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ORIGINAL PAPER

Numerical properties of isotrivial fibrations

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Abstract In this paper we investigate the numerical properties of relatively minimal isotrivial fibrations $\varphi : X \longrightarrow C$, where X is a smooth, projective surface and C is a curve. In particular we prove that, if $g(C) \geq 1$ and X is neither ruled nor isomorphic to a quasi-bundle, then $K_X^2 \leq 8\chi(\mathcal{O}_X) - 2$; this inequality is sharp and if equality holds then X is a minimal surface of general type whose canonical model has precisely two ordinary double points as singularities. Under the further assumption that K_X is ample, we obtain $K_X^2 \leq 8\chi(\mathcal{O}_X) - 5$ and the inequality is also sharp. This improves previous results of Serrano and Tan.

Keywords Isotrivial fibrations · Cyclic quotient singularities

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Introduction

One of the most useful tools in the study of algebraic surfaces is the analysis of *fibrations*, that is morphisms with connected fibres from a surface X onto a curve C. When all smooth fibres of a fibration $\varphi : X \longrightarrow C$ are isomorphic to each other, we call φ an *isotrivial fibration*. As far as we know, there is hitherto no systematic study of minimal models of isotrivial fibrations; the aim of the present paper is to shed some light on this problem.

A smooth, projective surface S is called a *standard isotrivial fibration* if there exists a finite group G, acting faithfully on two smooth projective curves C_1 and C_2, so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)/G$, where G acts diagonally on the product. When the action of G is free, then $S = T$ is called a *quasi-bundle*. These surfaces have been investigated in [1,2,8,9,18,22,24,26]. A monodromy argument shows that every isotrivial fibration $\varphi : X \longrightarrow C$ is birationally isomorphic to a standard one ([26,

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Sect. 2]); this means that there exist $T = (C_1 \times C_2)/G$, a birational map $T \dashrightarrow X$ and an isomorphism $C_2/G \longrightarrow C$ such that the diagram

$$\begin{array}{ccc} T & \dashrightarrow & X \\ \downarrow & & \downarrow \varphi \\ C_2/G & \xrightarrow{\cong} & C \end{array}$$

commutes.

If $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ is any standard isotrivial fibration, the two projections $\pi_1 : C_1 \times C_2 \longrightarrow C_1, \pi_2 : C_1 \times C_2 \longrightarrow C_2$ induce two morphisms $\alpha_1 : S \longrightarrow C_1/G, \alpha_2 : S \longrightarrow C_2/G$, whose smooth fibres are isomorphic to C_2 and C_1, respectively. Moreover $q(S) = g(C_1/G) + g(C_2/G)$. If S is a quasi-bundle, then all singular fibres of α_1 and α_2 are multiple of smooth curves. Otherwise, T contains some cyclic quotient singularities, and the invariants K_S^2 and $e(S)$ can be computed in terms of the number and type of such singularities. Moreover the corresponding fibres of α_1 and α_2 consist of an irreducible curve, called the central component, with at least two Hirzebruch-Jung strings attached. Assume that a fibre F of α_1 (or α_2) contains exactly r such strings, of type $\frac{1}{n_1}(1,\ q_1), \dots, \frac{1}{n_r}(1,\ q_r)$, respectively; therefore we say that F is of type $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$.

Now set $\mathfrak{g} := g(C_1)$ and consider a reducible fibre F of $\alpha_2 : S \longrightarrow C_2/G$. If $g(C_1/G) = 0$ then it may happen that the central component of F is a (-1)-curve; in this case we say that F is a (-1)-*fibre in genus* $\mathfrak{g}$. Moreover, if $g(C_2/G) \geq 1$ then the central components of (-1)-fibres of α_2 are the unique (-1)-curves on S.

Our first result provides a method to construct standard isotrivial fibrations with arbitrary many (-1)-fibres.

Theorem A (see Theorem 3.3) *Let* $\mathcal{S} := \left\{\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right\}$ *be a finite set of rational numbers, with* $(n_i, q_i) = 1$*, such that* $\sum_{i=1}^r \frac{q_i}{n_i} = 1$*. Set* $n :=$ l.c.m.$(n_1, \dots, n_r)$*. Then for any* $\mathfrak{q} \geq 0$ *there exists a standard isotrivial fibration* $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ *such that the following holds.*

(i) $\text{Sing}(T) = n \times \frac{1}{n_1}(1, q_1) + \dots + n \times \frac{1}{n_r}(1, q_r)$;
(ii) *the singular fibres of* $\alpha_2 : S \longrightarrow C_2/G$ *are exactly* n (-1)*-fibres, all of type* $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$;
(iii) $q(S) = \mathfrak{q}$.

Our second result deals with the "geography" of (minimal models of) isotrivial fibrations. It is straightforward to prove that every quasi-bundle S satisfies $K_S^2 = 8\chi(\mathcal{O}_S)$. In [26] Serrano extended this result, showing that any isotrivial fibred surface X satisfies $K_X^2 \leq 8\chi(\mathcal{O}_X)$; his proof is based on the properties of the projective cotangent bundle $\mathbb{P}(\Omega_X^1)$. Exploiting the fact that every isotrivial fibration is birationally isomorphic to a standard one, we obtain the following strengthening of Serrano's theorem. We want to emphasize that our method involves mostly arguments of combinatorial nature, and it is very different from Serrano's one.

Theorem B (see Theorem 4.22) *Let* $\varphi : X \longrightarrow C$ *be any relatively minimal isotrivial fibration, with* X *non ruled and* $g(C) \geq 1$*. If* X *is not isomorphic to a quasi-bundle, we have*

$$K_X^2 \leq 8\chi(\mathcal{O}_X) - 2 \tag{1}$$

and if equality holds then X *is a minimal surface of general type whose canonical model has precisely two ordinary double points as singularities.*

Moreover, under the further assumption that K_X is ample, we have

$$K_X^2 \leq 8\chi(\mathcal{O}_X) - 5. \tag{2}$$

Finally, both inequalities (1) *and* (2) *are sharp.*

We do not know whether Theorem B remains true if one drops the assumption $g(C) \geq 1$.

Let us now illustrate the structure of the paper and give a brief account of how the results are achieved.

In Sect. 1 we review some of the standard facts about group actions on Riemann surfaces and cyclic quotient singularities; in particular we recall the Riemann existence theorem and the Hirzebruch-Jung resolution in terms of continued fractions; furthermore, we make some computations that will be used in Sect. 4.

In Sect. 2 we summarize the basic properties of standard isotrivial fibrations. This section is strongly inspired by Serrano's papers [24] and [26], but our approach is different. In particular, we provide some results on the singular locus of T which one could not obtain by means of Serrano's techniques (Corollaries 2.9 and 2.10).

In Sect. 3 we start the analysis of the case where S is not a minimal surface. In particular we give necessary and sufficient conditions ensuring that a reducible fibre F is a (-1)-fibre (Proposition 3.2), and this allows us to prove Theorem A.

In Sect. 4 we look more closely at the relative minimal model $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$ of $\alpha_2 : S \longrightarrow C_2/G$. The main step is to define, for any reducible fibre F of α_2, an invariant $\delta(F) \in \mathbb{Q}$ such that

$$K_{\widehat{S}}^2 = 8\chi(\mathcal{O}_{\widehat{S}}) - \sum_{F \text{ reducible}} \delta(F). \tag{3}$$

We also obtain a combinatorial classification of (-1)-fibres. When $\mathfrak{g} = 0$, the so-called Riemenschneider's duality between the HJ-expansions of $\frac{n}{q}$ and $\frac{n}{n-q}$ implies $\delta(F) = 0$. If $\mathfrak{g} \geq 1$ one has instead $\delta(F) > 2$ for all reducible fibres F, with precisely three exceptions that we describe in detail (Corollary 4.14). Using these facts, together with relation (3) and some identities on continued fractions shown in Sect. 1, we prove Theorem B. In particular, the proof of inequality (2) uses the computer algebra program `GAP4`, whose database includes all groups of order less that 2000, with the exception of 1024 (see [14]). However, the computer can be replaced either by (tedious) hand-made computations or by the Atlas of Finite Groups ([10]).

In Appendix A we classify all possible types of (-1)-fibres for $\mathfrak{g} = 1, 2, 3$; we also relate this classification to those given by Kodaira (when $\mathfrak{g} = 1$) and Ogg (when $\mathfrak{g} = 2$).

Finally, in Appendix B we provide a list of all the cyclic quotient singularities $\frac{1}{n}(1, q)$ and their numerical invariants, for $2 \leq n \leq 14$. We hope that this will help the reader to check our computations.

Notations and conventions

All varieties in this article are defined over $\mathbb{C}$. If S is a projective, non-singular surface S then K_S denotes the canonical class, $p_g(S) = h^0(S, K_S)$ is the *geometric genus*, $q(S) = h^1(S, K_S)$ is the *irregularity* and $\chi(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ is the *Euler characteristic*. We denote by $\text{kod}(S)$ the Kodaira dimension of S and we say that S is *ruled* if $\text{kod}(S) = -\infty$. For every finite group G, the notation $G = G(|G|, *)$ indicates the label of G in the `GAP4` database of small groups. For instance, $D_4 = G(8, 3)$ means that D_4 is the third in the list of groups of order 8. If $x \in G$ the conjugacy class of x is denoted by $\mathrm{Cl}(x)$. If x and y are conjugate in G we write $x \sim_G y$. The commutator of x and y is defined as $[x, y] = xyx^{-1}y^{-1}$. The derived subgroup of G is denoted by $[G, G]$.

1 Preliminaries

1.1 Group actions on Riemann surfaces

Definition 1.1 Let G be a finite group and let

$$\mathfrak{g}' \geq 0, \quad m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$$

be integers. A *generating vector* for G of type $(\mathfrak{g}' \mid m_1, \ldots, m_r)$ is a $(2\mathfrak{g}' + r)$-tuple of elements

$$\mathcal{V} = \{g_1, \ldots, g_r;\ h_1, \ldots, h_{2\mathfrak{g}'}\}$$

such that the following conditions are satisfied:

- the set $\mathcal{V}$ generates G;
- the order of g_i is equal to m_i;
- $g_1 g_2 \ldots g_r \Pi_{i=1}^{\mathfrak{g}'}[h_i, h_{i+\mathfrak{g}'}] = 1.$

If such a $\mathcal{V}$ exists, then G is said to be $(\mathfrak{g}' \mid m_1, \ldots, m_r)$*-generated.*

Remark 1.2 If an *abelian* group G is $(\mathfrak{g}' \mid m_1, \ldots, m_r)$-generated then either $r = 0$ or $r \geq 2$. Moreover if $r = 2$ then $m_1 = m_2$.

For convenience we make abbreviations such as $(4 \mid 2^3, 3^2)$ for $(4 \mid 2, 2, 2, 3, 3)$ when we write down the type of the generating vector $\mathcal{V}$.

Proposition 1.3 *(*Riemann Existence Theorem*) A finite group G acts as a group of automorphisms of some compact Riemann surface C of genus $\mathfrak{g}$ if and only if there exist integers $\mathfrak{g}' \geq 0$ and $m_r \geq m_{r-1} \geq \cdots \geq m_1 \geq 2$ such that G is $(\mathfrak{g}' \mid m_1, \ldots, m_r)$-generated, with generating vector $\mathcal{V} = \{g_1, \ldots, g_r;\ h_1, \ldots, h_{2\mathfrak{g}'}\}$, and the Riemann-Hurwitz relation holds:*

$$2\mathfrak{g} - 2 = |G| \left(2\mathfrak{g}' - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) \right). \tag{4}$$

If this is the case, $\mathfrak{g}'$ is the genus of the quotient Riemann surface $D := C/G$ and the G-cover $C \longrightarrow D$ is branched in r points $P_1, \ldots, P_r$ with branching numbers $m_1, \ldots, m_r$, respectively. In addition, the subgroups $\langle g_i \rangle$ and their conjugates provide all the nontrivial stabilizers of the action of G on C.

In the situation of Proposition 1.3 we shall say that G *acts in genus* $\mathfrak{g}$ *with signature* $(\mathfrak{g}' \mid m_1, \ldots, m_r)$. We refer the reader to [7, Sect. 2], [6, Chapter 3], [15] and [22, Sect. 1], for more details.

Now let C be a compact Riemann surface of genus $\mathfrak{g} \geq 2$ and let $G \subseteq \mathrm{Aut}(C)$. For any $h \in G$ set $H := \langle h \rangle$ and define the set of fixed points of h as

$$\mathrm{Fix}_C(h) = \mathrm{Fix}_C(H) := \{x \in C \mid hx = x\}.$$

For our purposes it is also important to take into account how an automorphism acts in a neighborhood of each of its fixed points. We follow the exposition of [6, pp.17, 38]. Let $\mathscr{D} \subset \mathbb{C}$ be the unit disk and $h \in \mathrm{Aut}(C)$ of order $m > 1$ such that $hx = x$ for a point $x \in C$. Then there is a unique primitive complex m-th root of unity ξ such that any lift of h to $\mathscr{D}$ that fixes a point in $\mathscr{D}$ is conjugate to the transformation $z \longrightarrow \xi \cdot z$ in $\mathrm{Aut}(\mathscr{D})$. We write $\xi_x(h) = \xi$ and we call ξ^{-1} the *rotation constant* of h in x. Then for each integer $1 \leq q \leq m-1$ such that $(m, q) = 1$ we define

$$\mathrm{Fix}_{C,q}(h) = \{x \in \mathrm{Fix}_C(h) \mid \xi_x(h) = \xi^q\},$$

that is the set of fixed points of h with rotation constant ξ^{-q}. Clearly, we have

$$\mathrm{Fix}_C(h) = \biguplus_{\substack{1 \leq q \leq m-1 \\ (m,q)=1}} \mathrm{Fix}_{C,q}(h).$$

Proposition 1.4 *Assuming that we are in the situation of Proposition 1.3, let $h \in G$ be of order m, $H = \langle h \rangle$ and $(m, q) = 1$. Then*

$$|\mathrm{Fix}_C(h)| = |N_G(H)| \cdot \sum_{\substack{1 \leq i \leq r \\ m \mid m_i \\ H \sim_G \langle g_i^{m_i/m} \rangle}} \frac{1}{m_i}$$

and

$$|\mathrm{Fix}_{C,q}(h)| = |C_G(h)| \cdot \sum_{\substack{1 \leq i \leq r \\ m \mid m_i \\ h \sim_G g_i^{m_i q/m}}} \frac{1}{m_i} .$$

Proof See [6, Lemma 10.4 and 11.5]. □

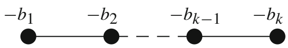
Corollary 1.5 *Assume that $h \sim_G h^q$. Then $|\mathrm{Fix}_{C,1}(h)| = |\mathrm{Fix}_{C,q}(h)|$.*

