The Fair Valuation of Defined Contribution Pension Funds in a Stochastic Mortality Environment

Massimiliano Politano

Abstract

This paper analyses the role of the term structure of interest and mortality rates for Defined Contribution Pension Schemes. In particular, the model suggested allows the actuary to determine the fair valuation of such a scheme by modelling both mortality and financial risk by means of diffusion processes. A numerical example illustrates the fair value accounting impact on reserve evaluations by comparing a traditional deterministic approach and a stochastic one for interest and mortality rates.

Key words: Pension Funding, Risk Neutral Valuation, Stochastic Mortality, CIR model.

Introduction

Traditionally actuaries have valued defined contribution pension schemes using a deterministic mortality intensity and some hypothesis on the dynamics of interest rates. However, since neither the interest rates nor the mortality intensity are deterministic, pension funds are essentially exposed to three types of risk: the financial risk, the systematic mortality risk, referring to the underlying mortality intensity, and the unsystematic mortality risk, referring to a possible adverse development of the scheme members mortality. It must be pointed out that only the third kind of risk can be controlled by means of portfolio diversification.

Since pension contracts often run for a very long time, a mortality intensity which seems to be prudential at the time of the issue, might turn out not to be so. An analogous phenomenon has been also observed for the interest rates during the last two decades, where we have experienced large drops in stock prices and low return on bonds. However, the systematic mortality risk is of different character than the financial risk. While the assets on the financial market are very volatile, changes in the mortality intensity seem to occur more slowly. Thus, the financial market poses an immediate problem, whereas the level of the mortality intensity poses a more long term, but also more permanent problem. This difference could be the reason why emphasis so far has been on the financial markets.

However, in recent years, some of this attention has shifted towards valuation models that fully capture the interest and mortality rates dynamics. In this context, the contribute of the International Accounting Standard Board was very important. It defines the Fair Value as “an estimate of an exit price determined by market interactions”: therefore, the problem of determining the market value of insurance liability is posed. In this field, it must be remembered the papers of Milevsky-Promislow (2001), Grosen-Jorgensen (2000), Ballotta-Haberman (2003), Biffis (2003).

The paper is organised as follows: section 2 develops the framework for the valuation of Defined Contribution Pension Schemes. In section 3, a stochastic model for the mortality risk is introduced, in section 4 the financial risk model is presented. A numerical evidence is proposed in section 5.

The model

Let us consider a defined contribution pension fund with an individual funding method, which liquidates a capital, resulting from a contribution accumulation process, to the subscriber in case of predecease, disability, old age. In presence of a guarantee of yield the liability b_h borne out by the fund in the year h with respect to a generic subscriber is given by

$$b_h = \max\left\{W_h^A, W_h^{GAR}\right\},$$

where W_h^A is equal to the share of the equivalent assets constituting the fund and W_h^{GAR} denotes the minimum guaranteed benefit.

Now, let us define $\{r_h; h = 1,2,...\}$ and $\{\mu_{x+h}; h = 1,2,...\}$ the random spot rate process and the mortality intensity process respectively, both of them measurable with respect to the filtrations F^r and F^μ. The aforementioned processes are defined on a unique probability space $(\Omega, F^{r,\mu}, P)$, such that $F^{r,\mu} = F^r \cup F^\mu$.

In this context, as the elimination of the state of active can happen by death (d), inability (i), old age (v), the liability of the fund towards a scheme member can be expressed in the following manner

$$W_0^L = \sum_{h=0}^{\xi} b_h \left({}_{h-1/1}Y_x^{(d)} + {}_{h-1/1}J_x^{(i)} \right) + b_\xi \, {}_\xi Z_x \,, \tag{1}$$

where

$${}_{h-1/1}Y_x^{(d)} = \begin{cases} e^{-\Delta(h)} & \text{if } h-1 < Tx \le h \\ 0 & \text{otherwise} \end{cases}, \quad {}_{h-1/1}J_x^{(i)} = \begin{cases} e^{-\Delta(h)} & \text{if disability occurs} \\ 0 & \text{otherwise} \end{cases},$$

$${}_\xi Z_x = \begin{cases} 0 & \text{if } 0 < Tx \le \xi \\ e^{-\Delta(\xi)} & T_x \ge \xi \end{cases}.$$

In the previous expressions T_x is a random variable which represents the remaining lifetime of a person aged x, $\Delta(h) = \int_0^h r_u du$ is the accumulation function of the spot rate, ξ denotes the maximum permanence of the generic subscriber.

