PHYSICAL REVIEW D **82,** 124069 (2010)

Quantum corrections in the group field theory formulation of the Engle-Pereira-Rovelli-Livine and Freidel-Krasnov models

Thomas Krajewski,[1,*] Jacques Magnen,[2,1,†] Vincent Rivasseau,[1,‡] Adrian Tanasa,[3,4,§] and Patrizia Vitale[1,5,‖]

[1]*Laboratoire de Physique Théorique, Université Paris XI, CNRS UMR 8627, 91405 Orsay Cedex, France*
[2]*Centre de Physique Théorique, Ecole Polytechnique, CNRS UMR 7644, 91128 Palaiseau Cedex, France*
[3]*LIPN, Institut Galilée, CNRS UMR 7030 Universite. Paris Nord, 99 avenue Clément, 93430 Villetaneuse, France*
[4]*Institutul de Fizică şi Inginerie Nucleară Horia Hulubei, Post Office Box MG-6, 077125 Măgurele, România*
[5]*Dipartimento di Scienze Fisiche, Università di Napoli Federico II and INFN, Sezione di Napoli, Via Cintia 80126 Napoli, Italy*

(Received 4 October 2010; published 30 December 2010)

We investigate the group field theory formulation of the Engle-Pereira-Rovelli-Livine/Freidel-Krasnov (EPRL/FK spin-foam models. These models aim at a dynamical, i.e., nontopological formulation of 4D quantum gravity. We introduce a saddle point method for general group field theory amplitudes and compare it with existing results, in particular, for a second order correction to the EPRL/FK propagator.

DOI: 10.1103/PhysRevD.82.124069 PACS numbers: 04.60.−m, 04.60.Pp

I. INTRODUCTION

Group field theories (GFTs) [1] are quantum field theories over group manifolds and can be also viewed as higher rank tensor field theories [2,3] which generalize matrix models. They provide one of the most promising frameworks for a background invariant theory of quantum gravity in which one sums both over topologies and geometries. Indeed, each Feynman graph of a D dimensional GFT can be dually associated with a discrete space-time via a specific triangulation and gluing rules given by the covariance and vertices of the theory. The functional integral formalism defines a weighted sum over triangulations with each weight (amplitude) related to a sum over geometries via a spin-foam formalism [4] (see [5,6] for results on power counting and nonperturbative resummation of such models).

Spin foams are the Feynman amplitudes of GFT. But GFT in addition specifies the class of graphs that should be summed, together with their combinatoric factors. This stems from Wick contractions rules, hence (perturbative) GFT requires to distinguish the nonquadratic part (interaction) from the quadratic part (propagator) in the field action.

The simplest group field theories correspond to quantization of the $B_{\mu\nu}F^{\mu\nu}$ (or BF) models, hence to topological versions of gravity. Recently new spin-foam rules have been proposed for the quantization of full fledged 4D gravity [7–10]. These models stem from an improved analysis of the Plebanski simplicity constraints. The corresponding so-called EPRL or FK models are neither of the BF nor of the Barret-Crane type. They mix the left and right part of $SO(4) \simeq SU(2) \times SU(2)$ in a new way which gives a central rôle to the Immirzi parameter. These new theories could be called dynamical since their propagators, combining two noncommuting projectors, have nontrivial spectrum.

Preliminary studies of the asymptotic large spin (also called ‘‘ultraspin’’) regime have been performed for the EPRL/FK amplitudes of the ‘‘self-energy’’ and the ‘‘starfish’’ graphs (see Figs. 4 and 3) [11]. These results are a first step towards a study of renormalizability of such theories.

In [12], a linearized approximation has been devised to investigate the ultraspin limit of BF spin-foam amplitudes (see also [13]). This approximation captures the correct power counting of some graphs, such as type 1 graphs in the Boulatov model [14], but it typically overestimates more general graphs.

