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Asymptotics of the s-fractional Gaussian perimeter as $s \to 0^+$

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Abstract

We study the asymptotic behaviour of the renormalised s-fractional Gaussian perimeter of a set E inside a domain Ω as $s \to 0^+$. Contrary to the Euclidean case, as the Gaussian measure is finite, the shape of the set at infinity does not matter, but, surprisingly, the limit set function is never additive.

Keywords Fractional Ornstein-Uhlenbeck operator · Fractional perimeters · Fractional Sobolev spaces · Gaussian analysis

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1 Introduction

In this paper we consider the fractional Gaussian perimeter

$$P_s^\gamma(E;\Omega) := \int_{E\cap\Omega} d\gamma(x)\int_{E^c\cap\Omega} K_s(x,y)d\gamma(y) \\ + \int_{E\cap\Omega} d\gamma(x)\int_{E^c\cap\Omega^c} K_s(x,y)d\gamma(y) + \int_{E\cap\Omega^c} d\gamma(x)\int_{E^c\cap\Omega} K_s(x,y)d\gamma(y), \tag{1.1}$$

where γ is the standard Gaussian measure in $\mathbb{R}^N$ defined in (2.1) and the kernel K_s is the jumping kernel defined in (2.3) and study the asymptotics of $sP_s^\gamma(E;\Omega)$ as $s \to 0^+$. In this sense this is a parallel study of our previous paper [5], where the Γ-limit of $(1-s)P_s^\gamma(E;\Omega)$ as $s \to 1^-$ is studied.

In the Euclidean setting the notion of s-fractional perimeter recovers the classical perimeter when $s \to 1^-$ in various senses as proved in [1, 2, 4, 7, 12, 17]. On the other side when $s \to 0^+$ one may wonder if there is convergence to some measure related to the Lebesgue one, and actually it holds true when considering the fractional perimeter of a set in the whole space (see [15]), but in a domain Ω the limit of $sP_s^\gamma(E;\Omega)$ does not always exist, and when it does, as a function of the set E it is not always a measure as proved in [9].

The main result of this paper consists in the computation of the limit

$$\mu(E) := \lim_{s\to 0^+} sP_s^\gamma(E;\Omega) \tag{1.2}$$

and the analysis of the set function μ. The Gaussian case is different from the Euclidean case treated in [9]. Indeed, the limit in (1.2) always exists under the only assumption that $P_{s_0}^\gamma(E;\Omega) < \infty$ for some $s_0 \in (0,1)$ and it is not affected neither by the behaviour at infinity of the set E nor on the unboundedness and $C^{1,\alpha}$ regularity of Ω. Nevertheless, in the limit cases $E \subset \Omega$ or $\Omega = \mathbb{R}^N$ we dot not recover at the limit the Gaussian measure of E, but rather $2\gamma(E)\gamma(E^c)$, a result that is is coherent with the fact that, whenever it exists, $\mu(E) = \mu(E^c)$. A related result in the Euclidean setting is the Maz'ya-Shaposhnikova approximation theorem proved in [15] in the framework of fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$. We prove in Theorem 2 an analogous result in the Gaussian case, $p = 2$. Our result is intrinsecally different with respect to its Euclidean counterpart concerning both the methods and the result, since in the Gaussian case the Ornstein-Uhlenbeck operator has compact resolvent (hence we can use a series expansion) and the constants are eigenfunctions relative to the 0 eigenvalue.

We point out that, when $\Omega = \mathbb{R}^N$, it is convenient to write the fractional Gaussian perimeter in terms of the $H_\gamma^{\frac{s}{2}}$-seminorm introduced in Definition 2, namely

$$P_s^\gamma(E;\mathbb{R}^N) = \frac{1}{2}[\chi_E]^2_{H_\gamma^{\frac{s}{2}}(\mathbb{R}^N)}.$$

This allows to prove Theorem 1 in the case $\Omega = \mathbb{R}^N$ as a straightforward consequence of Theorem 2. The approach via $H_\gamma^{\frac{s}{2}}$-seminorms has been useful to study the isoperimetric in this context. For instance we mention [16], where authors introduce a notion of fractional Gaussian perimeter via extension techniques (see [18]) in the more general setting of Wiener spaces and then prove that halfspaces are the unique volume-constrained isoperimetric sets by means of cylindrical approximation and Ehrhard symmetrization, and also [6] for a quantitative version of the isoperimetric inequality for $P_s^\gamma(\cdot;\mathbb{R}^N)$ in finite dimension.

In the following, we denote by $\mathcal{E}$ the family of sets $E\subset\mathbb{R}^N$ such that the limit in (1.2) exists which is defined as

$$\mathcal{E} := \left\{ E \in \mathcal{M}(\mathbb{R}^N) \quad \text{s.t.} \quad \exists s_0 \in (0,1) \quad \text{s.t.} \quad P_{s_0}^\gamma(E;\Omega) < \infty \right\}.$$

We stress that, differently from [9], we do not need to complement $\mathcal{E}$ with a control of the behaviour at infinity of its elements. Let us state the main result of the present paper.