A stochastic model for the mortality risk

For the dynamics of the process $\{\mu_{x+h}; h = 1,2,...\}$, we propose to choose a Mean Reverting Brownian Gompertz model (e.g. [6]); according to the traditional actuarial approach, the survival function of the random variable T_x is given by

$${}_s p_x = \Pr\left(T_x > s / F_0^{\mu}\right), \tag{12}$$

where F_0^μ represents the mortality informative structure available at time *0*.

If we make the hypotesis of time dependence of the mortality intensity and we define $\mu_{x+h:h}$ to be the mortality intensity for an individual aged $x+h$, observed in the year h, it is possible to express the previous formula as follows

$${}_s p_x = E\left(\exp\left\{ -\int_0^s \mu_{x+h:h} dt \right\} / F_0^{\mu} \right). \tag{13}$$

A widely used actuarial model for projecting mortality rates is the reduction factor model (e.g [7]). This model has traditionally been formulated with respect to the conditional probability of dying in a year

$$q(y,h) = q(y,0) RF(y,h),$$

where *q(y, 0)* represents the probability that a person aged y will die in the next year, based on the mortality experience for the base year 0, and correspondingly *q(y, h)* relates to future year t. Given the form of (3), it is considered a *reduction factor approach* for the mortality intensity so that

$$\mu_{y:h} = \mu_{y:0} RF(y,h),$$

where $\mu_{y:0}$ is the mortality intensity for a person aged y in the base year 0, $\mu_{y:h}$ is the mortality intensity for a person attaining age y in future year h, and the reduction factor $RF(y,h)$ is the ratio of the mortality intensity, whose form is given by

$$RF(y,h) = e^{(\alpha+\beta y)h}.$$

The parameter α represents the rate of change in the mortality on the logarithmic scale and β represents an offset term that reflects a rate of change that could differ with age y.

With $y = x+h$, it is possible to use the following model for the mortality intensity evolution

$$\mu_{x+h:h} = \mu_{x+h:0} \exp\{(\alpha + \beta x + \beta h)h + \sigma_x Y_h\}, \ \sigma_x, \mu_{x+h,0} > 0,$$

where σ_x is the standard deviation of the mortality intensity process and $\{Y_h\}$ is an Ornstein Uhlenbeck process whose dynamics is given by

$$\begin{cases} dY_h = -bY_h dh + dW_h \\ Y_0 = 0 \end{cases},$$

where $b>0$ is the mean reverting coefficient and $\{W_h\}$ is a standard Brownian motion. The solution of the previous system

$$Y_h = \int_0^h \exp\{-b(h-s)\} dW_s$$

implies that $Y_h \approx N(0, \varphi^2(h))$, with

$$\varphi^2(h) = \frac{1-e^{-2bh}}{2b}.$$

It must be pointed out that equation (2) does not lead to a closed form solution mainly due to the fact that, as well known, the sum of lognormal random variables is not lognormal (e.g. [6]). Thus, for valuing equation (1), there is the need for an appropriate numerical approximation.

Although approximations are available in the literature, for computational purposes, it has been conducted a simulation procedure based on the Monte Carlo technique (e.g. [6]).

A stochastic model for the financial risk

For the dynamics of the process $\{r_h; h = 1,2,...\}$, it is assumed a mean reverting square root dynamics

$$dr_h = f(r_h, h)dt + l(r_h, h)dZ_h,$$

where $f(r_h,h)$ is the drift of the process, $l^2(r_h,h)$ is the diffusion coefficient, $\{Z_h\}$ is a Standard Brownian Motion; in particular, in the CIR model for the interest rate dynamics (e.g. [3]), the spot rate evolution is given by

$$dr_h = k(\theta - r_h)dh + \sigma_r \sqrt{r_h} dZ_h,$$

where k is the mean reverting coefficient, θ is the long period "normal" rate, σ_r is the spot rate volatility. In a stochastic context, the bond price in h with maturity T is given by

$$B(h,T) = E\left(\exp\left\{-\int_h^T r_u du\right\} / F_h^{\ r}\right),$$

where F_h^r represents the interest informative structure available on the market at time h.
For the CIR model, it is known (e.g. [3]) that the solution is

$$B(h,T) = \exp\{-rC(h,T) - A(h,T)\} \quad 0 \le h < T \quad r \ge 0,$$

where

$$C(h,T) = \frac{sinh(\gamma(T-h))}{\gamma\cosh(\gamma(T-h)) + 1/2\,ksinh(\gamma(T-h))},$$

$$A(h,T) = \frac{2k\theta}{\sigma_r^{\ 2}}\log\left[\frac{\gamma\exp\{1/2\,k(T-h)\}}{\gamma\cosh(\gamma(T-h)) + 1/2\,ksinh(\gamma(T-h))}\right],$$

$$\gamma = \frac{1}{2}\sqrt{k^2 + 2\sigma_r^{\ 2}} \quad sinhu = \frac{e^u - e^{-u}}{2} \quad \cosh u = \frac{e^u + e^{-u}}{2}.$$