In this paper we push further the group field theory approach to the EPRL/FK models, first introduced in [9], and perform another step towards the general investigation of their renormalizability. We use a coherent state representation of the EPRL/FK propagator, as in [9], while other representations are exhibited mainly for comparison with other approaches. We introduce a general saddle point approximation, as in [15], which reproduces correctly the approximation [12] to the power counting of BF amplitudes for simply connected graphs and, for nondegenerate configurations, the EPRL/FK ‘‘self-energy graph’’ power counting of [11]. We discuss also the case of degenerate configurations, not studied in [11].

The plan is as follows: the next section is devoted to a review of the BF and EPRL/FK group field models in a field theoretical spirit. The following section presents the stationary phase method. Finally we remark that the sign of the self-energy graph points towards a singularity in the effective propagator of the EPRL/FK model, which could signal a phase transition. For completeness we included useful formulas and normalization conventions in the appendices.

*On leave, Centre de Physique Théorique, CNRS UMR 6207 CNRS Luminy, Case 907, 13288 Marseille Cedex 9. krajew@cpt.univ-mrs.fr
†magnen@cpht.polytechnique.fr
‡rivass@th.u-psud.fr
§adrian.tanasa@ens-lyon.org
‖vitale@na.infn.it

II. IMPLEMENTATION OF GFT

In GFT, the field arguments live on products of Lie groups. Feynman amplitudes are spin-foams, i.e., two-complexes with vertices, stranded lines (also called propagators) and faces (that is, closed circuits of strands).

A. Fields

Since GFT represents a quantum theory of space-time itself, the usual spin-statistic theorem may not apply. In this paper, we consider only Bosonic statistics. However, other choices have been considered [16]. We also work with an Euclidean signature.

The number of strands in the GFT lines encodes the space-time dimension D. The natural group associated to such a D dimensional GFT is $[SO(D)]^D$, hence a field ϕ is a function on $[SO(D)]^D$. We do not assume any symmetry under permutations of the arguments.

B. Vertices

In the spin-foam literature, the term "vertex" usually refers to the vertex *together with the square roots of its dressing propagators*. This terminology is not the standard one in quantum field theory. Further confusion often stems from the fact that in BF theory the propagator is a projector hence is equal to its square, and also to its square root.

To clarify the situation, let us return to ordinary field theory. In that case also the definition of the vertex could be considered ambiguous since one can dress it with a more or less arbitrary fraction of the propagator. What fixes this ambiguity is the usual requirement that vertices in field theory should obey a certain *locality* property in direct space. This allows to distinguish them from their dressing (half)-propagators, which are nonlocal operators.

Since GFT is nonlocal on the group, we cannot transpose directly this rule. To properly distinguish the vertex from the propagator we propose to use an extended notion of locality adapted to the GFT case, which we call *simpliciality* [17].

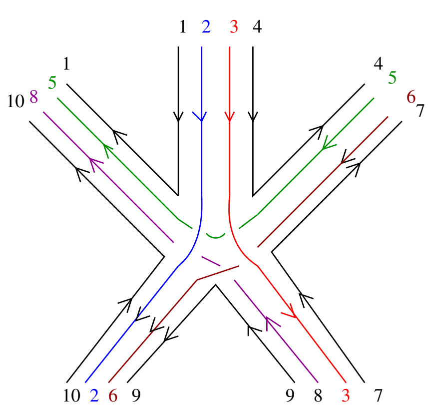

FIG. 1 (color online). A simplicial vertex of a four-dimensional GFT. We have chosen here a particular matching and orientation for each of the strands.

For consistency reasons, every vertex in GFT is required to have a total degree in the fields ensuring parity of the number of strands. In odd dimensions, this restricts the degree of the vertex to be even. Hence we propose the following definition:

Definition 2.1 A vertex joining $2p$ strands is called simplicial if it has for kernel in direct group space a product of p delta functions matching strand arguments, so that each delta function joins two strands in two different half-lines.