Theorem 1 *Let $\Omega\subset\mathbb{R}^N$ an open connected set with Lipschitz boundary. Then for any $E\subset\mathbb{R}^N$ measurable set such that $P_{s_0}^\gamma(E;\Omega)<\infty$ for some $s_0\in(0,1)$ the limit (1.2) exists and it holds*

$$\mu(E) = 2\left(\gamma(E)\gamma(\Omega\setminus E) + \gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)\right). \tag{1.3}$$

In Sect. 2 we introduce the main tools and definitions. In Sect. 3 we firstly prove Theorem 1 by stating and proving the ancillary Propositions 1 and 2 and we show some properties of the limit set function μ. In the last Sect. 4 we prove that for the Gaussian fractional perimeter defined and used in [8] the asymptotics for $s\to 0^+$ is trivial.

2 Notation and preliminary results

For $N\in\mathbb{N}$ we denote by γ the Gaussian measure on $\mathbb{R}^N$

$$\gamma := \frac{1}{(2\pi)^{N/2}} e^{-\frac{|\cdot|^2}{2}} \mathscr{L}^N, \tag{2.1}$$

where $\mathscr{L}^N$ is the Lebesgue measure. With a little abuse of notation we denote by γ both the measure and its density with respect to $\mathscr{L}^N$. Moreover, in the sequel we use the measure $\lambda := \frac{1}{(2\pi)^{N/2}} e^{-\frac{|\cdot|^2}{4}} \mathscr{L}^N$.

In order to define the fractional perimeter, we introduce the Ornstein-Uhlenbeck semigroup, its generator Δ_γ, the fractional powers of the generator and the functional setting.

Definition 1 Let $t>0$ and $x\in\mathbb{R}^N$. For $u\in L^1_\gamma(\mathbb{R}^N)$ we define the Ornstein-Uhlenbeck semigroup as

$$e^{t\Delta_\gamma} u(x) := \int_{\mathbb{R}^N} M_t(x,y) u(y)\, d\gamma(y),$$

where $M_t(x, y)$ denotes the Mehler kernel

$$M_t(x, y) := \frac{1}{(1-e^{-2t})^{N/2}} \exp\left(-\frac{e^{-2t}|x|^2 - 2e^{-t}x \cdot y + e^{-2t}|y|^2}{2(1-e^{-2t})}\right),$$

which satisfies

$$e^{t\Delta_\gamma}1 = \int_{\mathbb{R}^N} M_t(x, y)d\gamma(y) = 1,$$

for any $t > 0$ and any $x \in \mathbb{R}^N$.

The generator of $e^{t\Delta_\gamma}$ acts on sufficiently smooth functions as

$$\Delta_\gamma u = \Delta u - x \cdot Du$$

and is called Ornstein-Uhlenbeck operator; see e.g. [13] and the references therein for the main properties of $e^{t\Delta_\gamma}$ and Δ_γ.

Since $-\Delta_\gamma$ is a positive definite and selfadjoint operator which generates a C_0-semigroup of contractions in $L^2_\gamma(\mathbb{R}^N)$, we can define its fractional powers by means of spectral decomposition via the Bochner subordination formula. In particular, for $s \in (0, 1)$ and $x \in \mathbb{R}^N$ the fractional Ornstein-Uhlenbeck operator is defined as

$$\begin{aligned}(-\Delta_\gamma)^s u(x) :&= \frac{1}{\Gamma(-s)} \int_0^\infty \frac{e^{t\Delta_\gamma}u(x) - u(x)}{t^{s+1}} dt \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \frac{dt}{t^{s+1}} \int_{\mathbb{R}^N} M_t(x, y)(u(y) - u(x))d\gamma(y) \\ &= \frac{1}{|\Gamma(-s)|} \int_{\mathbb{R}^N} (u(x) - u(y))\, K_{2s}(x, y)d\gamma(y),\end{aligned} \tag{2.2}$$

where for $\sigma > 0$ we have set

$$K_\sigma(x, y) := \int_0^\infty \frac{M_t(x, y)}{t^{\frac{\sigma}{2}+1}}\, dt, \tag{2.3}$$

and the right-hand side in (2.2) has to be intended in the Cauchy principal value sense. Notice that the integrability of the function

$$(0, \infty) \ni t \mapsto \frac{M_t(x, y)}{t^{\frac{sp}{2}+1}}$$

near zero, for any $x, y \in \mathbb{R}^N$, $x \neq y$, is ensured by the fact that

$$\lim_{t \to 0^+} \frac{M_t(x, y)}{H_t(|x - y|)} = (2\pi)^{N/2} e^{\frac{|x|^2}{4}} e^{\frac{|y|^2}{4}} \quad \text{for any} \quad x, y \in \mathbb{R}^N, \tag{2.4}$$

where, for $r \geq 0$, H_t is the Gauss-Weierstrass kernel $H_t(r) := \frac{e^{-\frac{r^2}{4t}}}{(4\pi t)^{N/2}}$.

Definition 2 Let $s \in (0,1)$ and $1 \leq p < \infty$. We define the fractional Gaussian Sobolev space $W^{s,p}_\gamma(\mathbb{R}^N)$ as

$$W^{s,p}_\gamma(\mathbb{R}^N) := \left\{ u \in L^p_\gamma(\mathbb{R}^N);\ [u]_{W^{s,p}_\gamma(\mathbb{R}^N)} < \infty \right\},$$

where

$$[u]_{W^{s,p}_\gamma(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} d\gamma(x) \int_{\mathbb{R}^N} |u(x) - u(y)|^p K_{sp}(x,y) d\gamma(y) \right)^{1/p},$$

and K_{sp} is defined in (2.3) with $\sigma = sp$. When $p = 2$, as usual we use the notation $H^s_\gamma(\mathbb{R}^N)$ instead of $W^{s,2}_\gamma(\mathbb{R}^N)$.

