned}$$

Therefore

$$P(A_{\Sigma_1 \bar{\wedge} \Sigma_2};\lambda) = \lambda^{n_1}P\big(A_{\Sigma_2};\lambda\big)\det\Big(\lambda I_{n_1} - \overline{N}_{\Sigma_1,\lambda}\Big)\Big[1 - \chi_{A_{\Sigma_2}}(\lambda)\chi_{\overline{N}_{\Sigma_1,\lambda}}(\lambda)\Big]. \quad \blacksquare$$

3.1.2. Laplacian characteristic polynomials

Theorem 3.4. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be vertex-disjoint signed graphs on n_1 and n_2 vertices, respectively. Then we have the following.*

(a) *The Laplacian characteristic polynomial of the NS join $\Sigma_1 \veebar \Sigma_2$ is*

$$\begin{aligned}P(L_{\Sigma_1 \veebar \Sigma_2};\mu) = &\det\Big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\Big)\det\Big(\mu I_{n_1} - D_{G_1}\Big)\\ &\det\Big((\mu - n_2)I_{n_1} - M_{\Sigma_1,\mu}\Big)\Big[1 - \chi_{L_{\Sigma_2}}(\mu - n_1)\chi_{M_{\Sigma_1,\mu}}(\mu - n_2)\Big],\end{aligned}$$

where $M_{\Sigma_1,\mu} = L_{\Sigma_1} + D_{G_1} + A_{G_1}\big(\mu I_{n_1} - D_{G_1}\big)^{-1}A_{G_1}$.

(b) *The Laplacian characteristic polynomial of the NNS join $\Sigma_1 \bar{\wedge} \Sigma_2$ is*

$$\begin{aligned}P(L_{\Sigma_1 \bar{\wedge} \Sigma_2};\mu) = &\det\Big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\Big)\det\Big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\Big)\\ &\det\Big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\Big)\\ &\Big[1 - \chi_{L_{\Sigma_2}}(\mu - n_1)\chi_{\overline{M}_{\Sigma_1,\mu}}(\mu - n_1 - n_2 + 1)\Big],\end{aligned}$$

where $\overline{M}_{\Sigma_1,\mu} = L_{\Sigma_1} - D_{G_1} + A_{\overline{G}_1}((\mu - n_1 + 1)I_{n_1} + D_{G_1})^{-1}A_{\overline{G}_1}$.

Proof. By labelling the vertices of $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \bar{\wedge} \Sigma_2$ in a suitable order, we obtain the Laplacian matrices

$$L_{\Sigma_1 \veebar \Sigma_2} = \begin{pmatrix} n_2 I_{n_1} + D_{G_1} + L_{\Sigma_1} & -A_{G_1} & -J_{n_1 \times n_2}\\ -A_{G_1} & D_{G_1} & O_{n_1 \times n_2}\\ -J_{n_2 \times n_1} & O_{n_2 \times n_1} & n_1 I_{n_2} + L_{\Sigma_2}\end{pmatrix},$$

$$L_{\Sigma_1 \bar{\wedge} \Sigma_2} = \begin{pmatrix} (n_1 + n_2 - 1)I_{n_1} + L_{\Sigma_1} - D_{G_1} & -A_{\overline{G}_1} & -J_{n_1 \times n_2}\\ -A_{\overline{G}_1} & (n_1 - 1)I_{n_1} - D_{G_1} & O_{n_1 \times n_2}\\ -J_{n_2 \times n_1} & O_{n_2 \times n_1} & n_1 I_{n_2} + L_{\Sigma_2}\end{pmatrix}.$$

As in the proof of Theorem 3.3, we shall provide full details only for the NNS join.

Hence, by applying Proposition 2.6, we get the Laplacian characteristic polynomial

$$P(L_{\Sigma_1 \bar{\wedge} \Sigma_2};\mu) = \det((\mu - n_1)I_{n_2} - L_{\Sigma_2}) \det\begin{pmatrix} (\mu - n_1 - n_2 + 1)I_{n_1} - L_{\Sigma_1} + D_{G_1} - \chi_{L_{\Sigma_2}}(\mu - n_1)J_{n_1} & A_{\overline{G}_1} \\ A_{\overline{G}_1} & (\mu - n_1 + 1)I_{n_1} + D_{G_1} \end{pmatrix}.$$

Now we can apply Lemma 2.5 and make use of Proposition 2.7 to acquire

$$\begin{aligned} P(L_{\Sigma_1 \bar{\wedge} \Sigma_2};\mu) = {} & \det\big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\big) \det\big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\big) \\ & \det\big((\mu - n_1 - n_2 + 1)I_{n_1} - L_{\Sigma_1} + D_{G_1} - \chi_{L_{\Sigma_2}}(\mu - n_1)J_{n_1} \\ & - A_{\overline{G}_1}((\mu - n_1 + 1)I_{n_1} + D_{G_1})^{-1}A_{\overline{G}_1}\big) \\ = {} & \det\big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\big) \det\big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\big) \\ & \Big[\det\big((\mu - n_1 - n_2 + 1)I_{n_1} - L_{\Sigma_1} + D_{G_1} \\ & - A_{\overline{G}_1}((\mu - n_1 + 1)I_{n_1} + D_{G_1})^{-1}A_{\overline{G}_1}\big) \\ & - \chi_{L_{\Sigma_2}}(\mu - n_1) \cdot \mathbf{1}_{n_1}^T \operatorname{adj}\big((\mu - n_1 - n_2 + 1)I_{n_1} \\ & - L_{\Sigma_1} + D_{G_1} - A_{\overline{G}_1}((\mu - n_1 + 1)I_{n_1} + D_{G_1})^{-1}A_{\overline{G}_1}\big)\mathbf{1}_{n_1}\Big] \\ = {} & \det\big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\big) \det\big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\big) \\ & \Big[\det\big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\big) \\ & - \chi_{L_{\Sigma_2}}(\mu - n_1)\mathbf{1}_{n_1}^T \operatorname{adj}\big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\big)\mathbf{1}_{n_1}\Big] \\ = {} & \det\big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\big) \det\big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\big) \\ & \det\big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\big) \\ & \Big[1 - \chi_{L_{\Sigma_2}}(\mu - n_1)\mathbf{1}_{n_1}^T\big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\big)^{-1}\mathbf{1}_{n_1}\Big]. \end{aligned}$$

Therefore

$$\begin{aligned} P(L_{\Sigma_1 \bar{\wedge} \Sigma_2};\mu) = {} & \det\Big((\mu - n_1)I_{n_2} - L_{\Sigma_2}\Big) \det\Big((\mu - n_1 + 1)I_{n_1} + D_{G_1}\Big) \\ & \det\Big((\mu - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,\mu}\Big) \\ & \Big[1 - \chi_{L_{\Sigma_2}}(\mu - n_1)\chi_{\overline{M}_{\Sigma_1,\mu}}(\mu - n_1 - n_2 + 1)\Big]. \end{aligned}$$

■

3.2. Constructing nonregular cospectral signed pairs

Now that we have computed the characteristic polynomials for the NS and NNS joins, we can use the results to construct nonregular nonisomorphic cospectral pairs of signed graphs.