1.2 Surface cyclic quotient singularities and Hirzebruch-Jung resolutions

Let n and q be coprime natural numbers with $1 \leq q \leq n-1$, and let ξ_n be a primitive nth root of unity. Let us consider the action of the cyclic group $\mathbb{Z}_n = \langle \xi_n \rangle$ on $\mathbb{C}^2$ defined by $\xi_n \cdot (x, y) = (\xi_n x, \xi_n^q y)$. Then the analytic space $X_{n,q} = \mathbb{C}^2/\mathbb{Z}_n$ contains a cyclic quotient singularity of type $\frac{1}{n}(1, q)$. Denoting by q' the unique integer $1 \leq q' \leq n-1$ such that $qq' \equiv 1 \pmod{n}$, we have $X_{n_1,q_1} \cong X_{n,q}$ if and only if $n_1 = n$ and either $q_1 = q$ or $q_1 = q'$. The exceptional divisor on the minimal resolution $\tilde{X}_{n,q}$ of $X_{n,q}$ is a HJ-string (abbreviation of Hirzebruch-Jung string), that is to say, a connected union $\mathsf{E} = \bigcup_{i=1}^k Z_i$ of smooth rational curves $Z_1, \ldots, Z_k$ with self-intersection ≤ -2, and ordered linearly so that $Z_i Z_{i+1} = 1$ for all i, and $Z_i Z_j = 0$ if $|i - j| \geq 2$. More precisely, given the continued fraction

$$\frac{n}{q} = [b_1, \ldots, b_k] = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_k}}}, \quad b_i \geq 2 , \tag{5}$$

the dual graph of E is



(see [17, Chapter II]). Moreover

$$\frac{n}{q} = [b_1, \ldots, b_k] \quad \text{if and only if} \quad \frac{n}{q'} = [b_k, \ldots, b_1]. \tag{6}$$

In particular a rational double point of type A_n corresponds to the cyclic quotient singularity $\frac{1}{n+1}(1, n)$. A point of type $\frac{1}{2}(1, 1)$ is called an *ordinary double point* or a *node*. For any $1 \leq s \leq k$ set $\frac{n_s}{q_s} := [b_1, \ldots, b_s]$; then $\left\{\frac{n_s}{q_s}\right\}$ is called the sequence of *convergents* of the continued fraction (5). Its terms satisfy the recursive relation

$$\frac{n_s}{q_s} = \frac{b_s n_{s-1} - n_{s-2}}{b_s q_{s-1} - q_{s-2}}, \tag{7}$$

where $n_{-1} = 0,\ n_0 = 1,\ q_{-1} = -1,\ q_0 = 0$ (see Appendix to [21]).

Proposition 1.6 *The sequence* $\left\{\frac{n_s}{q_s}\right\}$ *is strictly decreasing, in fact*

$$\frac{n_{s-1}}{q_{s-1}} - \frac{n_s}{q_s} = \frac{1}{q_{s-1}q_s}. \tag{8}$$

Consequently, the sequence $\left\{\frac{q_s}{n_s}\right\}$ *is strictly increasing, in fact*

$$\frac{q_s}{n_s} - \frac{q_{s-1}}{n_{s-1}} = \frac{1}{n_s n_{s-1}}. \tag{9}$$

Proof Using (7) we can write

$$\begin{aligned} n_{s-1}q_s - n_s q_{s-1} &= n_{s-1}(b_s q_{s-1} - q_{s-2}) - (b_s n_{s-1} - n_{s-2})q_{s-1} \\ &= n_{s-2}q_{s-1} - n_{s-1}q_{s-2} = \ldots = n_1 q_2 - n_2 q_1 \\ &= b_1 b_2 - (b_1 b_2 - 1) = 1, \end{aligned}$$

so both (8) and (9) follow at once. □

Definition 1.7 Let x be a cyclic quotient singularity of type $\frac{1}{n}(1, q)$ and let E be the corresponding HJ-string. If $\frac{n}{q} = [b_1, \ldots, b_k]$, we write $\mathsf{E} : [b_1, \ldots, b_k]$ and we set

$$\begin{aligned} \ell_x &= \ell(\mathsf{E}) = \ell\left(\frac{q}{n}\right) := k, \\ h_x &= h(\mathsf{E}) = h\left(\frac{q}{n}\right) := 2 - \frac{2 + q + q'}{n} - \sum_{i=1}^{k}(b_i - 2), \\ e_x &= e(\mathsf{E}) = e\left(\frac{q}{n}\right) := k + 1 - \frac{1}{n}, \\ B_x &= B(\mathsf{E}) = B\left(\frac{q}{n}\right) := 2e_x - h_x = \frac{1}{n}(q + q') + \sum_{i=1}^{k} b_i. \end{aligned}$$

Remark 1.8 We have

$$\ell\left(\frac{q}{n}\right) = \ell\left(\frac{q'}{n}\right), \quad h\left(\frac{q}{n}\right) = h\left(\frac{q'}{n}\right), \quad e\left(\frac{q}{n}\right) = e\left(\frac{q'}{n}\right), \quad B\left(\frac{q}{n}\right) = B\left(\frac{q'}{n}\right).$$

Moreover $B\left(\frac{q}{n}\right) \geq 3$ and equality holds if and only if $\frac{q}{n} = \frac{1}{2}$.

For the reader's convenience, we listed in the Appendix B the cyclic quotient singularities $\frac{1}{n}(1,\ q)$ and the corresponding values of $h\left(\frac{q}{n}\right)$ and $B\left(\frac{q}{n}\right)$ for all $2 \leq n \leq 14$.

Proposition 1.9 *Let $\frac{n_s}{q_s}$, $\frac{n_t}{q_t}$ be two convergents of the continued fraction $\frac{n}{q} = [b_1, \ldots, b_k]$, with $s \geq t$. Then*

$$B\left(\frac{q_s}{n_s}\right) - B\left(\frac{q_t}{n_t}\right) \geq s - t$$

and equality holds if and only if $s = t$.

Proof It is sufficient to prove that $B\left(\frac{q_s}{n_s}\right) - B\left(\frac{q_{s-1}}{n_{s-1}}\right) > 1$. In fact we have

$$B\left(\frac{q_s}{n_s}\right) - B\left(\frac{q_{s-1}}{n_{s-1}}\right) = \frac{q_s}{n_s} - \frac{q_{s-1}}{n_{s-1}} + \frac{q'_s}{n_s} - \frac{q'_{s-1}}{n_{s-1}} + b_s,$$

that is, using (9),

$$B\left(\frac{q_s}{n_s}\right) - B\left(\frac{q_{s-1}}{n_{s-1}}\right) > \frac{1}{n_s n_{s-1}} - \frac{q'_{s-1}}{n_{s-1}} + b_s > b_s - 1 \geq 1.$$

□

Corollary 1.10 *Let $\frac{n}{q} = [b_1, \ldots, b_k]$ and let $c \in \mathbb{N}$ be such that $b_1 \geq c$. Then*

$$B\left(\frac{q}{n}\right) \geq B\left(\frac{1}{c}\right) = c + \frac{2}{c}$$

and equality holds if and only if $\frac{q}{n} = \frac{1}{c}$.

Proof Setting $s = k$ and $t = 1$ in Proposition 1.9 we obtain

$$B\left(\frac{q}{n}\right) \geq B\left(\frac{1}{b_1}\right) = b_1 + \frac{2}{b_1} \geq c + \frac{2}{c} = B\left(\frac{1}{c}\right)$$

and equality holds if and only if $k = 1$ and $c = b_1$. □

There is a duality between the HJ-expansions of $\frac{n}{q}$ and $\frac{n}{n-q}$, which comes from the Riemenschneider's point diagram ([23, p. 222]). It basically says that if $\frac{q}{n} \neq \frac{1}{2}$ then there exist nonnegative integers $k_1, \ldots, k_t, l_1, \ldots, l_{t-1}$ such that

$$\begin{aligned} \frac{n}{q} &= \left[(2)^{k_1},\ l_1 + 3,\ (2)^{k_2}, \ldots, (2)^{k_{t-1}},\ l_{t-1} + 3,\ (2)^{k_t}\right], \\ \frac{n}{n-q} &= \left[k_1 + 2,\ (2)^{l_1},\ k_2 + 3, \ldots, k_{t-1} + 3,\ (2)^{l_{t-1}},\ k_t + 2\right], \end{aligned} \qquad (10)$$

where $(2)^k$ means the constant sequence with k terms equal to 2, in particular the empty sequence if $k = 0$. It is important to notice that both the k_i or the l_j may actually be equal to zero; for instance, the case $q = 1$ (i.e. $\frac{n}{1} = [n]$, $\frac{n}{n-1} = [(2)^{n-1}]$) is obtained by setting $t = 2, k_1 = 0, l_1 = n - 3, k_2 = 0$. From a more geometric point of view, if N denotes a free abelian group of rank 2, then (10) reflects the duality between the oriented cone $\sigma_{n,\,q} \subset N_{\mathbb{R}}$ associated to $\frac{n}{q}$ and the oriented cone $\sigma_{n,\,n-q}$ associated to $\frac{n}{n-q}$ (see [19]). Now let us exploit Riemenschneider's duality in order to obtain some results on continued fractions that will be used in the proof of Proposition 4.13.

Proposition 1.11 *We have*

$$B\left(\frac{q}{n}\right) + B\left(\frac{n-q}{n}\right) = 3\sum_{i=1}^{t}(k_i+1) + 3\sum_{i=1}^{t-1}(l_i+1).$$

Proof Using (10) we obtain

$$B\left(\frac{q}{n}\right) = \frac{q}{n} + \frac{q'}{n} + 2\sum_{i=1}^{t} k_i + \sum_{i=1}^{t-1}(l_i+3),$$

$$B\left(\frac{n-q}{n}\right) = \frac{n-q}{n} + \frac{(n-q)'}{n} + \sum_{i=1}^{t}(k_i+3) + 2\sum_{i=1}^{t-1} l_i - 2.$$

Combining these relations and using $(n-q)' = n - q'$ we conclude the proof. □

Proposition 1.12 *Let n, q be positive, coprime integers and let a be such that $qq' = 1+an$. Assume moreover that*

$$\left[(2)^{k_1},\ l_1+3,\ (2)^{k_2},\dots,(2)^{k_{t-1}},\ l_{t-1}+3,\ (2)^{k_t}\right] = \frac{n}{n-q'}$$

for some non negative integers $k_1,\dots,k_t$, $l_1,\dots,l_{t-1}$. Then we have

$$\left[k_1+2,\ (2)^{l_1},\ k_2+3,\dots,k_{t-1}+3,\ (2)^{l_{t-1}},\ k_t+3\right] = \frac{n+q}{a+q'} \tag{11}$$

$$\left[k_1+2,\ (2)^{l_1},\ k_2+3,\dots,k_{t-1}+3,\ (2)^{l_{t-1}}\right] = \frac{q}{a}. \tag{12}$$

Proof Using (6) and (10) we can write

$$\begin{aligned}\left[k_t+3,\ (2)^{l_{t-1}},\dots,\ (2)^{l_1},\ k_1+2\right] &= 1 + \left[k_t+2,\ (2)^{l_{t-1}},\dots,(2)^{l_1},\ k_1+2\right]\\ &= 1 + \frac{n}{(q')'} = 1 + \frac{n}{q} = \frac{n+q}{q}.\end{aligned} \tag{13}$$

Since $q\cdot(a+q') \equiv 1 \pmod{n+q}$ and $1 \le a+q' < n+q$, from (6) we obtain (11). Now we have

$$\begin{aligned}\frac{n}{q} &= \left[k_t+2,\ (2)^{l_t-1},\ k_{t-1}+3,\dots,k_2+3,\ (2)^{l_1},\ k_1+2\right]\\ &= k_t+2-\left[(2)^{l_t-1}, k_{t-1}+3,\dots,k_2+3,\ (2)^{l_1},\ k_1+2\right]^{-1},\end{aligned}$$

which implies

$$\left[(2)^{l_t-1},\ k_{t-1}+3,\dots,k_2+3,\ (2)^{l_1},\ k_1+2\right] = \frac{q}{q(k_t+2)-n}.$$

Since $a\cdot(q(k_t+2)-n)) \equiv 1 \pmod{q}$ and $1 \le a < q$, by using (6) we obtain (12). □

Proposition 1.13 *With the notations of Proposition 1.12, we have*

$$B\left(\frac{n-q'}{n}\right) + B\left(\frac{a+q'}{n+q}\right) = 1 - \frac{1+q^2}{n(n+q)} + 3\sum_{i=1}^{t}(k_i+1) + 3\sum_{i=1}^{t-1}(l_i+1), \tag{14}$$

$$B\left(\frac{n-q'}{n}\right) + B\left(\frac{a}{q}\right) = -\frac{1+q^2+n^2}{nq} + 3\sum_{i=1}^{t}(k_i+1) + 3\sum_{i=1}^{t-1}(l_i+1). \tag{15}$$

Proof Write

$$B\left(\frac{n-q'}{n}\right) = \frac{n-q'}{n} + \frac{n-q}{n} + 2\sum_{i=1}^{t} k_i + \sum_{i=1}^{t-1}(l_i+3), \tag{16}$$

$$B\left(\frac{a+q'}{n+q}\right) = \frac{a+q'}{n+q} + \frac{q}{n+q} + \sum_{i=1}^{t}(k_i+3) + 2\sum_{i=1}^{t-1} l_i - 1, \tag{17}$$

$$B\left(\frac{a}{q}\right) = \frac{a}{q} + \frac{q(k_t+2)-n}{q} + \sum_{i=1}^{t-1}(k_i+3) + 2\sum_{i=1}^{t-1} l_i - 1. \tag{18}$$

Summing (16) and (17) we obtain (14), whereas summing (16) and (18) we obtain (15). □

2 Standard isotrivial fibrations

In this section we summarize the basic properties of standard isotrivial fibrations. Definition 2.1 and Theorem 2.3 can be found in [26].

Definition 2.1 We say that a projective surface S is a *standard isotrivial fibration* if there exists a finite group G acting faithfully on two smooth projective curves C_1 and C_2 so that S is isomorphic to the minimal desingularization of $T := (C_1 \times C_2)/G$, where G acts diagonally on the product. The two maps $\alpha_1 : S \longrightarrow C_1/G$, $\alpha_2 : S \longrightarrow C_2/G$ will be referred as the *natural projections*. If T is smooth then $S = T$ is called a *quasi-bundle*.

Remark 2.2 A monodromy argument shows that every isotrivial fibred surface X is birationally isomorphic to a standard isotrivial fibration ([26, Sect. 2]).

The stabilizer $H \subseteq G$ of a point $y \in C_2$ is a cyclic group ([12, p. 106]). If H acts freely on C_1, then T is smooth along the scheme-theoretic fibre of $\sigma : T \longrightarrow C_2/G$ over $\bar{y} \in C_2/G$, and this fibre consists of the curve C_1/H counted with multiplicity $|H|$. Thus, the smooth fibres of σ are all isomorphic to C_1. On the contrary, if $x \in C_1$ is fixed by some non-zero element of H, then one has a cyclic quotient singularity over the point $\overline{(x, y)} \in T$. These observations lead to the following statement, which describes the singular fibres that can arise in a standard isotrivial fibration (see [26, Theorem 2.1]).