The usual vertex for $D-$ dimensional GFT is a ϕ^{D+1} simplicial vertex in which the faces are glued in the pattern of a D-dimensional simplex. For instance, the ordinary Boulatov vertex in three dimensions is simplicial (with $p=6$) as it writes

$$S_{\rm int}[\phi]=\frac{\lambda}{4}\int\bigg(\prod_{i=1}^{12}dg_i\bigg)\phi(g_1,g_2,g_3)\phi(g_4,g_5,g_6)\times\phi(g_7,g_8,g_9)\phi(g_{10},g_{11},g_{12})K(g_1,..g_{12}),\quad(2.1)$$

with a kernel

$$K(g_1,..g_{12})=\delta(g_3g_4^{-1})\delta(g_2g_8^{-1})\delta(g_6g_7^{-1})\delta(g_9g_{10}^{-1})\times\delta(g_5g_{11}^{-1})\delta(g_1g_{12}^{-1})\quad(2.2)$$

satisfying to our definition. But remark that the "pillow term" [5]

$$S_{\rm int}^{\rm pillow}[\phi]=\frac{\lambda}{4}\int\bigg(\prod_{i=1}^{6}dg_i\bigg)\phi(g_1,g_2,g_3)\phi(g_3,g_4,g_5)\times\phi(g_5,g_4,g_6)\phi(g_6,g_2,g_1)\quad(2.3)$$

is also simplicial in $D=3$. Also in any dimension D there are infinitely many higher than order $D+1$ simplicial vertices according to our definition. A possible vertex of a four-dimensional GFT is represented in Fig. 1.

C. Propagators

We consider only field theories in which the propagator C is Hermitian. It can be considered either as an Hermitian operator $\phi\rightarrow C\phi$ acting on fields or as its Hermitian kernel $C(g_1,\ldots,g_D;g'_1,\ldots,g'_D)$:

$$[C\phi](g_1,\ldots,g_D)=\int dg'_1\ldots dg'_D C(g_1,\ldots,g_D;g'_1,\ldots,g'_D)\times\phi(g'_1,\ldots,g'_D).\quad(2.4)$$

The corresponding normalized Gaussian measure of covariance C is noted $d\mu_C$. Hence

$$C(g_1,\ldots,g_D;g'_1,\ldots,g'_D)=\int\phi(g_1,\ldots,g_D)\times\phi(g'_1,\ldots,g'_D)d\mu_C.\quad(2.5)$$

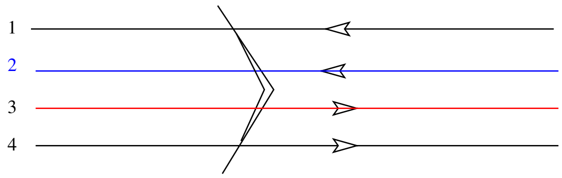

FIG. 2 (color online). A stranded propagator with particular orientation; two strands have $\eta_{\ell f} = 1$ and the other two have $\eta_{\ell f} = -1$.

D. Graphs

Graphs are generated by gluing together propagators and vertices, according to Wick contractions, hence to Feynman rules.

Definition 2.2 A stranded graph is called regular if it has no tadpoles (hence any line ℓ joins two distinct vertices) and no tadfaces (hence each face f goes at most once through any line of the graph).

It is convenient to introduce orientations on both lines and faces of stranded graphs. Regular oriented graphs are natural since they are conveniently described by two matrices

(i) the ordinary incidence matrix $\epsilon_{v,\ell}$ which has value $+1$ if the edge ℓ enters the vertex v, -1 if the edge ℓ exits vertex v and 0 otherwise. Hence $\sum_v |\epsilon_{v,\ell}| = 2$ for each ℓ.