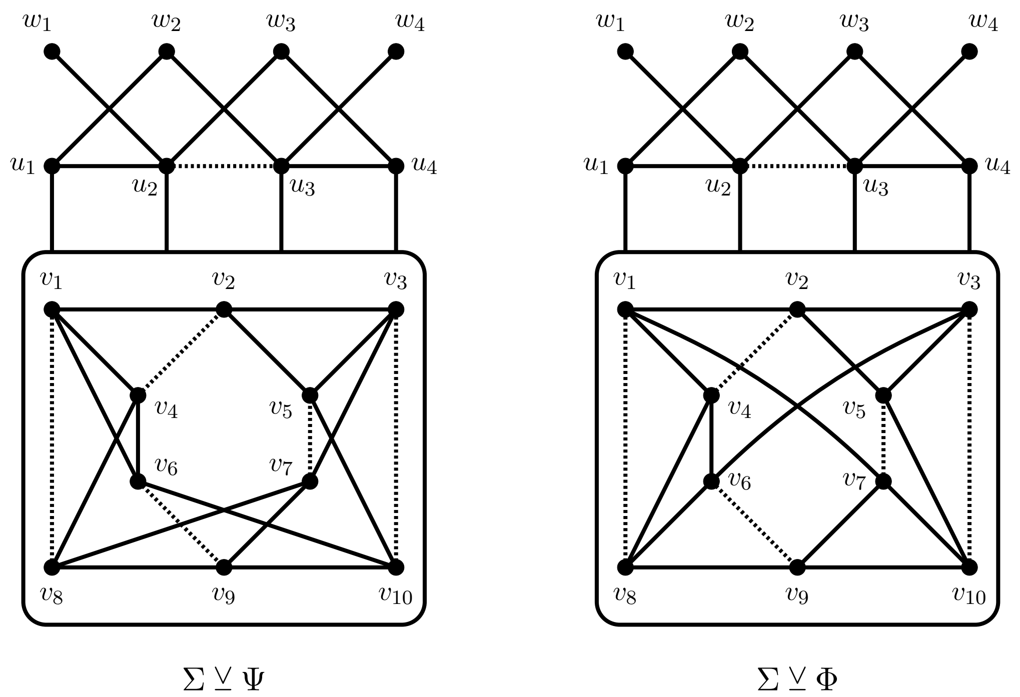


Figure 3. The NS joins $\Sigma \veebar \Psi$ and $\Sigma \veebar \Phi$, where $\Sigma = (P_4, \sigma)$ and Ψ, Φ are from Figure 1. A line from a vertex to a box indicates edges connecting the single vertex to all vertices in the box.

Corollary 3.5. *Let $\Psi = (F, \psi)$ and $\Phi = (H, \phi)$ be (r, k)-co-regular signed graphs that are cospectral mates. Then for every signed graph $\Sigma = (G, \sigma)$*

(a) *the graphs $\Sigma \veebar \Psi$ and $\Sigma \veebar \Phi$ are $\{A, L\}$-cospectral mates,*

(b) *the graphs $\Sigma \barwedge \Psi$ and $\Sigma \barwedge \Phi$ are $\{A, L\}$-cospectral mates.*

Proof. We shall consider (a). The NS join graphs $\Sigma \veebar \Psi$ and $\Sigma \veebar \Phi$ are nonisomorphic since Ψ and Φ are nonisomorphic. By Theorems 3.3 and 3.4, we know that

$$P(A_{\Sigma \veebar \Psi}; \lambda) = P(A_{\Sigma \veebar \Phi}; \lambda) \quad \text{and} \quad P(L_{\Sigma \veebar \Psi}; \mu) = P(L_{\Sigma \veebar \Phi}; \mu).$$

This is because from Proposition 2.4 the adjacency matrices A_Ψ and A_Φ have the same coronal by their shared k-net regularity and the Laplacian matrices L_Ψ and L_Φ have the same coronal by their shared (r, k)-co-regularity. By the cospectrality of Ψ and Φ they also give the same characteristic polynomials. This completes the proof of (a) and the proof of (b) is similar. ∎

Corollary 3.6. *Let $\Psi = (F, \psi)$ and $\Phi = (H, \phi)$ be (r, k)-co-regular signed graphs that are cospectral mates. Then for every signed graph $\Sigma = (G, \sigma)$*

(a) *the graphs* $\Psi \veebar \Sigma$ *and* $\Phi \veebar \Sigma$ *are* $\{A, L\}$*-cospectral mates,*

(b) *the graphs* $\Psi \barwedge \Sigma$ *and* $\Phi \barwedge \Sigma$ *are* $\{A, L\}$*-cospectral mates.*

Proof. The proof is similar to that of the previous corollary. ■

Remark 3.7. It is necessary to restrict the graphs Ψ and Φ in Corollaries 3.5 and 3.6 to being (r, k)-co-regular. To see this, consider the cospectral mates $K_{1,4}$ and $C_4 \cup K_1$, each with all-positive signature. By definition these share the same spectrum and hence their characteristic polynomials are equal. However, the values of their coronals are not. Thus, in general, for $\Sigma \veebar K_{1,4}$ and $\Sigma \veebar (C_4 \cup K_1)$, the characteristic polynomials of these join graphs are not necessarily equal, and therefore they may not be $\{A, L\}$-cospectral mates.

Example 3.8. The signed graphs depicted in Figure 1 are $(4, 2)$-co-regular and not switching isomorphic, recall that in Ψ there are edges that lie in three triangles while there are no such edges in Φ.

Let $\Sigma = (P_4, \sigma)$, with $\sigma(u_1u_2) = \sigma(u_3u_4) = +1$ and $\sigma(u_2u_3) = -1$. Then, taking the NS joins $\Sigma \veebar \Psi$ and $\Sigma \veebar \Phi$ gives the two signed graphs depicted in Figure 3. These graphs share the following adjacency characteristic polynomial.

$$\begin{aligned} P(A_{\Sigma\veebar\Psi}; \lambda) = P(A_{\Sigma\veebar\Phi}; \lambda) &= \lambda^{18} - 69\lambda^{16} - 96\lambda^{15} + 1182\lambda^{14} + 1936\lambda^{13} \\ &\quad - 8771\lambda^{12} - 14524\lambda^{11} + 32278\lambda^{10} + 49960\lambda^{9} \\ &\quad - 59603\lambda^{8} - 77616\lambda^{7} + 51914\lambda^{6} + 45824\lambda^{5} \\ &\quad - 17185\lambda^{4} - 7076\lambda^{3} + 1569\lambda^{2} + 312\lambda - 36. \end{aligned}$$

Therefore, $\Sigma \veebar \Psi$ and $\Sigma \veebar \Phi$ form a pair of cospectral mates.

3.3. Spectra of the signed NS and NNS joins

In this section we shall compute the adjacency and Laplacian spectra of graphs produced by the two join operations. Our analysis is restricted to joins of two (r, k)-co-regular signed graphs, as this allows for the application of Proposition 2.4, simplifying the computations. Without this restriction the calculations become considerably more difficult.

3.3.1. Adjacency spectrum of the signed NS join

Theorem 3.9. *Let* $\Sigma_1 = (G_1, \sigma_1)$ *be an* (r_1, k_1)*-co-regular signed graph on* n_1 *vertices and let* $\Sigma_2 = (G_2, \sigma_2)$ *be an* (r_2, k_2)*-co-regular signed graph on* n_2 *vertices. Then the adjacency spectrum of* $\Sigma_1 \veebar \Sigma_2$ *consists of*

(i) $\lambda_j(\Sigma_2)$ *for each* $j = 2, 3, \ldots, n_2$*,*

(ii) *the two roots of the quadratic equation* $x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1) = 0$ *for each* $i = 2, 3, \ldots, n_1$*,*

(iii) *the three roots of the cubic equation* $x^3 - (k_1 + k_2)x^2 - (r_1^2 - k_1k_2 + n_1n_2)x + k_2r_1^2 = 0$.

Proof. By Theorem 3.3, the adjacency characteristic polynomial of $\Sigma_1 \veebar \Sigma_2$ is

$$P(A_{\Sigma_1 \veebar \Sigma_2}; x) = x^{n_1} P\big(A_{\Sigma_2}; x\big) \det\Big(xI_{n_1} - N_{\Sigma_1,x}\Big)\Big[1 - \chi_{A_{\Sigma_2}}(x)\chi_{N_{\Sigma_1,x}}(x)\Big],$$

where $N_{\Sigma_1,x} = A_{\Sigma_1} + \frac{1}{x}A_{G_1}^2$.