Theorem 2.3 *Let $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ be a standard isotrivial fibration and let us consider the natural projection $\alpha_2 : S \longrightarrow C_2/G$. Take any point over $\bar{y} \in C_2/G$ and let F denote the schematic fibre of α_2 over $\bar{y}$. Then*

(i) *The reduced structure of F is the union of an irreducible curve Y, called the central component of F, and either none or at least two mutually disjoint HJ-strings, each meeting Y at one point, and each being contracted by λ to a singular point of T. These strings are in one-to-one correspondence with the branch points of $C_1 \longrightarrow C_1/H$, where $H \subseteq G$ is the stabilizer of y.*
(ii) *The intersection of a string with Y is transversal, and it takes place at only one of the end components of the string.*
(iii) *Y is isomorphic to C_1/H, and has multiplicity equal to $|H|$ in F.*

An analogous statement holds if one considers the natural projection $\alpha_1 : S \longrightarrow C_1/G$.

In the sequel we denote by $\mathscr{H}(F)$ the set of the HJ-strings contained in F and we say that F is a *reducible fibre* if $\mathscr{H}(F) \neq \emptyset$. Theorem 2.3 therefore implies

Remark 2.4 For every reducible fibre F, the cardinality of $\mathscr{H}(F)$ is at least two.

For a proof of the following result, see [4,13,18, pp. 509–510].

Proposition 2.5 *Let $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ be a standard isotrivial fibration. Then the invariants of S are given by*

(i) $K_S^2 = \frac{8(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \text{Sing } T} h_x$;
(ii) $e(S) = \frac{4(g(C_1)-1)(g(C_2)-1)}{|G|} + \sum_{x \in \text{Sing } T} e_x$;
(iii) $q(S) = g(C_1/G) + g(C_2/G)$.

Corollary 2.6 *Let $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ be a standard isotrivial fibration. Then*

$$K_S^2 = 8\chi(\mathcal{O}_S) - \frac{1}{3} \sum_{x \in \text{Sing } T} B_x. \tag{19}$$

Proof Proposition 2.5 yields $K_S^2 = 2e(S) - \sum_{x \in \text{Sing } T}(2e_x - h_x)$. By Noether's formula we have $K_S^2 = 12\chi(\mathcal{O}_S) - e(S)$, so (19) follows. □

Let us consider now the minimal resolution of a cyclic quotient singularity $x \in T$. If Y_1 and Y_2 are the strict transforms of C_1 and C_2, by Theorem 2.3 we obtain the situation illustrated in Fig. 1.

The curves Y_1 and Y_2 are the central components of two reducible fibres F_1 and F_2 of $\alpha_2 : S \longrightarrow C_2/G$ and $\alpha_1 : S \longrightarrow C_1/G$, respectively. Then there exist $\lambda_1, \ldots, \lambda_k, \mu_1, \ldots, \mu_k \in \mathbb{N}$ such that

$$\begin{aligned} F_1 &= \rho_1 Y_1 + \sum_{i=1}^{k} \lambda_i Z_i + \Gamma_1, \\ F_2 &= \rho_2 Y_2 + \sum_{i=1}^{k} \mu_i Z_i + \Gamma_2, \end{aligned} \tag{20}$$

where the supports of both divisors Γ_1 and Γ_2 are union of HJ-strings disjoint from the Z_i; moreover if x is of type $\frac{1}{n}(1, q)$, then n divides both ρ_1 and ρ_2. Now we have

$$\begin{cases} 0 = F_1 Z_k = -\lambda_k b_k + \lambda_{k-1} \\ 0 = F_1 Z_{k-1} = \lambda_k - b_{k-1}\lambda_{k-1} + \lambda_{k-2} \\ \cdots \\ 0 = F_1 Z_2 = \lambda_3 - b_2 \lambda_2 + \lambda_1 \\ 0 = F_1 Z_1 = \lambda_2 - b_1 \lambda_1 + \rho_1, \end{cases} \tag{21}$$

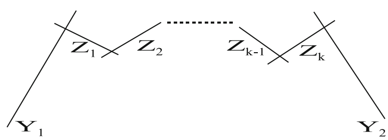


Fig. 1 Resolution of a cyclic quotient singularity $x \in T$

which gives

$$\begin{cases} \lambda_{k-1}/\lambda_k = b_k \\ \lambda_{k-2}/\lambda_{k-1} = [b_{k-1},\ b_k] \\ \cdots \\ \lambda_1/\lambda_2 = [b_2,\ b_3, \ldots, b_k] \\ \rho_1/\lambda_1 = [b_1,\ b_2, \ldots, b_k]. \end{cases}$$

In particular

$$\lambda_1 = \frac{\rho_1}{[b_1, b_2,\ \ldots b_k]} = \frac{\rho_1 q}{n}. \tag{22}$$

Analogously, we have

$$\begin{cases} \mu_2/\mu_1 = b_1 \\ \mu_3/\mu_2 = [b_2,\ b_1] \\ \cdots \\ \mu_k/\mu_{k-1} = [b_{k-1},\ b_{k-2}, \ldots, b_1] \\ \rho_2/\mu_k = [b_k,\ b_{k-1}, \ldots, b_1], \end{cases}$$

hence

$$\mu_k = \frac{\rho_2}{[b_k,\ b_{k-1}, \ldots, b_1]} = \frac{\rho_2 q'}{n}. \tag{23}$$

Definition 2.7 We say that a reducible fibre F_1 of $\alpha_2 : S \longrightarrow C_2/G$ is of type $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$ if it contains exactly r HJ-strings $\mathsf{E}_1, \ldots, \mathsf{E}_r$, where each E_i is of type $\frac{1}{n_i}(1, q_i)$. The same definition holds for a reducible fibre F_2 of $\alpha_1 : S \longrightarrow C_1/G$.

Proposition 2.8 *Let* F_1 *be of type* $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$ *and let* Y_1 *be its central component. Then*

$$(Y_1)^2 = -\sum_{i=1}^{r} \frac{q_i}{n_i}. \tag{24}$$

Analogously, if F_2 *is of type* $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$ *then*

$$(Y_2)^2 = -\sum_{i=1}^{r} \frac{q_i'}{n_i}. \tag{25}$$

Proof If F_1 is of type $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$, set $\rho_1 = \text{l.c.m.}(n_1, \ldots, n_r)$ and

$$\mathsf{E}_i := \bigcup_{j=1}^{k_i} Z_{j,\,i} \qquad i = 1, \ldots, r.$$

Then we can write

$$F_1 = \rho_1 Y_1 + \sum_{i=1}^{r} \sum_{j=1}^{k_i} \lambda_{j,\,i}\ Z_{j,\,i}.$$

By using (22), we have

$$0 = F_1 Y = \rho_1 (Y_1)^2 + \sum_{i=1}^{r} \lambda_{1,i} = \rho_1 (Y_1)^2 + \rho_1 \sum_{i=1}^{r} \frac{q_i}{n_i}$$

and this proves (24). Analogously, one can use (23) in order to prove (25). □

Corollary 2.9 *Assume* $\mathrm{Sing}(T) = \frac{1}{n_1}(1, q_1) + \cdots + \frac{1}{n_r}(1, q_r)$. *Then both*

$$\sum_{i=1}^{r} \frac{q_i}{n_i} \quad \text{and} \quad \sum_{i=1}^{r} \frac{q_i'}{n_i}$$

are integers.

Corollary 2.10 *Assume that T contains exactly r ordinary double points as singularities. Then r is even.*

3 The non-minimal case

Let $\lambda : S \longrightarrow T := (C_1 \times C_2)/G$ be a standard isotrivial fibration. If $g(C_1/G) \geq 1$ and $g(C_2/G) \geq 1$ then S is necessarily a minimal model. If instead $g(C_1/G) = 0$, it may happen that the central component of some reducible fibre F_1 of $\alpha_2 : S \longrightarrow C_2/G$ is a (-1)-curve. Analogously, if $g(C_2/G) = 0$ it may happen that the central component of some reducible fibre F_2 of $\alpha_1 : S \longrightarrow C_1/G$ is a (-1)-curve.

Definition 3.1 We say that a reducible fibre F_1 of $\alpha_2 : S \longrightarrow C_2/G$ is a (-1)-*fibre* if its central component Y_1 is a (-1)-curve. If $g(C_1) = \mathfrak{g}$, we will also say that F is a (-1)-*fibre in genus* $\mathfrak{g}$. The same definitions hold for a reducible fibre F_2 of $\alpha_1 : S \longrightarrow C_1/G$.

Proposition 3.2 *Assume that F_1 is a reducible fibre of $\alpha_2 : S \longrightarrow C_2/G$, of type* $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$. *Set* $\rho := \mathrm{l.c.m.}(n_1, \dots, n_r)$. *Then F_1 is a (-1)-fibre if and only if*

$$\sum_{i=1}^{r} \frac{q_i}{n_i} = 1 \quad \text{and} \quad 2g(C_1) - 2 = \rho \left(-2 + \sum_{i=1}^{r} \left(1 - \frac{1}{n_i} \right) \right).$$

Assume that F_2 is a reducible fibre of $\alpha_1 : S \longrightarrow C_1/G$, of type $\left(\frac{q_1}{n_1}, \dots, \frac{q_r}{n_r}\right)$. *Then F_2 is a (-1)-fibre if and only if*

$$\sum_{i=1}^{r} \frac{q_i'}{n_i} = 1 \quad \text{and} \quad 2g(C_2) - 2 = \rho \left(-2 + \sum_{i=1}^{r} \left(1 - \frac{1}{n_i} \right) \right).$$

Proof Let us consider first F_1. By Proposition 2.8 and Theorem 2.3 the two conditions are equivalent to $(Y_1)^2 = -1$ and $g(Y_1) = 0$, respectively. If we consider F_2 the proof is analogous. □

The following result provides a method to construct non-minimal standard isotrivial fibrations with arbitrarily many (-1)-fibres.

Theorem 3.3 *Let* $\mathcal{S} := \left\{ \frac{q_1}{n_1}, \dots, \frac{q_r}{n_r} \right\}$ *be a finite set of rational numbers, with* $(n_i, q_i) = 1$, *such that* $\sum_{i=1}^{r} \frac{q_i}{n_i} = 1$. *Set* $n := \mathrm{l.c.m.}(n_1, \dots, n_r)$. *Then for any* $\mathfrak{q} \geq 0$ *there exists a standard isotrivial fibration* $\lambda : S \longrightarrow T := (C_1 \times C_2)/G$ *such that the following holds.*

(i) $\mathrm{Sing}(T) = n \times \frac{1}{n_1}(1, q_1) + \cdots + n \times \frac{1}{n_r}(1, q_r)$;
(ii) *the singular fibres of the natural projection* $\alpha_2 : S \longrightarrow C_2/G$ *are exactly* n (-1)-*fibres, all of type* $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$;

(iii) $q(S) = \mathfrak{q}$.

Proof For all $i \in \{i, \ldots, r\}$ set $t_i := q_i n/n_i$. Set moreover $G := \langle \xi \mid \xi^n = 1\rangle \cong \mathbb{Z}_n$. Since $(n, t_i) = n/n_i$, the element ξ^{t_i} has order $n/(n, t_i) = n_i$ in G. It follows that G is both $(0 \mid n_1, \ldots, n_r)$ and $(\mathfrak{q} \mid n^n)$-generated, with generating vectors given by

$$\mathcal{V}_1 = \{g_1, \ldots, g_r\} := \{\xi^{t_1}, \ldots, \xi^{t_r}\} \quad \text{and}$$
$$\mathcal{V}_2 = \{\ell_1, \ldots, \ell_n;\ \ h_1, \ldots, h_{2\mathfrak{q}}\} := \{\underbrace{\xi, \ldots, \xi}_{n \text{ times}};\ \underbrace{\xi, \ldots, \xi}_{2\mathfrak{q} \text{ times}}\},$$

respectively. Therefore by Proposition 1.3 we obtain two G-covers

$$C_1 \longrightarrow C_1/G \cong \mathbb{P}^1, \quad C_2 \longrightarrow C_2/G,$$

where $g(C_2/G) = \mathfrak{q}$. By using Proposition 1.4 we see that

- for all $i \in \{1, \ldots, r\}$, there are n/n_i fixed points on C_1 with stabilizer $\langle \xi^{t_i}\rangle \cong \mathbb{Z}_{n_i}$; if P_i is the set of these fixed points, we have

$$|\mathrm{Fix}_{C_1, q}(\xi^{t_i}) \cap P_i| = \begin{cases} n/n_i & \text{if } q = 1 \\ 0 & \text{otherwise;} \end{cases}$$

- there are n fixed points on C_2, whose stabilizer is the whole G; for all $i \in \{1, \ldots, r\}$ we have

$$|\mathrm{Fix}_{C_2, q}(\xi^{t_i})| = \begin{cases} n & \text{if } q = q_i \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the standard isotrivial fibration $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ has all the desired properties. □

In the sequel we will focus our attention on the natural projection $\alpha_2 : S \longrightarrow C_2/G$; this involves no loss of generality and similar results hold if one considers instead the projection $\alpha_1 : S \longrightarrow C_1/G$. For abbreviation, we simply write "(-1)-fibre" instead of "(-1)-fibre of $\alpha_2 : S \longrightarrow C_2/G$".

Corollary 3.4 *The classification of* (-1)-*fibres in genus* $\mathfrak{g}$ *is equivalent to the classification of pairs* $(G, \mathcal{S})$, *where* G *is a finite group and* $\mathcal{S} := \left\{\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right\}$ *is a set of rational numbers, with* $(n_i, q_i) = 1$ *for all* i, *such that*

(i) G *acts in genus* $\mathfrak{g}$ *with rational quotient and signature* $(0 \mid n_1, \ldots, n_r)$*;*
(ii) $\sum_{i=1}^{r} \frac{q_i}{n_i} = 1$.

Proof Immediate by Proposition 3.2 and Theorem 3.3. □

Corollary 3.5 *The following are equivalent:*

(i) F *is a* (-1)-*fibre in genus* $\mathfrak{g} = 0$*;*
(ii) F *is a reducible fibre in genus* $\mathfrak{g} = 0$*;*
(iii) F *is a reducible fibre of type* $\left(\frac{q}{n}, \frac{n-q}{n}\right)$ *whose central component is rational.*

Proof $(i) \Rightarrow (ii)$. Obvious.

