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Twisted noncommutative field theory with the Wick-Voros and Moyal products

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We present a comparison of the noncommutative field theories built using two different star products: Moyal and Wick-Voros (or normally ordered). For the latter we discuss both the classical and the quantum field theory in the quartic potential case and calculate the Green's functions up to one loop, for the two- and four-point cases. We compare the two theories in the context of the noncommutative geometry determined by a Drinfeld twist, and the comparison is made at the level of Green's functions and S matrix. We find that while the Green's functions are different for the two theories, the S matrix is the same in both cases and is different from the commutative case.

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I. INTRODUCTION

It is likely that at short distances spacetime has to be described by different geometrical structures and that the very concept of point and localizability may no longer be adequate. This is one of the main motivations for the introduction of noncommutative geometry [1–3]. The simplest kind of noncommutative geometry is the so-called *canonical* one [4,5]. What is usually done for the construction of a field theory on a noncommutative space is to deform the product among functions (and hence among fields) with the introduction of a noncommutative $\star$ product, so that for the coordinate functions one has

$$[x^i, x^j]_\star \equiv x^i \star x^j - x^j \star x^i = \mathrm{i}\theta^{ij}. \tag{1.1}$$

In the simplest case θ^{ij} is constant; i.e. it does not depend on the x's. The choice of the $\star$ product compatible with (1.1) is not unique; in the following we will introduce two different products, the Moyal [6,7] and Wick-Voros [8–11], and compare their "physical predictions."

There are several reasons to consider field theories on a noncommutative space equipped with the standard canonical noncommutativity, ranging from intrinsic motivations to the localizability of events [4,5] to string theory [12] to constructive field theories [13]. Field theories on noncommutative spaces have interesting renormalization properties [14,15]. For a review see [16,17], their references and their citations. What we will compare are field theories in which the product among fields is substituted by the two different $\star$ products. This leads to an action in which arbitrary degree derivatives of the fields are present, as a series in θ. Written in terms of derivatives the two actions with the Moyal and Wick-Voros products are different. There is however a map which renders equivalent the algebras generated by the two products. Field theory with the Wick-Voros product has been discussed in [18] as a regularizing model, and their conclusion (that ultraviolet divergences persist) is in agreement with ours.

This paper originates from the consideration that one can reason in two ways: one point of view is to say that what counts is the noncommutative structure of spacetime, and the $\star$ product is just a way to express this intimate structure, and therefore one chooses the most convenient product. As long as one is describing the same field theory, the results should be the same, a fact noted already in [19]. Another view is to claim a total lack of interest in the noncommutativity of spacetime. What counts is the fact that one has a field theory on ordinary spacetime, whose action contains an infinity of derivatives of arbitrary order. With this second point of view one would not in principle expect the same physical results from the two theories. In this paper we calculated the Green's functions of the Wick-Voros field theory and found them to be different from the Moyal case. This leads to a contradiction. We will see that the contradiction is only apparent. Green's functions are not observable quantities; what is observable is the S matrix.

Discussions of the properties of the S matrix often go together with the issue of Poincaré invariance. Relation (1.1) is not Poincaré invariant, and this casts doubts on its being fundamental. It is however possible to build a theory which is invariant under a *deformation* of the Poincaré Lie algebra, so that the theory becomes a *twisted* theory. This theory has a symmetry described by a noncommutative, noncocommutative Hopf algebra. In particular the kind of noncommutativity described by the two $\star$ products is the one generated by a *twist* [20–22]. Then the theory has a twisted Poincaré symmetry [23–25].

The presence of a twist forces one to reconsider all of the steps in a field theory, which has to be built in a coherent "twisted" way. We will see that there is equivalence between the two theories at the very end, where by "very end" we mean the calculation of the S matrix. Prior to this, vertex, propagators and Green's functions are in fact differ-

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ent. Moreover the equivalence is only obtained if a consistent procedure of *twisting all products* is applied. In this way the Poincaré symmetry, which appears to be broken in (1.1), is preserved, albeit in a deformed way, as a noncommutative, noncocommutative Hopf algebra.

There is some ambiguity in the issue of twisting, and some results have been somewhat controversial [26–35]. In an optimal world one should let experiments resolve these ambiguities. Unfortunately the noncommutative structure of spacetime is not yet mature for a confrontation with experiments at such a level. What we do in this paper is to use the field theories built with the Wick-Voros and Moyal products to check each other. We will see that using a consistent twisting procedure we obtain that at the level of the S matrix the two theories are equivalent. This gives us the indication on the procedure to follow for noncommutative theories coming from a twist.

In this paper we will consider exclusively spatial noncommutativity; i.e. time is a commuting variable. The matrix θ therefore is of the form

$$\theta^{ij} = \theta\varepsilon^{ij}, \tag{1.2}$$

with ε the antisymmetric tensor of order two.

The paper is organized as follows. In Sec. II we introduce the two products. In Sec. III we discuss the classical free field theory for the Wick-Voros product. In Sec. IV we calculate the Green's functions for the two theories for the two- and four-point case to one loop and compare the two cases. In Sec. V we describe the two products as twisted noncommutative geometries. In Sec. VI we describe the relevant twisted products which we then use in Sec. VII to calculate the S matrix. A final short section contains the conclusions.

II. THE WICK-VOROS AND MOYAL PRODUCTS

In this section we describe in a comparative way the two $\star$ products we are using in this paper. The most well known product is the Moyal product [6,7]

$$f(\vec{x}) \star_M g(\vec{x}) = f(\vec{x}) e^{(i/2)\theta^{ij}\overleftarrow{\partial}_i\vec{\partial}_j} g(\vec{x}), \tag{2.1}$$

where the operator $\overleftarrow{\partial}_i$ (respectivly, $\vec{\partial}_j$) acts on the left (respectively, the right). This product comes from a Weyl map which associates to a function on the plane an operator according to

$$\hat{\Omega}_M(f) = \frac{1}{2\pi}\int d^2\eta \tilde{f}(\eta_1, \eta_2) e^{i\theta_{ij}\hat{X}^i\eta^j}, \tag{2.2}$$

where $\tilde{f}$ is the symplectic Fourier transform of the function f:

$$\tilde{f} = \frac{1}{\theta\pi}\int d^2x \tilde{f}(x^1, x^2) e^{-i\theta_{ij}x^i\eta^j}, \tag{2.3}$$

θ_{ij} is the inverse of θ^{ij}, and the $\hat{X}$ are operators which satisfy the commutation relation

$$[\hat{X}^i, \hat{X}^j] = i\theta^{ij}. \tag{2.4}$$

It is useful to think of the operators $\hat{X}$ in two dimensions in an abstract way, without reference to spacetime, and define them as

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^\dagger}{\sqrt{2}}, \qquad \hat{X}_2 = \frac{\hat{a} - \hat{a}^\dagger}{i\sqrt{2}}, \tag{2.5}$$

and $\hat{a}$ is an operator which we define on a certain basis by its components as

$$\hat{a}_{nm} = \sqrt{\theta n}\delta_{m,n+1}, \tag{2.6}$$

with $m, n \geq 0$. Of course we are using the analogy of the commutation relations (2.4) with the usual quantum mechanical commutation relations and using the $|n\rangle$'s as a convenient basis.

The Moyal product is then defined as

$$f \star_M g = \Omega_M^{-1}(\hat{\Omega}_M(f)\hat{\Omega}_M(g)). \tag{2.7}$$

From this expression it is not difficult (see for example [36]) to obtain integral expressions for the product, a few of which are collected in the appendix of [37]. The standard expression is then an asymptotic expansion of the integral expressions [37].