Since we are working with co-regular signed graphs, we can apply Proposition 2.4 to obtain the adjacency characteristic polynomial as

$$\begin{aligned}
P(A_{\Sigma_1 \veebar \Sigma_2}; x) = & x^{n_1} \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \prod_{i=1}^{n_1} \Big(x - \lambda_i(\Sigma_1) - \frac{1}{x}\lambda_i^2(G_1)\Big) \\
& \left[1 - \frac{n_2}{x - k_2} \cdot \frac{n_1}{x - k_1 - \frac{1}{x}r_1^2}\right] \\
= & \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \prod_{i=1}^{n_1} \Big(x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1)\Big) \\
& \left[1 - \frac{n_1n_2x}{\big(x - k_2\big)\big(x^2 - k_1x - r_1^2\big)}\right] \\
= & \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \prod_{i=1}^{n_1} \Big(x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1)\Big) \\
& \left[\frac{\big(x - k_2\big)\big(x^2 - k_1x - r_1^2\big) - n_1n_2x}{\big(x - k_2\big)\big(x^2 - k_1x - r_1^2\big)}\right] \\
= & \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \prod_{i=1}^{n_1} \Big(x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1)\Big) \\
& \left[\frac{x^3 - \big(k_1 + k_2\big)x^2 - \big(r_1^2 + n_1n_2 - k_1k_2\big)x + r_1^2k_2}{\big(x - k_2\big)\big(x^2 - k_1x - r_1^2\big)}\right].
\end{aligned}$$

Now we can use Remark 2.1 to get

$$\begin{aligned}
P(A_{\Sigma_1 \veebar \Sigma_2}; x) = & \prod_{j=2}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \prod_{i=2}^{n_1} \Big(x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1)\Big) \\
& \Big[x^3 - \big(k_1 + k_2\big)x^2 - \big(r_1^2 - k_1k_2 + n_1n_2\big)x + k_2r_1^2\Big].
\end{aligned}$$

Hence we have the desired result. ■

3.3.2. Adjacency spectrum of the signed NNS join

Theorem 3.10. *Let $\Sigma_1 = (G_1, \sigma_1)$ be an (r_1, k_1)-co-regular signed graph on n_1 vertices and let $\Sigma_2 = (G_2, \sigma_2)$ be an (r_2, k_2)-co-regular signed graph on n_2 vertices. Then the adjacency spectrum of $\Sigma_1 \overline{\barwedge} \Sigma_2$ consists of*

(i) $\lambda_j(\Sigma_2)$ *for each* $j = 2, 3, \dots, n_2$,

(ii) *the two roots of the quadratic equation* $x^2 - \lambda_i(\Sigma_1)x - \lambda_i^2(G_1) - 2\lambda_i(G_1) - 1 = 0$ *for each* $i = 2, 3, \dots, n_1$,

(iii) *the three roots of the cubic equation* $x^3 - \big(k_1 + k_2\big)x^2 + \big(k_1k_2 - n_1n_2 - (n_1 - r_1 - 1)^2\big)x + k_2(n_1 - r_1 - 1)^2 = 0$.

Proof. By Theorem 3.3, the adjacency characteristic polynomial of $\Sigma_1 \overline{\barwedge} \Sigma_2$ is

$$P(A_{\Sigma_1 \overline{\barwedge} \Sigma_2}; x) = x^{n_1} P\big(A_{\Sigma_2}; x\big) \det\Big(xI_{n_1} - \overline{N}_{\Sigma_1, x}\Big)\Big[1 - \chi_{A_{\Sigma_2}}(x)\chi_{\overline{N}_{\Sigma_1,x}}(x)\Big],$$

where $\overline{N}_{\Sigma_1,x} = A_{\Sigma_1} + \frac{1}{x}A^2_{\overline{G_1}}$.

We can use the substitution $A_{\overline{G_1}} = J_{n_1} - I_{n_1} - A_{G_1}$ and, since we are working with co-regular signed graphs, we can apply Proposition 2.4 to obtain the adjacency characteristic polynomial as

$$\begin{aligned}
P(A_{\Sigma_1 \overline{\barwedge} \Sigma_2}; x) &= x^{n_1} \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \det\Big(xI_{n_1} - A_{\Sigma_1} - \frac{1}{x}(J_{n_1} - I_{n_1} - A_{G_1})^2\Big) \\
&\quad \Big[1 - \chi_{A_{\Sigma_2}}(x)\chi_{A_{\Sigma_1} + \frac{1}{x}(J_{n_1} - I_{n_1} - A_{G_1})^2}(x)\Big] \\
&= x^{n_1} \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \\
&\quad \det\Big(xI_{n_1} - A_{\Sigma_1} - \frac{1}{x}(J_{n_1}^2 - 2J_{n_1} + I_{n_1} - 2r_1J_{n_1} + 2A_{G_1} + A_{G_1}^2)\Big) \\
&\quad \Big[1 - \frac{n_2}{x - k_2} \cdot \frac{n_1}{x - k_1 - \frac{1}{x}(n_1 - r_1 - 1)^2}\Big] \\
&= x^{n_1} \prod_{j=1}^{n_2} \Big(x - \lambda_j(\Sigma_2)\Big) \\
&\quad \det\Big(\Big(x - \frac{1}{x}\Big) I_{n_1} - A_{\Sigma_1} - \frac{1}{x}(2A_{G_1} + A_{G_1}^2) - \frac{1}{x}(J_{n_1}^2 - 2J_{n_1} - 2r_1J_{n_1})\Big) \\
&\quad \Big[1 - \frac{n_1n_2x}{\big(x - k_2\big)\big(x^2 - k_1x - (n_1 - r_1 - 1)^2\big)}\Big].
\end{aligned}$$

Using Proposition 2.7, we can see that

$$\det\left(\left(x-\frac{1}{x}\right)I_{n_1}-A_{\Sigma_1}-\frac{1}{x}(2A_{G_1}+A_{G_1}^2)-\frac{1}{x}(J_{n_1}^2-2J_{n_1}-2r_1J_{n_1})\right)$$
$$=\det(B)-\frac{1}{x}(n_1-2-2r_1)\mathbf{1}_{n_1}^T\operatorname{adj}(B)\mathbf{1}_{n_1}$$

where

$$B=\left(x-\frac{1}{x}\right)I_{n_1}-A_{\Sigma_1}-\frac{1}{x}(2A_{G_1}+A_{G_1}^2).$$

Making use of this along with Proposition 2.4 again, we have

$$\begin{aligned}
P(A_{\Sigma_1\bar{\barwedge}\Sigma_2};x) = {} & x^{n_1}\prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x-\frac{1}{x}-\lambda_i(\Sigma_1)-\frac{2}{x}\lambda_i(G_1)-\frac{1}{x}\lambda_i^2(G_1)\Big)\\
& \left[1-\frac{1}{x}(n_1-2-2r_1)\chi_{\frac{1}{x}I_{n_1}+A_{\Sigma_1}+\frac{2}{x}A_{G_1}+\frac{1}{x}A_{G_1}^2}(x)\right]\\
& \left[1-\frac{n_1n_2x}{(x-k_2)\big(x^2-k_1x-(n_1-r_1-1)^2\big)}\right]\\
= {} & x^{n_1}\prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x-\frac{1}{x}-\lambda_i(\Sigma_1)-\frac{2}{x}\lambda_i(G_1)-\frac{1}{x}\lambda_i^2(G_1)\Big)\\
& \left[1-\frac{n_1-2-2r_1}{x}\cdot\frac{n_1}{x-\frac{1}{x}-k_1-\frac{2r_1}{x}-\frac{r_1^2}{x}}\right]\\
& \left[1-\frac{n_1n_2x}{(x-k_2)\big(x^2-k_1x-(n_1-r_1-1)^2\big)}\right]\\
= {} & x^{n_1}\prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x-\frac{1}{x}-\lambda_i(\Sigma_1)-\frac{2}{x}\lambda_i(G_1)-\frac{1}{x}\lambda_i^2(G_1)\Big)\\
& \left[1-\frac{n_1(n_1-2-2r_1)}{x^2-k_1x-r_1^2-2r_1-1}\right]\\
& \left[1-\frac{n_1n_2x}{(x-k_2)\big(x^2-k_1x-(n_1-r_1-1)^2\big)}\right]\\
= {} & x^{n_1}\prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x-\frac{1}{x}-\lambda_i(\Sigma_1)-\frac{2}{x}\lambda_i(G_1)-\frac{1}{x}\lambda_i^2(G_1)\Big)\\
& \left[\frac{x^2-k_1x-r_1^2-2r_1-1-n_1(n_1-2-2r_1)}{x^2-k_1x-r_1^2-2r_1-1}\right]\\
& \left[\frac{(x-k_2)\big(x^2-k_1x-(n_1-r_1-1)^2\big)-n_1n_2x}{(x-k_2)\big(x^2-k_1x-(n_1-r_1-1)^2\big)}\right]
\end{aligned}$$