$(ii) \Rightarrow (iii)$. Assume $g(C_1) = 0$. For all $n \geq 2$, the cyclic group $\mathbb{Z}_n$ acts on $\mathbb{P}^1$, and the only possible signature is $(0 \mid n, n)$ ([6, p. 9]). Therefore every reducible fibre of $\alpha_2 : S \longrightarrow C_2/G$ is of type $\left(\frac{q_1}{n}, \frac{q_2}{n}\right)$ for some positive integers n, q_1, q_2. On the other hand we have seen that $\frac{q_1}{n} + \frac{q_2}{n}$ must be integer, so F is of type $\left(\frac{q}{n}, \frac{n-q}{n}\right)$. Finally, the central component of F is rational since it is a quotient of C_1 (Theorem 2.3).

$(iii) \Rightarrow (i)$. This follows from Proposition 3.2. □

Corollary 3.5 shows that there are infinitely many types of (-1)-fibres in genus $\mathfrak{g} = 0$. On the other hand, for all genera $\mathfrak{g} \geq 1$ there are only finitely many types, since there are only finitely many cyclic groups of automorphisms; the cases where $\mathfrak{g} = 1, 2, 3$ are described in detail in Appendix A.

Example 3.6 Let $n \geq 2$ be any positive integer and take $\mathcal{S} = \left\{\frac{1}{n}, \frac{n-1}{n}\right\}$, $\mathfrak{q} = 1$. Using the construction given in Theorem 3.3, we obtain a standard isotrivial fibration $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ with

$$g(C_1) = 0, \quad 2g(C_2) - 2 = n^2 - n, \quad \text{Sing}(T) = n \times \frac{1}{n}(1,\ 1) + n \times \frac{1}{n}(1,\ n-1).$$

For all n, S is a ruled surface whose invariants are $p_g(S) = 0$, $q(S) = 1$, $K_S^2 = -n^2$. Hence every minimal model $\widehat{S}$ of S satisfies $K_{\widehat{S}}^2 = 0$.

Example 3.7 Take $\mathcal{S} = \left\{\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ times}}\right\}$ and $\mathfrak{q} = 1$. We obtain a standard isotrivial fibration with

$$2g(C_1) - 2 = n^2 - 3n, \quad 2g(C_2) - 2 = n^2 - n, \quad \text{Sing}(T) = n^2 \times \frac{1}{n}(1,\ 1).$$

Thus Proposition 2.5 yields

$$K_S^2 = n^3 - 4n^2 + 2n, \quad e(S) = n^3 - 2n^2 + 2n,$$
$$\chi(\mathcal{O}_S) = \frac{n(n-1)(n-2)}{6}, \quad q(S) = 1.$$

For $n = 2$, S is a ruled surface. Now we assume $n \geq 3$. Since $\mathfrak{q} > 0$, the minimal model $\widehat{S}$ of S is obtained by contracting n disjoint (-1)-curves. Hence its invariants are

$$K_{\widehat{S}}^2 = n(n-1)(n-3), \quad e(\widehat{S}) = n(n-1)^2.$$

For $n = 3$ we obtain an elliptic surface with $\text{kod}(\widehat{S}) = 1$ and $p_g(\widehat{S}) = q(\widehat{S}) = 1$, whose elliptic fibration $\alpha_2 : S \longrightarrow C_2/G$ contains exactly three singular fibres, all of type $IV(\widetilde{A}_2)$ according to Kodaira classification ([3, Chapter V]); for $n \geq 4$ we have a surface of general type. Taking $\mathfrak{q} > 1$ leads to similar results: for $n = 3$ the surface $\widehat{S}$ is elliptic and satisfies $p_g(\widehat{S}) = q(\widehat{S}) = \mathfrak{q}$, whereas for $n \geq 4$ it is of general type.

Remark 3.8 Under the assumptions of Theorem 3.3, one may ask whether there exists a standard isotrivial fibration such that $\text{Sing}(T) = \frac{1}{n_1}(1,\ q_1) + \cdots + \frac{1}{n_r}(1,\ q_r)$. In general the answer is negative, in fact further necessary conditions are

$$\frac{1}{3}\sum_{i=1}^{r} B\left(\frac{q_i}{n_i}\right) \in \mathbb{Z} \quad \text{and} \quad \sum_{i=1}^{r} \frac{q_i'}{n_i} \in \mathbb{Z},$$

see Corollaries 2.6 and 2.9. For example, there are no standard isotrivial fibrations with $\text{Sing}(T) = 3 \times \frac{1}{3}(1,\ 1)$ or with $\text{Sing}(T) = 2 \times \frac{1}{5}(1,\ 1) + \frac{1}{5}(1, 3)$. In some cases, however, the question above has an affirmative answer. For instance, in [18] there are examples of standard isotrivial fibrations with $\text{Sing}(T) = 4 \times \frac{1}{4}(1, 1)$ and with $\text{Sing}(T) = \frac{1}{7}(1, 1) + \frac{1}{7}(1, 2) + \frac{1}{7}(1, 4)$.

4 The relatively minimal model

4.1 Contractible components

Let $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ be a standard isotrivial fibration. If F is any (-1)-fibre of $\alpha_2 : S \longrightarrow C_2/G$, with $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$, we consider the following procedure:

Step 0: contract the central component Y of F;
Step 1: make all possible contractions in the image of E_1;
Step 2: make all possible contractions in the image of E_2;
...
Step r: make all possible contractions in the image of E_r;
Step r + 1: go back to Step 1 and repeat.

Applying this algorithm to all (-1)-fibres, we obtain a relative minimal fibration $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$. If $g(C_1) \geq 1$ this is the unique relative minimal model of α_2 ([3, Chapter III, Proposition 8.4]); by abuse of terminology, we will say that $\hat{\alpha}_2$ is *the* relative minimal model of α_2 also when $g(C_1) = 0$. If $g(C_2/G) \geq 1$, then $\widehat{S}$ is obviously a minimal surface. If $g(C_2/G) = 0$ this is not true in general, as following example illustrates.

Example 4.1 The group $G = \text{PSL}_2(\mathbb{F}_7)$ has order 168 and it is $(0\,|\,2, 3, 7)$-generated ([16, pp. 265–266]). Then there exists a genus 3 curve C and a G-cover $C \longrightarrow \mathbb{P}^1$, branched in three points with branching numbers 2,3 and 7, respectively. Set $C_1 = C_2 = C$ and consider the standard isotrivial fibration $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$; standard computations as in [18] show that

$$\text{Sing}(T) = 4 \times \frac{1}{2}(1,\ 1) + \frac{1}{3}(1,\ 1) + \frac{1}{3}(1,\ 2) + \frac{1}{7}(1,\ 1) + \frac{1}{7}(1,\ 2) + \frac{1}{7}(1,\ 4).$$

By using Proposition 2.5 we obtain

$$K_S^2 = -6, \quad e(S) = 18, \quad q(S) = 0,$$

hence $\chi(\mathcal{O}_S) = 1$ and $p_g(S) = 0$. The natural projection $\alpha_2 : S \longrightarrow C/G \cong \mathbb{P}^1$ contains precisely three reducible fibres F_2, F_3, F_7 and moreover:

- F_2 is of type $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$;
- F_3 is of type $\left(\frac{1}{3}, \frac{2}{3}\right)$;
- F_7 is of type $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$.

Out of these, the unique (-1)-fibre is F_7, in fact the central components of F_2 and F_3 are not rational curves. The surface $\widehat{S}$ is therefore obtained by blowing down two curves (see Example 4.2 below), hence $K^2_{\widehat{S}} = -4$ and consequently $\widehat{S}$ is *not* a minimal surface. It is no difficult to check that in this example $\text{kod}(S) = -\infty$.

Now let $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ be a standard isotrivial fibration and $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$ the relative minimal model of α_2. Let F be a reducible fibre of α_2 and let $\mathsf{E} = \bigcup_{i=1}^k Z_i \in \mathscr{H}(F)$ be a HJ-string contained in F. We say that an irreducible component $Z_i \subset \mathsf{E}$ is *contractible* if it is contracted by the natural map $\pi : S \longrightarrow \widehat{S}$. By definition it follows that if both Z_i and Z_j are contractible, then Z_l is also contractible for any $i \leq l \leq j$. Now we define

$$\mathfrak{c}(F) := \text{number of irreducible components of } F \text{contracted by } \pi.$$

Obviously, $\mathfrak{c}(F) \geq 0$; moreover $\mathfrak{c}(F) > 0$ if and only if F is a (-1)-fibre, and $\mathfrak{c}(F) = 1$ if and only if F is a (-1)-fibre and none of its HJ-strings contains contractible components.

Example 4.2 If F is a (-1)-fibre of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then $\mathfrak{c}(F) = 1$. If F is a (-1)-fibre of type $\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$, then $\mathfrak{c}(F) = 2$.

For any $\tau \in C_2/G$, let F_τ and $(F_\tau)_{\text{red}}$ be the fibre and the reduced fibre of $\alpha_2 : S \longrightarrow C_2/G$ over τ, respectively, and set

$$\begin{aligned}
\text{Crit}(\alpha_2) &:= \{\tau \in C_2/G \mid F_\tau \text{ is singular}\};\\
\mathcal{R}(\alpha_2) &:= \{\tau \in \text{Crit}(\alpha_2) \mid (F_\tau)_{\text{red}} \text{ is smooth}\};\\
\text{Crit}(\alpha_2)' &:= \text{Crit}(\alpha_2) \setminus \mathcal{R}(\alpha_2) = \{\tau \in C_2/G \mid F_\tau \text{ is reducible}\}.
\end{aligned}$$

Moreover, given any reducible fibre F, let us define

$$\delta(F) := \frac{1}{3} \sum_{\mathsf{E} \in \mathscr{H}(F)} B(\mathsf{E}) - \mathfrak{c}(F).$$

Example 4.3 If F is of type $\left(\underbrace{\frac{1}{n}, \ldots, \frac{1}{n}}_{n \text{ times}}\right)$, with $n \geq 3$, then $\delta(F) = \frac{1}{3}(n^2 - 1)$ if F is a (-1)-fibre and $\delta(F) = \frac{1}{3}(n^2 + 2)$ otherwise. If $n = 2$ then $\delta(F) = 0$ if F is a (-1)-fibre and $\delta(F) = 2$ otherwise.

The rational number $\delta(F)$ plays an important role in the sequel, because of the following result.

Proposition 4.4 *With the above notations we have*

$$K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}}) - \sum_{\tau \in \text{Crit}(\alpha_2)'} \delta(F_\tau). \tag{26}$$

Proof Immediate by using (19) and the definition of $\delta(F)$. □

Remark 4.5 S.L. Tan pointed out that one has the equality

$$\delta(F) = \frac{1}{3}\left(2c_2(F) - c_1^2(F)\right),$$

where $c_1^2(F)$ and $c_2(F)$ are the invariants defined in [27].

The behaviour of $\delta(F)$ when F is not a (-1)-fibre is quite simple.

Lemma 4.6 *Let F be a reducible fibre which is* not *a* (-1) *fibre. Then* $\delta(F) \geq 2$ *and equality holds if and only if F is of type* $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Proof Since F is not a (-1)-fibre we have $\mathfrak{c}(F) = 0$; moreover $\mathscr{H}(F)$ contains at least two HJ-strings (Remark 2.4), so Remark 1.8 yields

$$\delta(F) = \frac{1}{3} \sum_{\mathsf{E} \in \mathscr{H}(F)} B(\mathsf{E}) \geq 2$$

and equality holds if and only if $\mathscr{H}(F)$ contains exactly two HJ-strings, both of type $\frac{1}{2}(1, 1)$. □

Now we start the analysis of the case where F is a (-1)-fibre. If $\mathsf{E} \in \mathscr{H}(F)$ is a HJ-string of type $\frac{1}{n}(1, q)$, with $\frac{n}{q} = [b_1, \ldots, b_k]$, we define $b_i(\mathsf{E}) := b_i$ for all $1 \leq i \leq k$. In particular, $-b_1(\mathsf{E}) = -\lceil \frac{n}{q} \rceil$ equals the self-intersection of the unique curve in E which meets the central component Y of F.

Lemma 4.7 *Assume that F is a* (-1)*-fibre of* $\alpha_2 : S \longrightarrow C_2/G$ *and set* $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$. *Then*

(i) $\mathfrak{c}(F) = 1$ *if and only if* $b_1(\mathsf{E}_i) \geq 3$ *for all* i;
(ii) *the set* $\{i \mid b_1(\mathsf{E}_i) = 2\}$ *has cardinality at most two, and it has cardinality two if and only if F is of type* $\left(\frac{1}{2}, \frac{1}{2}\right)$. *If this happens, then S is ruled*;
(iii) *if* $r = 2$ *and F is not of type* $\left(\frac{1}{2}, \frac{1}{2}\right)$, *we may assume* $b_1(\mathsf{E}_1) = 2$ *and* $b_1(\mathsf{E}_2) \geq 3$.

Proof We have $\mathfrak{c}(F) = 1$ if and only if no further (-1)-curves arise in F after contracting its central component; this is in turn equivalent to say that $b_1(\mathsf{E}_i) \geq 3$ for all i, so our first claim is proven.

Now let us assume $b_1(\mathsf{E}_1) = 2$; hence $\frac{n_1}{q_1} \leq 2$, that is $\frac{q_1}{n_1} \geq \frac{1}{2}$. Therefore by using (24) we obtain $\sum_{i \geq 2} \frac{q_i}{n_i} = 1 - \frac{q_1}{n_1} \leq \frac{1}{2}$, which in turn implies $b_1(\mathsf{E}_i) = \lceil \frac{n_i}{q_i} \rceil \geq 3$ for all $i \geq 2$, unless F contains exactly two strings $\mathsf{E}_1, \mathsf{E}_2$, both of type $\frac{1}{2}(1, 1)$. In this case, contracting the central component we obtain two (-1)-curves intersecting transversally in a point; therefore by [3, Proposition 4.6 p. 79] it follows $\mathrm{kod}(S) = -\infty$, that is S is ruled. This proves (ii).

Finally, assume $\mathscr{H}(F) = \{\mathsf{E}_1, \mathsf{E}_2\}$. In this case $\mathfrak{g} = 0$ and, by Corollary 3.5, F is of type $\left(\frac{q}{n}, \frac{n-q}{n}\right)$. We may assume $\frac{q}{n} > \frac{1}{2}$; hence $\frac{n}{q} < 2$ and $b_1(\mathsf{E}_1) = \lceil \frac{n}{q} \rceil = 2$. Now part (ii) gives $b_1(\mathsf{E}_2) \geq 3$. □

Proposition 4.8 *All* (-1)*-fibres in genus* 0 *satisfy* $\delta(F) = 0$.

Proof By Corollary 3.5, any (-1)-fibre F in genus 0 is of type $\left(\frac{q}{n}, \frac{n-q}{n}\right)$. If $\frac{q}{n} = \frac{n-q}{n} = \frac{1}{2}$ the result is clear. Otherwise by Riemenschneider's duality (10) it follows

$$\mathfrak{c}(F) = \ell\left(\frac{q}{n}\right) + \ell\left(\frac{n-q}{n}\right) = \sum_{i=1}^{t}(k_i + 1) + \sum_{i=1}^{t-1}(l_i + 1),$$

hence Proposition 1.11 implies $\delta(F) = 0$. □

Remark 4.9 If $g(C_1) = 0$ then Proposition 4.8 and relation (26) imply $K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}})$, according to the fact that $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$ is a relatively minimal rational fibration.