One important property of the Moyal product is that

$$\int d^2x f \star_M g = \int d^2x f g \tag{2.8}$$

and obviously

$$x^1 \star_M x^2 - x^2 \star_M x^1 = i\theta. \tag{2.9}$$

We now proceed to the definition of the Wick-Voros product. For the following it is useful to consider the space as a complex plane defining:

$$z_\pm = \frac{x^1 \pm ix^2}{\sqrt{2}}, \tag{2.10}$$

where of course $z_+^* = z_-$. With this substitution we define the Wick-Voros product as

$$f \star_V g = \sum_n \left(\frac{\theta^n}{n!}\right)\partial_+^n f \partial_-^n g = f e^{\theta\overleftarrow{\partial}_+\vec{\partial}_-} g, \tag{2.11}$$

where

$$\partial_\pm = \frac{\partial}{\partial z_\pm} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x^1} \mp i\frac{\partial}{\partial x^2}\right). \tag{2.12}$$

Notice that the Moyal product (2.7) may be rewritten in these coordinates as

$$f \star_M g = f e^{(\theta/2)(\overleftarrow{\partial}_+\vec{\partial}_- - \overleftarrow{\partial}_-\vec{\partial}_+)} g. \tag{2.13}$$

It results

$$z_+ \star_V z_- = z_+z_- + \theta, \qquad z_- \star_V z_+ = z_+z_-, \tag{2.14}$$

and therefore

$$[z_+, z_-]_{\star_V} = \theta. \tag{2.15}$$

Going back to the x's, it is possible to see that this relation gives rise again to the standard commutator among the x's:

$$x^1 \star_V x^2 - x^2 \star_V x^1 = \mathrm{i}\theta. \tag{2.16}$$

With the $z_\pm$ coordinates the Laplacian and the d'Alembertian are, respectively, $\nabla^2 = 2\partial_+\partial_-$ and $\Box = \partial_0^2 - \nabla^2$. The integral on the plane is still a trace, but the strong condition of (2.8) is not valid any more:

$$\int d^2 z f \star_V g = \int d^2 z g \star_V f \neq \int d^2 z f g, \tag{2.17}$$

where by d^2z we mean the usual measure on the plane $dz_+ dz_-$. We will also use the notation

$$k_\pm = \frac{k_1 \pm \mathrm{i}k_2}{\sqrt{2}} \tag{2.18}$$

for a generic vector $\vec{k}$.

The Wick-Voros and Moyal products can be cast in the same general framework in that they are both coming from a generalized "Weyl map." More precisely, as we saw in (2.2) the Moyal product comes from a map which associates operators to functions, with symmetric ordering. The Wick-Voros product comes from a similar map, a *weighted* Weyl map as follows:

$$\hat{\Omega}_V(f) = \frac{1}{2\pi}\int d^2\eta \tilde{f}(\eta, \bar{\eta}) \mathrm{e}^{\theta\eta a^\dagger} \mathrm{e}^{-\theta\bar{\eta}a}. \tag{2.19}$$

An equivalent way to associate the operators $\hat{\Omega}_V(f)$ to a function $f = \sum_{mn} f_{mn} z_+^m z_-^n$ analytic on the plane is

$$\hat{\Omega}_V(f) = \sum_{mn} f_{mn} \hat{a}^{\dagger m} \hat{a}^n, \tag{2.20}$$

where $\hat{a}$ has been defined in (2.6). Thus effectively the map (2.20) corresponds to the normal (or Wick) ordering (and is sometimes called normal ordered product). In this sense the two maps correspond to two different quantization procedures (see for example [38,39]).

III. CLASSICAL FREE FIELD THEORY

Although the main interest of this paper is in the interacting quantum field theory, we start the discussion from the classical free case. In this section we discuss the kind of field theory one obtains from a deformation of the free Klein-Gordon action based on the Wick-Voros product. Such analysis is unnecessary in the Moyal case, because in that case the action, being quadratic in the fields, is the same as in the commutative case.

We consider a field theory described by an action which is a Wick-Voros deformation of a scalar field theory action, obtained inserting the star product. Consider a classical free theory and its action, Lagrangian density and Lagrangian defined as:

$$S_0 = \int dt L_0 = \int dt d^2 z \mathcal{L}_0$$
$$= \int dt d^2 z \frac{1}{2}(\partial_\mu \varphi \star_V \partial_\mu \varphi - m^2 \varphi \star_V \varphi). \tag{3.1}$$

With the help of (2.11) it may be rewritten as

$$S_0 = \int dt d^2 z \frac{1}{2}(\partial_\mu \varphi \mathrm{e}^{\theta \overleftarrow{\partial}_+ \overrightarrow{\partial}_-} \partial_\mu \varphi - m^2 \varphi \mathrm{e}^{\theta \overleftarrow{\partial}_+ \overrightarrow{\partial}_-} \varphi)$$
$$= \int dt d^2 z \frac{1}{2}\varphi[\mathrm{e}^{-(\theta/2)\nabla^2}(-\partial_\mu^2 - m^2)]\varphi. \tag{3.2}$$

This is a theory which contains an infinite number of the derivatives of the fields, and in principle even the Cauchy problem would not be well defined. This appears to be the biggest difference with respect to the noncommutative field theory defined via the Moyal product. In the latter case the action being the same as in the commutative case, the solution of the free theory is still given by plane waves, and upon quantization the propagator is the same as in the commutative case. This time instead the action is different, the theory is nonlocal as it contains derivatives of arbitrary order.

The two products are equivalent in a precise technical sense: there is an invertible map [40,41]

$$T(f) = \sum_n \theta^n t_n(f) \tag{3.3}$$

with the t_n differential operators, such that

$$T(f \star_M g) = T(f) \star_V T(g), \tag{3.4}$$

where

$$T = \mathrm{e}^{(\theta/4)\nabla^2}. \tag{3.5}$$

Therefore the two products define the same deformed algebra. This is certainly true if we consider functions as formal power series in the generators. The issue can be more complicated in the realm of C^* algebras. Starting from the same set of functions the completion in the supremum norm of the two products could in principle be different.

The fact that the algebra is the same does not mean that the two deformations of an action are the same. Therefore let us map the free action S_0 (3.1) written with the Wick-Voros product to the corresponding action with the Moyal product, using (3.5), to find which Moyal theory corresponds to it.

The action (3.1) is mapped into

$$\begin{aligned} S_0' &= \int dtd^2z T^{-1}(\mathcal{L}_0) \\ &= \int dtd^2z \frac{1}{2}((\mathrm{e}^{-(\theta/4)\nabla^2}\partial_\mu\varphi)\star_M(\mathrm{e}^{-(\theta/4)\nabla^2}\partial_\mu\varphi) \\ &\quad - m^2(\mathrm{e}^{-(\theta/4)\nabla^2}\varphi)\star_M(\mathrm{e}^{-(\theta/4)\nabla^2}\varphi)) \\ &= \int dtd^2z \frac{1}{2}((\mathrm{e}^{-(\theta/4)\nabla^2}\partial_\mu\varphi)(\mathrm{e}^{-(\theta/4)\nabla^2}\partial_\mu\varphi) \\ &\quad - m^2(\mathrm{e}^{-(\theta/4)\nabla^2}\varphi)(\mathrm{e}^{-(\theta/4)\nabla^2}\varphi)) \\ &= \int dtd^2z \frac{1}{2}(\partial_\mu\varphi \mathrm{e}^{-(\theta/2)\nabla^2}\partial_\mu\varphi - m^2\varphi\mathrm{e}^{-(\theta/2)\nabla^2}\varphi), \end{aligned} \tag{3.6}$$

which is not the free action with the Moyal product. In fact in the latter case the noncommutative product could be eliminated from the integral leaving just the free commutative action. Therefore, the two actions being different, they could in principle give different equations of motion.

Since we are dealing with a theory involving an infinite number of derivatives we can ask whether we would need an infinity of boundary conditions to solve the classical theory. This is not so, as the higher derivatives appear as analytic functions of the Laplacian, and in this case the boundary problem is the same as in the standard case. Note that with our choice of $\theta^{\mu\nu}$ our equation is second order in the time derivatives, so that the initial value problem requires knowledge of the field and its derivative as an initial condition. But also if we had deformed the time derivatives, the initial data for the Cauchy problem would have been the same if the higher derivative had been an analytic function of the d'Alembertian. For more details and references see the recent paper [42].