For the sake of completeness we recall that the Gaussian perimeter of a measurable set E in a Lipschitz open connected set Ω is defined by

$$P^\gamma(E;\Omega) = \sqrt{2\pi} \sup\left\{ \int_E (\mathrm{div}\varphi - \varphi \cdot x) d\gamma(x) : \varphi \in C^\infty_c(\Omega;\mathbb{R}^N), \|\varphi\|_\infty \leq 1 \right\}. \tag{2.5}$$

Now, we make more precise the definition of Gaussian fractional perimeter (1.1) given in Sect. 1.

Definition 3 Let $\Omega \subset \mathbb{R}^N$ be a connected open set with Lipschitz boundary, and $E \subset \mathbb{R}^N$ a measurable set. We define the Gaussian s-perimeter of E in Ω as

$$P^\gamma_s(E;\Omega) := P^{\gamma,L}_s(E;\Omega) + P^{\gamma,NL}_s(E;\Omega),$$

where the *local part* is

$$P^{\gamma,L}_s(E;\Omega) := \int_{E\cap\Omega} d\gamma(x) \int_{E^c\cap\Omega} K_s(x,y) d\gamma(y),$$

and the *nonlocal part* is

$$\begin{aligned} P^{\gamma,NL}_s(E;\Omega) := &\int_{E\cap\Omega} d\gamma(x) \int_{E^c\cap\Omega^c} K_s(x,y) d\gamma(y) \\ &+ \int_{E\cap\Omega^c} d\gamma(x) \int_{E^c\cap\Omega} K_s(x,y) d\gamma(y). \end{aligned}$$

Using the symmetry of the kernel K_s we immediately notice that $P^\gamma_s(E^c;\Omega) = P^\gamma_s(E;\Omega)$ for any measurable set E. If $\Omega = \mathbb{R}^N$ we simply write $P^\gamma_s(E)$ instead of $P^\gamma_s(E;\mathbb{R}^N)$. We notice that if $E \subset \Omega$ or $E^c \subset \Omega$ we have that $P^\gamma_s(E;\Omega) = P^\gamma_s(E)$.

In the sequel, for A, B measurable and disjoint sets, we denote with $L^\gamma_s(A,B)$ the (s-Gaussian) interaction functional

$$L^\gamma_s(A,B) := \int_A d\gamma(x) \int_B K_s(x,y) d\gamma(y). \tag{2.6}$$

Using this notation we have

$$P_s^\gamma(E;\Omega) = L_s^\gamma(E\cap\Omega, E^c\cap\Omega) + L_s^\gamma(E\cap\Omega, E^c\cap\Omega^c) + L_s^\gamma(E\cap\Omega^c, E^c\cap\Omega).$$

It is useful the following integration by parts formula proved for instance in [5]

$$\frac{1}{2}[u]^2_{H^s_\gamma(\mathbb{R}^N)} = \int_{\mathbb{R}^N} u(-\Delta_\gamma)^s u \, d\gamma. \tag{2.7}$$

The kernel K_s satisfies the following estimates (see [5, Lemmas 2.8, 2.9]).

Lemma 1 *For any $x, y \in \mathbb{R}^N$ and for any $s \in (0,1)$ we have*

$$K_s(x,y) \geq \frac{C_{N,s}}{|x-y|^{N+s}}, \tag{2.8}$$

where $C_{N,s} := 2^{s+\frac{N}{2}}\Gamma\left(\frac{s+N}{2}\right)$, and

$$K_s(x,y) \leq e^{\frac{|x|^2}{4}} e^{\frac{|y|^2}{4}} \tilde{K}_s(|x-y|), \tag{2.9}$$

where, for any $r \geq 0$, $\tilde{K}_s$ denotes the decreasing kernel

$$\tilde{K}_s(r) := \int_0^\infty \exp\left(-\frac{e^t r^2}{2(e^{2t}-1)}\right) \frac{dt}{t^{\frac{s}{2}+1}(1-e^{-2t})^{N/2}}.$$

3 Main Results

We begin this section by proving the analogue of [15, Theorem 3] in the case $p = 2$ in the Gaussian setting. Notice that our proof exploits the Hilbert structure of $H^s_\gamma(\mathbb{R}^N)$ and the compactness of the resolvent of Δ_γ. For $p \neq 2$ the proof is more delicate and requires explicit estimates on the kernel joint with a Hardy-type inequality (see [11, Subsection 2.1]).