Lemma 4.10 *Let F be a* (-1)*-fibre in genus* $\mathfrak{g} \geq 1$. *Then* $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$ *with* $r \geq 3$. *Moreover we may assume that* E_i *contains no contractible components for* $i \geq 3$.

Proof If $\mathscr{H}(F) = \{\mathsf{E}_1, \mathsf{E}_2\}$ then $\mathfrak{g} = 0$ by Corollary 3.5. Then $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$ with $r \geq 3$. Suppose $r = 3$ and put

$$\mathsf{E}_1 = \bigcup Z_i, \quad \mathsf{E}_2 = \bigcup W_j, \quad \mathsf{E}_3 = \bigcup T_h.$$

If $\mathfrak{c}(F) = 1$ there is nothing to prove. Thus we can assume $\mathfrak{c}(F) > 1$ and $b_1(\mathsf{E}_1) = -(Z_1)^2 = 2$; by Lemma 4.7 we have $b_1(E_2) \geq 3$ and $b_1(E_3) \geq 3$. Let us write

$$\mathsf{E}_1 : \left[(2)^k, \, l+3, \ldots\right] = \frac{n_1}{q_1}, \quad k > 0, \; l \geq 0; \tag{27}$$

therefore we can contract the central component Y of F and the images of $Z_1, \ldots, Z_k$, but not the image of Z_{k+1}. After these contractions, the images of the curves W_1 and T_1 are tangent at one point. If also W_1 can be contracted, then the image of T_1 becomes *singular*, hence E_3 contains no contractible components. If $r \geq 4$ the argument is the same. □

Proposition 4.11 *Let F be a (-1)-fibre such that $\mathfrak{c}(F) = 1$. Then $\delta(F) \geq 2 + \frac{2}{3}$ and equality holds if and only if F is of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.*

Proof Since $\mathfrak{c}(F) = 1$ we have $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$, with $r \geq 3$ and $b_1(\mathsf{E}_i) \geq 3$ for all i (Lemma 4.7). Thus Corollary 1.10 implies

$$\delta(F) = \frac{1}{3}\sum_{i=1}^{r} B(\mathsf{E}_i) - 1 \geq \frac{1}{3} \cdot 3 \cdot B\left(\frac{1}{3}\right) - 1 = 2 + \frac{2}{3}$$

and equality holds if and only if $\mathscr{H}(F) = \{\mathsf{E}_1, \mathsf{E}_2, \mathsf{E}_3\}$ and all E_i are of type $\frac{1}{3}(1, 1)$. □

Proposition 4.12 *Let F be a (-1)-fibre in genus $\mathfrak{g} \geq 1$ such that $\mathfrak{c}(F) \geq 2$. If $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$, then E_1 and E_2 belong to one of the following cases.*

Case 1. $\mathsf{E}_1 : [(2)^{k_1}, ***]$
$\mathsf{E}_2 : [***]$

Case 2. $\mathsf{E}_1 : [(2)^{k_1}, ***]$
$\mathsf{E}_2 : [k_1 + 2, ***]$

Case 3. $\mathsf{E}_1 : \left[(2)^{k_1}, \; l_1 + 3, \; (2)^{k_2}, \ldots, (2)^{k_{t-1}}, \; l_{t-1} + 3, \; (2)^{k_t}, ***\right]$
$\mathsf{E}_2 : \left[k_1 + 2, \; (2)^{l_1}, \; k_2 + 3, \ldots, k_{t-1} + 3, \; (2)^{l_{t-1}}, \; k_t + 3, ***\right] \quad t \geq 1,$

Case 4. $\mathsf{E}_1 : \left[(2)^{k_1}, \; l_1 + 3, \; (2)^{k_2}, \ldots, (2)^{k_{t-1}}, \; l_{t-1} + 3, \; (2)^{k_t}, ***\right]$
$\mathsf{E}_2 : \left[k_1 + 2, \; (2)^{l_1}, \; k_2 + 3, \ldots, k_{t-1} + 3, \; (2)^{l_{t-1}}, ***\right] \quad t \geq 2,$

*where $k_i, l_j \geq 0$ and "***" denotes the non-contractible part of the HJ-string.*

Proof By Lemma 4.10 we may assume that all contractible components of F, different from the central component Y, belong to $\mathsf{E}_1 \cup \mathsf{E}_2$. Moreover, since $\mathfrak{c}(F) \geq 2$ and $\mathfrak{g} \geq 1$, we can suppose $b_1(\mathsf{E}_1) = 2$ and $b_1(\mathsf{E}_2) \geq 3$ (Lemma 4.7). Set

$$\mathsf{E}_1 = \bigcup Z_i, \quad \mathsf{E}_2 = \bigcup W_j$$

and

$$\mathsf{E}_1 : \; \left[(2)^{k_1}, \, l_1 + 3, \; (2)^{k_2}, \, l_2 + 3, \ldots\right] \quad k_i, l_j \geq 0, \; k_1 > 0,$$
$$\mathsf{E}_2 : \; \left[u_1 + 3, \; (2)^{v_1}, \, u_2 + 3, \; (2)^{v_2}, \ldots\right] \quad u_i, v_j \geq 0.$$

Now we start the contraction process described in Sect. 4.1; since $\mathfrak{g} \geq 1$, it never gives rise to rational curves with self-intersection equal to 0. First, we can contract the central component Y and the images of the curves $Z_1, \ldots, Z_{k_1}$, but not the image of Z_{k_1+1}; then either we stop or the image of W_1 has self-intersection (-1), that forces $u_1 = k_1 - 1$. In this case we can contract the images of $W_1, \ldots, W_{v_1+1}$, but not the image of W_{v_1+2}; then either we stop or the image of Z_{k_1+1} has self-intersection (-1), which gives $v_1 = l_1$. In the same way we obtain

$$u_i = k_i \quad \text{and} \quad v_i = l_i \quad \text{for all } i \geq 2.$$

Repeated application of this argument yields either one of Cases $1, \ldots, 4$ described in the statement or one of Cases $3'$,$4'$ below.

Case 3$'$. $\mathsf{E}_1 : \left[(2)^{k_1},\ l_1 + 3,\ (2)^{k_2}, \ldots, (2)^{k_t},\ l_t + 3,\ *\!*\!*\right]$
$\mathsf{E}_2 : \left[k_1 + 2,\ (2)^{l_1},\ k_2 + 3, \ldots, k_t + 3,\ (2)^{l_t},\ *\!*\!*\right]$

Case 4$'$. $\mathsf{E}_1 : \left[(2)^{k_1},\ l_1 + 3,\ (2)^{k_2}, \ldots, (2)^{k_t},\ *\!*\!*\right]$
$\mathsf{E}_2 : \left[k_1 + 2,\ (2)^{l_1},\ k_2 + 3, \ldots, k_t + 3,\ (2)^{l_t},\ *\!*\!*\right].$

Finally we observe that Case $3'$ (resp. Case $4'$) is obtained by putting $k_1 = 0$ in Case 3 (resp. in Case 4) and interchanging E_1 and E_2. This completes the proof. □

Proposition 4.13 *Let F be a (-1)-fibre in genus $\mathfrak{g} \geq 1$. Then $\delta(F) > 2$ with exactly the following two exceptions:*

(i) *F is of type $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$; in this case $\mathfrak{g} = 1$ and $\delta(F) = 1 + \frac{1}{3}$.*

(ii) *F is of type $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$; in this case $\mathfrak{g} = 1$ and $\delta(F) = 2$.*

Proof Set $\mathscr{H}(F) = \{\mathsf{E}_1, \ldots, \mathsf{E}_r\}$, where each E_i is of type $\frac{1}{n_i}(1, q_i)$; by Lemma 4.10 we have $r \geq 3$. Since we dealt with the case $\mathfrak{c}(F) = 1$ in Proposition 4.11, we may assume $\mathfrak{c}(F) \geq 2$. Moreover by Lemma 4.10 we can suppose that E_i contains no contractible components for $i \geq 3$. We will discuss Cases $1, \ldots, 4$ of Proposition 4.12 separately.

Case 1. $\mathsf{E}_1 : \left[(2)^{k_1},\ *\!*\!*\right]$
$\mathsf{E}_2 : [*\!*\!*].$

In this case

$$\mathfrak{c}(F) = \ell\left(\frac{k_1}{k_1 + 1}\right) + 1 = k_1 + 1. \tag{28}$$

By Propositions 1.6 and 1.9 it follows

$$\frac{q_1}{n_1} \geq \frac{k_1}{k_1 + 1} \quad \text{and } B\left(\frac{q_1}{n_1}\right) \geq B\left(\frac{k_1}{k_1 + 1}\right) = 2k_1 + \frac{2k_1}{k_1 + 1}.$$

Moreover

$$\sum_{i=2}^{r} \frac{q_i}{n_i} = 1 - \frac{q_1}{n_1} \leq 1 - \frac{k_1}{k_1 + 1} = \frac{1}{k_1 + 1}.$$

Then we may assume

$$\frac{q_2}{n_2} \leq \frac{1}{(r-1)(k_1 + 1)} \leq \frac{1}{2(k_1 + 1)}$$

hence $b_1(\mathsf{E}_2) = \lceil \frac{n_2}{q_2} \rceil \geq 2(k_1+1)$; moreover $b_1(\mathsf{E}_3) \geq k_1+3$ since E_3 contains no contractible components. Thus Corollary 1.10 implies

$$B\left(\frac{q_2}{n_2}\right) \geq 2(k_1+1) + \frac{1}{k_1+1}, \quad B\left(\frac{q_3}{n_3}\right) \geq k_1+3+\frac{2}{k_1+3}.$$

Then

$$\begin{aligned}\delta(F) &\geq \frac{1}{3}B\,(\mathsf{E}_1) + \frac{1}{3}B\,(\mathsf{E}_2) + \frac{1}{3}B\,(\mathsf{E}_3) - \mathfrak{c}(F)\\ &\geq \frac{1}{3}\left(2k_1+4+\frac{k_1-1}{(k_1+1)(k_1+3)}\right) \geq 2 \end{aligned} \tag{29}$$

and equality holds if and only if F is of type $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$.

Case 2. $\mathsf{E}_1 : \left[(2)^{k_1}, *\,*\,*\right]$
$\mathsf{E}_2 : [k_1+2, *\,*\,*] \quad k_1 \geq 1.$

In this case

$$\mathfrak{c}(F) = \ell\left(\frac{k_1}{k_1+1}\right) + \ell\left(\frac{1}{k_1+2}\right) + 1 = k_1+2. \tag{30}$$

By Proposition 1.6 it follows

$$\frac{q_1}{n_1} \geq \frac{k_1}{k_1+1}, \quad \frac{q_2}{n_2} \geq \frac{1}{k_1+2}$$

and Proposition 1.9 implies

$$\begin{aligned} B\left(\frac{q_1}{n_1}\right) &\geq B\left(\frac{k_1}{k_1+1}\right) = 2k_1 + \frac{2k_1}{k_1+1},\\ B\left(\frac{q_2}{n_2}\right) &\geq B\left(\frac{1}{k_1+2}\right) = k_1+2+\frac{2}{k_1+2}. \end{aligned}$$

Moreover

$$\sum_{i=3}^{r} \frac{q_i}{n_i} = 1 - \frac{q_1}{n_1} - \frac{q_2}{n_2} \leq 1 - \frac{k_1}{k_1+1} - \frac{1}{k_1+2} = \frac{1}{(k_1+1)(k_1+2)}.$$

Then we may assume

$$\frac{q_3}{n_3} \leq \frac{1}{(r-2)(k_1+1)(k_1+2)} \leq \frac{1}{(k_1+1)(k_1+2)}$$

hence $b_1(\mathsf{E}_3) = \lceil \frac{n_3}{q_3} \rceil \geq (k_1+1)(k_1+2)$. Thus Corollary 1.10 yields

$$B\left(\frac{q_3}{n_3}\right) \geq (k_1+1)(k_1+2) + \frac{2}{(k_1+1)(k_1+2)}.$$

Therefore we obtain

$$\begin{aligned}\delta(F) &\geq \frac{1}{3}B\,(\mathsf{E}_1) + \frac{1}{3}B\,(\mathsf{E}_2) + \frac{1}{3}B\,(\mathsf{E}_3) - \mathfrak{c}(F)\\ &\geq \frac{1}{3}k_1(k_1+3) \geq 1 + \frac{1}{3} \end{aligned} \tag{31}$$

and equality holds if and only if F is of type $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$.

Case 3. $\mathsf{E}_1 : \left[(2)^{k_1},\ l_1+3,\ (2)^{k_2},\ldots,(2)^{k_{t-1}},\ l_{t-1}+3,\ (2)^{k_t}, ***\right]$
$\mathsf{E}_2 : \left[k_1+2,\ (2)^{l_1},\ k_2+3,\ldots,k_{t-1}+3,\ (2)^{l_{t-1}},\ k_t+3, ***\right],\quad t\geq 2.$

Let $n,\ q$ be coprime integers such that

$$\left[(2)^{k_1},\ l_1+3,\ (2)^{k_2},\ldots,(2)^{k_{t-1}},\ l_{t-1}+3,\ (2)^{k_t}\right]=\frac{n}{n-q'}$$

and let a be such that $qq'=1+an$. Then Proposition 1.12 yields

$$\left[k_1+2,\ (2)^{l_1},\ k_2+3,\ldots,k_{t-1}+3,\ (2)^{l_{t-1}},\ k_t+3\right]=\frac{n+q}{a+q'}.$$

Notice that if $\frac{n}{n-q'}=2$ then $\frac{n+q}{a+q'}=3$, and by interchanging E_1 and E_2 we are in Case 2; hence we may assume $n\geq 3$. We have

$$\mathfrak{c}(F)=\ell\left(\frac{n-q'}{n}\right)+\ell\left(\frac{a+q'}{n+q}\right)+1=\sum_{i=1}^{t}(k_i+1)+\sum_{i=1}^{t-1}(l_i+1)+1. \tag{32}$$

By Proposition 1.6 it follows

$$\frac{q_1}{n_1}\geq\frac{n-q'}{n},\quad \frac{q_2}{n_2}\geq\frac{a+q'}{n+q}$$

and Proposition 1.9 gives

$$B\left(\frac{q_1}{n_1}\right)\geq B\left(\frac{n-q'}{n}\right),\quad B\left(\frac{q_2}{n_2}\right)\geq B\left(\frac{a+q'}{n+q}\right). \tag{33}$$

Moreover

$$\sum_{i=3}^{r}\frac{q_i}{n_i}\leq 1-\frac{n-q'}{n}-\frac{a+q'}{n+q}=\frac{1}{n(n+q)}.$$

Then we may assume

$$\frac{q_3}{n_3}\leq\frac{1}{(r-2)n(n+q)}\leq\frac{1}{n(n+q)}$$

hence $b_1(\mathsf{E}_3)=\lceil\frac{n_3}{q_3}\rceil\geq n(n+q)$. By Corollary 1.10 this implies

$$B\left(\frac{q_3}{n_3}\right)\geq n(n+q)+\frac{2}{n(n+q)}. \tag{34}$$

Estimates (33) and (34) together with (14) now yield

$$\begin{aligned}\delta(F)&\geq\frac{1}{3}B\left(\mathsf{E}_1\right)+\frac{1}{3}B\left(\mathsf{E}_2\right)+\frac{1}{3}B\left(\mathsf{E}_3\right)-\mathfrak{c}(F)\\&\geq\frac{1}{3}\left(1-\frac{1+q^2}{n(n+q)}+n(n+q)+\frac{2}{n(n+q)}\right)-1.\end{aligned} \tag{35}$$

Since $n\geq 3$ we obtain $\delta(F)>3$.