(ii) the incidence matrix $\eta_{\ell,f}$ between faces and edges, which has value $+1$ if the face f goes through edge ℓ in the same direction, -1 if the face f goes through edge ℓ in the opposite direction and 0 otherwise. Hence $\sum_f |\eta_{\ell,f}| = D$ for each ℓ (see Fig. 2).

These orientations are useful to write down the *integrand* of the Feynman amplitudes. However the *integrals*, that is the spin-foam amplitudes themselves, do not depend on these orientations, at least for the class of theories considered in this paper.

From now on we consider only amplitudes for *regular* graphs. This is for convenience, as generalization to any graph of our formulas is possible. It has been argued that GFT should in fact be restricted to colored models, which generate only regular graphs [16,18,19]. Remark that every colorable stranded graph is regular, but the converse is not true; colorable graphs, in particular, have all their faces of even length, hence the starfish graph of Fig. 3 with ten faces of length 3, although regular, is not colorable.

E. The BF theory

1. Propagator in direct space

By direct space we mean the representation which uses the group elements.

In the case of the BF theory the propagator,[1] here noted $\mathbb{P}$, is just the projection on gauge invariant fields

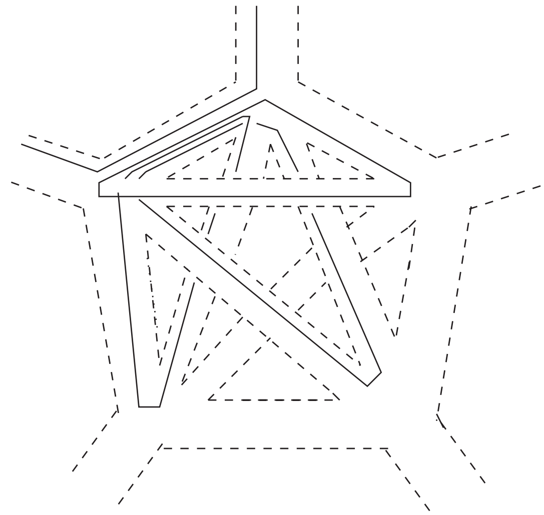

FIG. 3. The "starfish" graph, quantum correction to the vertex. The dashed lines represent the edges (they do not correspond to strands). Each edge contains 4 strands, there are 40 such strands, forming 10 closed faces and 10 open faces. We have shown four faces: three closed and one open which take into account all four" strands of one particular edge (upper left).

$$\mathbb{P}(\phi)(g_1, \ldots, d_D) = \int dh \phi(g_1 h, \ldots, g_D h), \tag{2.6}$$

where the integral is performed over the group $SO(D)$ with respect to its Haar measure. Let us remark that $\mathbb{P}^2 = \mathbb{P}$ so that the only eigenvalues are 0 and 1 (which means that the BF theory has no dynamics). The operator $\mathbb{P}$ is Hermitian with kernel

$$\mathbb{P}(g_1, \ldots, g_D; g'_1, \ldots g'_D) = \int dh \prod_{i=1}^{D} \delta(g_i h (g'_i)^{-1}). \tag{2.7}$$

2. Amplitudes in direct space

Suppose that we choose an arbitrary orientation of the lines and faces of a graph $\mathcal{G}$ (which for simplicity has no external legs).

Combining together the vertices (2.1) (or the generalization to dimension D) and the propagators (2.7) of the graph, the integration over all g variables can be explicitly performed, leading to the direct space representation of the BF Feynman amplitude as an integral over line variables h

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} dh_\ell \prod_{f \in F_{\mathcal{G}}} \delta\Big(\vec{\prod}_{\ell \in f} h_\ell^{\eta_{\ell f}}\Big), \tag{2.8}$$

where $L_{\mathcal{G}}$, $F_{\mathcal{G}}$ are the set of lines and faces of $\mathcal{G}$, respectively. The oriented product $\vec{\prod}_{l \in f} h^{\eta_{\ell f}}$ means that the product of the variables h_ℓ has to be taken in the cyclic ordering corresponding to the face orientation (starting anywhere on the cycle).