Theorem 2 *(Maz'ya-Shaposhnikova approximation in $H^s_\gamma(\mathbb{R}^N)$) Let $s_0 \in (0,1)$ and $u \in H^{s_0}_\gamma(\mathbb{R}^N)$. Then it holds that*

$$\lim_{s\to 0^+} s[u]^2_{H^s_\gamma(\mathbb{R}^N)} = 2\left(\|u\|^2_{L^2_\gamma(\mathbb{R}^N)} - \left|\int_{\mathbb{R}^N} u \, d\gamma\right|^2\right).$$

Proof Let us notice that since $u \in L^2_\gamma(\mathbb{R}^N)$, we can write it in terms of the orthonormal basis $\mathcal{B}$ of eigenfunctions of $(-\Delta_\gamma)^s$ given by Hermite polynomials (see for instance [10]), i.e. $\mathcal{B} = \{H_n\}_{n\in\mathbb{N}_0}$, with $H_0 \equiv 1$ on $\mathbb{R}^N$. We recall that on the whole of $\mathbb{R}^N$ the spectral fractional Ornstein-Uhlenbeck operator coincides with the integro-differential operator in (2.2), and so, by the spectral mapping Theorem, see e.g. [14, Theorem 5.3.1], the latter has discrete spectrum given by $\sigma((-\Delta_\gamma)^s) = \sigma((-\Delta_\gamma))^s = \{n^s\}_{n\in\mathbb{N}_0}$. With these ideas in mind we have that

$$u = \sum_{n=0}^{\infty}(u, H_n)H_n,$$
$$(-\Delta_\gamma)^s u = |\Gamma(-s)| \sum_{n=1}^{\infty} n^s (u, H_n)H_n.$$

We use the integration by parts formula (2.7)

$$s[u]^2_{H^s_\gamma(\mathbb{R}^N)} = 2s\int_{\mathbb{R}^N} u(-\Delta_\gamma)^s u d\gamma = 2\Gamma(1-s)\sum_{n=1}^{\infty} n^s|(u, H_n)|^2, \tag{3.1}$$

where the right-hand side in (3.1) is finite for any $s \in (0, s_0)$ thanks to the assumption $u \in H^{s_0}_\gamma(\mathbb{R}^N)$. Passing to the limit for $s \to 0^+$ in (3.1) we have

$$\begin{aligned}\lim_{s\to 0^+} s[u]^2_{H^s_\gamma(\mathbb{R}^N)} &= 2\sum_{n=1}^{\infty}|(u, H_n)|^2\\ &= 2\left[\left(\sum_{n=0}^{\infty}|(u, H_n)|^2\right) - |(u, H_0)|^2\right]\\ &= 2\left(\|u\|^2_{L^2_\gamma(\mathbb{R}^N)} - \left|\int_{\mathbb{R}^N} u\, d\gamma\right|^2\right),\end{aligned}$$

concluding the proof. □

Remark 1 We point out that Theorem 2 is sufficient to prove Theorem 1 when $\Omega = \mathbb{R}^N$. Indeed, by choosing $u = \chi_E$, where E is a measurable set with $P^\gamma_{s_0}(E) < \infty$ for some $s_0 \in (0, 1)$, we get

$$\begin{aligned}\lim_{s\to 0^+} sP^\gamma_s(E) &= \lim_{s\to 0^+}\frac{s}{2}[\chi_E]^2_{H^{\frac{s}{2}}_\gamma(\mathbb{R}^N)} = \lim_{\sigma\to 0^+}\sigma[\chi_E]^2_{H^\sigma_\gamma(\mathbb{R}^N)} = 2\left(\gamma(E) - \gamma(E)^2\right)\\ &= 2\gamma(E)(1-\gamma(E)) = 2\gamma(E)\gamma(E^c).\end{aligned}$$

The remaining part of this section is devoted to the proof of Theorem 1 in the general case.

Proposition 1 *Let $\Omega \subset \mathbb{R}^N$ be an open connected set with Lipschitz boundary and let $E \subset \mathbb{R}^N$ be measurable. If $P^\gamma_{s_0}(E;\Omega) < \infty$ for some $s_0 \in (0, 1)$, then*

$$\limsup_{s\to 0^+} sP^\gamma_s(E;\Omega) \le 2\big[\gamma(E)\gamma(\Omega\setminus E) + \gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)\big]. \tag{3.2}$$

Proof We split

$$K_s(x,y) = \int_0^1 \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt + \int_1^\infty \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt.$$

For the first term we have

$$\int_0^1 \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt \le \int_0^1 \frac{M_t(x,y)}{t^{\frac{s_0}{2}+1}}dt \le \int_0^\infty \frac{M_t(x,y)}{t^{\frac{s_0}{2}+1}}dt = K_{s_0}(x,y), \tag{3.3}$$

for any $x, y \in \mathbb{R}^N$ and $s \le s_0$. To handle the second term, we write

$$M_t(x,y) = \frac{\exp\left(\phi_t(x,y)\right)}{(1-e^{-2t})^{N/2}}.$$

and estimate

$$\begin{aligned}
&\exp\left(\phi_t(x,y)\right)\gamma(x)\gamma(y) \\
&= \exp\left(-\frac{e^{-t}|x-y|^2 + (|x|^2+|y|^2)(e^{-2t}-e^{-t})}{2(1-e^{-2t})}\right)\gamma(x)\gamma(y) \\
&= \exp\left(-\frac{e^{-t}|x-y|^2}{2(1-e^{-2t})}\right)\exp\left(-\frac{(|x|^2+|y|^2)(e^{-2t}-e^{-t})}{2(1-e^{-2t})}\right)\gamma(x)\gamma(y) \\
&\le \frac{1}{(2\pi)^N}\exp\left(-\frac{|x|^2+|y|^2}{2}\left(\frac{e^{-2t}-e^{-t}}{1-e^{-2t}}+1\right)\right) \\
&= \frac{1}{(2\pi)^N}\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-t}}\right),
\end{aligned} \tag{3.4}$$

for any $t > 0$ and $x, y \in \mathbb{R}^N$. Now, we split again

$$\int_1^\infty \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt = \int_1^{1/s} \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt + \int_{1/s}^\infty \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt.$$