Case 4. $\mathsf{E}_1 : \left[(2)^{k_1},\ l_1+3,\ (2)^{k_2},\ldots,(2)^{k_{t-1}},\ l_{t-1}+3,\ (2)^{k_t}, ***\right]$
$\mathsf{E}_2 : \left[k_1+2,\ (2)^{l_1},\ k_2+3,\ldots,k_{t-1}+3,\ (2)^{l_{t-1}}, ***\right],\quad t\geq 2.$

Let $n,\ q$ be coprime integers such that

$$\left[(2)^{k_1},\ l_1+3,\ (2)^{k_2},\dots,(2)^{k_{t-1}},\ l_{t-1}+3,\ (2)^{k_t}\right]=\frac{n}{n-q'}$$

and let a be such that $qq'=1+an$. Then Proposition 1.12 yields

$$\left[k_1+2,\ (2)^{l_1},\ k_2+3,\dots,k_{t-1}+3,\ (2)^{l_{t-1}}\right]=\frac{q}{a}.$$

Notice that $q\geq 2$. If $n=3, q=2$ we obtain $\frac{n}{n-q'}=3, \frac{q}{a}=2$, so by interchanging E_1 and E_2 we are in Case 2; analogously if $n=4, q=3$. Hence we can suppose $n\geq 5$. We have

$$\mathfrak{c}(F)=\ell\left(\frac{n-q'}{n}\right)+\ell\left(\frac{a}{q}\right)+1=\sum_{i=1}^{t}(k_i+1)+\sum_{i=1}^{t-1}(l_i+1). \tag{36}$$

By Proposition 1.6 it follows

$$\frac{q_1}{n_1}\geq\frac{n-q'}{n},\quad \frac{q_2}{n_2}\geq\frac{a}{q}$$

and Proposition 1.9 gives

$$B\left(\frac{q_1}{n_1}\right)\geq B\left(\frac{n-q'}{n}\right),\quad B\left(\frac{q_2}{n_2}\right)\geq B\left(\frac{a}{q}\right). \tag{37}$$

Moreover

$$\sum_{i=3}^{r}\frac{q_i}{n_i}=1-\frac{q_1}{n_1}-\frac{q_2}{n_2}\leq 1-\frac{n-q'}{n}-\frac{a}{q}=\frac{1}{nq}.$$

Then we may assume

$$\frac{q_3}{n_3}\leq\frac{1}{(r-2)nq}\leq\frac{1}{nq}$$

hence $b_1(\mathsf{E}_3)=\lceil\frac{n_3}{q_3}\rceil\geq nq$. By Corollary 1.10 this implies

$$B\left(\frac{q_3}{n_3}\right)\geq nq+\frac{2}{nq}. \tag{38}$$

Estimates (37) and (38) together with (15) now yield

$$\begin{aligned}\delta(F)&\geq\frac{1}{3}B\left(\mathsf{E}_1\right)+\frac{1}{3}B\left(\mathsf{E}_2\right)+\frac{1}{3}B\left(\mathsf{E}_3\right)-\mathfrak{c}(F)\\&\geq\frac{1}{3}\left(n-\frac{1}{n}\right)\left(q-\frac{1}{q}\right).\end{aligned} \tag{39}$$

Since $n\geq 5$ and $q\geq 2$ it follows $\delta(F)\geq 2+\frac{2}{5}$. □

Summarizing Lemma 4.6, Proposition 4.8 and Proposition 4.13 we obtain

Corollary 4.14 *Let F be a reducible fibre of $\alpha_2: S\longrightarrow C_2/G$. Then $\delta(F)\geq 0$ and moreover the following holds.*

- *If $g(C_1)=0$ then F is a (-1)-fibre and $\delta(F)=0$. Conversely, if $\delta(F)=0$ then F is a (-1)-fibre and $g(C_1)=0$.*

- *If* $g(C_1) \geq 1$ *then* $\delta(F) > 2$, *with precisely three exceptions:*

(i) $g(C_1) = 1$ *and* F *is a* (-1)*-fibre of type* $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$. *In this case* $\delta(F) = 1 + \frac{1}{3}$;
(ii) $g(C_1) = 1$ *and* F *is a* (-1)*-fibre of type* $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$. *In this case* $\delta(F) = 2$;
(iii) F *is of type* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *but it is* not *a* (-1)*-fibre. In this case* $\delta(F) = 2$.

In particular, if S *is of general type then the only possible exception is* (iii).

Notice that in case (iii) the central component Y of F satisfies $Y^2 = -1$ but it is *not* a rational curve.

Proposition 4.15 *Let* $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ *be a standard isotrivial fibration and let* $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$ *be the relatively minimal model of* $\alpha_2 : S \longrightarrow C_2/G$. *Then*

$$K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}})$$

and equality holds if and only if either S *is a quasi-bundle or* $g(C_1) = 0$. *Otherwise we have*

$$K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}}) - 2$$

and equality holds if and only if α_2 *contains exactly one reducible fibre* F, *which is of type* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *and which is* not *a* (-1)*-fibre (in particular this implies* $\widehat{S} = S$).

Proof By using formula (26) and Corollary 4.14 we obtain $K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}})$, and equality holds if and only if either

(i) $\text{Crit}(\alpha_2)' = \emptyset$, that is S is a quasi-bundle, or
(ii) $g(C_1) = 0$.

Otherwise, since both $K^2_{\widehat{S}}$ and $\chi(\mathcal{O}_{\widehat{S}})$ are integers, we have $K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}}) - 2$, and equality holds if and only if α_2 contains exactly one reducible fibre F and either

(i') $g(C_1) = 1$ and F is a (-1) fibre of type $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$, or
(ii') F is of type $\left(\frac{1}{2}, \frac{1}{2}\right)$, but it is not a (-1)-fibre.

Assume now that case (i') occurs. Therefore, by using Proposition 2.5 we would obtain $K^2_S = -2$ and $e(S) = 5$, contradicting Noether's formula. Therefore the only possibility is (ii'). □

Proposition 4.16 *Assume that* $q(S) \geq 1$ *and that* S *is neither ruled nor a quasi-bundle. Then, up to interchanging* C_1 *and* C_2, *the surface* $\widehat{S}$ *is the minimal model of* S *and we have*

$$K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}}) - 2. \tag{40}$$

Equality holds if and only if $\text{Sing}(T) = 2 \times \frac{1}{2}(1, 1)$, *and in this case* $S = \widehat{S}$ *is a minimal surface of general type.*

Proof Consider the relatively minimal fibration $\hat{\alpha}_2 : \widehat{S} \longrightarrow C_2/G$. Since $q(S) \geq 1$, up to interchanging C_1 and C_2 we can suppose $g(C_2/G) \geq 1$, hence $\widehat{S}$ is the minimal model of S. We are also assuming that S is not ruled, so $g(C_1) \geq 1$ and Proposition 4.15 gives $K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}}) - 2$. Equality occurs if and only if α_2 contains exactly one reducible fibre, which is of type $\left(\frac{1}{2}, \frac{1}{2}\right)$ and which is not a (-1)-fibre; this implies $S = \widehat{S}$, hence S is minimal and consequently K_S is nef. Therefore relation $K^2_S = 8\chi(\mathcal{O}_S) - 2$ yields $K^2_S \geq 6$, that is S is of general type. □

Corollary 4.17 *Let S be a standard isotrivial fibration, with* $\mathrm{kod}(S) = 0$ *or* 1 *and* $\chi(\mathcal{O}_S) = 0$. *Then S is a quasi-bundle.*

Proof Since $\chi(\mathcal{O}_S) = 0$ we obtain $q(S) \geq 1$, hence $\widehat{S}$ is the minimal model of S. Now $\mathrm{kod}(S) = 0$ or 1 yields $0 = K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}})$, so Proposition 4.15 implies that S is a quasi-bundle. □

Remark 4.18 If $\mathrm{kod}(S) = 0$, then Corollary 4.17 applies when S is either abelian or bielliptic. If instead $\mathrm{kod}(S) = 1$, it applies when S is any properly elliptic surface with $\chi(\mathcal{O}_S) = 0$ (examples of such surfaces are described in [25]).

Finally, observe that there exist (non-minimal) properly elliptic surfaces with $\chi(\mathcal{O}_S) = 1$ that are standard isotrivial fibrations but not quasi-bundles, see Example 3.7. This show that the assumption $\chi(\mathcal{O}_S) = 0$ in Corollary 4.17 cannot be dropped.

Under the further assumption that $K_{\widehat{S}}$ is ample, we can improve inequality (40) as follows.

Proposition 4.19 *Assume that $q(S) \geq 1$, S is not a quasi-bundle and $K_{\widehat{S}}$ is ample. Then, up to interchanging C_1 and C_2, the surface $\widehat{S}$ is the minimal model of S and we have*

$$K^2_{\widehat{S}} \leq 8\chi(\mathcal{O}_{\widehat{S}}) - 5. \tag{41}$$

Proof By Proposition 4.16 we must show that, if $K_{\widehat{S}}$ is ample, the two cases $K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}})-3$ and $K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}}) - 4$ do not occur. This will be consequence of Lemmas 4.20 and 4.21 below.

Lemma 4.20 *If $K_{\widehat{S}}$ is ample, then $K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}}) - 3$ does not occur.*

By contradiction, assume that this case occurs. Since $\widehat{S}$ is of general type, by formula (26) and Corollary 4.14 it follows that $\alpha_2 : S \longrightarrow C_2/G$ contains exactly one reducible fibre F, which satisfies $\delta(F) = 3$. Assuming that F is of type $\left(\frac{q_1}{n_1}, \ldots, \frac{q_r}{n_r}\right)$, there are two subcases.

Subcase (1). F is not a (-1)-fibre. This implies $S = \widehat{S}$ and $\sum_{i=1}^r B\left(\frac{q_i}{n_i}\right) = 9$. Since $\sum_{i=1}^r \frac{q_i}{n_i} \in \mathbb{Z}$, by looking at the table in Appendix B we see that the only possibility for the type of F is $\left(\frac{1}{3}, \frac{2}{3}\right)$, see also [18, Proposition 4.1], and this contradicts the ampleness of the canonical bundle. Hence (1) does not occur.
Subcase (2). F is a (-1)-fibre. By using estimates (29), (31), (35), (39), we can check that the only possibilities for the type of F are $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$ and $\left(\frac{1}{2}, \frac{1}{8}, \frac{3}{8}\right)$, see also Appendix A. But in the latter case $K_{\widehat{S}}$ would not be ample, hence F is necessarily of type $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$. Therefore $g(C_1) = 2$. Moreover, since F is a (-1)-fibre, we have $g(C_1/G) = 0$; setting $\mathfrak{g}' := g(C_2/G)$, it follows that G is both $(0 \,|\, \mathbf{m})$-generated and $(\mathfrak{g}' \,|\, \mathbf{n})$-generated, where $\mathbf{m} := (m_1, \ldots, m_r)$ and $\mathbf{n} := (n_1, \ldots, n_s)$; we will denote by

$$\mathcal{V} := \{g_1, \ldots, g_r\} \quad \text{and} \quad \mathcal{W} := \{\ell_1, \ldots, \ell_s;\ h_1, \ldots, h_{2\mathfrak{g}'}\} \tag{42}$$

the corresponding generating vectors, see Sect. 1. The group G acts in genus 2 with rational quotient; moreover, since

$$\mathrm{Sing}(T) = \frac{1}{3}(1,\ 2) + 2 \times \frac{1}{6}(1,\ 1), \tag{43}$$

at least one of the m_i must be divisible by 6. Looking at [7, p. 252] and [22, Appendix A], we see that there are at most two possibilities:

(2*a*) $G = \mathbb{Z}_2 \times \mathbb{Z}_6, \quad \mathbf{m} = (2,\, 6^2)$;
(2*b*) $G = \mathbb{Z}_2 \ltimes ((\mathbb{Z}_2)^2 \times \mathbb{Z}_3) = G(24,\, 8), \quad \mathbf{m} = (2,\, 4,\, 6)$,

where $G = G(24,\, 8)$ means that G has the label number 8 in the `GAP4` list of groups of order 24, see [22]. Let us analyze (2*a*) and (2*b*) separately.

Assume (2*a*) *occurs*. Set $G = \langle x,\, y \,|\, x^2 = y^6 = [x,\, y] = 1\rangle$. Up to automorphisms, we may suppose

$$\begin{aligned} &g_1 = x, \quad g_2 = xy^{-1}, \quad g_3 = y, \\ &\ell_1 = y. \end{aligned}$$

Set $\mathscr{S} := \langle g_1\rangle \cup \langle g_2\rangle \cup \langle g_3\rangle$. Since G is abelian, $s \geq 2$ (Remark 1.2); moreover there is just one reducible fibre, so we must have

$$\langle \ell_2\rangle \cup \cdots \cup \langle \ell_s\rangle \subseteq G\backslash\mathscr{S} = \{xy^2,\, xy^4\}.$$

But this is impossible, since $(xy^2)^2 = y^4 \in \mathscr{S}$ and $(xy^4)^2 = y^2 \in \mathscr{S}$. Therefore (2*a*) does not occur.
Assume (2*b*) *occurs*. The presentation of $G = G(24,\, 8)$ is

$$\begin{aligned} G = \langle x,\, y,\, z,\, w \,|\, & x^2 = y^2 = z^2 = w^3 = 1, \\ & [y,\, z] = [y,\, w] = [z,\, w] = 1, \\ & xyx^{-1} = y,\, xzx^{-1} = zy,\, xwx^{-1} = w^{-1}\rangle. \end{aligned}$$

It is no difficult to check that this group contains exactly one conjugacy class of elements of order 3, namely $\mathrm{Cl}(w) = \{w,\, w^{-1}\}$. In particular every element of order 3 is conjugate to its inverse, hence Corollary 1.5 implies that if T contains some singularity of type $\frac{1}{3}(1,\, 2)$, it must also contain some singularity of type $\frac{1}{3}(1,\, 1)$. But this contradicts (43), hence (2*b*) must be excluded too.