Let us derive the classical equations of motion starting from the variation of the action. Since the standard techniques have been developed for a theory with a finite number of derivatives, we will proceed from first principles and start from the infinitesimal variation of the field:

$$\varphi \rightarrow \varphi + \delta\varphi. \tag{3.7}$$

It is not difficult to see that the corresponding infinitesimal variation of the action under such a transformation is given by

$$\begin{aligned} \delta S_0 = \int dtd^2z((\partial_0\varphi)\star_V(\partial_0\delta\varphi) - (\partial_+\varphi)\star_V(\partial_-\delta\varphi) \\ - (\partial_+\delta\varphi)\star_V(\partial_-\varphi) - m^2\varphi\star_V\delta\varphi), \end{aligned} \tag{3.8}$$

where we have used the trace property of the integrals with Wick-Voros products. By integrating by parts and using once again the trace property we obtain, up to boundary terms

$$\delta S_0 = -\int dtd^2z(\delta\varphi)\star_V(\partial_0\partial_0\varphi - 2\partial_+\partial_-\varphi + m^2\varphi), \tag{3.9}$$

namely,

$$\begin{aligned} \delta S_0 = -\int dtd^2z \sum_{n=0}^{\infty}\frac{\theta^n}{n!}\partial_+^n(\delta\varphi) \\ \times \partial_-^n(\partial_0\partial_0\varphi - 2\partial_+\partial_-\varphi + m^2\varphi). \end{aligned} \tag{3.10}$$

By integrating once again by parts we obtain

$$\begin{aligned} \delta S_0 = -\int dtd^2z \sum_{n=0}^{\infty}\frac{(-\theta)^n}{n!}(\delta\varphi)\partial_+^n \\ \times \partial_-^n(\partial_0\partial_0\varphi - 2\partial_+\partial_-\varphi + m^2\varphi), \end{aligned} \tag{3.11}$$

that is,

$$\delta S_0 = -\int dtd^2z(\delta\varphi)\mathrm{e}^{-\theta\partial_+\partial_-}(\partial_0\partial_0 - 2\partial_+\partial_- + m^2)\varphi. \tag{3.12}$$

Since the variation of the action δS must be vanishing for any variation of the field $\delta\varphi$, we obtain that the equation of motion is given by

$$\mathrm{e}^{-\theta\partial_+\partial_-}(\partial_0\partial_0 - 2\partial_+\partial_- + m^2)\varphi = 0. \tag{3.13}$$

Equivalently, it can be written as

$$\mathrm{e}^{-(\theta/2)\nabla^2}(\Box + m^2)\varphi = 0. \tag{3.14}$$

As we can see, the equation of motion (3.14) differs from the classical Klein-Gordon equation

$$(\Box + m^2)\varphi = 0 \tag{3.15}$$

only by the presence of the exponential of the Laplacian, an invertible operator. It is immediate to see that all solutions of the commutative theory are still solutions of the noncommutative one. It is in principle possible that there can be solutions of the noncommutative equation (3.14) which are not solutions of the commutative one. This is not the case, due to the invertibility of the operator $\mathrm{e}^{-(\theta/2)\nabla^2}$.

Notice that the on shell condition is not altered by the presence of the deformation factor. In other words the dispersion relation is the same in the two cases. In fact in Fourier transform

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3} e^{\mathrm{i}k\cdot x}\tilde{\varphi}(k), \tag{3.16}$$

and then Eq. (3.14) becomes

$$\begin{aligned} \mathrm{e}^{-(\theta/2)\nabla^2}(\Box + m^2)\int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{\mathrm{i}k\cdot x}\tilde{\varphi}(k) \\ = \int\frac{d^3k}{(2\pi)^3}\mathrm{e}^{(\theta/2)k^2}(-k^2 + m^2)\mathrm{e}^{\mathrm{i}k\cdot x}\tilde{\varphi}(k) = 0. \end{aligned} \tag{3.17}$$

The relation

$$e^{(\theta/2)k^2}(k^2 - m^2)\tilde{\varphi}(k) = 0 \tag{3.18}$$

gives the same on shell relation of the classical case since the exponential never vanishes.

Classically therefore the two theories have the same solutions of the equations of the motion, despite the fact that the action, the Lagrangian and the equations of motion are different.

IV. GREEN'S FUNCTIONS

Let us consider a field theory described by the action:

$$S = S_0 + \frac{g}{4!}\int dt d^2z \varphi \star \varphi \star \varphi \star \varphi, \tag{4.1}$$

where $\star$ is either $\star_M$ or $\star_V$. In the following we will use a generic $\star$ for all relations and formulas valid for both products. We now calculate the Feynman rules for these field theories.

Because of property (2.8) the free theory is unchanged for the Moyal case. Therefore the Moyal propagator is the same as in the undeformed case. In the Wick-Voros case [18] there are differences.

To this purpose let us rewrite the action S_0 in Eq. (3.2) in the form

$$S_0 = \int dt d^2z dt' d^2z' \varphi(t, z) K(t, t', z, z') \varphi(t', z'), \tag{4.2}$$

with

$$K(t, t', z, z') = e^{-(\theta/2)\nabla^2}(-\partial_\mu^2 - m^2)\delta(t - t')\delta^2(z - z'). \tag{4.3}$$

The quantum propagator $\Delta_{\star_V}(x, y)$ is the two-point Green's function of the free theory, that is, the inverse of the operator K,

$$\Delta_{\star_V}(x_a, x_b) = \int \frac{d^3p}{(2\pi)^3} e^{ip\cdot(x_a - x_b)} \frac{e^{-(\theta/2)|\vec{p}|^2}}{p^2 - m^2}. \tag{4.4}$$

We can read off the propagator in momentum space and compare it with the one in the Moyal (and undeformed) case

$$G^{(2)}_{0_M}(p) = \frac{1}{p^2 - m^2}, \qquad G^{(2)}_{0_V}(p) = \frac{e^{-(\theta/2)|\vec{p}|^2}}{p^2 - m^2}. \tag{4.5}$$

Since the poles in the propagator in momentum space are the same as in the commutative theory, despite the change in the propagator, the free field theory with the Wick-Voros product is the same as in the commutative (and Moyal noncommutative) case. This is the quantum counterpart of the previous result that the solutions of the classical equations of the motion are the same. Nevertheless the two propagators are not identical, and we will have to take this into account in the following. Note however that for infinite momentum there is an essential singularity, or a zero, of the propagator, according to the sign of θ. The meaning of the essential singularity is not clear, but the oddity is that the sign of θ has no physical meaning since it can be changed by an exchange of the sign of one of the two coordinates, in a theory which appears to be parity invariant. We will see later that, with the proper twisting of the theory, also this paradox is solved.

We now proceed to the calculation of the interaction vertex in the Wick-Voros case, comparing it with the theory obtained with the Moyal product. In this latter case the difference with respect to the commutative case resides in the fact that the vertex acquires a phase [43]. In order to see the corrections let us write down the Moyal product as a convolution twist in momentum space[1]:

$$(f \star_M g)(x) = \int \frac{d^3k}{(2\pi)^3} \times \frac{d^3k'}{(2\pi)^3} \tilde{f}(k)\tilde{g}(k') e^{i(k+k')\cdot x} e^{-(i/2)\theta \vec{k}\wedge\vec{k}'}, \tag{4.6}$$

where $\tilde{f}$ and $\tilde{g}$ are the Fourier transforms of f and g and

$$\vec{k} \wedge \vec{k}' = \varepsilon^{ij} k_i k'_j. \tag{4.7}$$

We see that in momentum space the Moyal product is the standard convolution of Fourier transforms, twisted by a phase. For the Wick-Voros product, defining $k_\pm = (k_1 \pm ik_-)/\sqrt{2}$ in a way analogous to (2.10) we have

$$(f \star_V g)(z_+, z_-, t) = \int \frac{d^3k}{(2\pi)^3} \times \frac{d^3k'}{(2\pi)^3} \tilde{f}(k)\tilde{g}(k') e^{i(k+k')\cdot x} e^{-\theta k_- k'_+}. \tag{4.8}$$

Explicitly the exponent of the twist in the convolution can be expressed as

$$k_- k'_+ = \tfrac{1}{2}(\vec{k}\cdot\vec{k}' + i\vec{k}\wedge\vec{k}') \tag{4.9}$$

with the same imaginary part as in the Moyal case (4.6) plus a real part.

For a φ^4 theory in ordinary space the four-point vertex in momentum space is the coupling constant multiplying the δ of momentum conservation:

$$V = -i\frac{g}{4!}(2\pi)^3\delta^3\left(\sum_{a=1}^4 k_a\right). \tag{4.10}$$

In the Moyal case we have that the vertex acquires a phase factor due to the twist in the product (4.6):

$$V_{\star_M} = V\prod_{a<b} e^{-(i/2)\theta^{ij}k_{ai}k_{bj}}. \tag{4.11}$$

[1]Some of the formulas of this section are specific to our 2 + 1 case, but the results are more general.