Using (3.4), we have

$$\begin{aligned}
&s\gamma(x)\gamma(y)\int_1^{1/s}\frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt \\
&\le \frac{s}{(2\pi)^N}\frac{1}{(1-e^{-2})^{N/2}}\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-1}}\right)\int_1^{1/s}\frac{dt}{t^{\frac{s}{2}+1}} \\
&= \frac{1}{(2\pi)^N}\frac{2}{(1-e^{-2})^{N/2}}\left(1-s^{s/2}\right)\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-1}}\right)
\end{aligned} \tag{3.5}$$

and

$$
\begin{aligned}
&s\gamma(x)\gamma(y)\int_{1/s}^{\infty}\frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt\\
&\leq \frac{s}{(2\pi)^N}\int_{1/s}^{\infty}\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-t}}\right)\frac{dt}{t^{\frac{s}{2}+1}(1-e^{-2t})^{N/2}}\\
&\leq \frac{s}{(2\pi)^N}\frac{1}{(1-e^{-\frac{2}{s}})^{N/2}}\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-\frac{1}{s}}}\right)\int_{1/s}^{\infty}\frac{dt}{t^{\frac{s}{2}+1}}\\
&= \frac{1}{(2\pi)^N}\frac{2}{(1-e^{-\frac{2}{s}})^{N/2}}\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-\frac{1}{s}}}\right)s^{s/2}.
\end{aligned}
\tag{3.6}
$$

By using (3.3), (3.5) and (3.6), for any $s\in(0,s_0)$ we obtain

$$
\begin{aligned}
sP_s^{\gamma}(E;\Omega) \leq{}& sP_{s_0}^{\gamma}(E;\Omega)\\
&+\frac{1}{(2\pi)^N}\frac{2}{(1-e^{-2})^{N/2}}\left(1-s^{s/2}\right)L_f(E\cap\Omega,E^c\cap\Omega)\\
&+\frac{1}{(2\pi)^N}\frac{2}{(1-e^{-2})^{N/2}}\left(1-s^{s/2}\right)L_f(E\cap\Omega,E^c\cap\Omega^c)\\
&+\frac{1}{(2\pi)^N}\frac{2}{(1-e^{-2})^{N/2}}\left(1-s^{s/2}\right)L_f(E\cap\Omega^c,E^c\cap\Omega)\\
&+\frac{s^{s/2}}{(2\pi)^N}\frac{2}{(1-e^{-\frac{2}{s}})^{N/2}}L_{g_s}(E\cap\Omega,E^c\cap\Omega)\\
&+\frac{s^{s/2}}{(2\pi)^N}\frac{2}{(1-e^{-\frac{2}{s}})^{N/2}}L_{g_s}(E\cap\Omega,E^c\cap\Omega^c)\\
&+\frac{s^{s/2}}{(2\pi)^N}\frac{2}{(1-e^{-\frac{2}{s}})^{N/2}}L_{g_s}(E\cap\Omega^c,E^c\cap\Omega),
\end{aligned}
\tag{3.7}
$$

where for A, B measurable and disjoint sets and for $0\leq h\in L^1(A\times B)$ we have used the notation

$$
L_h(A,B)=\int_A dx\int_B h(x,y)dy,
$$

with $f(x,y):=\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-1}}\right)$ and $g_s(x,y):=\exp\left(-\frac{|x|^2+|y|^2}{2}\frac{1}{1+e^{-\frac{1}{s}}}\right)$. To conclude, passing to the lim sup as $s\to 0^+$ in (3.7) it is easily seen that the first four terms in the right hand-side in (3.7) vanish, and, using the dominated convergence Theorem, the last three ones approach exactly the right-hand side in (3.2). □

To complete the asymptotic estimate, we need an estimate from below for the liminf.

Proposition 2 *Let $\Omega \subset \mathbb{R}^N$ be an open connected set with Lipschitz boundary. Then for any measurable set $E \subset \mathbb{R}^N$ it holds*

$$\liminf_{s\to 0^+} s P_s^\gamma(E;\Omega) \geq 2\big[\gamma(E)\gamma(\Omega\setminus E) + \gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)\big]. \tag{3.8}$$

Proof Let $\delta > 0$ and let $R > 0$ be such that

$$\begin{aligned}
&\gamma\left((E\cap\Omega)\cap B_R(0)\right) \geq \gamma\left(E\cap\Omega\right) - \delta,\\
&\gamma\left((E^c\cap\Omega)\cap B_R(0)\right) \geq \gamma\left(E^c\cap\Omega\right) - \delta,\\
&\gamma\left((E\cap\Omega^c)\cap B_R(0)\right) \geq \gamma\left(E\cap\Omega^c\right) - \delta,\\
&\gamma\left((E^c\cap\Omega^c)\cap B_R(0)\right) \geq \gamma\left(E^c\cap\Omega^c\right) - \delta.
\end{aligned} \tag{3.9}$$