This completes the proof of Lemma 4.20.

Lemma 4.21 *If* $K_{\widehat{S}}$ *is ample, then* $K^2_{\widehat{S}} = 8\chi(\mathcal{O}_{\widehat{S}}) - 4$ *does not occur.*

Again, assume by contradiction that this case occurs. As in the proof of Lemma 4.20, we see that $\alpha_2 : S \longrightarrow C_2/G$ contains just one reducible fibre, which must be a (-1)-fibre with $\delta(F) = 4$. By using estimates (29), (31), (35), (39), we see that the only possibilities for the type of F are $\left(\frac{3}{4},\, \frac{1}{8},\, \frac{1}{8}\right)$ and $\left(\frac{1}{2},\, \frac{1}{12},\, \frac{5}{12}\right)$. One immediately checks that in the latter case $K_{\widehat{S}}$ would not be ample, hence F is necessarily of type $\left(\frac{3}{4},\, \frac{1}{8},\, \frac{1}{8}\right)$. Therefore $g(C_1) = 3$. Moreover, since F is a (-1)-fibre we have $g(C_1/G) = 0$; setting $\mathfrak{g}' := g(C_2/G)$, it follows that G is both $(0\,|\,\mathbf{m})$-generated and $(\mathfrak{g}'\,|\,\mathbf{n})$-generated, where $\mathbf{m} := (m_1, \ldots, m_r)$ and $\mathbf{n} := (n_1, \ldots, n_s)$; we will denote the corresponding generating vectors as in (42). The group G acts in genus 3 with rational quotient; moreover, since

$$\mathrm{Sing}(T) = \frac{1}{4}(1,\, 3) + 2 \times \frac{1}{8}(1,\, 1), \tag{44}$$

at least one of the m_i must be divisible by 8. Looking at [7, p. 252] and [22, Appendix A], we see that there are at most five possibilities:

(*a*) $G = \mathbb{Z}_2 \times \mathbb{Z}_8, \quad \mathbf{m} = (2,\, 8^2)$,
(*b*) $G = D_{2,8,5} = G(16,\, 6), \quad \mathbf{m} = (2,\, 8^2)$,

(c) $G = \mathbb{Z}_2 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_8) = G(32,\ 9)$, $\quad \mathbf{m} = (2,\ 4,\ 8)$,
(d) $G = \mathbb{Z}_2 \ltimes D_{2,8,5} = G(32,\ 11)$, $\quad \mathbf{m} = (2,\ 4,\ 8)$,
(e) $G = \mathcal{S}_3 \ltimes (\mathbb{Z}_4)^2 = G(96,\ 64)$, $\quad \mathbf{m} = (2,\ 3,\ 8)$.

We first rule out Case (a). Set

$$G = \langle x,\ y,\ | \, x^2 = y^8 = [x,\ y] = 1\rangle.$$

Up to automorphisms, we may assume

$$g_1 = x, \quad g_2 = xy^{-1}, \quad g_3 = y,$$
$$\ell_1 = y.$$

Set $\mathscr{S} := \langle g_1\rangle \cup \langle g_2\rangle \cup \langle g_3\rangle$. Since G is abelian, $s \geq 2$. Moreover there is just one reducible fibre, so we must have

$$\langle \ell_2\rangle \cup \cdots \cup \langle \ell_s\rangle \subseteq G \backslash \mathscr{S} = \{xy^2,\ xy^4,\ xy^6\}.$$

But $(xy^2)^2 = (xy^6)^2 = y^4 \in \mathscr{S}$, so we obtain $\ell_2 = \ldots = \ell_s = xy^4$. On the other hand,

$$1 = \ell_1\ell_2 \ldots \ell_s \cdot \Pi_{i=1}^{\mathfrak{g}'}[h_i, h_{i+\mathfrak{g}'}] = \ell_1\ell_2 \ldots \ell_s,$$

so $y = \ell_1 \in \langle xy^4\rangle$ which is a contradiction. Hence (a) must be excluded.

Now we rule out Cases $(b), \ldots, (e)$. Notice that (44) implies that the group G must satisfy the following condition:

$(*)$ there exists an element $g \in G$ such that $|g| = 8$ and g is *not* conjugate to g^3, g^5, g^7.

By using GAP4 (or by means of tedious hand-made computations) we can easily check that in Cases (b), (d) and (e) every $g \in G$ with $|g| = 8$ is conjugate to g^5, so condition $(*)$ is not satisfied. Therefore we are only left to exclude (c). In Case (c) the presentation of $G = G(32,\ 9)$ is

$$G = \langle x,\ y,\ z,\ | \, x^2 = y^2 = z^8 = 1,\ [x, y] = [y, z] = 1,\ xzx^{-1} = yz^3\rangle.$$

By simple GAP4 scripts one checks that the automorphism group $\mathrm{Aut}(G)$ has order 64, and that G admits precisely 64 generating vectors $\mathcal{V} = \{g_1,\ g_2,\ g_3\}$ of type $(0\,|\,2,\ 4,\ 8)$, which form a unique orbit for the action of $\mathrm{Aut}(G)$. Hence, up to automorphisms, we may assume that $\mathcal{V}$ is as follows:

$$g_1 = x, \quad g_2 = xz^{-1}, \quad g_3 = z.$$

Set $\mathfrak{g}' = g(C_2/G)$ and let $\mathcal{W} := \{\ell_1, \ldots, \ell_s;\ h_1, \ldots, h_{2\mathfrak{g}'}\}$ be the generating vector of type $(\mathfrak{g}' \,|\, n_1, \ldots, n_s)$ inducing the covering $C_2 \longrightarrow C_2/G$. The group G contains no elements of order greater than 8, so by (44) we may assume $\ell_1 = z$, and since $z \notin [G,\ G] = \langle yz^2\rangle$, we have $s \geq 2$. Put

$$\mathscr{S} := \bigcup_{\sigma \in G} \bigcup_{i=1}^{3} \langle \sigma g_i \sigma^{-1}\rangle;$$

since $\alpha_2 : S \longrightarrow C_2/G$ contains exactly one reducible fibre, we obtain

$$\langle \ell_2\rangle \cup \cdots \cup \langle \ell_s\rangle \subseteq G \backslash \mathscr{S} = \{yz^2,\ xz^2,\ xyz^2x,\ zxz,\ z^2x,\ y,\ xy\} \subset \langle x,\ y,\ z^2\rangle.$$

In particular this implies

$$\ell_2\ell_3 \ldots \ell_s \in \langle x,\ y,\ z^2\rangle. \tag{45}$$

On the other hand

$$\ell_1\ell_2\ldots\ell_s = \left(\Pi_{i=1}^{\mathfrak{g}'}[h_i, h_{i+\mathfrak{g}'}]\right)^{-1} \in [G, G] = \langle yz^2\rangle \subset \langle x, y, z^2\rangle, \tag{46}$$

hence (45) and (46) together imply $z = \ell_1 \in \langle x, y, z^2\rangle$, a contradiction. Therefore Case (c) does not occur, and this shows Lemma 4.21.

The proof of Proposition 4.19 is now complete. □

In [26] Serrano showed that any isotrivial fibred surface X satisfies $K_X^2 \leq 8\chi(\mathcal{O}_X)$. Moreover, S. L. Tan proved in [27] that equality holds if and only if X is either ruled or isomorphic to a quasi-bundle. By using Propositions 4.16 and 4.19, we are led to the following strengthening of Serrano's and Tan's results.

Theorem 4.22 *Let $\varphi : X \longrightarrow C$ be any relatively minimal isotrivial fibration, with X non ruled and $g(C) \geq 1$. If X is not isomorphic to a quasi-bundle, we have*

$$K_X^2 \leq 8\chi(\mathcal{O}_X) - 2 \tag{47}$$

and if equality holds then X is a minimal surface of general type whose canonical model has precisely two ordinary double points as singularities.

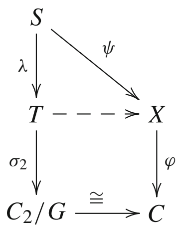
Moreover, under the further assumption that K_X is ample, we have

$$K_X^2 \leq 8\chi(\mathcal{O}_X) - 5. \tag{48}$$

Finally, both inequalities (47) and (48) are sharp.

Proof By Remark 2.2 there exist a standard isotrivial fibration $\lambda : S \longrightarrow T = (C_1 \times C_2)/G$ and a birational map $T \dashrightarrow X$ such that the diagram

(49)



commutes. Since φ is relatively minimal and $g(C) \geq 1$, the surface X is a minimal model. As X is not ruled K_X is nef, so the rational map $\psi : S \longrightarrow X$ is actually a morphism, which induces an isomorphism $\hat{\psi} : \widehat{S} \longrightarrow X$. Thus Propositions 4.16 and 4.19 imply inequalities (47) and (48). Finally, both these inequalities are sharp, in fact:

- there exist examples of relatively minimal isotrivial fibrations $X \longrightarrow C$ with $g(C) = 1$, $p_g(X) = q(X) = 1$ and $K_X^2 = 6$, see [22, Section 7.1];
- there exist examples of relatively minimal isotrivial fibrations with $g(C) = 1$, $p_g(X) = q(X) = 1$, $K_S^2 = 3$ and K_S ample, see [18, Section 5.5]. The fibres have genus 3 and there is a unique singular fibre, composed of four (-3) curves intersecting in one single point.

This concludes the proof of Theorem 4.22. □

Remark 4.23 If K_X is not ample, then both cases $K_X^2 = 8\chi(\mathcal{O}_X) - 3$ and $K_X^2 = 8\chi(\mathcal{O}_X) - 4$ actually occur. For instance, there are examples of relatively minimal isotrivial fibrations with $g(C) = 1$, $p_g(X) = q(X) = 1$ and $K_X^2 = 5, 4$, see [18] and [22].

We end this section with an open problem.

Problem 4.24 *What happens if one drops the assumptions $q(S) \geq 1$ in Proposition 4.16 and $g(C) \geq 1$ in Theorem 4.22?*

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Appendix A: The classification of (−1)-fibres for low values of $\mathfrak{g}$

For low values of $\mathfrak{g}$ there exists a complete classification of cyclic groups acting in genus $\mathfrak{g}$ with rational quotient; by Corollary 3.4 this provides in turn a complete classification of the corresponding (-1)-fibres. Since Corollary 3.5 settles the case $\mathfrak{g} = 0$, we may assume $\mathfrak{g} \geq 1$. If F is any (-1)-fibre of $\alpha_2 : S \longrightarrow C_2/G$, we denote by $F_{\min} := \pi(F)$ the image of F in the relative minimal model $\widehat{S}$.

4.2 The case $\mathfrak{g} = 1$

Proposition 4.25 *There are precisely three types of (-1)-fibres F in genus $\mathfrak{g} = 1$. The type of F, the values of $\mathfrak{c}(F)$ and $\delta(F)$ and the type of $F_{\min}$ in the Kodaira classification of elliptic singular fibres are as in the table below.*

Type of F	$\mathfrak{c}(F)$	$\delta(F)$	*Type of* $F_{\min}$
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	1	$2 + \frac{2}{3}$	$IV(\widetilde{A}_2)$
$\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$	2	2	$III(\widetilde{A}_1)$
$\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$	3	$1 + \frac{1}{3}$	II

Proof The cyclic groups G acting in genus 1 with rational quotient and the corresponding signatures are as follows ([6, p. 9]):

(i) $G = \mathbb{Z}_2$, $(0 \,|\, 2^4)$;
(ii) $G = \mathbb{Z}_3$, $(0 \,|\, 3^3)$;
(iii) $G = \mathbb{Z}_4$, $(0 \,|\, 2, 4^2)$;
(iv) $G = \mathbb{Z}_6$, $(0 \,|\, 2, 3, 6)$.

In case (i) we cannot have a (-1)-fibre.
In case (ii) a (-1)-fibre F is necessarily of type $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; $F_{\min}$ is obtained by contracting only the central component, hence $\mathfrak{c}(F) = 1$ and $\delta(F) = B\left(\frac{1}{3}\right) - 1 = 2 + \frac{2}{3}$.
In case (iii) a (-1)-fibre is necessarily of type $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{4}\right)$; $F_{\min}$ is obtained by performing two blow-downs, hence $\mathfrak{c}(F) = 2$ and $\delta(F) = \frac{1}{3}(B(\frac{1}{2}) + 2B(\frac{1}{4})) - 2 = 2$.
In case (iv) a (-1)-fibre is necessarily of type $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right)$; $F_{\min}$ is obtained by performing three blow-downs, hence $\mathfrak{c}(F) = 3$ and $\delta(F) = \frac{1}{3}\left(B\left(\frac{1}{2}\right) + B\left(\frac{1}{3}\right) + B\left(\frac{1}{6}\right)\right) - 3 = 1 + \frac{1}{3}$.
In each case the blow-down process and the type of $F_{\min}$ are illustrated in Fig. 2. This completes the proof. □

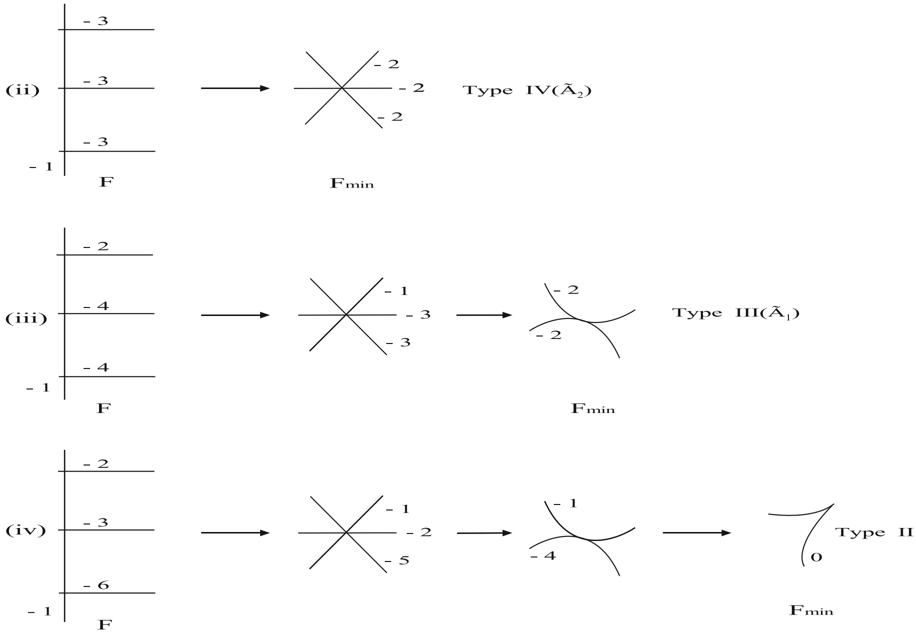

Fig. 2 (-1)-fibres and their minimal models in genus 1

Remark 4.26 Proposition *4.25* generalizes Serrano's example of a nonstandard elliptic isotrivial fibration having a singular fibre of type II (see [24, Proposition 2.5]). A strictly related result, namely the existence of isotrivial elliptic fibrations $f : X \longrightarrow \mathscr{D}$ over an open disk having the central fibre of type $IV(\widetilde{A}_2)$, $III(\widetilde{A}_1)$ or II, appears in [3, Chapter V, pp. 137–138].