[1] Beware that this propagator is called C in [6,20].

As announced, the amplitude (2.8) neither depends on the arbitrary orientation of the lines, nor on those of the faces. A pedestrian way to see this is to exploit carefully the parity of the δ function and of the Haar measure under $h \to h^{-1}$. Beware that formula (2.8) may be formal as this amplitude can be infinite for many graphs.

3. Amplitudes in the angular momentum basis

The angular momentum basis uses the irreducible unitary representation spaces V^j of dimension $d_j \equiv 2j + 1$. In this space there is a standard decomposition of unity

$$\mathbf{1}_j = \sum_m |j, m\rangle\langle j, m|, \tag{2.9}$$

where $|j, m\rangle$, $m \in [-j, j]$ is the usual orthonormal basis in V^j.

In dimension 3, using the Peter-Weyl theorem, we can transform (2.8) to get the representation

$$A_{\mathcal{G}} = \prod_f \sum_{j_f} d_{j_f} \prod_v \{6j\}, \tag{2.10}$$

where the Ponzano-Regge vertex is the $6j$ symbol (according to Appendix A) corresponding to the six-face indices meeting at the vertex.

In dimension 4, for simplicity we work with the covering group $SU(2) \times SU(2)$ of $SO(4)$ and we decompose the group elements as $g = (g_+, g_-)$, with $g_\pm \in SU(2)$. Moreover, we write $j \equiv (j_+, j_-)$ for the eigenvalues of the angular momentum J in each $SU(2)$ component. The Peter-Weyl decomposition leads to the similar angular momentum representation:

$$A_{\mathcal{G}} = \prod_f \sum_{j_{f+}, j_{f-}} d_{j_{f+}} d_{j_{f-}} \prod_v \{15j_+\}\{15j_-\}, \tag{2.11}$$

where $\{15j\}$ are the $15j$ symbols (see for example [11] for the definition and normalization conventions).

4. Coherent states

Consider $R_k^{(j)m}(g)$, the matrix element of the group element g in the representation j, computed between the states $\langle j, m|$ and $|j, k\rangle$. We have

$$\begin{aligned}\mathbf{1}_j &= d_j \sum_{mm'} |j, m\rangle\langle j, m'| \int_{SU(2)} dg R_j^{(j)m}(g) \bar{R}_j^{(j)m'}(g) \\ &= d_j \int_{SU(2)} dg |j, g\rangle\langle j, g|,\end{aligned} \tag{2.12}$$

where we have introduced the notation

$$|j, g\rangle \equiv g|j, j\rangle = \sum_m |j, m\rangle R_j^{(j)m}(g). \tag{2.13}$$

The states $|j, g\rangle$ are the coherent states [21], and the last expression in (2.12) is a decomposition of the identity in terms of these coherent states.

Let us recall that the decomposition of the identity (2.12) can be further simplified and taken over the coset G/H, $G = SU(2)$, $H = \mathrm{U}(1)$:

$$\mathbf{1}_j = d_j \int_{G/H=S_2} dn |j, n\rangle\langle j, n| \tag{2.14}$$

with $|j, n\rangle = g_n|j, j\rangle$ and g_n defined in (A14). We suppress the domain of integration G/H in what follows.

The states $|j, n\rangle$ form a generating set in V^j sometimes called "overcomplete basis."

Let us now turn to the coherent states of the group $SU(2) \times SU(2)$. In fact, one has four possible such coherent states which are given by acting with the same group element $(g_+, g_-) \in SU(2) \times SU(2)$ on either of the following states:

$$\begin{aligned} &|j, j\rangle \otimes |j, j\rangle, \qquad |j, j\rangle \otimes |j, -j\rangle, \\ &|j, -j\rangle \otimes |j, j\rangle, \qquad |j, -j\rangle \otimes |j, -j\rangle. \end{aligned} \tag{2.15}$$

Note that these four states can be obtained from one another by the action of an $SO(4)$ group element. However, if one considers only the action of the diagonal $SU(2)$ subgroup of elements of the form (g, g), then there are two inequivalent states that cannot be related by such a transformation.