The presence of the phase in the vertex makes it noninvariant for a generic exchange of the momenta. This is a consequence of noncommutativity and of the fact that the integral of Moyal product of more than two functions is not invariant for an exchange of the functions. Invariance for a cyclic rotation of the factors still survives. This gives rise to a difference between planar and nonplanar graphs and ultimately to the well known phenomenon of infrared-ultraviolet mixing [44].

In the Wick-Voros case the correction, due to (4.9), is not just a phase, but it contains a real exponent:

$$V_{\star_V} = V\prod_{a<b} \mathrm{e}^{-\theta k_{a-}k_{b+}} = V\prod_{a<b}\mathrm{e}^{-(\theta/2)(\vec{k}_a\cdot\vec{k}_b + i\vec{k}_a\wedge\vec{k}_b)}. \quad (4.12)$$

The exponent can have both signs, and in some case it could diverge exponentially for large external momenta. The divergence is however compensated by the fact that, to the four-point function, there must be added the contribution of the four propagators, each of which comes with an exponentially convergent part. These convergent parts compensate the possibly divergent contributions of the vertex for positive θ.

We can write the vertices with an unified notation as

$$V_\star = V\prod_{a<b}\mathrm{e}^{k_a\bullet k_b}, \quad (4.13)$$

where

$$k_a \bullet k_b = \begin{cases} -\frac{i}{2}\theta^{ij}k_{ai}k_{bj} & \text{Moyal,} \\ -\theta k_{a-}k_{b+} & \text{Wick-Voros.}\end{cases} \quad (4.14)$$

To calculate the four-point Green's function in the Wick-Voros case at the tree level we must attach to the vertex four propagators (4.5), each carrying an exponential. The four-point Green's function therefore is

$$G^{(4)}_{0_V} = -\mathrm{i}g(2\pi)^3 \frac{\mathrm{e}^{-\theta(\sum_{a=1}^4 k_{a-}k_{a+} + \sum_{a<b} k_{a-}k_{b+})}}{\prod_{a=1}^4 (k_a^2 - m^2)}\delta^{(3)}\Big(\sum_{a=1}^4 k_a\Big). \quad (4.15)$$

With some simple algebraic passages we can express the exponent as

$$\sum_{a=1}^4 k_{a-}k_{a+} + \sum_{a<b} k_{a-}k_{b+} = \frac{1}{4}\Big(|\vec{k}_1|^2 + |\vec{k}_2|^2 + |\vec{k}_3|^2 + |\vec{k}_4|^2 + 2\mathrm{i}\sum_{a<b}\vec{k}_a\wedge\vec{k}_b + |\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4|^2\Big). \quad (4.16)$$

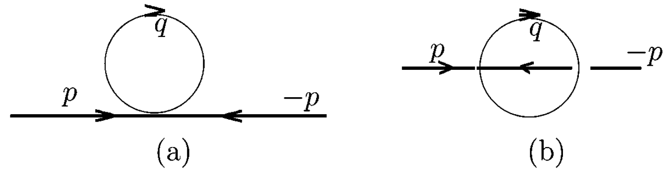

FIG. 1. The (a) planar and (b) nonplanar one-loop correction to the propagator.

The δ of conservation of momentum effectively kills the last term, so that the four-point function, at tree level, is

$$G^{(4)}_{0_V} = -\mathrm{i}g(2\pi)^3 \times \frac{\mathrm{e}^{-(\theta/4)\sum_{a=1}^4|\vec{k}_a|^2}\prod_{a<b}\mathrm{e}^{-(\mathrm{i}/2)\theta^{ij}k_{ai}k_{bj}}}{\prod_{a=1}^4(k_a^2-m^2)}\delta^{(3)}\Big(\sum_{a=1}^4 k_a\Big). \quad (4.17)$$

Noticing that in the Moyal case, because of antisymmetry, it results $p\bullet p = 0$, we can express in the unified notation the Green's functions as

$$G^{(4)}_0 = -\mathrm{i}g(2\pi)^3\frac{\mathrm{e}^{\sum_{a\le b}k_a\bullet k_b}}{\prod_{a=1}^4(k_a^2-m^2)}\delta\Big(\sum_{a=1}^4 k_a\Big). \quad (4.18)$$

The presence of a real exponent for the Wick-Voros case could signify that the ultraviolet behavior of the theory could be different from the Moyal (and the commutative) case. Hence we calculate the one-loop correction to the propagator and verify the ultraviolet behavior of the theory under renormalization. The presence of the phase in the four-point function in the complete vertex (4.17) makes it noninvariant for a generic permutation of the external momenta, and this in turn implies that the planar and nonplanar cases are to be treated differently; this is what happens in the Moyal case as well. Consider first the planar case in Fig. 1(a). The amplitude is obtained using three propagators (4.5), two with momentum p, one with momentum q, and the vertex (4.12) with assignments $k_1 = -k_4 = p$ and $k_2 = -k_3 = q$, and of course the integration in q and the proper symmetry factor:

$$G^{(2)}_{\mathrm{P}} = -\mathrm{i}\frac{g}{3}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{-\theta(2p_-p_++q_-q_+)}\mathrm{e}^{-\theta(p_-q_+-p_-q_+-p_-p_+-q_-q_+-q_-p_++q_-p_+)}}{(p^2-m^2)^2(q^2-m^2)} = -\mathrm{i}\frac{g}{3}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{-\theta p_-p_+}}{(p^2-m^2)^2(q^2-m^2)}, \quad (4.19)$$

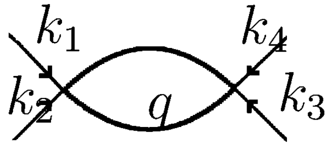

FIG. 2. The planar one-loop four-point diagram.

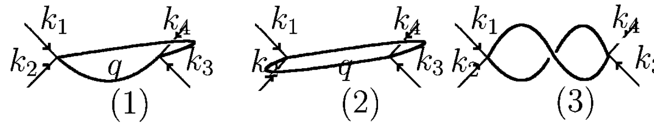

FIG. 3. The nonplanar one-loop four-point diagrams.

where the first exponential is due to the propagators and the second to the vertex. In this case all of the contributions due to q cancel, so that there is no change in the convergence of the integral.

We now proceed to the discussion of the nonplanar case, in Fig. 1(b). The structure is the same as in the planar case, but this time the assignments are instead $k_1 = -k_3 = p$ and $k_2 = -k_4 = q$, and we have

$$G^{(2)}_{\mathrm{NP}} = -\mathrm{i}\frac{g}{6}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{-\theta(2p_-p_+ + q_-q_+)}\mathrm{e}^{-\theta(p_-q_+ - p_-p_+ - p_-q_+ - q_-p_+ - q_-q_+ + p_-q_+)}}{(p^2-m^2)^2(q^2-m^2)} = -\mathrm{i}\frac{g}{6}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{-\theta(p_-p_+ + \mathrm{i}\vec{p}\wedge\vec{q})}}{(p^2-m^2)^2(q^2-m^2)}. \tag{4.20}$$

This time the q contribution does not cancel completely, and there remains the factor

$$p_-q_+ - q_-p_+ = \mathrm{i}\vec{p}\wedge\vec{q} \tag{4.21}$$

so that the phase factor of the Moyal case is reproduced. We can express in the unified notation:

$$\begin{aligned} G^{(2)}_{\mathrm{P}} &= -\mathrm{i}\frac{g}{3}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{p\bullet p}}{(p^2-m^2)^2(q^2-m^2)}, \\ G^{(2)}_{\mathrm{NP}} &= -\mathrm{i}\frac{g}{6}\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{p\bullet p + p\bullet q - q\bullet p}}{(p^2-m^2)^2(q^2-m^2)}. \end{aligned} \tag{4.22}$$

The ultraviolet divergence of the diagram is unchanged, with respect to the commutative case, for the planar diagram. In the nonplanar case there is the presence of the oscillating factor $\mathrm{i}\vec{p}\wedge\vec{q}$ in the exponential. This factor softens the ultraviolet divergence, since it dampens the functions for high q, but is responsible for infrared divergences. We can conclude at this level that, while the Green's functions are different, between the Moyal and Wick-Voros case, their ultraviolet behavior is the same as far as the momentum in the loop is concerned.[2] This indicates that the noncommutative geometry, at the ultraviolet level, is basically described by the uncertainty principle, a consequence of the commutator (1.1), which is unchanged between Wick-Voros and Moyal cases. Nevertheless the two Green's functions are not the same because of the $p \bullet p$ term which vanishes in the Moyal case but not in the Wick-Voros one.

