

Abhyankar-Moh Semigroups for arbitrary hypersurfaces

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February 12, 2026

Abstract

For an arbitrary hypersurface singularity, we construct a family of semigroups associated with algebraically closed fields that arise as an infinite union of rings of series. These semigroups extend the value semigroup of a plane curve studied by Abhyankar and Moh [4, 2, 3]. The algebraically closed fields under consideration possess a natural valuation that induces a corresponding value semigroup. We establish the necessary conditions under which these semigroups are independent of the choice of the root. Moreover, the extensions proposed by P. González and Kiyek-Micus [11, 12], where González specifically addresses the case of quasi-ordinary singularities, and the extension introduced by Abbas-Assi [5], can be understood as particular instances within our constructed family.

MSC [2000]: 05E40; 20M14; 13F25.

Keywords: Semigroup; Formal power series; Newton-Puiseux expansions

Introduction

Let K be an algebraically closed field of characteristic zero, let $K[[x]]$ denote the ring of formal power series in x over K and let $K((x))$ be its field of fractions.

Newton-Puiseux theorem [21], asserts that, if $f \in K[[x]][y]$ is a monic irreducible polynomial of degree d , then it factors in $K[[x^{\frac{1}{d}}]][y]$ as

$$f = \prod_{\eta^d=1} (y - \xi(\eta x^{\frac{1}{d}})) \quad (1)$$

where $\xi \in K[[x^{\frac{1}{d}}]]$.

Given a series $\xi = \sum_{i=\alpha}^{\infty} a_i x^{\frac{i}{d}} \in K((x^{\frac{1}{d}}))$ we denote

$$ord_x(\xi) := \min_{a_i \neq 0} \frac{i}{d}. \quad (2)$$

In 1973, Abhyankar and Moh [4, 2, 3] studied the structure of the semigroup

$$\Gamma := \{ord_x h(\xi); h \in K[[x]][y] \setminus (f)\}. \quad (3)$$

In 1948, Cahit Arf [6] had already introduced a semigroup similar to the one in (3), defined for spaces of any dimension. Du Val [13] discussed Cahit's semigroup results, providing an alternative interpretation. A more recent discussion on the study of semigroups in spaces of any dimension can be found in [20].

As a consequence of (1), the semigroup Γ does not depend on the chosen root of f and it makes sense to say that Γ is the **value semigroup of the plane curve defined by f** .

Most texts consider the semigroup Γ as a subset of \mathbb{Z} instead as a subset of \mathbb{Q} .

The value semigroup is a useful tool to study and classify plane curve singularities (see for example [27]). In particular, it determines the topological type of the singularity. Moreover, the structure of this semigroup is a useful tool in coding theory (see for example [11, 12]).

To extend the concept of "value semigroup" to hypersurfaces it is needed:

- A suitable field \mathcal{S} containing $K[[x_1, \dots, x_n]]$ such that $f \in K[[x_1, \dots, x_n]][y]$ factors as an element of $\mathcal{S}[y]$ (So that we can choose ξ).
- A mapping $\nu : \mathcal{S} \rightarrow \mathbb{Q}^n$ analogous to the mapping $ord_x : K((x^{\frac{1}{d}})) \rightarrow \mathbb{Q}$.
- A subring $\mathcal{A} \subset \mathcal{S}$ to consider the values of $h(\xi)$ with $h \in \mathcal{A}[y]$.

In 1983, Abhyakar's student, A. Sathaye [24], gave a generalization of Abhyankar-Moh results when x is replaced by an n -tuple (x_1, \dots, x_n) . A. Sathaye's definition uses the field of iterated Puiseux series as \mathcal{S} , the minimum of the support with the rev-lex order as ν and, the ring $K[[x_1, \dots, x_n]]$ as \mathcal{A} . P. González and Kiyek-Micus gave, independently, an extension for quasiordinary singularities [14, 15]. P. González's construction has been extended to σ -free singularities by Abbas-Assi [5] using the ideas of J.M. Tornero presented in [25, 26].

In this paper we construct, for an arbitrary hypersurface singularity, a family of semigroups, defined in terms of the family of algebraically closed fields constructed in [7]. These algebraically closed fields have a natural valuation that induces a value semigroup. We give the necessary conditions so that these semigroups do not depend on the chosen root ξ .