For any $x, y \in B_R(0)$ it holds

$$\exp(\phi_t(x,y)) \geq \exp\left(-\frac{e^{-2t}|x-y|^2}{2(1-e^{-2t})}\right) \geq \exp\left(-\frac{2e^{-2t}}{1-e^{-2t}}R^2\right), \tag{3.10}$$

where ϕ_t is as in (3.4) and we used that $|x-y|^2 \leq 4R^2$. Since

$$\frac{1}{(1-e^{-2t})^{N/2}} > 1$$

and the map

$$t \mapsto \exp\left(-\frac{2e^{-2t}}{1-e^{-2t}}R^2\right)$$

is increasing in $(0,+\infty)$ and by (3.10) we get, for any $x, y \in B_R(0)$,

$$\begin{aligned}
K_s(x,y) &\geq \int_{1/s}^{\infty} \frac{M_t(x,y)}{t^{\frac{s}{2}+1}}dt \\
&\geq \int_{1/s}^{\infty} \frac{1}{t^{\frac{s}{2}+1}(1-e^{-2t})^{N/2}} \exp\left(-\frac{2e^{-2t}}{1-e^{-2t}}R^2\right)dt \\
&\geq \exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)\int_{1/s}^{\infty}\frac{dt}{t^{\frac{s}{2}+1}} = \frac{2}{s}\exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)s^{s/2}.
\end{aligned} \tag{3.11}$$

We can now estimate from below $sP_s^\gamma(E;\Omega)$

$$\begin{aligned}
&sP_s^\gamma(E;\Omega)\\
&\quad\geq s\int_{(E\cap\Omega)\cap B_R(0)} d\gamma(x)\int_{(E^c\cap\Omega)\cap B_R(0)} K_s(x,y)d\gamma(y)\\
&\qquad + s\int_{(E\cap\Omega)\cap B_R(0)} d\gamma(x)\int_{(E^c\cap\Omega^c)\cap B_R(0)} K_s(x,y)d\gamma(y)
\end{aligned}$$

$$
\begin{aligned}
&+ s\int_{(E\cap\Omega^c)\cap B_R(0)} d\gamma(x)\int_{(E^c\cap\Omega)\cap B_R(0)} K_s(x,y)d\gamma(y)\\
\geq\ & 2\exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)s^{s/2}\Big[\gamma\left((E\cap\Omega)\cap B_R(0)\right)\gamma\left((E^c\cap\Omega)\cap B_R(0)\right)\\
&+\gamma\left((E\cap\Omega)\cap B_R(0)\right)\gamma\left((E^c\cap\Omega^c)\cap B_R(0)\right)\\
&+\gamma\left((E\cap\Omega^c)\cap B_R(0)\right)\gamma\left((E^c\cap\Omega)\cap B_R(0)\right)\Big]\\
\geq\ & 2\exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)s^{s/2}\Big[\left(\gamma\left(E\cap\Omega\right)-\delta\right)\left(\gamma\left(E^c\cap\Omega\right)-\delta\right)\\
&+\left(\gamma\left(E\cap\Omega\right)-\delta\right)\left(\gamma\left(E^c\cap\Omega^c\right)-\delta\right)\\
&+\left(\gamma\left(E\cap\Omega^c\right)-\delta\right)\left(\gamma\left(E^c\cap\Omega\right)-\delta\right)\Big]\\
=\ & 2\exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)s^{s/2}\\
&\times\left[\gamma(E)\gamma(\Omega\setminus E)+\gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)+3\delta^2-(1+\gamma(\Omega))\delta\right]\\
\geq\ & 2\exp\left(-\frac{2e^{-2/s}}{1-e^{-2/s}}R^2\right)s^{s/2}\\
&\times\left[\gamma(E)\gamma(\Omega\setminus E)+\gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)+3\delta^2-2\delta\right].
\end{aligned}
$$

By letting $s\to 0^+$ we obtain

$$
\liminf_{s\to 0^+} sP_s^\gamma(E;\Omega)\geq 2\left[\gamma(E)\gamma(\Omega\setminus E)+\gamma(E\cap\Omega)\gamma(E^c\cap\Omega^c)+3\delta^2-2\delta\right],
$$

thus we get (3.8) in view of the arbitrariness of $\delta>0$. □

Proof of Theorem 1 It is an immediate consequence of Proposition 1 and Proposition 2.

In the proof of Theorem 1, the hypothesis $P_{s_0}^\gamma(E;\Omega)<+\infty$ for some $s_0\in(0,1)$ is crucial (it is required to prove Proposition 1). Adapting [9, Example 2.10], we show that there are measurable sets that do not satisfy that requirement.

Example 1 (A measurable set with $P_s^\gamma(E;\Omega)=+\infty$ for any $s\in(0,1)$) Let us consider a decreasing sequence $(\beta_k)_k\subset\mathbb{R}$ with $\beta_k>0$ for any $k\in\mathbb{N}$ such that

$$
M:=\sum_{k=1}^{+\infty}\beta_k<+\infty
$$

but

$$
\sum_{k=1}^{+\infty}\beta_k^{1-s}=+\infty
$$

for every $s \in (0, 1)$ (in [9, Example 2.10] the authors suggest the possible choice $\beta_1 = \frac{1}{\log^2 2}$ and $\beta_k = \frac{1}{k \log^2 k}$ for any $k \geq 2$). Let us define

$$\Omega := (0, M) \subset \mathbb{R}, \quad \sigma_m := \sum_{k=1}^{m} \beta_k, \quad I_m := (\sigma_m, \sigma_{m+1}), \quad E := \bigcup_{j=1}^{+\infty} I_{2j}.$$