We now proceed to the one-loop Green's functions corresponding to the planar case of Fig. 2. In the NC case the Green's function correspondent to it can easily be calculated by properly joining two vertices. It turns out that we have for the two cases

$$G^{(4)}_{\mathrm{P}} = \frac{(-\mathrm{i}g)^2}{8}(2\pi)^3\int\frac{d^3q}{(2\pi)^3}\frac{\mathrm{e}^{\sum_{a\le b}k_a\bullet k_b}\delta(\sum_{a=1}^4 k_a)}{(q^2-m^2)((k_1+k_2-q)^2-m^2)\prod_{a=1}^4(k_a^2-m^2)}. \tag{4.23}$$

The exponent in the numerator can be rewritten as in (4.17), and we see that the internal momentum q appears only in the denominator, so that also in this case the planar diagram has the same renormalization property of the undeformed theory. In the Moyal case the real part exponent of the numerator is not present. In the Wick-Voros case instead there is the same correction encountered at tree level.

The three nonplanar cases are shown in Fig. 3. The calculation of their contribution is straightforward and gives, in momentum space,

$$G^{(4)}_{\mathrm{P}} = \frac{(-\mathrm{i}g)^2}{8}\int d^3q\frac{\mathrm{e}^{\sum_{a\le b}k_a\bullet k_b}\delta(\sum_{a=1}^4 k_a)}{(q^2-m^2)((k_1+k_2-q)^2-m^2)\prod_{a=1}^4(k_a^2-m^2)}, \tag{4.24}$$

[2]The convergence properties can however be changed by going to a different noncommutative space, such as a torus [18]

$$G^{(4)}_{\mathrm{NP_a}} = \frac{(-\mathrm{i}g)^2}{8} \int d^3q \frac{\mathrm{e}^{\sum_{a\leq b} k_a \bullet k_b + E_a} \delta(\sum_{a=1}^4 k_a)}{(q^2 - m^2)((k_1 + k_2 - q)^2 - m^2) \prod_{a=1}^4 (k_a^2 - m^2)}, \tag{4.25}$$

with

$$\begin{aligned} E_1 &= q \bullet k_1 - k_1 \bullet q = \mathrm{i}\vec{q} \wedge \vec{k}_1, \\ E_2 &= k_2 \bullet q - q \bullet k_2 + k_3 \bullet q - q \bullet k_3 \\ &= \mathrm{i}(\vec{k}_2 \wedge \vec{q} + \vec{k}_3 \wedge \vec{q}), \\ E_3 &= k_1 \bullet q - q \bullet k_1 + k_2 \bullet q - q \bullet k_2 \\ &= \mathrm{i}(\vec{k}_1 \wedge \vec{q} + \vec{k}_2 \wedge \vec{q}). \end{aligned} \tag{4.26}$$

Contrary to our expectations we find that the Green's functions are different. The Green's functions are not however directly measurable quantities, and the S matrix is. We will calculate it in the twist-deformed framework in Sec. VII.

V. THE WICK-VOROS AND MOYAL PRODUCTS AS TWISTED NONCOMMUTATIVE GEOMETRIES

The main physical motivation to study field theory equipped with a $\star$ product is the belief that, at very short distances, the geometry of spacetime is deformed, with the deformation dictated by a small parameter, θ in our case. In the presence of noncommutativity the concept of point is not well defined, and in fact the proper mathematical formalism should use the theory of C^* algebras and the language of spectral triples (see for example [1–3]). A star product deforms the commutative algebra of functions on a space into a noncommutative algebra. The proper formal definition of the mathematical objects involved in the definition is beyond the scope of this article. For us it suffices to know that the plane equipped Moyal product can be made into a spectral triple [45,46].

As we have discussed in Sec. III the two products come from a different quantization of the same Poisson structure, which classifies $\star$ products up to equivalences [47]. They can also be seen as gauge equivalent for the (infinite rank) group of gauge transformations given by field redefinition of the kind (3.3). Note however that the action is not invariant under the action of this gauge group.

With the introduction of a different, but equivalent, product one can heuristically reason as follows. The presence of the noncommutativity described by (1.1) gives the noncommutative structure of space, regardless of the realization of the product one uses. In fact one could avoid the use of a $\star$ product altogether, by considering the fields to be infinite matrices function of the operators X defined in (2.5) and solving, for example, with a path integration, this matrix model. We tested this conjecture for a bosonic quantum field theory with a φ^4 interaction and found that the two deformations of the action give different Green's functions. Interestingly however the ultraviolet structure of the two theories remains the same. We are nevertheless in front of a puzzle.

The element that we need to consider to solve this puzzle is symmetries. The commutation relation (1.1) breaks Poincaré symmetry, and this is not a desirable feature for a fundamental theory. The symmetry can be reinstated however at a deformed level, considering the fact that both products can be seen as coming from a Drinfeld twist [20,21]. The noncommutative geometry described by either $\star$ product is therefore a *twisted* noncommutativity.

Given the Poincaré Lie algebra Ξ and its universal enveloping algebra $U\Xi$, the twist $\mathcal{F}$ which we will consider is an element of $U\Xi \otimes U\Xi$. For the Moyal and Wick-Voros case it is, respectively,

$$\mathcal{F}_M = \exp\left[-\mathrm{i}\frac{\theta^{ij}}{2}\partial_i \otimes \partial_j\right], \tag{5.1}$$

$$\mathcal{F}_V = \exp[-\theta\partial_+ \otimes \partial_-], \tag{5.2}$$

where partial derivatives stand for translation generators and have to be appropriately realized when acting on a given space. Following [25,48–50] we will consider the following point of view: *the noncommutative geometry is a consequence of a twist of all products of the theory.* Then every bilinear map μ defined as

$$\mu\colon X \otimes Y \to Z \tag{5.3}$$

(where X, Y, and Z are vector spaces) is consistently deformed by composing it with the appropriate realization of the twist $\mathcal{F}$. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ:

$$\mu_\star := \mu \circ \mathcal{F}^{-1}. \tag{5.4}$$

The $\star$ product on the space of functions is recovered setting $X = Y = Z = \mathrm{Fun}(M)$. That is, if we indicate with m_0 the usual pointwise product between functions[3]:

$$m_0\colon \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \to \mathrm{Fun}(M), \qquad m_0(f \otimes g) = f \cdot g, \tag{5.5}$$

the noncommutative product can be seen as the composition of m_0 with the twist:

$$\mathcal{F}\colon \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \to \mathrm{Fun}(M) \otimes \mathrm{Fun}(M) \tag{5.6}$$

so that

[3]At this level we need not specify which kind of algebra of functions we are considering. The algebra of formal series of the generators is adequate, but more restricted algebras such as Schwarzian functions can also be considered.

$$f \star_M g = m_0 \circ \mathcal{F}_{\star_M}^{-1}(f \otimes g) f \star_V g = m_0 \circ \mathcal{F}_{\star_V}^{-1}(f \otimes g). \tag{5.7}$$

Associativity of the product is ensured by normalization and cocycle conditions (see [48,49] for a short introduction; see also the book [51]). We also introduce the universal $\mathcal{R}$ matrix which represents the permutation group in noncommutative space

$$\mathcal{R} := \mathcal{F}_{21}\mathcal{F}^{-1}, \tag{5.8}$$

with

$$\mathcal{F}_{21}(a \otimes b) = \tau \circ \mathcal{F} \circ \tau(a \otimes b) \tag{5.9}$$

and τ the usual exchange operator

$$\tau(a \otimes b) = b \otimes a. \tag{5.10}$$

For the cases at hand with the two twists given by (5.1) and (5.2), it results:

$$\mathcal{R}_{\star_V} = \mathcal{R}_{\star_M} = \mathcal{F}_{\star_M}^{-2}; \tag{5.11}$$

that is, the exchange operator, and therefore the statistics, are the same in the two cases.