The constructions of A. Sathaye, P. González, and Abbas-Assi's semigroups naturally arise as specific examples within our defined family.

Some of the results that we present in this article are also presented in [9].

A family of algebraically closed fields

As we pointed out in the introduction, to extend the concept of Abhyankar-Moh semigroup from plane curves to hypersurfaces $V(f)$, we need to be able to produce a root of $f \in K[[x_1, \dots, x_n]][y]$.

González's extension uses Abhyankar-Jung Theorem [1] that guaranties, for quasiordinary singularities, the existence of roots in $K[[x_1^{\frac{1}{d}}, \dots, x_n^{\frac{1}{d}}]]$. To extend P. González's construction, Abbas-Assi uses a theorem due to J. McDonald [17] that assures the existence of a cone σ such that f factors in $K_\sigma[[x_1^{\frac{1}{d}}, \dots, x_n^{\frac{1}{d}}]][y]$. A. Sathaye's construction relies on the fact that the field of iterated Puiseux series is algebraically closed.

In this section we recall the construction of a family of algebraically closed fields presented in [7, 8]. For a detailed discussion of the fields $K_\sigma[[X]]$ and $K_{\leq}[[X]]$ we refer the reader to the beautiful article written by A.A. Monforte and M. Kauers [16].

A subset $\sigma \subset \mathbb{R}^n$ is a (convex polyhedral rational) **cone** when

$$\sigma = \langle u_1, u_2, \dots, u_s \rangle = \{ \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_s u_s ; \lambda_i \in \mathbb{R}_{\geq 0} \}$$

for some $u_1, u_2, \dots, u_s \in \mathbb{Q}^n$.

Let \preceq be a total order on \mathbb{R}^n compatible with the group structure. If $\sigma \subset \mathbb{R}^n_{\preceq 0}$ then σ doesn't contain any nontrivial linear subspace and the set of formal series

$$K_\sigma[[X]] := \left\{ \sum_{\gamma \in \sigma \cap \mathbb{Z}^n} a_\gamma x^\gamma ; a_\gamma \in K \right\}$$

has a natural ring structure. The ring $K_\sigma[[X]]$ is the completion of the coordinate ring of an affine toric variety. In some texts, for example P. González [15], $K_\sigma[[X]]$ is denoted by $K[[\sigma]]$.

We will be using the following rings $K[[X]] \subset K_{\preceq}[[X]] \subset K_{\preceq}[[X]]_k \subset \mathcal{A}_{\preceq}$,

$$\begin{aligned} K_{\preceq}[[X]] &:= \bigcup_{\sigma \subset (\mathbb{R}^n)_{\preceq 0}} K_\sigma[[X]] \\ K_{\preceq}[[X]]_k &:= K_{\preceq}[[x_1^{\frac{1}{k}}, \dots, x_n^{\frac{1}{k}}]] \\ \mathcal{A}_{\preceq} &:= \bigcup_{k \in \mathbb{Z}_{>0}} K_{\preceq}[[X]]_k \end{aligned}$$

and their corresponding fields of fractions $K((X)) \subset K_{\preceq}((X)) \subset K_{\preceq}((X))_k \subset \mathcal{S}_{\preceq}$,

$$K_{\preceq}((X)) := \{ \varphi ; \exists \gamma \in \mathbb{Z}^n, x^\gamma \varphi \in K_{\preceq}[[X]] \} \quad (4)$$

$$K_{\preceq}((X))_k := \{ \varphi ; \exists \gamma \in \mathbb{Z}^n, x^\gamma \varphi \in K_{\preceq}[[X]]_k \} \quad (5)$$

$$\mathcal{S}_{\preceq} := \{ \varphi ; \exists \gamma \in \mathbb{Z}^n, x^\gamma \varphi \in \mathcal{A}_{\preceq} \}. \quad (6)$$

Let $R = \mathcal{A}_{\preceq}, K_{\preceq}[[X]]$ or $K_{\preceq}[[X]]_k$, an element $\sum_{\gamma \in \Lambda} a_\gamma x^\gamma \in R$ is a unit of R if and only if $a_{(0, \dots, 0)} \neq 0$.