We claim that $P_s^{\gamma}(E; \Omega) = +\infty$ for any $s \in (0, 1)$. By recalling that $E \subset \Omega$ it holds

$$\begin{aligned}
&P_s^{\gamma}(E; \Omega) = P_s^{\gamma}(E) \geq C_{1,s} \sum_{j=1}^{+\infty} \int_{\sigma_{2j}}^{\sigma_{2j+1}} d\gamma(x) \int_{\sigma_{2j+1}}^{\sigma_{2j+2}} \frac{d\gamma(y)}{|x-y|^{1+s}} \\
&\geq \frac{1}{2\pi} \frac{e^{-M^2}}{s(1-s)} \sum_{j=1}^{+\infty} \Big[(\sigma_{2j+2} - \sigma_{2j+1})^{1-s} + (\sigma_{2j+1} - \sigma_{2j})^{1-s} - (\sigma_{2j+2} - \sigma_{2j})^{1-s} \Big] \\
&= \frac{1}{2\pi} \frac{e^{-M^2}}{s(1-s)} \sum_{j=1}^{+\infty} \Big[\beta_{2j+2}^{1-s} + \beta_{2j+1}^{1-s} - (\beta_{2j+2} + \beta_{2j+1})^{1-s} \Big],
\end{aligned}$$

where in the first inequality we used (2.8), while in the second inequality we used that $C_{1,s} \geq 1$, the boundedness from below of the Gaussian weights in $(\sigma_{2j}, \sigma_{2j+1}) \times (\sigma_{2j+1}, \sigma_{2j+2})$ for any $j \geq 1$ and that for $a < b \leq c < d$

$$\begin{aligned}
&\int_a^b dx \int_c^d \frac{dy}{|x-y|^{1+s}} \\
&= \frac{1}{s(1-s)} \Big[(c-a)^{1-s} + (d-b)^{1-s} - (c-b)^{1-s} - (d-a)^{1-s} \Big].
\end{aligned}$$

Since the map $t \mapsto (1+t)^{1-s}$ is concave in $[0, 1)$, it holds

$$1 + t^{1-s} - (1+t)^{1-s} \geq s t^{1-s}.$$

By the choice $t = \frac{\beta_{2j+2}}{\beta_{2j+1}}$ we get

$$\beta_{2j+2}^{1-s} + \beta_{2j+1}^{1-s} - (\beta_{2j+2} + \beta_{2j+1})^{1-s} \geq s \beta_{2j+2}^{1-s}$$

and so,

$$P_s^{\gamma}(E; \Omega) \geq \frac{1}{2\pi} \frac{e^{-M^2}}{1-s} \sum_{j=1}^{+\infty} \beta_{2j+2}^{1-s} = +\infty,$$

concluding the proof of the claim.

Now we state some properties of the set function μ.

Proposition 3 *μ is subadditive on $\mathcal{E}$, i.e. $\mu(E \cup F) \leq \mu(E) + \mu(F)$ for any $E, F \in \mathcal{E}$; μ is not monotone with respect to inclusions.*

Proof To show the subadditivity, we proceed as in the proof of [9, Proposition 2.1]; to show the lack of monotonicity, it is sufficient to choose as E a small ball contained in Ω or a halfspace such that $\mathcal{H}^{N-1}(\partial E \cap \Omega) > 0$ and $F = \mathbb{R}^N$. □

Notice that μ is not additive. Indeed, if $\Omega = \mathbb{R}^N$, then, for any pair of measurable disjoint sets $A, B \subset \mathbb{R}^N$

$$\begin{aligned}
\mu(A \cup B) &= 2\gamma(A \cup B)\gamma(A^c \cap B^c) = 2\,(\gamma(A) + \gamma(B))\,(1 - \gamma(A) - \gamma(B)) \\
&= 2\gamma(A)\,(1 - \gamma(A)) + 2\gamma(B)\,(1 - \gamma(B)) - 4\gamma(A)\gamma(B) \\
&= 2\gamma(A)\gamma(A^c) + 2\gamma(B)\gamma(B^c) - 4\gamma(A)\gamma(B) \\
&= \mu(A) + \mu(B) - 4\gamma(A)\gamma(B).
\end{aligned}$$

Otherwise, if $\Omega \neq \mathbb{R}^N$, we proceed as in the proof [9, Proposition 2.3] by using the following result.

Lemma 2 *For any $A, B \subset \mathbb{R}^N$ measurable disjoint sets there exists $C = C(A, B) > 0$ such that*

$$sL_s^\gamma(A, B) \geq C,$$

for any $s \in (0, 1)$.