$$= x^{n_1}\prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x-\frac{1}{x}-\lambda_i(\Sigma_1)-\frac{2}{x}\lambda_i(G_1)-\frac{1}{x}\lambda_i^2(G_1)\Big)$$
$$\Bigg[\frac{x^2-k_1x-(n_1-r_1-1)^2}{x^2-k_1x-r_1^2-2r_1-1}$$
$$\cdot\frac{\big(x-k_2\big)\big(x^2-k_1x-(n_1-r_1-1)^2\big)-n_1n_2x}{\big(x-k_2\big)\big(x^2-k_1x-(n_1-r_1-1)^2\big)}\Bigg]$$
$$= \prod_{j=1}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x^2-\lambda_i(\Sigma_1)x-\lambda_i^2(G_1)-2\lambda_i(G_1)-1\Big)$$
$$\Bigg[\frac{\big(x-k_2\big)\big(x^2-k_1x-(n_1-r_1-1)^2\big)-n_1n_2x}{\big(x-k_2\big)\big(x^2-k_1x-r_1^2-2r_1-1\big)}\Bigg].$$

Now we can use Remark 2.1 to get

$$\begin{aligned}P(A_{\Sigma_1\bar{\wedge}\Sigma_2};x) = &\prod_{j=2}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=2}^{n_1}\Big(x^2-\lambda_i(\Sigma_1)x-\lambda_i^2(G_1)-2\lambda_i(G_1)-1\Big)\\ &\Big[\big(x-k_2\big)\big(x^2-k_1x-(n_1-r_1-1)^2\big)-n_1n_2x\Big]\\ = &\prod_{j=2}^{n_2}\Big(x-\lambda_j(\Sigma_2)\Big)\prod_{i=2}^{n_1}\Big(x^2-\lambda_i(\Sigma_1)x-\lambda_i^2(G_1)-2\lambda_i(G_1)-1\Big)\\ &\Big[x^3-(k_1+k_2)x^2+\big(k_1k_2-n_1n_2-(n_1-r_1-1)^2\big)x+k_2(n_1-r_1-1)^2\Big].\end{aligned}$$

Hence we have the desired result. ■

3.3.3. Laplacian spectrum of the signed NS join

Theorem 3.11. *Let $\Sigma_1=(G_1,\sigma_1)$ be an (r_1,k_1)-co-regular signed graph on n_1 vertices and let $\Sigma_2=(G_2,\sigma_2)$ be an (r_2,k_2)-co-regular signed graph on n_2 vertices. Then the Laplacian spectrum of $\Sigma_1 \veebar \Sigma_2$ consists of*

(i) $n_1+\mu_j(\Sigma_2)$ *for each* $j=2,3,\ldots,n_2$,

(ii) *the two roots of the quadratic equation*

$$x^2-\big(n_2+2r_1+\mu_i(\Sigma_1)\big)x+r_1\mu_i(\Sigma_1)-\mu_i^2(G_1)+2r_1\mu_i(G_1)+n_2r_1=0$$

for each $i=2,3,\ldots,n_1$,

(iii) *the three roots of the cubic equation*

$$\begin{gathered}x^3+\big(k_1+k_2-n_1-n_2-3r_1-r_2\big)x^2+\big(r_1^2+k_1k_2-k_1n_1-k_1r_1\\ -k_1r_2-k_2n_2-3k_2r_1+3n_1r_1+n_2r_1+n_2r_2+3r_1r_2\big)x+k_2r_1^2\\ -n_1r_1^2-r_1^2r_2-k_1k_2r_1+k_1r_1r_2+k_2n_2r_1-n_2r_1r_2=0.\end{gathered}$$

Proof. By Theorem 3.4, the Laplacian characteristic polynomial of $\Sigma_1 \veebar \Sigma_2$ is

$$P(L_{\Sigma_1 \veebar \Sigma_2}; x) = \det\Big((x-n_1)I_{n_2} - L_{\Sigma_2}\Big) \det\Big(xI_{n_1} - D_{G_1}\Big)$$
$$\det\Big((x-n_2)I_{n_1} - M_{\Sigma_1,x}\Big)\Big[1 - \chi_{L_{\Sigma_2}}(x-n_1)\chi_{M_{\Sigma_1,x}}(x-n_2)\Big],$$

where $M_{\Sigma_1,x} = L_{\Sigma_1} + D_{G_1} + A_{G_1}\big(xI_{n_1} - D_{G_1}\big)^{-1}A_{G_1}$.

Since we are working with co-regular signed graphs, we can use the substitution $D_{G_1} = r_1 I_{n_1}$ and then apply Proposition 2.4 to obtain the Laplacian characteristic polynomial as

$$P(L_{\Sigma_1 \veebar \Sigma_2}; x) = \prod_{j=1}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)\det\Big((x-r_1)I_{n_1}\Big)$$
$$\det\Big((x-r_1-n_2)I_{n_1} - L_{\Sigma_1} - A_{G_1}((x-r_1)I_{n_1})^{-1}A_{G_1}\Big)$$
$$\Big[1 - \chi_{L_{\Sigma_2}}(x-n_1)\chi_{L_{\Sigma_1}+r_1I_{n_1}+A_{G_1}((x-r_1)I_{n_1})^{-1}A_{G_1}}(x-n_2)\Big]$$
$$= (x-r_1)^{n_1}\prod_{j=1}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)$$
$$\det\Big((x-r_1-n_2)I_{n_1} - L_{\Sigma_1} - \frac{1}{x-r_1}A_{G_1}^2\Big)$$
$$\Big[1 - \chi_{L_{\Sigma_2}}(x-n_1)\chi_{L_{\Sigma_1}+\frac{1}{x-r_1}A_{G_1}^2}(x-r_1-n_2)\Big]$$
$$= (x-r_1)^{n_1}\prod_{j=1}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)$$
$$\prod_{i=1}^{n_1}\Big(x - r_1 - n_2 - \mu_i(\Sigma_1) - \frac{1}{x-r_1}(r_1 - \mu_i(G_1))^2\Big)$$
$$\left[1 - \frac{n_2}{x-n_1-(r_2-k_2)}\cdot\frac{n_1}{x-r_1-n_2-(r_1-k_1)-\frac{r_1^2}{x-r_1}}\right]$$
$$= \prod_{j=1}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)\prod_{i=1}^{n_1}\Big(x(x-r_1) - r_1(x-r_1) - n_2(x-r_1)$$
$$- \mu_i(\Sigma_1)(x-r_1) - (r_1 - \mu_i(G_1))^2\Big)$$
$$\left[1 - \frac{n_1n_2(x-r_1)}{(x-n_1-(r_2-k_2))}\right.$$
$$\left.\cdot\frac{1}{(x(x-r_1) - r_1(x-r_1) - n_2(x-r_1) - (r_1-k_1)(x-r_1) - r_1^2)}\right]$$

$$= \prod_{j=1}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)$$
$$\prod_{i=1}^{n_1}\Big(x^2 - (2r_1 + n_2 + \mu_i(\Sigma_1))x + n_2r_1 + r_1\mu_i(\Sigma_1) - \mu_i^2(G_1) + 2r_1\mu_i(G_1)\Big)$$
$$\Bigg[1 - \frac{n_1n_2(x - r_1)}{\big(x - n_1 - (r_2 - k_2)\big)}$$
$$\cdot\frac{1}{\big(x^2 - (2r_1 + n_2 + (r_1 - k_1))x + n_2r_1 + r_1(r_1 - k_1)\big)}\Bigg].$$