When K is a zero characteristic algebraically closed field, the field \mathcal{S}_{\preceq} is algebraically closed [7, 8, Theorem 1, Theorem 4.5].

Note that the ring of Puiseux power series (as defined, for example, in [26]) is contained in \mathcal{S}_{\preceq} if and only if the first orthant is non negative for \preceq . Therefore, by Abhyakar-Jung Theorem [1], the roots of a quasiordinary polynomial in \mathcal{S}_{\preceq} will coincide for any order \preceq with $\mathbb{R}_{\geq 0}^n \subset \mathbb{R}^n_{\preceq 0}$. The same applies for Puiseux hypersurfaces.

Given a vector $\omega \in \mathbb{R}_{>0}^n$ of rationally independent coordinates. The vector ω induces a total order on \mathbb{Q}^n compatible with the group structure given by

$$\alpha \leq_\omega \beta \text{ if and only if } \omega \cdot \alpha \leq \omega \cdot \beta. \quad (7)$$

The order \leq_ω may be extended to a total order \preceq on \mathbb{R}^n [22]. The field \mathcal{S}_{\preceq} is the same independently of the extension and we may denote $\mathcal{S}_\omega := \mathcal{S}_{\preceq}$.

Manfred Buchacher [10] has implemented an algorithm using Mathematica that computes the first terms of the roots in \mathcal{S}_ω of polynomials $f \in K[X][y]$. We have used his implementation for the examples presented in this paper.

Example 1. Set $f(y) := y^2 - 2(x_2 + 1)y + (x_2 + 1) - x_1 \in \mathbb{C}[[x_1, x_2]][y]$. We have that $\Delta_y : (f) = -4x_1$, so f is quasiordinary, its roots are $\xi = \pm x_1^{\frac{1}{2}} + x_2 + 1 \in \mathbb{C}[[x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}]]$.

Example 2. Set $g(y) := y^4 - 2(x_1 + x_2)y^2 + (x_1 - x_2)^2 \in \mathbb{C}[[x_1, x_2]][y]$. The roots of f are:

$$y = \pm x_1^{\frac{1}{2}} \pm x_2^{\frac{1}{2}},$$

belong to $\mathbb{C}[[x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}]]$.

Note that $\Delta_y(f) = -256(x_2 - x_1)(x_2 + x_2(2x_1 + 1) + x_1^2 - x_1)^2$, so f is not quasiordinary

Example 3. The roots of $f := z^2 - (x + y^2)$, in the field \mathcal{S}_ω with $\omega := (\sqrt{2}, 1)$ are

$$\xi_1 = -x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y^2 + \frac{1}{8}x^{-\frac{3}{2}}y^4 + \frac{1}{16}x^{-\frac{5}{2}}y^6 + \frac{5}{128}x^{-\frac{7}{2}}y^8 + \frac{7}{256}x^{-\frac{9}{2}}y^{10}$$

and

$$\xi_2 = x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{1}{2}}y^2 - \frac{1}{8}x^{-\frac{3}{2}}y^4 - \frac{1}{16}x^{-\frac{5}{2}}y^6 - \frac{5}{128}x^{-\frac{7}{2}}y^8 - \frac{7}{256}x^{-\frac{9}{2}}y^{10}$$

Taking $\omega := (4, \sqrt{2})$ the roots of f in \mathcal{S}_ω are:

$$\tilde{\xi}_1 = \frac{7i}{256}x^5y^{-9} + \frac{5i}{128}x^4y^{-7} + \frac{i}{16}x^3y^{-5} + \frac{i}{8}x^2y^{-3} + \frac{i}{2}xy^{-1} - iy$$

and

$$\tilde{\xi}_2 = -\frac{7i}{256}x^5y^{-9} - \frac{5i}{128}x^4y^{-7} - \frac{i}{16}x^3y^{-5} - \frac{i}{8}x^2y^{-3} - \frac{i}{2}xy^{-1} + iy.$$

Dominating exponent

Given a series $\xi \in K((x))$ and ord_x as in (2) we have that

$$\xi = x^{ord_x(\xi)} \bar{\xi} \tag{8}$$

where $\bar{\xi} = x^{-ord_x(\xi)}\xi$ is a unit of $K[[x]]$. The term $ax^{ord_x(\xi)}$ in the series ξ is called the dominating term. The exponent of the dominating term is called the dominating exponent.