Proof We firstly assume that A, B are bounded and fix $R > 0$ sufficiently large such that $A, B \subset B_R$. We have

$$\begin{aligned}
sL_s^\gamma(A, B) &\geq s\int_A d\gamma(x)\int_B d\gamma(y)\int_1^\infty \frac{M_t(x, y)}{t^{\frac{s}{2}+1}}dt \\
&\geq s\int_A d\gamma(x)\int_B d\gamma(y)\int_1^\infty \exp\left(-\frac{e^{-2t}(|x|^2 + |y|^2) - 2e^{-t}(x, y)}{2(1 - e^{-2t})}\right)\frac{dt}{t^{\frac{s}{2}+1}} \\
&\geq s\int_A d\gamma(x)\int_B d\gamma(y)\int_1^\infty \exp\left(-\frac{e^{-2t}|x - y|^2}{2(1 - e^{-2t})}\right)\frac{dt}{t^{\frac{s}{2}+1}} \\
&\geq s\exp\left(-\frac{2R^2}{(e^2 - 1)}\right)\int_A d\gamma(x)\int_B d\gamma(y)\int_1^\infty \frac{dt}{t^{\frac{s}{2}+1}} \\
&= 2\exp\left(-\frac{2R^2}{(e^2 - 1)}\right)\gamma(A)\gamma(B) =: C(A, B).
\end{aligned} \tag{3.12}$$

If A, B are unbounded we simply have

$$sL_s^\gamma(A, B) \geq sL_s^\gamma(A \cap B_R, B \cap B_R) \geq C$$

for any $s \in (0, 1)$ and $R > 0$. □

Remark 2 We notice that, even if we add in Lemma 2 the hypothesis of strictly positive distance between A and B, the result is left unchanged.

4 Final remarks

We conclude by studying the asymptotics for $s \to 0^+$ even for the fractional perimeter defined in [8]

$$\begin{aligned}\mathcal{J}_s^\lambda(E;\Omega) := &\int_{E\cap\Omega} d\lambda(x) \int_{E^c\cap\Omega} \frac{d\lambda(y)}{|x-y|^{N+s}}\\ &+ \int_{E\cap\Omega} d\lambda(x) \int_{E^c\cap\Omega^c} \frac{d\lambda(y)}{|x-y|^{N+s}}\\ &+ \int_{E\cap\Omega^c} d\lambda(x) \int_{E^c\cap\Omega} \frac{d\lambda(y)}{|x-y|^{N+s}}.\end{aligned} \tag{4.1}$$

We recall that the functional (4.1) is linked to (1.1) by the fact that they have the same Γ-limit by multiplying by $1-s$ and letting $s \to 1^-$ ([5, Main Theorem]); this depends on the fact that $K_s(x,y)\gamma(x)\gamma(y)$ and $\frac{\lambda(x)\lambda(y)}{|x-y|^{N+s}}$ approach the Dirac delta in the same way, up to a multiplicative constant, when $|x-y| \to 0$. Nevertheless, definition (4.1) is somehow unnatural, because it is not linked to functional calculus as (1.1). Therefore, we can say that (1.1) is the fractional counterpart of the Gaussian perimeter (2.5), and we can refer to it as "Fractional Gaussian perimeter", while (4.1) is a weighted version of the fractional perimeter defined in [3], and we can refer to it as "Gaussian fractional perimeter". As already said in Sect. 1 for the Gaussian fractional perimeter the asymptotics for $s \to 0^+$ is not meaningful. Indeed the following proposition holds.

Proposition 4 *For any measurable set E such that $\mathcal{J}_{s_0}^\lambda(E;\Omega) < \infty$ for some $s_0 \in (0,1)$ we have*

$$\lim_{s\to 0^+} s\mathcal{J}_s^\lambda(E;\Omega) = 0.$$

Proof Let A, B be measurable and disjoint sets such that $L_{s_0}^\lambda(A,B) < \infty$ for some $s_0 \in (0,1)$, where

$$L_\sigma^\lambda(A,B) := \int_A d\lambda(x) \int_B \frac{d\lambda(y)}{|x-y|^{N+\sigma}}.$$

Then, for any $s \in (0,s_0)$ we have

$$\begin{aligned}L_s^\lambda(A,B) = &\iint_{(A\times B)\cap\{|x-y|\geq 1\}} \frac{d\lambda(y)}{|x-y|^{N+s}} d\lambda(x)\\ &+ \iint_{(A\times B)\cap\{|x-y|<1\}} \frac{d\lambda(y)}{|x-y|^{N+s}} d\lambda(x)\\ \leq &\lambda(A)\lambda(B) + \iint_{(A\times B)\cap\{|x-y|<1\}} \frac{d\lambda(y)}{|x-y|^{N+s_0}} d\lambda(x)\\ \leq &\lambda(A)\lambda(B) + L_{s_0}^\lambda(A,B) < \infty.\end{aligned} \tag{4.2}$$

Therefore

$$\lim_{s\to 0^+} sL_s^\lambda(A,B)=0. \tag{4.3}$$

By applying (4.3) to the couples of sets $(E\cap\Omega, E^c\cap\Omega)$, $(E\cap\Omega, E^c\cap\Omega^c)$, $(E\cap\Omega^c, E^c\cap\Omega)$, we completely prove the claim. □

Remark 3 We notice that even in this case we cannot drop the condition $\mathcal{J}_{s_0}^\lambda(E;\Omega)<\infty$ for some $s_0\in(0,1)$. Indeed [9, Example 2.10] still works with

$$\mathcal{J}_s^\lambda(E;\Omega)\geq \frac{1}{2\pi}\frac{e^{-\frac{M^2}{2}}}{1-s}\sum_{j=1}^{+\infty}\beta_{2j+2}^{1-s}=+\infty,$$

for any $s\in(0,1)$.

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Conflict of interest The authors declare that they have no conflict of interest.

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