Now we can use Remark 2.2 to get

$$\begin{aligned}
P(L_{\Sigma_1\veebar\Sigma_2}; x) = {} & \prod_{j=2}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)\prod_{i=2}^{n_1}\Big(x^2 - (2r_1 + n_2 + \mu_i(\Sigma_1))x + n_2r_1 \\
& + r_1\mu_i(\Sigma_1) - \mu_i^2(G_1) + 2r_1\mu_i(G_1)\Big) \\
& \quad\Big[\big(x - n_1 - (r_2 - k_2)\big)\big(x^2 - (2r_1 + n_2 + (r_1 - k_1))x \\
& + n_2r_1 + r_1(r_1 - k_1)\big) - n_1n_2(x - r_1)\Big] \\
= {} & \prod_{j=2}^{n_2}\Big(x - n_1 - \mu_j(\Sigma_2)\Big)\prod_{i=2}^{n_1}\Big(x^2 - (n_2 + 2r_1 + \mu_i(\Sigma_1))x \\
& + r_1\mu_i(\Sigma_1) - \mu_i^2(G_1) + 2r_1\mu_i(G_1) + n_2r_1\Big) \\
& \quad\Big[x^3 + \big(k_1 + k_2 - n_1 - n_2 - 3r_1 - r_2\big)x^2 + \big(r_1^2 + k_1k_2 \\
& - k_1n_1 - k_1r_1 - k_1r_2 - k_2n_2 - 3k_2r_1 + 3n_1r_1 + n_2r_1 \\
& + n_2r_2 + 3r_1r_2\big)x + k_2r_1^2 - n_1r_1^2 - r_1^2r_2 - k_1k_2r_1 \\
& + k_1r_1r_2 + k_2n_2r_1 - n_2r_1r_2\Big].
\end{aligned}$$

Hence we have the desired result. ■

3.3.4. Laplacian spectrum of the signed NNS join

Theorem 3.12. *Let $\Sigma_1 = (G_1, \sigma_1)$ be an (r_1, k_1)-co-regular signed graph on n_1 vertices and let $\Sigma_2 = (G_2, \sigma_2)$ be an (r_2, k_2)-co-regular signed graph on n_2 vertices. Then the Laplacian spectrum of $\Sigma_1 \overline{\barwedge} \Sigma_2$ consists of*

(i) *$n_1 + \mu_j(\Sigma_2)$ for each $j = 2, 3, \ldots, n_2$,*

(ii) *the two roots of the quadratic equation*

$$\begin{aligned} x^2 &- \big(2n_1 + n_2 - 2r_1 - 2 + \mu_i(\Sigma_1)\big)x + (n_1 - r_1 - 1)\mu_i(\Sigma_1) - \mu_i^2(G_1) \\ &+ (2r_1 + 2)\mu_i(G_1) + n_1^2 - 2n_1 - n_2 + n_1n_2 - 2n_1r_1 - n_2r_1 = 0, \end{aligned}$$

for each $i = 2, 3, \ldots, n_1$,

(iii) *the three roots of the cubic equation*

$$\begin{aligned} x^3 &+ \big(k_1 + k_2 - 3n_1 - n_2 + r_1 - r_2 + 2\big)x^2 + \big(2n_1^2 - r_1^2 + k_1 \\ &+ 2k_2 - 2n_1 - n_2 - r_1 - 2r_2 + k_1k_2 - 2k_1n_1 + k_1r_1 - k_1r_2 - 2k_2n_1 \\ &- k_2n_2 + k_2r_1 + n_1n_2 + 2n_1r_2 - n_2r_1 + n_2r_2 - r_1r_2\big)x + k_1n_1^2 - k_2r_1^2 \\ &- n_1^2r_1 + n_1r_1^2 + r_1^2r_2 + k_1k_2 - k_1n_1 - k_1r_2 - k_2n_2 - k_2r_1 + n_1r_1 \\ &+ n_2r_2 + r_1r_2 - k_1k_2n_1 + k_1k_2r_1 - k_1n_1r_1 + k_1n_1r_2 - k_1r_1r_2 \\ &+ k_2n_1n_2 + k_2n_1r_1 - k_2n_2r_1 - n_1n_2r_2 - n_1r_1r_2 + n_2r_1r_2 = 0. \end{aligned}$$

Proof. By Theorem 3.4, the Laplacian characteristic polynomial of $\Sigma_1 \barwedge \Sigma_2$ is

$$\begin{aligned} P(L_{\Sigma_1 \barwedge \Sigma_2}; x) = {} & \det\Big((x - n_1)I_{n_2} - L_{\Sigma_2}\Big) \det\Big((x - n_1 + 1)I_{n_1} + D_{G_1}\Big) \\ & \det\Big((x - n_1 - n_2 + 1)I_{n_1} - \overline{M}_{\Sigma_1,x}\Big) \\ & \Big[1 - \chi_{L_{\Sigma_2}}(x - n_1)\chi_{\overline{M}_{\Sigma_1,x}}(x - n_1 - n_2 + 1)\Big], \end{aligned}$$

where $\overline{M}_{\Sigma_1,x} = L_{\Sigma_1} - D_{G_1} + A_{\overline{G}_1}((x - n_1 + 1)I_{n_1} + D_{G_1})^{-1}A_{\overline{G}_1}$.

Using the substitution $D_{G_1} = r_1 I_{n_1}$ and then Proposition 2.4 yields

$$\begin{aligned} P(L_{\Sigma_1 \barwedge \Sigma_2}; x) = {} & \det\Big((x - n_1)I_{n_2} - L_{\Sigma_2}\Big) \det\Big((x - n_1 + r_1 + 1)I_{n_1}\Big) \\ & \det\Big((x - n_1 - n_2 + r_1 + 1)I_{n_1} - L_{\Sigma_1} - \frac{1}{x - n_1 + r_1 + 1}A_{\overline{G}_1}^2\Big) \\ & \Big[1 - \chi_{L_{\Sigma_2}}(x - n_1)\chi_{L_{\Sigma_1} + \frac{1}{x - n_1 + r_1 + 1}A_{\overline{G}_1}^2}(x - n_1 - n_2 + r_1 + 1)\Big] \\ = {} & (x - n_1 + r_1 + 1)^{n_1} \det\Big((x - n_1)I_{n_2} - L_{\Sigma_2}\Big) \\ & \det\Big((x - n_1 - n_2 + r_1 + 1)I_{n_1} - L_{\Sigma_1} - \frac{1}{x - n_1 + r_1 + 1}A_{\overline{G}_1}^2\Big) \\ & \Bigg[1 - \frac{n_2}{x - n_1 - (r_2 - k_2)} \\ & \cdot \frac{n_1}{x - n_1 - n_2 + r_1 + 1 - (r_1 - k_1) - \frac{(n_1 - r_1 - 1)^2}{x - n_1 + r_1 + 1}}\Bigg]. \end{aligned}$$