The semigroup (3) of a plane curve is the semigroup of the exponents of the dominating terms of elements of $K[[x]][y]$ evaluated on a root of its defining polynomial.

We quote Patrick Popescu [19]:

”A difficulty for extending the plane branch definition of the semigroup is that in dimension > 1 , fractional series may have no dominating term. *One way to force the existence of a dominating term is to restrict to those functions which do have one.*”

Given $f \in K[[x_1, \dots, x_n]][y]$ let Δf denotes the discriminant of f with respect to y . The hypersurface singularity $V(f)$ is said to be *quasiordinary* when

$$\Delta f = X^\gamma u$$

with $\gamma \in (\mathbb{Z}_{\geq 0})^n$ and u a unit in $K[[x_1, \dots, x_n]]$. P. González’s construction relies on this fact.

Instead of restricting the ring of functions to those that do have a dominating term, our approach is to enlarge our ring of functions so that every element can be written as (9) in a unique way.

Lemma 4. Given an element $\xi \in \mathcal{S}_{\preceq}$ there exists unique $\gamma \in \mathbb{Z}^n$ such that

$$\xi = X^\gamma u,$$

with $u \in \mathcal{A}_{\preceq}$ invertible.

Proof. Let $\xi = \sum a_\alpha X^\alpha$ and set $\gamma := \min_{\preceq} \{\alpha; a_\alpha \neq 0\}$.

$$\xi = \sum a_\alpha X^\alpha = X^\gamma \left(\sum a_\alpha X^{\alpha-\gamma} \right). \quad (9)$$

Now $\phi := \sum a_\alpha X^{\alpha-\gamma} = \sum b_\alpha X^\alpha$ where $b_\alpha = a_{\alpha+\gamma}$. Since, $b_{(0,\dots,0)} = a_\gamma \neq 0$, we have that ϕ is a unit. \square

To make explicit the ring we are working on, we will say that γ as in lemma 4 is the \preceq -dominating exponent of ξ . When the order \preceq is an extension of \leq_ω , the n -tuple γ will be called the ω -dominating exponent.

Example 5. Set $\xi = x_1^{\frac{1}{2}} + x_2 + 1 \in \mathbb{C}[[x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}]]$. The \preceq -dominating exponent of ξ is $(\frac{1}{2}, 0)$ for any order \preceq such that the first orthant is non negative.

Example 6. Set $\xi := x_1^{\frac{1}{2}} + x_2^{\frac{1}{2}} \in \mathbb{C}[[x_1^{\frac{1}{2}}, x_2^{\frac{1}{2}}]]$.

The $(\sqrt{2}, 1)$ -dominating exponent of ξ is $(0, \frac{1}{2})$ whilst the $(1, \sqrt{2})$ -dominating exponent of ξ is $(\frac{1}{2}, 0)$

Example 7. The $(\sqrt{2}, 1)$ -dominating exponent of

$$\xi_1 = -x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}y^2 + \frac{1}{8}x^{-\frac{3}{2}}y^4 + \frac{1}{16}x^{-\frac{5}{2}}y^6 + \frac{5}{128}x^{-\frac{7}{2}}y^8 + \frac{7}{256}x^{-\frac{9}{2}}y^{10}$$

is $(\frac{1}{2}, 0)$.

And the $(4, \sqrt{2})$ -dominating exponent of

$$\tilde{\xi}_1 = \frac{7i}{256}x^5y^{-9} + \frac{5i}{128}x^4y^{-7} + \frac{i}{16}x^3y^{-5} + \frac{i}{8}x^2y^{-3} + \frac{i}{2}xy^{-1} - iy$$

is $(0, 1)$.

Order and branches

We quote Shreeram S. Abhyankar [4]

"If $z(t)$ is any element of $k((t))$ such that $f(tn, z(t)) = 0$ then $z(t) = y(wt)$ for some $w \in \mu_n(k)$. In particular, we have $\text{Suppt } z(t) = \text{Suppt } y(t)$. Thus the set $\text{Suppt } y(t)$ depends only on f and not on a root $y(t)$ of $f(tn, Y) = 0$. Therefore we can make ..."