The presence of the twist deforms the structure of the universal enveloping algebra of the Poincaré Lie algebra, rendering it a noncocommutative Hopf algebra. The analysis of [23–25], made for the Moyal case, can be repeated in the Wick-Voros case with similar conclusions. Therefore the representations of the undeformed Poincaré algebra can still be used. We will see later in the paper the important role that the twisted Poincaré symmetry will play in the equivalence between Moyal and Wick-Voros theories.

VI. TWIST-DEFORMED PRODUCTS

We have now the necessary ingredients to calculate a physical process, like the S matrix for the elastic scattering of two particles. We recall that one of the crucial ingredients in the importance of the S matrix in physics is the issue of Lorentz and Poincaré invariance. If we naively insert the Green's functions of Sec. IV into the calculation of the S matrix, we would find a dependence of it from the external momenta, something like a momentum dependence of the coupling constant. What is more relevant for our purposes, we find that the result would be different for the Moyal and the Wick-Voros case, in contradiction with the heuristic reasoning we made in the introduction. We would also find a breaking of Poincaré invariance.[4]

The reason for the breaking of Poincaré invariance is that the commutator (1.1) apparently breaks this invariance. However the invariance can be reinstated if one

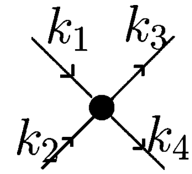

FIG. 4. The two-particle elastic scattering.

considers it to be a *quantum symmetry*; i.e. the Poincaré algebra is not a cocommutative Hope algebra, but it has a nontrivial coproduct [23,24].

Our purpose is to show, with an explicit calculation of scattering amplitudes, that the naive procedure which leads to a difference between the two cases can be corrected by a coherent twisting procedure. We will see that, if the twisted symmetry is properly implemented, the final, "physical" result will be the same in the Wick-Voros and Moyal cases, despite the presence of different propagators and vertices. We will consider the elastic scattering of two particles, as described in Fig. 4.

The first consequence of noncommutativity is the fact that, since the vertex is noninvariant for noncyclic exchange of the particles, we have to twist-symmetrize the incoming and outgoing states using the $\mathcal{R}$ matrix. Several aspects of this twist symmetrization and the consequences for spin and statistics have been discussed in [27,54,55]. In the commutative case the order of the propagators into the vertex is irrelevant, and therefore this discussion is redundant. Here there are several twists at work and we have to be careful in considering all of them.

Since we have to properly define multiparticle states as twisted tensor products, and accordingly modify the definition of their scalar product, for the remaining part of the section we will only deal with free fields solution of the Klein-Gordon equation, and free states. In the next section these will serve to define the asymptotic states.

Let us consider the two-particle state:

$$|k_a, k_b\rangle = |k_a\rangle \otimes |k_b\rangle. \tag{6.1}$$

Although for the comparison we are going to make later we will not actually use the fact that the state has to be symmetrized, we will discuss the symmetrization of the states for completeness. Consider the exchange operator

$$\tau|k_a\rangle \otimes |k_b\rangle = |k_b\rangle \otimes |k_a\rangle. \tag{6.2}$$

The symmetrized state, the eigenvector of the exchange operator τ with eigenvalue $+1$, is

$$|k_a, k_b\rangle_{\rm simm} = \frac{|k_a\rangle \otimes |k_b\rangle + |k_b\rangle \otimes |k_a\rangle}{2}, \tag{6.3}$$

and inserting the two expressions (6.1) or (6.3) does not make a difference in the calculation (of the connected diagrams) because of the invariance for exchange on the incoming momenta. The symmetries for identical particles change for the noncommutative case [27,54,55], we have

[4]We are considering θ^{ij} to be constant. Another solution which preserves Poincaré invariance is to have it a tensor [4,52] or to have it transform together with the product [53]. The residual rotational invariance is an artifact of the two-dimensionality of the model. In higher dimensions also this invariance is broken.

to take into account the fact that the tensor product is twisted, and moreover that the exchange is twisted as well. Therefore we define

$$|k_a, k_b\rangle_\star = \tilde{\mathcal{F}}^{-1}|k_a, k_b\rangle, \tag{6.4}$$

where by $\tilde{\mathcal{F}}$ we indicate the twist that this time acts in momentum space:

$$\begin{aligned}\tilde{\mathcal{F}}^{-1}_{\star_M}|k_a\rangle\otimes|k_b\rangle &= \mathrm{e}^{-(\mathrm{i}/2)\theta^{ij}k_{a_i}\otimes k_{b_j}}|k_a\rangle\otimes|k_b\rangle,\\ \tilde{\mathcal{F}}^{-1}_{\star_V}|k_a\rangle\otimes|k_b\rangle &= \mathrm{e}^{\theta k_{a-}\otimes k_{b+}}|k_a\rangle\otimes|k_b\rangle.\end{aligned} \tag{6.5}$$

This is not the only change we have to make to the state (6.3): the state has to be the eigenvalue of the twist-exchange, given by the $\mathcal{R}$ matrix acting in momentum space. The properly symmetrized state is therefore

$$\begin{aligned}|k_a, k_b\rangle_{\mathrm{simm}_\star} &= \tfrac{1}{2}(\tilde{\mathcal{F}}^{-1}|k_a\rangle\otimes|k_b\rangle + \tilde{\mathcal{F}}^{-1}\tilde{\mathcal{R}}^{-1}|k_a\rangle\otimes|k_b\rangle)\\ &= \tfrac{1}{2}(\tilde{\mathcal{F}}^{-1}|k_a\rangle\otimes|k_b\rangle + \tilde{\mathcal{F}}^{-1}\tilde{\mathcal{F}}\tilde{\mathcal{F}}_{21}^{-1}|k_a\rangle\otimes|k_b\rangle).\end{aligned} \tag{6.6}$$

We define as usual the momentum eigenstates as created by the creation operators a_k and $a_k^\dagger$:

$$|k\rangle = a_k^\dagger|0\rangle, \tag{6.7}$$

where a_k and $a_k^\dagger$ are obtained in terms of the free field

$$\varphi(x) = \int\frac{d^2k}{\sqrt{(2\pi)^2 2\omega_k}}(a_k\mathrm{e}^{-\mathrm{i}k\cdot x} + a_k^\dagger\mathrm{e}^{\mathrm{i}k\cdot x}) \tag{6.8}$$

to be

$$\begin{aligned}a_k &= \frac{\mathrm{i}}{\sqrt{(2\pi)^2 2\omega_k}}\int d^2x\mathrm{e}^{\mathrm{i}k\cdot x}\overleftrightarrow{\partial}_0\varphi_{\mathrm{in}}(x),\\ a_k^\dagger &= -\frac{\mathrm{i}}{\sqrt{(2\pi)^2 2\omega_k}}\int d^2x\mathrm{e}^{-\mathrm{i}k\cdot x}\overleftrightarrow{\partial}_0\varphi_{\mathrm{in}}(x).\end{aligned} \tag{6.9}$$

The operators a_k and $a_k^\dagger$ may be regarded, for fixed k, as functionals of the fields, and therefore their $\star$ product may be obtained as in [50]:

$$\begin{aligned}a(k)\star_M a(k') &= \mathrm{e}^{-(\mathrm{i}/2)\theta^{ij}k_ik'_j}a(k)a(k'),\\ a(k)\star_M a^\dagger(k') &= \mathrm{e}^{(\mathrm{i}/2)\theta^{ij}k_ik'_j}a(k)a^\dagger(k'),\\ a^\dagger(k)\star_M a(k') &= \mathrm{e}^{-(\mathrm{i}/2)\theta^{ij}k_ik'_j}a^\dagger(k)a(k'),\end{aligned} \tag{6.10}$$

$$\begin{aligned}a(k)\star_V a(k') &= \mathrm{e}^{-\theta k_-k'_+}a(k)a^\dagger(k'),\\ a(k)\star_V a^\dagger(k') &= \mathrm{e}^{\theta k_-k'_+}a(k)a^\dagger(k'),\\ a^\dagger(k)\star_V a(k') &= \mathrm{e}^{-\theta k_-k'_+}a^\dagger(k)a(k').\end{aligned} \tag{6.11}$$