Since $A_{G_1} = J_{n_1} - I_{n_1} - A_{\overline{G}_1}$ and $L_{G_1} = D_{G_1} - A_{G_1}$, we can rewrite the adjacency matrix of the complement of G_1 as $A_{\overline{G}_1} = L_{G_1} + J_{n_1} - D_{G_1} - I_{n_1}$. Using this, we acquire

$$\begin{aligned}
P(L_{\Sigma_1 \bar{\wedge} \Sigma_2}; x) = {} & (x - n_1 + r_1 + 1)^{n_1} \prod_{j=1}^{n_2} \Big(x - n_1 - \mu_j(\Sigma_2)\Big) \\
& \prod_{i=1}^{n_1} \Big(x - n_1 - n_2 + r_1 + 1 - \mu_i(\Sigma_1) - \frac{(\mu_i(G_1) - r_1 - 1)^2}{x - n_1 - r_1 + 1}\Big) \\
& \Bigg[1 - \frac{n_1 n_2}{\big(x - n_1 - (r_2 - k_2)\big)} \\
& \cdot \frac{1}{\big(x - n_1 - n_2 + r_1 + 1 - (r_1 - k_1) - \frac{(n_1 - r_1 - 1)^2}{x - n_1 + r_1 + 1}\big)}\Bigg] \\
= {} & \prod_{j=1}^{n_2} \Big(x - n_1 - \mu_j(\Sigma_2)\Big) \\
& \prod_{i=1}^{n_1} \Big((x - n_1 - n_2 + r_1 + 1 - \mu_i(\Sigma_1))(x - n_1 + r_1 + 1) \\
& \qquad - (\mu_i(G_1) - r_1 - 1)^2\Big) \\
& \Bigg[1 - \frac{n_1 n_2 (x - n_1 + r_1 + 1)}{\big(x - n_1 - (r_2 - k_2)\big)} \\
& \cdot \frac{1}{\big((x - n_1 - n_2 + r_1 + 1 - (r_1 - k_1)\big)} \\
& \cdot \frac{1}{(x - n_1 + r_1 + 1) - (n_1 - r_1 - 1)^2\big)}\Bigg].
\end{aligned}$$

Now we can use Remark 2.2 to get

$$\begin{aligned}
P(L_{\Sigma_1 \bar{\wedge} \Sigma_2}; x) = {} & \prod_{j=2}^{n_2} \Big(x - n_1 - \mu_j(\Sigma_2)\Big) \\
& \prod_{i=2}^{n_1} \Big((x - n_1 - n_2 + r_1 + 1 - \mu_i(\Sigma_1))(x - n_1 + r_1 + 1) \\
& - (\mu_i(G_1) - r_1 - 1)^2\Big) \\
& \Big[\big(x - n_1 - (r_2 - k_2)\big)\big((x - n_1 - n_2 + r_1 + 1 - (r_1 - k_1)\big)
\end{aligned}$$

$$
\begin{aligned}
&\cdot (x - n_1 + r_1 + 1) - (n_1 - r_1 - 1)^2\Big) \\
&- n_1 n_2 (x - n_1 + r_1 + 1)\Big] \\
= &\prod_{j=2}^{n_2} \Big(x - n_1 - \mu_j(\Sigma_2)\Big) \\
&\prod_{i=2}^{n_1} \Big(x^2 - \big(2n_1 + n_2 - 2r_1 - 2 + \mu_i(\Sigma_1)\big)x \\
&+ (n_1 - r_1 - 1)\mu_i(\Sigma_1) - \mu_i^2(G_1) + (2r_1 + 2)\mu_i(G_1) \\
&+ n_1^2 - 2n_1 - n_2 + n_1 n_2 - 2n_1 r_1 - n_2 r_1\Big) \\
&\Big[x^3 + \big(k_1 + k_2 - 3n_1 - n_2 + r_1 - r_2 + 2\big)x^2 + \big(2n_1^2 - r_1^2 + k_1 \\
&+ 2k_2 - 2n_1 - n_2 - r_1 - 2r_2 + k_1 k_2 - 2k_1 n_1 + k_1 r_1 - k_1 r_2 \\
&- 2k_2 n_1 - k_2 n_2 + k_2 r_1 + n_1 n_2 + 2n_1 r_2 - n_2 r_1 + n_2 r_2 - r_1 r_2\big)x \\
&+ k_1 n_1^2 - k_2 r_1^2 - n_1^2 r_1 + n_1 r_1^2 + r_1^2 r_2 + k_1 k_2 - k_1 n_1 - k_1 r_2 \\
&- k_2 n_2 - k_2 r_1 + n_1 r_1 + n_2 r_2 + r_1 r_2 - k_1 k_2 n_1 + k_1 k_2 r_1 \\
&- k_1 n_1 r_1 + k_1 n_1 r_2 - k_1 r_1 r_2 + k_2 n_1 n_2 + k_2 n_1 r_1 - k_2 n_2 r_1 \\
&- n_1 n_2 r_2 - n_1 r_1 r_2 + n_2 r_1 r_2\Big].
\end{aligned}
$$

Hence we have the desired result. ■

4. Switching-Stable Signed NS and NNS Joins

It can be shown, see, for example, [15], where the result is established in the more general setting of complex unit gain graphs, that a signed graph $\Sigma = (G, \sigma)$ is balanced if and only if it admits a potential function, that is, a function $U\colon V_G \to \{+1, -1\}$ satisfying $\sigma(uv) = U(u)U(v)$ for every pair of adjacent vertices $u, v \in G$.

While the joins studied in the previous section achieve the goal of enabling constructions of nonregular cospectral signed graphs, the operations themselves are not stable under switchings of their component parts. In this section we use pseudo-potential functions defined by Cavaleri *et al.* [3] to define versions of the NS and NNS joins which are switching stable.

4.1. Pseudo-potential functions

Pseudo-potential functions on signed graphs were defined by Cavaleri *et al.* [3] as follows.

Definition 4.1 [3, Definition 2.1]**.** Let $\Sigma = (G, \sigma)$ be a connected signed graph. Fix a spanning tree $\mathcal{T}$ of G and fix a vertex v in V_G. The pseudo-potential $U_{\mathcal{T},v,\sigma}\colon V_G \to \{+1, -1\}$ is the function defined, for each w in V_G, as

$$U_{\mathcal{T},v,\sigma}(w) = \sigma(\gamma_{vw}),$$

where γ_{vw} is any walk contained in $\mathcal{T}$ from v to w.

For a signed graph Σ that is not connected, the definition of a pseudo-potential requires a maximal spanning forest rather than a single spanning tree, and a set of fixed vertices, $\overline{v} = \{v_1, v_2, \ldots, v_r\} \subseteq V_G$ (one chosen from each connected component) instead of a single vertex.

With a slight abuse of notation, denoting by the same symbol $U_{\mathcal{T},v,\sigma}$ the diagonal matrix such that $(U_{\mathcal{T},v,\sigma})_{ww} = U_{\mathcal{T},v,\sigma}(w)$, we have the following lemma.

Lemma 4.2 [3, Lemma 2.3]**.** *Let $\Sigma = (G, \sigma)$ be a signed graph and let $\mathcal{T}$ be a spanning tree of G. Then*

(a) *if G is connected, then either $U_{\mathcal{T},v,\sigma} = U_{\mathcal{T},v',\sigma}$ or $U_{\mathcal{T},v,\sigma} = -U_{\mathcal{T},v',\sigma}$ for all $v, v' \in V_G$,*

(b) *if Σ is balanced, then for any spanning tree $\mathcal{T}'$ of G we have $U_{\mathcal{T},v,\sigma} = U_{\mathcal{T}',v,\sigma}$,*

(c) *if σ and σ' are two switching-equivalent signatures on G and v is a fixed vertex in V_G, then there exists a diagonal matrix S, whose entries are either $+1$ or -1, such that $A_{\sigma'} = SA_\sigma S$ and $U_{\mathcal{T},v,\sigma'} = SU_{\mathcal{T},v,\sigma} = U_{\mathcal{T},v,\sigma}S$.*

The authors also define what it means for a signed graph with a spanning tree to be $\mathcal{T}$-positive.

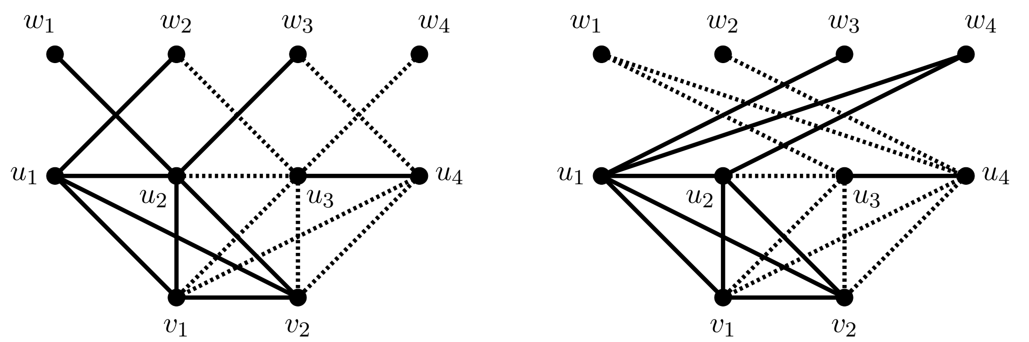
Definition 4.3 [3, Definition 2.5]**.** Let $\Sigma = (G, \sigma)$ be a signed graph and let $\mathcal{T}$ be a spanning tree of G. Then Σ is $\mathcal{T}$-positive if $\sigma(uv) = 1$ for each edge uv in $\mathcal{T}$.