The above property does not hold anymore when we consider several variables: The polynomial $f(y) := y^5 + x_1^2x_2^2y^2 + x_2^5$ is irreducible as an element of $K[[x_1, x_2]][y]$ and the roots of f in $\mathcal{S}_{(1, \sqrt{5})}$ are

$$\begin{aligned} \xi_1 &= -x_1^{\frac{2}{3}}x_2^{\frac{2}{3}} - \frac{1}{3}x_1^{-\frac{8}{3}}x_2^{\frac{7}{3}} + \frac{1}{3}x_1^{-6}x_2^4 - \frac{44}{81}x_1^{-\frac{28}{3}}x_2^{\frac{17}{3}} + \dots \\ \xi_2 &= \frac{1-i\sqrt{3}}{2}x_1^{\frac{2}{3}}x_2^{\frac{2}{3}} + \frac{2i}{3i+3\sqrt{3}}x_1^{-\frac{8}{3}}x_2^{\frac{7}{3}} + \frac{1}{3}x_1^{-6}x_2^4 - \frac{-44i+44\sqrt{3}}{81i+81\sqrt{3}}x_1^{-\frac{28}{3}}x_2^{\frac{17}{3}} + \dots \end{aligned}$$

$$\begin{aligned}\xi_3 &= \frac{1+i\sqrt{3}}{2}x_1^{\frac{2}{3}}x_2^{\frac{2}{3}} - \frac{2i}{-3i+3\sqrt{3}}x_1^{-\frac{8}{3}}x_2^{\frac{7}{3}} + \frac{1}{3}x_1^{-6}x_2^4 - \frac{44i+44\sqrt{3}}{-81i+81\sqrt{3}}x_1^{-\frac{28}{3}}x_2^{\frac{17}{3}} + \dots \\ \xi_4 &= -ix_1^{-1}x_2^{\frac{3}{2}} - \frac{1}{2}x^{-6}x_2^4 + \frac{9i}{8}x_1^{11}x_2^{\frac{13}{2}} + \frac{7}{2}x_1^{-16}x_2^9 + \dots \\ \xi_5 &= ix_1^{-1}x_2^{\frac{3}{2}} - \frac{1}{2}x^{-6}x_2^4 - \frac{9i}{8}x_1^{11}x_2^{\frac{13}{2}} + \frac{7}{2}x_1^{-16}x_2^9 + \dots\end{aligned}$$

do not have the same support.

Another example of this phenomenon may be found in the introduction of Guillaume Rond's paper [23].

Let $f \in K[[X]][y]$ be an irreducible (as an element of $K[[X]][y]$) monic polynomial and let $g_1, \dots, g_l \in K_{\preceq}[[X]][y]$ be irreducible (as elements of $K_{\preceq}[[X]][y]$) with $f = g_1 \cdots g_l$. Since the extension $K((X))/\mathcal{S}_{\preceq}$ is separable, the set of roots of f in \mathcal{S}_{\preceq} is the disjoint union of the sets of roots of the g_i 's.

Definition 8. Given $f \in K[[X]][y]$ and a total order \preceq on \mathbb{R}^n , compatible with the group structure. A \preceq -branch of f is the set of roots in \mathcal{S}_{\preceq} of an irreducible element $g \in K_{\preceq}[[X]][y]$ that divides f as element of $K_{\preceq}[[X]][y]$.

As a consequence of the following proposition we have that the branches of $f(y) := y^5 + x_1^2x_2^2y^2 + x_2^5$ are the sets $\{\xi_1, \xi_2, \xi_3\}$ and $\{\xi_4, \xi_5\}$.

The field $K_{\preceq}((X))_k = K_{\preceq}((X))[X^{\frac{1}{k}}]$ is the root field of the separable polynomial $(y^k - x_1)(y^k - x_2) \cdots (y^k - x_n)$, then, it is a finite Galois extension of $K_{\preceq}((X))$. The elements of the Galois group of this extension are given by

$$\tau_{\eta, \mu} : \varphi(x_1^{\frac{1}{k}}, \dots, x_n^{\frac{1}{k}}) \mapsto \varphi(\eta^{\mu_1}x_1^{\frac{1}{k}}, \dots, \eta^{\mu_n}x_n^{\frac{1}{k}}) \quad (10)$$

where η is a k -th primitive root of unity and $\mu = (\mu_1, \dots, \mu_n) \in \{0, \dots, k-1\}^n$.