5. The BF propagator and amplitudes using coherent states

To prepare for the EPRL/FK propagator, we rewrite the BF propagator inserting the coherent state decomposition of identity on each strand. Let us consider $SU(2)$ BF first. Since $\mathbb{P}^2 = \mathbb{P}$ we can introduce two distinct $SU(2)$ gauge-averaging variables, u and v at both ends of the propagator, instead of the single variable h (e.g., u on the side where $\epsilon_{v,\ell} = -1$ and v on the side where $\epsilon_{v,\ell} = +1$). Between these two variables we insert the partition of unity (2.14). This does not modify the propagator. Working out the algebra, we find

$$\mathbb{P}(g; g') = \int du dv \prod_{f=1}^4 \sum_{j_f} d_{j_f} \mathrm{Tr}_{V_{j_f}}(u g_f (g'_f)^{-1} v^{-1} \mathbf{1}_{j_f}), \tag{2.16}$$

with g_f, g'_f, u and v elements of $SU(2)$. The index f labels the four strands of the propagator, which belong to four different faces (since we consider only regular graphs).

To write down the amplitudes, we need to introduce some notations. There are now group variables $2|L_{\mathcal{G}}|$, u_ℓ and v_ℓ, and $D|L_{\mathcal{G}}|$ variables n. The amplitude is again factorized over faces:

$$A_{\mathcal{G}} = \int \prod_{\ell \in L_{\mathcal{G}}} du_\ell dv_\ell \prod_{f \in F_{\mathcal{G}}} \mathcal{A}_f. \tag{2.17}$$

To write down $\mathcal{A}_f$, let us number the vertices and lines in the (anti)-cyclic order along a face f of length p as $\ell_1, v_1 \cdots \ell_p, v_p$, with by definition $\ell_{p+1} = \ell_1$. We have then

$$\mathcal{A}_f = \sum_j d_j^{p+1} \int \prod_{a=1}^{p} dn_{\ell_a f} \langle j, n_{\ell_a f} | h_{\ell_a, v_a}^{\eta_{\ell_a f}} h_{\ell_{a+1}, v_a}^{\eta_{\ell_{a+1} f}} | j, n_{\ell a+1, f} \rangle, \tag{2.18}$$

where

$$\begin{aligned} h_{\ell_a, v_a} &= v_{\ell_a} \quad \text{if } \epsilon_{v_a, \ell_a} = +1 \\ h_{\ell_a, v_a} &= u_{\ell_a} \quad \text{if } \epsilon_{v_a, \ell_a} = -1. \end{aligned} \tag{2.19}$$

6. *The $D = 4$ BF case*

In $D = 4$ we work with $SU(2) \times SU(2)$, the covering group of $SO(4)$; we have the similar system $|j_+, n_+\rangle \otimes |j_-, n_-\rangle$ of coherent states and the partition of unity on the space V_j, with $j = (j_+, j_-)$

$$\begin{aligned} \mathbf{1}_{j_+} \otimes \mathbf{1}_{j_-} &= \mathbf{1}_j \\ &= d_{j_+} d_{j_-} \int dn_+ dn_- |j_+, n_+\rangle \otimes |j_-, n_-\rangle \\ &\quad \times \langle j_+, n_+ | \langle j_-, n_- |. \end{aligned} \tag{2.20}$$

The gauge-averaging variables, $u = (u_+, u_-)$ and $v = (v_+, v_-)$ at both ends of the propagator are now elements of $SU(2) \times SU(2)$. Between these two variables we insert the partition of unity (2.20) and we find

$$\begin{aligned} \mathbb{P}(g; g') = \int dudv \prod_{f=1}^{4} \sum_{j_{f+}, j_{f-}} d_{j_{f+}} d_{j_{f-}} \\ \times \mathrm{Tr}_{V_{j_{f+}} \otimes V_{j_{f-}}} (u g_f (g'_f)^{-1} v^{-1} \mathbf{1}_{j_f}) \end{aligned} \tag{2.21}$$

with g_f, g'_f, u and v elements of $SU(2) \times SU(2)$, and we have formulas similar to (2.17) and (2.18) for the amplitudes.