Lemma 4.4 [16, Lemma 3.1]**.** *Let G be a graph and let $\mathcal{T}$ be a maximal forest. Each switching equivalence class of signed graphs on G has a unique representative which is $\mathcal{T}$-positive.*

4.2. Switching-stable definitions

Following the work done in [3], where pseudo-potentials are applied to build a definition of the lexicographic product of signed graphs which is stable under switching equivalence, we can develop a method for defining switching-stable versions of the signed NS and NNS joins.

Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two vertex-disjoint signed graphs of order n_1 and n_2, respectively, and write $V_{\Sigma_1} = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_{\Sigma_2} = \{v_1, v_2, \ldots, v_{n_2}\}$.



$(P_4, \sigma_1) \veebar_{\mathcal{T},\overline{v}} (P_2, \sigma_2)$ $(P_4, \sigma_1) \barwedge_{\mathcal{T},\overline{v}} (P_2, \sigma_2)$

Figure 4. The switching-stable NS and NNS joins $(P_4, \sigma_1) \veebar_{\mathcal{T},\overline{v}} (P_2, \sigma_2)$ and $(P_4, \sigma_1) \barwedge_{\mathcal{T},\overline{v}} (P_2, \sigma_2)$, where $\mathcal{T}_1 = P_4$, $\mathcal{T}_2 = P_2$, and $\overline{v} = \{u_1, v_1\}$.

Fix spanning trees $\mathcal{T}_1$ of Σ_1 and $\mathcal{T}_2$ of Σ_2 and set $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$, fix vertices $\overline{v} = \{u_i, v_j\}$ with u_i in V_{Σ_1} and v_j in V_{Σ_2}, and take the pseudo-potential functions $U_1 = U_{\mathcal{T}_1,u_i,\sigma_1} \colon V_{\Sigma_1} \to \{+1, -1\}$ as $U_1(u_k) = \sigma_1(\gamma_{u_i u_k})$ for each u_k in V_{Σ_1} and $U_2 = U_{\mathcal{T}_2,v_j,\sigma_2} \colon V_{\Sigma_2} \to \{+1, -1\}$ as $U_2(v_l) = \sigma_2(\gamma_{v_j v_l})$ for each v_l in V_{Σ_2}.

Introduce the edgeless graph on n_1 vertices, which we shall label $\Sigma_3 = (\overline{K}_{n_1}, \sigma_3)$, where σ_3 is the empty function and $V_{\Sigma_3} = \{w_1, w_2, \ldots, w_{n_1}\}$. As there are no edges in this graph, we shall take the pseudo-potential values to be $U_3(w_j) = 1$ for each $j = 1, 2, \ldots, n_1$.

Definition 4.5. The switching-stable signed neighbours splitting (ss-NS) join of Σ_1 and Σ_2, denoted $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$, is the signed graph obtained by taking the same underlying graph as the standard NS join and applying the signature

$$\sigma_{\mathcal{T},\overline{v}}(e) = \begin{cases} \sigma_1(e), & \text{if } e \in E_{\Sigma_1}, \\ \sigma_2(e), & \text{if } e \in E_{\Sigma_2}, \\ U_1(u_i)U_2(v_j), & \text{if } e \in \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\}, \\ U_1(u_i)U_3(w_j), & \text{if } e \in \{u_i w_j : u_i u_j \in E_{\Sigma_1}\}. \end{cases}$$

Definition 4.6. The switching-stable signed non-neighbours splitting (ss-NNS) join of Σ_1 and Σ_2, denoted $\Sigma_1 \barwedge_{\mathcal{T},\overline{v}} \Sigma_2$, is the signed graph obtained by taking the same underlying graph as the standard NNS join and applying the signature

$$\sigma_{\mathcal{T},\overline{v}}(e) = \begin{cases} \sigma_1(e), & \text{if } e \in E_{\Sigma_1}, \\ \sigma_2(e), & \text{if } e \in E_{\Sigma_2}, \\ U_1(u_i)U_2(v_j), & \text{if } e \in \{u_i v_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\}, \\ U_1(u_i)U_3(w_j), & \text{if } e \in \{u_i w_j : i \neq j \text{ and } u_i u_j \notin E_{\Sigma_1}\}. \end{cases}$$

Now that we have the definitions, we can present the main theorem of this section.

Theorem 4.7. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two vertex-disjoint signed graphs on n_1 and n_2 vertices, respectively. Then the ss-NS and ss-NNS join operations of Σ_1 and Σ_2 are switching stable. That is, replacing either Σ_1 or Σ_2 with some switching-equivalent Σ_1' or Σ_2' leaves the switching equivalence classes of $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$ and $\Sigma_1 \barwedge_{\mathcal{T},\overline{v}} \Sigma_2$ unchanged.*

Proof. Since the proofs for the two joins are quite similar, we shall present a full proof only for the ss-NS join.

Suppose that we replace σ_1 and σ_2 with switching-equivalent signatures σ_1' and σ_2', respectively. Then there exist functions $s_1\colon V_{\Sigma_1} \to \{+1, -1\}$ and $s_2\colon V_{\Sigma_2} \to \{+1, -1\}$ such that

$$\sigma_1'(u_iu_j) = s_1(u_i)\sigma_1(u_iu_j)s_1(u_j) \quad \text{and} \quad \sigma_2'(v_iv_j) = s_2(v_i)\sigma_2(v_iv_j)s_2(v_j).$$

Since there are no edges to consider in $\Sigma_3 = (\overline{K}_{n_1}, \sigma_3)$, we shall define the arbitrary function $s_3\colon V_{\Sigma_3} \to \{+1, -1\}$.

Lemma 4.2(c) shows that the pseudo-potentials, which are determined by the signatures on the spanning trees, change under switching as follows

$$\begin{aligned} U_1'(u) &= s_1(u)U_1(u) = U_1(u)s_1(u), \\ U_2'(v) &= s_2(v)U_2(v) = U_2(v)s_2(v), \\ U_3'(w) &= s_3(w)U_3(w) = U_3(w)s_3(w). \end{aligned}$$

With this, we have the new signature of the ss-NS join as

$$\sigma'_{\mathcal{T},\overline{v}}(e) = \begin{cases} s_1(u_i)\sigma_1(u_iu_j)s_1(u_j), & \text{if } e = u_iu_j \in E_{\Sigma_1}, \\ s_2(v_i)\sigma_2(v_iv_j)s_2(v_j), & \text{if } e = v_iv_j \in E_{\Sigma_2}, \\ s_1(u_i)U_1(u_i)U_2(v_j)s_2(v_j), & \text{if } e \in \{u_iv_j : u_i \in V_{\Sigma_1}, v_j \in V_{\Sigma_2}\}, \\ s_1(u_i)U_1(u_i)U_3(w_j)s_3(w_j), & \text{if } e \in \{u_iw_j : u_iu_j \in E_{\Sigma_1}\}. \end{cases}$$

So, the signature of each edge in the ss-NS join transforms according to

$$\sigma'_{\mathcal{T},\overline{v}}(xy) = s(x)\sigma_{\mathcal{T},\overline{v}}(xy)s(y),$$

where

$$s(x) = \begin{cases} s_1(x), & \text{if } x \in V_{\Sigma_1}, \\ s_2(x), & \text{if } x \in V_{\Sigma_2}, \\ s_3(x), & \text{if } x \in V_{\Sigma_3}. \end{cases}$$