Proposition 9. Given $f \in K[[X]][y]$ and a total order \preceq on \mathbb{R}^n , compatible with the group structure. Let ξ be a root of f in \mathcal{S}_{\preceq} , let k be such that f factorizes in $K_{\preceq}[[X]]_k[y]$ and let η be a primitive k -th root of unity. The \preceq -branch of f containing ξ is the set

$$B(\xi) = \{\tau_{\eta, \mu}(\xi); \mu = (\mu_1, \dots, \mu_n) \in \{0, \dots, k-1\}^n\}$$

where $\tau_{\eta, \mu}$ is as in (10).

Proof.

Let $g \in K_{\preceq}[[X]][y]$ be an irreducible element that divides f and such that $g(\xi) = 0$. The elements of the \preceq -branch of f containing ξ are the roots of g in \mathcal{S}_{\preceq} .

The morphisms $\tau_{\eta, \mu} : K_{\preceq}[[X]]_k \rightarrow K_{\preceq}[[X]]_k$ respect the ring structure, then $g(\tau_{\eta, \mu}(\xi)) = \tau_{\eta, \mu}(g(\xi)) = 0$ therefore $\tau_{\eta, \mu}(\xi)$ is in the \preceq -branch of f containing ξ .

Let L be the splitting field of g and let H be the Galois group of the extension $L/K_{\preceq}((X))$.

Since g is irreducible in $K_{\preceq}[[X]][y]$ then, by Gauss Lemma, it is irreducible in $K_{\preceq}((X))[y]$. Since L is a splitting field for g , the group H acts transitively on the roots of g .

Moreover, since $K_{\preceq}((X)) \subset L \subset K_{\preceq}((X))_k$, for each $\tau \in H$, there exists $\vartheta \in \text{Gal}(K_{\preceq}((X))_k/K_{\preceq}((X)))$ such that $\tau = \vartheta|_L$. And the result follows from the fact that the elements of $\text{Gal}(K_{\preceq}((X))_k/K_{\preceq}((X)))$ are given by (10). \square

Example 10. The polynomial $f(y) := y^2 - 2(x_2 + 1)y + (x_2 + 1) - x_1 \in \mathbb{C}[[x_1, x_2]][y]$ has only one \preceq -branch, for any order \preceq such that the first orthant is non negative.

Example 11. The polynomial $g(y) := y^4 - 2(x_1 + x_2)y^2 + (x_1 - x_2)^2 \in \mathbb{C}[[x_1, x_2]][y]$ has only one \preceq -branch, for any order \preceq such that the first orthant is non negative.

Example 12. The polynomial $f := z^2 - (x + y^2)$ has only one $(\sqrt{2}, 1)$ -branch whilst it has two $(4, \sqrt{2})$ -branches.

The family of semigroups of values of a hypersurface singularity

A valuation on a field K is a mapping $\nu : K^* \rightarrow G$, where (G, \preceq) is a totally ordered abelian group and, for $a, b \in K^*$, $\nu(ab) = \nu(a) + \nu(b)$ and $\nu(a + b) \succeq \min_{\preceq} \{\nu(a), \nu(b)\}$. The image of a valuation is a subgroup of G . The mapping ord_x in (3) is an example of a valuation.

In what follows, the symbol \preceq will stand for a total order on \mathbb{R}^n compatible with the group structure for which the first orthant is non negative.

Given a series $\varphi = \sum_{\gamma \in \Lambda} a_\gamma x^\gamma$, the support of φ is the set $Supp(\varphi) := \{\gamma \in \Lambda; a_\gamma \neq 0\}$. Note that

$$\mathcal{S}_{\preceq} = \left\{ \varphi; \exists \sigma \subset (\mathbb{R}^n)_{\succeq 0}, \gamma \in \mathbb{Z}^n \text{ and } k \in \mathbb{N} \text{ with } Supp(\varphi) \subset (\gamma + \sigma) \cap \frac{1}{k} \mathbb{Z}^n \right\},$$

therefore, \mathcal{S}_{\preceq} has a natural valuation with value group (\mathbb{Q}^n, \preceq) given by:

$$\nu_{\preceq}(\varphi) := \min_{\preceq} Supp(\varphi).$$

The above definition extends the definition of order in (2). Also, $\nu_{\preceq}(\varphi)$ is the \preceq -dominant exponent of φ .