4.3 The cases $\mathfrak{g} = 2$ and $\mathfrak{g} = 3$

Ogg classified in [20] all singular fibres that may occur in pencils of genus 2 curves; in particular he showed that they are either irreducible or belong to 44 reducible types. In the following proposition we classify all (-1) fibres F in genus 2 and we give the corresponding type of $F_{\min}$ according to Ogg's classification.

Proposition 4.27 *There are precisely six types of* (-1)*-fibres* F *in genus* $\mathfrak{g} = 2$*. The type of* F*, the values of* $\mathfrak{c}(F)$ *and* $\delta(F)$ *and the type of* $F_{\min}$ *are as in the table below.*

Type of F	$\mathfrak{c}(F)$	$\delta(F)$	*Type of* $F_{\min}$
$\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)$	2	$3 + \frac{3}{5}$	Type 36
$\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$	1	$4 + \frac{4}{5}$	Type 8
$\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$	3	3	Type 34
$\left(\frac{1}{2}, \frac{1}{8}, \frac{3}{8}\right)$	3	3	Type 1
$\left(\frac{1}{2}, \frac{1}{5}, \frac{3}{10}\right)$	2	$3 + \frac{4}{5}$	Type 16
$\left(\frac{1}{2}, \frac{2}{5}, \frac{1}{10}\right)$	4	$2 + \frac{2}{5}$	Irreducible

Proof The cyclic groups G acting in genus 2 with rational quotient and the respective signatures are as follows ([7, p. 252]):

(i) $G = \mathbb{Z}_2$, $(0 \mid 2^6)$;
(ii) $G = \mathbb{Z}_3$, $(0 \mid 3^4)$;
(iii) $G = \mathbb{Z}_4$, $(0 \mid 2^2, 4^2)$;
(iv) $G = \mathbb{Z}_5$, $(0 \mid 5^3)$;
(v) $G = \mathbb{Z}_6$, $(0 \mid 3, 6^2)$;
(vi) $G = \mathbb{Z}_6$, $(0 \mid 2^2, 3^2)$;
(vii) $G = \mathbb{Z}_8$, $(0 \mid 2, 8^2)$;
(viii) $G = \mathbb{Z}_{10}$, $(0 \mid 2, 5, 10)$.

In cases (i), (ii), (iii) and (vi) we cannot have any (-1)-fibre.
In case (iv), if F is a (-1)-fibre there are two possibilities:

(iv_a) F is of type $\left(\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\right)$; we have $\mathfrak{c}(F) = 2$ and $\delta(F) = \frac{1}{3}\left(2B\left(\frac{1}{5}\right) + B\left(\frac{3}{5}\right)\right) - 2 = 3 + \frac{3}{5}$;
(iv_b) F is of type $\left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}\right)$; we have $\mathfrak{c}(F) = 1$, hence $\delta(F) = \frac{1}{3}\left(B\left(\frac{1}{5}\right) + 2B\left(\frac{2}{5}\right)\right) - 1 = 4 + \frac{4}{5}$.

In case (v) a (-1)-fibre is necessarily of type $\left(\frac{2}{3}, \frac{1}{6}, \frac{1}{6}\right)$; we have $\mathfrak{c}(F) = 3$, hence $\delta(F) = \frac{1}{3}\left(B\left(\frac{2}{3}\right) + 2B\left(\frac{1}{6}\right)\right) - 3 = 3$.
In case (vii) a (-1)-fibre is necessarily of type $\left(\frac{1}{2}, \frac{1}{8}, \frac{3}{8}\right)$; we have $\mathfrak{c}(F) = 3$, hence $\delta(F) = \frac{1}{3}\left(B\left(\frac{1}{2}\right) + B\left(\frac{1}{8}\right) + B\left(\frac{3}{8}\right)\right) - 3 = 3$.
In case $(viii)$ there are again two possibilities:

$(viii_a)$ F is of type $\left(\frac{1}{2}, \frac{1}{5}, \frac{3}{10}\right)$; we have $\mathfrak{c}(F) = 2$, hence $\delta(F) = \frac{1}{3}\left(B\left(\frac{1}{2}\right) + B\left(\frac{1}{5}\right) + B\left(\frac{3}{10}\right)\right) - 2 = 3 + \frac{4}{5}$;
$(viii_b)$ F is of type $\left(\frac{1}{2}, \frac{2}{5}, \frac{1}{10}\right)$; we have $\mathfrak{c}(F) = 4$, hence $\delta(F) = \frac{1}{3}\left(B\left(\frac{1}{2}\right) + B\left(\frac{2}{5}\right) + B\left(\frac{1}{10}\right)\right) - 4 = 2 + \frac{2}{5}$.

By looking at the classification of singular fibres in [20], one sees that the types of $F_{\min}$ are precisely those in our table and this completes the proof. □

In the same way, we can give the following list of (-1)-fibres in genus 3.

Proposition 4.28 *There are precisely* 17 *types of* (-1)*-fibres* F *in genus* $\mathfrak{g} = 3$. *The type of* F *and the corresponding values of* $\mathfrak{c}(F)$ *and* $\delta(F)$ *are as in the table below.*

Type of F	$\mathfrak{c}(F)$	$\delta(F)$
$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$	1	5
$\left(\frac{1}{7}, \frac{1}{7}, \frac{5}{7}\right)$	3	$4 + \frac{4}{7}$
$\left(\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\right)$	2	5
$\left(\frac{1}{7}, \frac{3}{7}, \frac{3}{7}\right)$	1	$6 + \frac{6}{7}$
$\left(\frac{2}{7}, \frac{2}{7}, \frac{3}{7}\right)$	1	$6 + \frac{2}{7}$

$\left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right)$	2	5
$\left(\frac{1}{4}, \frac{3}{8}, \frac{3}{8}\right)$	1	5
$\left(\frac{3}{4}, \frac{1}{8}, \frac{1}{8}\right)$	4	4
$\left(\frac{1}{3}, \frac{1}{9}, \frac{5}{9}\right)$	3	$3+\frac{8}{9}$
$\left(\frac{1}{3}, \frac{2}{9}, \frac{4}{9}\right)$	1	$6+\frac{2}{9}$
$\left(\frac{2}{3}, \frac{1}{9}, \frac{2}{9}\right)$	3	$4+\frac{4}{9}$
$\left(\frac{1}{2}, \frac{1}{12}, \frac{5}{12}\right)$	4	4
$\left(\frac{1}{3}, \frac{1}{4}, \frac{5}{12}\right)$	1	$4+\frac{2}{3}$
$\left(\frac{2}{3}, \frac{1}{4}, \frac{1}{12}\right)$	4	$3+\frac{1}{3}$
$\left(\frac{1}{2}, \frac{1}{7}, \frac{5}{14}\right)$	3	$3+\frac{2}{7}$
$\left(\frac{1}{2}, \frac{2}{7}, \frac{3}{14}\right)$	2	$4+\frac{1}{7}$
$\left(\frac{1}{2}, \frac{3}{7}, \frac{1}{14}\right)$	5	$3+\frac{3}{7}$

Proof The cyclic groups G acting in genus 3 with rational quotient and the corresponding signatures are as follows ([7, pp. 254–255]):

(i) $G = \mathbb{Z}_2$, $(0 \mid 2^8)$;
(ii) $G = \mathbb{Z}_3$, $(0 \mid 3^5)$;
(iii) $G = \mathbb{Z}_4$, $(0 \mid 4^4)$;
(iv) $G = \mathbb{Z}_4$, $(0 \mid 2^3, 4^2)$;
(v) $G = \mathbb{Z}_6$, $(0 \mid 2^2, 6^2)$;
(vi) $G = \mathbb{Z}_6$, $(0 \mid 2, 3^2, 6)$;
(vii) $G = \mathbb{Z}_7$, $(0 \mid 7^3)$;
(viii) $G = \mathbb{Z}_8$, $(0 \mid 4, 8^2)$;
(ix) $G = \mathbb{Z}_9$, $(0 \mid 3, 9^2)$;
(x) $G = \mathbb{Z}_{12}$, $(0 \mid 2, 12^2)$;
(xi) $G = \mathbb{Z}_{12}$, $(0 \mid 3, 4, 12)$;
(xii) $G = \mathbb{Z}_{14}$, $(0 \mid 2, 7, 14)$.

In cases (i), (ii), (iv), (v) and (vi) we cannot have any (-1)-fibre, whereas the remaining possibilities give the occurrences in the table. The details are as in the proof of Proposition 4.27 and they are left to the reader, who can check them by using the table in Appendix B. □

Appendix B: List of cyclic quotient singularities $x = \frac{1}{n}(1, q)$ with $2 \leq n \leq 14$

$\frac{1}{n}(1, q)$	$n/q = [b_1, \ldots, b_s]$	$\frac{1}{n}(1, q')$	$B\left(\frac{q}{n}\right)$	$h\left(\frac{q}{n}\right)$
$\frac{1}{2}(1, 1)$	[2]	$\frac{1}{2}(1, 1)$	3	0
$\frac{1}{3}(1, 1)$	[3]	$\frac{1}{3}(1, 1)$	$3 + 2/3$	$-1/3$
$\frac{1}{3}(1, 2)$	[2, 2]	$\frac{1}{3}(1, 2)$	$5 + 1/3$	0
$\frac{1}{4}(1, 1)$	[4]	$\frac{1}{4}(1, 1)$	$4 + 1/2$	-1

$\frac{1}{4}(1,3)$	[2, 2, 2]	$\frac{1}{4}(1,3)$	$7+1/2$	0
$\frac{1}{5}(1,1)$	[5]	$\frac{1}{5}(1,1)$	$5+2/5$	$-9/5$
$\frac{1}{5}(1,2)$	[3, 2]	$\frac{1}{5}(1,3)$	6	$-2/5$
$\frac{1}{5}(1,4)$	[2, 2, 2, 2]	$\frac{1}{5}(1,4)$	$9+3/5$	0
$\frac{1}{6}(1,1)$	[6]	$\frac{1}{6}(1,1)$	$6+1/3$	$-8/3$
$\frac{1}{6}(1,5)$	[2, 2, 2, 2, 2]	$\frac{1}{6}(1,5)$	$11+2/3$	0
$\frac{1}{7}(1,1)$	[7]	$\frac{1}{7}(1,1)$	$7+2/7$	$-25/7$
$\frac{1}{7}(1,2)$	[4, 2]	$\frac{1}{7}(1,4)$	$6+6/7$	$-8/7$
$\frac{1}{7}(1,3)$	[3, 2, 2]	$\frac{1}{7}(1,5)$	$8+1/7$	$-3/7$
$\frac{1}{7}(1,6)$	[2, 2, 2, 2, 2, 2]	$\frac{1}{7}(1,6)$	$13+5/7$	0
$\frac{1}{8}(1,1)$	[8]	$\frac{1}{8}(1,1)$	$8+1/4$	$-9/2$
$\frac{1}{8}(1,3)$	[3, 3]	$\frac{1}{8}(1,3)$	$6+3/4$	-1
$\frac{1}{8}(1,5)$	[2, 3, 2]	$\frac{1}{8}(1,5)$	$8+1/4$	$-1/2$
$\frac{1}{8}(1,7)$	[2, 2, 2, 2, 2, 2, 2]	$\frac{1}{8}(1,7)$	$15+3/4$	0
$\frac{1}{9}(1,1)$	[9]	$\frac{1}{9}(1,1)$	$9+2/9$	$-49/9$
$\frac{1}{9}(1,2)$	[5, 2]	$\frac{1}{9}(1,5)$	$7+7/9$	-2
$\frac{1}{9}(1,4)$	[3, 2, 2, 2]	$\frac{1}{9}(1,7)$	$10+2/9$	$-4/9$
$\frac{1}{9}(1,8)$	[2, 2, 2, 2, 2, 2, 2, 2]	$\frac{1}{9}(1,8)$	$17+7/9$	0
$\frac{1}{10}(1,1)$	[10]	$\frac{1}{10}(1,1)$	$10+1/5$	$-32/5$
$\frac{1}{10}(1,3)$	[4, 2, 2]	$\frac{1}{10}(1,7)$	9	$-6/5$
$\frac{1}{11}(1,1)$	[11]	$\frac{1}{11}(1,1)$	$11+2/11$	$-81/11$
$\frac{1}{11}(1,2)$	[6, 2]	$\frac{1}{11}(1,6)$	$8+8/11$	$-32/11$
$\frac{1}{11}(1,3)$	[4, 3]	$\frac{1}{11}(1,4)$	$7+7/11$	$-20/11$
$\frac{1}{11}(1,5)$	[3, 2, 2, 2, 2]	$\frac{1}{11}(1,9)$	$12+3/11$	$-5/11$
$\frac{1}{11}(1,7)$	[2, 3, 2, 2]	$\frac{1}{11}(1,8)$	$10+4/11$	$-6/11$
$\frac{1}{12}(1,1)$	[12]	$\frac{1}{12}(1,1)$	$12+1/6$	$-25/3$
$\frac{1}{12}(1,5)$	[3, 2, 3]	$\frac{1}{12}(1,5)$	$8+5/6$	-1
$\frac{1}{12}(1,7)$	[2, 4, 2]	$\frac{1}{12}(1,7)$	$9+1/6$	$-4/3$
$\frac{1}{13}(1,1)$	[13]	$\frac{1}{13}(1,1)$	$13+2/13$	$-121/13$
$\frac{1}{13}(1,2)$	[7, 2]	$\frac{1}{13}(1,7)$	$9+9/13$	$-50/13$
$\frac{1}{13}(1,3)$	[5, 2, 2]	$\frac{1}{13}(1,9)$	$9+12/13$	$-27/13$
$\frac{1}{13}(1,4)$	[4, 2, 2, 2]	$\frac{1}{13}(1,10)$	$11+1/13$	$-16/13$

$\frac{1}{13}(1,5)$	[3, 3, 2]	$\frac{1}{13}(1,8)$	9	$-15/13$
$\frac{1}{13}(1,6)$	[3, 2, 2, 2, 2, 2]	$\frac{1}{13}(1,11)$	$14+4/13$	$-6/13$
$\frac{1}{14}(1,1)$	[14]	$\frac{1}{14}(1,1)$	$14+1/7$	$-72/7$
$\frac{1}{14}(1,3)$	[5, 3]	$\frac{1}{14}(1,5)$	$8+4/7$	$-19/7$
$\frac{1}{14}(1,9)$	[2, 3, 2, 2, 2]	$\frac{1}{14}(1,11)$	$12+3/7$	$-4/7$

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