Let $W = x_1x_2\cdots x_{k+1}$ be any closed walk of length k in $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$. Its sign is given by

$$\sigma_{\mathcal{T},\overline{v}}(W) = \sigma_{\mathcal{T},\overline{v}}(x_1x_2)\sigma_{\mathcal{T},\overline{v}}(x_2x_3)\cdots\sigma_{\mathcal{T},\overline{v}}(x_kx_{k+1}) = \prod_{i=1}^{k}\sigma_{\mathcal{T},\overline{v}}(x_ix_{i+1}).$$

After switching, we have

$$\begin{aligned}\sigma'_{\mathcal{T},\overline{v}}(W) &= \sigma'_{\mathcal{T},\overline{v}}(x_1x_2)\sigma'_{\mathcal{T},\overline{v}}(x_2x_3)\cdots\sigma'_{\mathcal{T},\overline{v}}(x_kx_{k+1})\\ &= (s(x_1)\sigma_{\mathcal{T},\overline{v}}(x_1x_2)s(x_2))(s(x_2)\sigma_{\mathcal{T},\overline{v}}(x_2x_3)s(x_3))\\ &\quad\cdots(s(x_{k-1})\sigma_{\mathcal{T},\overline{v}}(x_{k-1}x_k)s(x_k))(s(x_k)\sigma_{\mathcal{T},\overline{v}}(x_kx_{k+1})s(x_{k+1}))\\ &= \prod_{i=1}^{k}s(x_i)\sigma_{\mathcal{T},\overline{v}}(x_ix_{i+1})s(x_{i+1}) = \sigma_{\mathcal{T},\overline{v}}(W)\prod_{i=1}^{k}s(x_i)s(x_{i+1}).\end{aligned}$$

Because the walk is closed, the switching factor $s(x)$ appears twice for each vertex x in the walk (once upon arrival and once again upon leaving) and so they multiply to give $\prod_{i=1}^{k}s(x_i)s(x_{i+1}) = 1$. Hence $\sigma'_{\mathcal{T},\overline{v}}(W) = \sigma_{\mathcal{T},\overline{v}}(W)$ for all closed walks in $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$. Therefore, the two signatures are switching equivalent and the operation is switching stable. ■

Since the operations are switching stable, to compute the spectra of the ss-NS and ss-NNS joins, one can, without loss of generality, consider the graphs obtained by replacing Σ_1 and Σ_2 with the $\mathcal{T}$-positive representatives of their switching equivalence classes, which we know exist (and are unique) thanks to Lemma 4.4. In this specific case the switching stable and standard variations coincide and as a result the spectral analysis done in the previous section also applies in this case.

4.3. Balancedness of the various joins

We now present characterisations of balancedness for the neighbours splitting and non-neighbours splitting joins, considering both the standard and switching-stable variations. Consequently, we see that the conditions are less restrictive in the cases of the switching-stable signatures.

Theorem 4.8. *Let $\Sigma_1 = (G_1, \sigma_1)$ and $\Sigma_2 = (G_2, \sigma_2)$ be two vertex-disjoint graphs such that the union $E_{G_1} \cup E_{G_2}$ is non-empty. Then the join graphs $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \barwedge \Sigma_2$ are balanced if and only if the signatures on Σ_1 and Σ_2 are both all-positive.*

Proof. Suppose that the signatures on Σ_1 and Σ_2 are both all-positive. Then all of the edges in the resulting products $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \barwedge \Sigma_2$ are positive and so the graphs are clearly balanced.

Conversely, suppose that $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \barwedge \Sigma_2$ are balanced signed graphs. Then the product of the edge signs in any closed walk is positive. Since $E_{G_1} \cup E_{G_2}$ is nonempty, assume without loss of generality that E_{G_1} is nonempty (the case E_{G_2} is non-empty follows analogously). Take any edge $u_iu_j \in E_{G_1}$ and any vertex $v_k \in V_{G_2}$. Both $\Sigma_1 \veebar \Sigma_2$ and $\Sigma_1 \barwedge \Sigma_2$ contain the triangle $u_iu_jv_k$, whose sign is

$$\sigma(u_iu_j)\sigma(u_jv_k)\sigma(v_ku_i) = \sigma_1(u_iu_j)(+1)(+1).$$

Hence $\sigma_1(u_iu_j)$ must be equal to $+1$ for all u_iu_j in E_{G_1} and by the same argument $\sigma_2(v_iv_j)$ must be equal to $+1$ for all v_iv_j in E_{G_2}. Therefore both Σ_1 and Σ_2 must be all-positive. ∎

Theorem 4.9. *The join graphs $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$ and $\Sigma_1 \barwedge_{\mathcal{T},\overline{v}} \Sigma_2$ are balanced signed graphs, whose signatures are independent of the choice of $\mathcal{T}$, if and only if Σ_1 and Σ_2 are balanced.*

Proof. It is straightforward to see that if either Σ_1 or Σ_2 is unbalanced, their presence in the join graphs means that those products are also unbalanced.

Replacing Σ_1 and Σ_2 with some switching-equivalent Σ_1' and Σ_2' leaves the switching equivalence classes of $\Sigma_1 \veebar_{\mathcal{T},\overline{v}} \Sigma_2$ and $\Sigma_1 \barwedge_{\mathcal{T},\overline{v}} \Sigma_2$ unchanged. By Lemma 4.2(b), since the graphs are balanced, different choices of $\mathcal{T}$ give the same values for the pseudo-potentials. Accordingly, we can consider the all-positive signatures of Σ_1 and Σ_2, which clearly give balanced components and balanced products. ∎

Consider the graphs depicted in Figures 2 and 4. The same constituent parts $\Sigma_1 = (P_4, \sigma_1)$ and $\Sigma_2 = (P_2, \sigma_2)$ are used in each case, and they are both balanced, though Σ_1 does not have an all-positive signature. In Figure 2 the two products are unbalanced, conforming with Theorem 4.8, whereas the products in Figure 4 are balanced, as predicted by Theorem 4.9.

5. Open Problems

Taking the definition provided by Chung [4] to signed graphs, the normalised Laplacian matrix $\mathcal{L}_\Sigma$ of a signed graph $\Sigma = (G, \sigma)$ is defined by $\mathcal{L}_\Sigma = D_G^{-\frac{1}{2}} L_\Sigma D_G^{-\frac{1}{2}}$ with the convention that the ith diagonal entry of $D_G^{-\frac{1}{2}}$ is $(d_i)^{-\frac{1}{2}}$ and if the degree of vertex v_i in G is zero then $(d_i)^{-\frac{1}{2}} = 0$. In other words

$$(\mathcal{L}_\Sigma)_{ij} = \begin{cases} 1, & \text{if } i = j \text{ and } d_i \neq 0, \\ -\sigma(e_{ij})\frac{1}{\sqrt{d_id_j}}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

If the signed graph Σ in question is without isolated vertices, then

$$\mathcal{L}_\Sigma = D_G^{-\frac{1}{2}} L_\Sigma D_G^{-\frac{1}{2}} = I - D_G^{-\frac{1}{2}} A_\Sigma D_G^{-\frac{1}{2}}.$$

Applying the same methods used Section 2, it is possible to compute the normalised Laplacian characteristic polynomials of the NS and NNS joins when Σ_1 and Σ_2 are on regular underlying graphs. When considering two arbitrary signed graphs, the calculations become much more involved.

Open problem 5.1. Compute the characteristic polynomials of the normalised Laplacian matrices of the signed NS and NNS joins of any two arbitrary signed graphs.

The spectra provided in this paper are restricted to joins of (r, k)-co-regular signed graphs. This allows for the use of Proposition 2.4 when doing the computations and thus makes the work simpler to carry out. It would be ideal if these, along with their normalised Laplacian analogues, could be generalised completely.

Open problem 5.2. Compute the adjacency, Laplacian, and normalised Laplacian spectra of the signed NS and NNS joins of any two arbitrary signed graphs.

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