Therefore we may reexpress Eqs. (6.5) and (6.6) as

$$|k_a, k_b\rangle_\star = a_{k_a}^\dagger\star a_{k_b}^\dagger|0\rangle \tag{6.12}$$

and

$$|k_a, k_b\rangle_{\mathrm{simm}_\star} = \frac{a_{k_a}^\dagger\star a_{k_b}^\dagger + a_{k_b}^\dagger\star a_{k_a}^\dagger}{2}|0\rangle. \tag{6.13}$$

The next step is the twist of the inner product. We consider it as a map from two copies of the Fock space of states into complex numbers. In the commutative case, for the momentum one-particle states we have

$$\langle\cdot|\cdot\rangle\colon |k\rangle\otimes|k'\rangle \to \langle k|k'\rangle = \langle 0|a_k a_{k'}^\dagger|0\rangle = \delta(k-k'). \tag{6.14}$$

We twist this product in the usual way composing it with the twist operator:

$$\begin{aligned}\langle\cdot\,|^\star\,\cdot\rangle\colon |k\rangle\otimes|k'\rangle &\to \langle\cdot|\cdot\rangle\circ\mathcal{F}^{-1}\colon |k\rangle\otimes|k'\rangle\\ &= \tilde{\mathcal{F}}^{-1}(k,k')\langle k|k'\rangle = \langle 0|a_k\star a_{k'}^\dagger|0\rangle,\end{aligned} \tag{6.15}$$

with $\tilde{\mathcal{F}}^{-1}(k,k')$ given by the exponential factor in Eqs. (6.5) for the Moyal and Wick-Voros case, respectively.

We are not yet finished twisting. Let us consider the inner product in the commutative case:

$$\langle k_1, k_2|k_3, k_4\rangle = \delta(k_1-k_3)\delta(k_2-k_4). \tag{6.16}$$

In the noncommutative case we have to twist the two-particle state according to (6.5), and then we have to twist the inner product according to the two-particle generalization of (6.15). In order to realize such a generalization we must consider the action of the twist on two-particle states. This is done, in canonical form, via the coproduct of the Hope algebra. Given a representation of an element of the Hope algebra on a space, the representation of the element on the product of states is given (in the undeformed case) by

$$\Delta_0(u)(f\otimes g) = (1\otimes u + u\otimes 1)(f\otimes g). \tag{6.17}$$

The coproduct is responsible, for example, for the Leibnitz rule. For the twisted Hope algebra the coproduct is deformed according to the fact that it is the $\mathcal{R}$ matrix which realizes the permutations:

$$\Delta_\star(u)(f\otimes g) = (1\otimes u + \mathcal{R}^{-1}(u\otimes 1))(f\otimes g). \tag{6.18}$$

However the twists we are considering are built out of translations, whose coproduct is undeformed:

$$\Delta_0(\partial_i) = \Delta_{\star_M}(\partial_i) = \Delta_{\star_V}(\partial_i) = 1\otimes\partial_i + \partial_i\otimes 1. \tag{6.19}$$

Since we are acting on two-particle states we need to define also

$$\begin{aligned}\Delta_0(\partial_i\otimes\partial_j) &= \Delta_\star(\partial_i\otimes\partial_j)\\ &= 1\otimes 1\otimes\partial_i\otimes\partial_j + \partial_i\otimes\partial_j\otimes 1\otimes 1.\end{aligned} \tag{6.20}$$

Therefore the twisted inner product among two-particle states

$$\langle k_1k_2\,|^\star\,k_3k_4\rangle = \langle\cdot|\cdot\rangle\circ\Delta_\star(\mathcal{F}^{-1})(|k_1k_2\rangle\otimes|k_3k_4\rangle) \tag{6.21}$$

may be easily computed to be

$$\langle k_1, k_2 |^{\star_M} k_3, k_4\rangle = e^{(i/2)\theta^{ij}(k_{1_i}+k_{2_i})(k_{3_j}+k_{4_j})}\langle k_1, k_2|k_3, k_4\rangle,$$
$$\langle k_1, k_2 |^{\star_V} k_3, k_4\rangle = e^{\theta(k_{1-}+k_{2-})(k_{3+}+k_{4-})}\langle k_1, k_2|k_3, k_4\rangle. \quad (6.22)$$

We can now calculate the *twisted* inner product of *twisted* states. Combining (6.22) with (6.5) we obtain the simple expression

$$_{\star_M}\langle k_1, k_2 |^{\star_M} k_3, k_4\rangle_{\star_M} = e^{(i/2)\theta^{ij}\sum_{a<b} k_{a_i}k_{b_j}}\langle k_1, k_2|k_3, k_4\rangle,$$
$$_{\star_V}\langle k_1, k_2 |^{\star_V} k_3, k_4\rangle_{\star_V} = e^{\theta\sum_{a<b} k_{a-}k_{b+}}\langle k_1, k_2|k_3, k_4\rangle, \quad (6.23)$$

that is,

$$_{\star}\langle k_1, k_2 |^{\star} k_3, k_4\rangle_{\star} = e^{-\sum_{a<b} k_a \bullet k_b}\langle k_1, k_2|k_3, k_4\rangle, \quad (6.24)$$

with $k_a \bullet k_b$ defined in (4.14).

Recalling the results (6.10) and (6.11), we can cast the previous expression in the form:

$$_{\star}\langle k_1, k_2 |^{\star} k_3, k_4\rangle_{\star} = \langle 0|a_{k_1} \star a_{k_2} \star a^\dagger_{k_3} \star a^\dagger_{k_4}|0\rangle. \quad (6.25)$$

This is in some sense also a consistency check. We could have started with the commutative expression $\langle k_1, k_2|k_3, k_4\rangle = \langle 0|a_{k_1}a_{k_2}a^\dagger_{k_3}a^\dagger_{k_4}|0\rangle$ and twisted the product among the creation and annihilation operators, obtaining the above result. We decided to follow a longer procedure to highlight the appearance of the various twists.

VII. THE TWISTED *S* MATRIX

Let $|f\rangle$ and $|i\rangle$ denote a collection of free asymptotic states at $t = \pm\infty$, respectively. We also assume that we can define in some way the one-particle incoming and outgoing states. This is a nontrivial assumption,[5] that in a theory in which localization is impossible the concept of localization may not be well defined. Nevertheless it is reasonable to expect that also in this theory, for small θ for large distances and times it will be possible to talk on incoming and outgoing states, expandable in the eigenvalues of momentum $|k\rangle$.

As in standard books in quantum field theory we define the S matrix as the matrix which describes the scattering of the initial $|i\rangle$ states into the final $|f\rangle$ states

$$S_{fi} = {}_{\text{in}\star}\langle f |^{\star} i\rangle_{\star\text{out}} = {}_{\text{out}\star}\langle f \star |S \star |i\rangle_{\star\text{out}} = {}_{\text{in}\star}\langle f |^{\star} S |^{\star} i\rangle_{\star\text{in}}, \quad (7.1)$$

where the twisted inner product of twisted states (6.23) is understood. The one-particle asymptotic state is defined as in (6.7) to be

$$|k\rangle_{\text{in}} = N_\star(k)a^\dagger_k|0\rangle_{\text{in}} = -N_\star(k)\frac{i}{\sqrt{(2\pi)^2 2\omega_k}}\int d^2x e^{-ik\cdot x}\overleftrightarrow{\partial}_0\varphi_{\text{in}}(x)|0\rangle_{\text{in}}, \quad (7.2)$$

with $N_\star(k)$ a normalization factor to be determined for the Moyal and Wick-Voros cases separately and analogously for the out states. Moreover we assume, as in the commutative case, that the matrix elements of the interacting field $\varphi(x)$ approach those of the free asymptotic field as time goes to $\mp\infty$

$$\lim_{x^0\to\pm\infty}\langle f|\varphi(x)|i\rangle = Z^{1/2}\langle f|\varphi_{\substack{\text{out}\\ \text{in}}}(x)|i\rangle, \quad (7.3)$$

with Z a renormalization factor. To be definite let us consider an elastic process of two particles in two particles. According to the previous section we have then

$$S_{fi\star}(k_1, \ldots, k_4) = {}_{\text{in}\star}\langle k_1k_2 |^{\star} k_3k_4\rangle_{\text{in}\star} = e^{\sum_{a<b} k_a\bullet k_b}{}_{\text{in}}\langle k_1k_2|k_3k_4\rangle_{\text{out}}, \quad (7.4)$$

which can be expressed in terms of Green's functions, following the same procedure as in the commutative case (see for example [56]). On repeatedly using (7.2) and (7.3) we arrive at

$$S_{fi} = {}_{\text{in}\star}\langle k_1k_2 |^{\star} k_3k_4\rangle_{\text{out}\star} = \text{disconnected graphs} + \bar{N}_\star(k_1)\bar{N}_\star(k_2)N_\star(k_3)N_\star(k_4)(iZ^{-1/2})^2 e^{-\sum_{a<b} k_a\bullet k_b}\int\frac{\Pi_a d^2x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \times e^{-ik_a x^a}(\partial^2_\mu + m^2)_a G(x_1, x_2, x_3, x_4), \quad (7.5)$$

where $G(x_1, x_2, x_3, x_4)$ is the four-point Green's function.