An application

The model described has been applied in order to analyse the temporal profile of the fund liability. In the numerical application, we compare the values of the fund liability using, for a scheme member aged 40, a technical basis given by a constant interest rate of 4% and the Italian life table SIM92 with the values obtained in a fair value context according to the models of previous sections. In particular the Mean Reverting Brownian Gompertz process modelling the mortality risk is generated on the Italian Institute of Statistics life tables 1931/1991, according to the projection of the demographic factor. Moreover, the parameters of the CIR process, modelling the financial risk, are obtained on the basis of the data related to and the 3-month Italian T-Bill January 1996-January 2004, with long term mean of 4%. For the elimination cause disability, according to its purely accidental nature, we have chosen the projected Italian INPS tables both for deterministic and stochastic case. The results of the analysis are summarised in the following table.

Table 1

Deterministic vs. Stochastic Interests and Mortality Rates

Year	W_t^L	$FV(W_t^L)$	ΔW_t^L	$\Delta W_t^L/W_t^L$
1	2	3	4	5
0	0,00	0,00	0,00	0,00%
1	434,56	401,34	33,22	7,64%
2	450,67	418,35	32,32	7,17%
3	467,23	436,06	31,17	6,67%
4	484,33	454,49	29,84	6,16%
5	502,05	473,66	28,39	5,65%
6	520,30	493,59	26,71	5,13%
7	539,22	514,33	24,89	4,62%
8	558,77	535,93	22,84	4,09%
9	578,78	558,41	20,37	3,52%

Table 1 (continuous)

1	2	3	4	5
10	601,31	581,83	19,48	3,24%
11	629,85	611,84	18,01	2,86%
12	654,02	637,41	16,61	2,54%
13	678,96	663,95	15,01	2,21%
14	705,91	691,46	14,45	2,05%
15	733,98	719,85	14,13	1,93%
16	763,19	749,12	14,07	1,84%
17	792,43	779,18	13,25	1,67%
18	820,60	809,87	10,73	1,31%
19	850,80	841,01	9,79	1,15%
20	880,21	873,10	7,61	0,86%
21	913,92	906,75	7,17	0,78%
22	939,15	934,25	4,90	0,52%
23	965,16	962,46	2,70	0,28%
24	989,23	987,52	1,71	0,17%
25	1000,00	1000,00	0,00	0,00%

The table above puts in evidence a decrement in the level of the liability borne out by the fund for all the balance years. This difference, that decreases as the maturity is drawing on, is mainly caused by the financial and demographic conditions in force at the moment of the parameters estimate. In particular, the values obtained in a fair value context strictly depend on the shape of the term structure of interest rates: the mortality rates, that also are important for correctly determining the fair valuation of such a scheme, play a subordinate role.

In conclusion, the main target of the paper was to analyse the role of the interest and mortality rates for the valuation of Defined Contribution Pension Schemes. At this proposal the model suggested allows for not only determining the fair valuation of such a scheme but it can also be applied for measuring the impact of a change in the interest and mortality rates on the mathematical reserve value.

References

1. Ballotta L., S. Haberman. Valuation of guaranteed annuity conversion options // Insurance: Mathematics and Economics, 2003. – Vol. 33, N° 1 – pp. 71-91.
2. Biffis E. Affine processes in mortality modelling // Proceedings of the 6th Italian Spanish meeting on financial Mathematics, Trieste 2003. – pp. 83-103.
3. Cox J., J. Ingersoll, S. Ross. A theory of the term structure of interest rates // Econometrica, 1985.- N° 53 – pp. 385-408.
4. Grosen A., P.L. Jorgensen. Fair valuation of life insurance liabilities: the impact of interest rate guarantees, surrender option and bonus policies // Insurance: Mathematics and Economics, 2000. – Vol. 26, N° 1 – pp. 37-57.
5. International Accounting Standard Board. Draft Statement of Principles, 2001. Available at www.iasb.org.uk.
6. Milevsky M.A., S.D. Promislow. Mortality Derivatives and the option to annuitise // Insurance: Mathematics and Economics, 2001. – Vol. 29, N° 3 – pp. 299-318.
7. Renshaw A.E., S. Haberman. On the forecasting of mortality reduction factors // Insurance: Mathematics and Economics, 2003. – Vol. 32, N° 3 – pp. 379-401.