Note that, since $Supp(\varphi) = Supp(\tau_{\eta, \mu}(\varphi))$, for $\varphi \in K_{\preceq}((X))_k$ and $\tau_{\eta, \mu}$ as in (10), we have

$$\nu_{\preceq}(\varphi) = \nu_{\preceq}(\tau_{\eta, \mu}(\varphi)). \quad (11)$$

That is, the elements of the Galois group of the extension $K_{\preceq}((X)) \subset K_{\preceq}((X))_k$ are automorphisms of the valued field $(K_{\preceq}((X))_k, \nu_{\preceq})$.

An element $\xi \in \mathcal{S}_{\preceq}$ induces a mapping that extends ν_{\preceq} to $\mathcal{S}_{\preceq}[y] \setminus (y - \xi)$ given by

$$\nu_{\preceq}^{\xi}(P(y)) := \nu_{\preceq}(P(\xi))$$

where $(y - \xi)$ denotes the ideal of $\mathcal{S}_{\preceq}[y]$ generated by $y - \xi$.

Given a subring $\mathcal{A} \subset K_{\preceq}[[X]]$ the subset of \mathbb{Q}^n

$$\Gamma_{\xi, \mathcal{A}}^{\preceq} := \{\nu_{\preceq}(h(\xi)); h \in \mathcal{A}[y] \setminus (y - \xi)\}$$

is a semigroup.

The semigroup considered by Sathaye is obtained by taking the reverse lexicographical order as \preceq . When f is a quasiordinary singularity, and $\mathcal{A} = K[[x_1, \dots, x_n]]$ the semigroup $\Gamma_{\xi, \mathcal{A}}^{\preceq}$ does not depend on the order \preceq and is the semigroup studied by P. González. Even though it is not made explicit in their paper, Abbas-Assi construction depends both of the chosen cone σ and on the chosen order needed to do the construction. Fixed a cone σ , the semigroups considered by Abbas-Assi for a σ -free singularity are obtained taking $\mathcal{A} = K_{\sigma}[[X]]$ and \preceq compatible with σ .

Theorem 13. *Given an order \preceq and a subring $\mathcal{A} \subset K_{\preceq}((X))$. If ξ and ξ' are roots of $f \in K[[X]][y]$ in \mathcal{S}_{\preceq} that belong to the same branch, then $\Gamma_{\xi, \mathcal{A}}^{\preceq} = \Gamma_{\xi', \mathcal{A}}^{\preceq}$.*

Proof. Given $f \in K[[X]][y]$ and a total order \preceq on \mathbb{R}^n , compatible with the group structure. Let ξ and ξ' be roots of f in \mathcal{S}_{\preceq} in the same \preceq -branch, let k be such that f factorizes in $K_{\preceq}[[X]]_k[y]$. By proposition 9, there exists η , a primitive k -th root of unity, and $\mu = (\mu_1, \dots, \mu_n) \in \{0, \dots, k-1\}^n$ such that

$$\xi' = \tau_{\eta, \mu}(\xi)$$

where $\tau_{\eta, \mu}$ is as in (10).

Given $h \in K_{\preceq}((X))[y]$ we have $h(\xi') = h(\tau_{\eta, \mu}(\xi)) = \tau_{\eta, \mu}(h(\xi))$.

Now, for any $\varsigma \in K[[X]]_k$, $Supp(\varsigma) = Supp(\tau_{\eta, \mu}(\varsigma))$, and then $\nu_{\preceq}(\varsigma) = \nu_{\preceq}(\tau_{\eta, \mu}(\varsigma))$. This implies that

$$\nu_{\preceq}(h(\xi)) = \nu_{\preceq}(h(\xi')).$$

□

Acknowledgments

We extend our gratitude to Dr. Fabiola Manjarrez Gutiérrez for her valuable comments and suggestions, which have significantly enhanced this work.

This project was supported by DGAPA-UNAM through PAPIIT IN113323.

Giovanna Ilardi was partially supported by GNSAGA-INDAM.

We would like to thank Patrick Popescu-Pampu for providing us with the references by Arf and Du Val.

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