In order to fix the normalization of the asymptotic states let us compute the scattering amplitude for one particle going into one particle, at zeroth order. Up to the undeformed normalization factors $N(p_a)$, this has to give a delta function

$$\bar{N}(k)N(p)\delta^2(k-p) = N^*_\star(k)N_\star(p)_{\text{in}\star}\langle k |^{\star} p\rangle_{\text{out}\star} = N^*_\star(k)N_\star(p)e^{-k\bullet p}{}_{\text{in}}\langle k|p\rangle_{\text{out}} = N^*_\star(k)N_\star(p)e^{-k\bullet p}\delta^2(k-p) \quad (7.6)$$

[5] We thank Harald Grosse for bringing this fact to our attention.

from which

$$N_{\star_M}(p) = N(p), \qquad N_{\star_V}(p) = \mathrm{e}^{-(\theta/4)|\vec{p}|^2} N(p). \quad (7.7)$$

Let us now compute the scattering amplitude for the process above (two particles in two particles) at one loop. We have two kinds of contribution to (7.5), one coming from the planar terms (4.24), which in spatial coordinates read

$$G_{\mathrm{P}}(x_1, x_2, x_3, x_4) = \int \Pi_a \frac{d^2 k_a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \times \mathrm{e}^{ik_a x^a} G_{\mathrm{P}}^{(4)}(k_1, k_2, k_3, k_4), \quad (7.8)$$

the other coming from nonplanar terms (4.25)

$$G_{\mathrm{NP}}(x_1, x_2, x_3, x_4) = \int \Pi_a \frac{d^2 k_a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \times \mathrm{e}^{ik_a x^a} G_{\mathrm{NP}}^{(4)}(k_1, k_2, k_3, k_4). \quad (7.9)$$

Let us do the computation for the planar case first. Substituting in (7.5) we find the same result in Moyal and Wick-Voros case; moreover they coincide with the undeformed result:

$$S_{fi_{\star P}}(k_1, \ldots, k_4) = \frac{(-\mathrm{i}g)^2}{8} (2\pi)^3 \bar{N}(k_1) \bar{N}(k_2) N(k_3) N(k_4) \Pi_a \mathrm{e}^{(\theta/4)|\vec{k}_a|^2} \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2 x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-\mathrm{i}k_a x^a} \times \int \Pi_a \frac{d^2 p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} \mathrm{e}^{\mathrm{i}p_a x^a} (-p_a^2 + m^2) \int \frac{d^3 q}{(2\pi)^3} \frac{\mathrm{e}^{\sum_{a\le b} p_a \bullet p_b} \delta(\sum_{a=1}^4 p_a)}{(q^2 - m^2)((p_1 + k_2 - q)^2 - m^2) \prod_{a=1}^4 (p_a^2 - m^2)}. \quad (7.10)$$

The integration over the x^a variables yields factors of $(2\pi)^2 \delta^{(2)}(k_a - p_a)$; therefore the propagators of the external legs cancel as in the standard case, as well as the factor

$$\Pi_a \mathrm{e}^{(\theta/4)|\vec{k}_a|^2} \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \times \mathrm{e}^{\sum_{a\le b} p_a \bullet p_b} \delta^{(2)}(k_a - p_a) \to 1, \quad (7.11)$$

so that we are left with the usual commutative expression

$$S_{fi_{\star P}}(k_1, \ldots, k_4) = S_{fi}(k_1, \ldots, k_4). \quad (7.12)$$

In the NP case instead we find

$$S_{fi_{\star \mathrm{NP}}}(k_1, \ldots, k_4) = \frac{(-\mathrm{i}g)^2}{8} (2\pi)^3 \bar{N}(k_1) \bar{N}(k_2) N(k_3) N(k_4) \Pi_a \mathrm{e}^{(\theta/4)|\vec{k}_a|^2} \mathrm{e}^{-\sum_{a<b} k_a \bullet k_b} \int \Pi_a \frac{d^2 x^a}{\sqrt{(2\pi)^2 2\omega_{k_a}}} \mathrm{e}^{-\mathrm{i}k_a x^a} \times \int \Pi_a \frac{d^2 p_a}{\sqrt{(2\pi)^2 2\omega_{p_a}}} \mathrm{e}^{\mathrm{i}p_a x^a} (-p_a^2 + m^2) \int \frac{d^3 q}{(2\pi)^3} \frac{\mathrm{e}^{\sum_{a\le b} p_a \bullet p_b + E_a} \delta(\sum_{a=1}^4 p_a)}{(q^2 - m^2)((p_1 + k_2 - q)^2 - m^2) \prod_{a=1}^4 (p_a^2 - m^2)}. \quad (7.13)$$

After integrating over x^a the propagators of the external legs cancel and the simplification (7.11) continues to hold, but we are left with the exponential of E_a which does not simplify. Its explicit expression is given in (4.26), as we can see it is an imaginary phase, and it has *the same expression in the Moyal and Wick-Voros case*. It depends on the q variable, and therefore it gets integrated and modifies the IR and UV behavior of the loop: this is the correction responsible for the UV/IR mixing [44]. Therefore we can conclude that

$$S_{fi_{\star_M}\mathrm{NP}}(k_1, \ldots, k_4) = S_{fi_{\star_V}\mathrm{NP}}(k_1, \ldots, k_4) \neq S_{fi}(k_1, \ldots, k_4). \quad (7.14)$$

VIII. CONCLUSIONS

In Giuseppe Tomasi di Lampedusa's novel *Il Gattopardo* (translated *The Leopard*) [57] the Prince of Salina says: "Change everything so that nothing changes." This sums up the situation that we faced in our analysis of the field theory in the presence of the Wick-Voros and Moyal products. We started with different actions, coming from different Lagrangian densities already at the level of the free theory. The free propagator for the Wick-Voros is different from the Moyal case, but the classical theory has no new solutions, and at the quantum level the poles of the propagators are the same. Then we found a different vertex for the quartic theory, which led to a different four-point function. But the differences are reabsorbed in the S matrix, *provided one recognizes the properly normalized asymptotic states*. It is not anymore enough to think of a flux of particles to be identified by ordinary plane waves described by the usual exponential wave with the customary dispersion relation. In a field theory with a different propagator such as the one considered here the asymptotic states change. The noncommutative cases are however different from the commutative one (something has to change), but they describe the same ''physics'' among themselves.

The two noncommutative products (Moyal and Wick-Voros) are different realizations of the same algebra and as such describe the same noncommutative geometry, and it would have been curious to find different physical consequences. In fact one could have studied the noncommutative geometry exclusively at the operatorial level, without the need for a deformed product. But at the end of the day, to confront with a physical theory, one has to map the states into physically observable states that an experimenter (at least an ideal one) can prepare. The correspondence between states and real objects is not immediate in noncommutative geometry and has to be handled with extreme care. This is the moral of this tale. In noncommutative geometry, the different structure of spacetime forces to change the correspondence between mathematical objects and physical observables. This should lead to the formulation of a coherent theory of observables and measurements on noncommutative spaces.

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