F. The EPRL/FK GFT

The EPRL/FK model introduces a modification of the propagator of the BF model, while the vertex remains the same. The EPFL/FK propagator has a structure similar to (2.21) but with replacement of $\mathbf{1}_j$ by a nontrivial projector. We notice at this point that since this projector does not commute with $\mathbb{P}$, it is not possible to recombine u and v in a single gauge-averaging variable h.

It implements in two steps the Plebanski constraints with a nontrivial value of the Immirzi parameter γ. Starting from the (2.21) expression of the BF propagator in the coherent states representation, the first step adds the constraint $j_+/j_- = (1+\gamma)/(1-\gamma)$ on the representations summed. Remark however that this equation may have no solution (e.g., if γ is irrational) and should be true only in an asymptotic sense in the ultraspin limit where j_+ and j_- are both very large.

More precisely, this constraint reads

$$\gamma > 1 \qquad j_\pm = \frac{\gamma \pm 1}{2} j, \qquad n_+ = n_- \tag{2.22}$$

$$\gamma < 1 \qquad j_\pm = \gamma_\pm j = \frac{1 \pm \gamma}{2} j, \qquad n_+ = n_-, \tag{2.23}$$

where $j_\pm$, j are half-integers.[2]

From now on we consider only the case $0 < \gamma \leq 1$ where the EPRL and FK models coincide. At $\gamma = 1$, the EPRL/FK model reduces to a single $SU(2)$ BF theory (see below).

The second step replaces in each strand of (2.21) the identity $\mathbf{1}_j$ by a projector T_j^γ whose definition is

$$\begin{aligned} T_j^\gamma = d_{j_+ + j_-} [\delta_{j_{f-}/j_{f+} = (1-\gamma)/(1+\gamma)}] \int dn |j_+, n\rangle \otimes |j_-, n\rangle \\ \times \langle j_+, n | \otimes \langle j_-, n |. \end{aligned} \tag{2.24}$$

Let us notice here that, in the angular momentum basis, the operator T_j^γ takes the form

$$\begin{aligned} T_j^\gamma = \sum_{k, \tilde{k}, m, \tilde{m}} & (j_+, k; j_-, \tilde{k} | j_+ + j_-, k + \tilde{k}) \\ & \times (j_+ + j_-, m + \tilde{m} | j_+, m; j_-, \tilde{m}) |j_+ k\rangle \otimes |j_- \tilde{k}\rangle \\ & \times \langle j_+ m | \otimes \langle j_- \tilde{m} | \delta_{m+\tilde{m}, k+\tilde{k}}, \end{aligned} \tag{2.25}$$

where $(.|.)$ denotes the Clebsch-Gordan coefficients.

Grouping the four strands of a line defines a $\mathbb{T}^\gamma$ operator that acts separately and independently on each strand of the propagator

$$\mathbb{T}^\gamma = \oplus_{j_f} \otimes_{f=1}^4 T_{j_f}^\gamma \tag{2.26}$$

so that the EPRL/FK propagator is

$$\begin{aligned} C &= \mathbb{P}\mathbb{T}^\gamma\mathbb{P}; C(g, g') \\ &= \int dudv \prod_{f=1}^{4} \sum_{j_f} [\delta_{j_{f-}/j_{f+} = (1-\gamma)/(1+\gamma)}] \alpha_{j_f} \beta_{j_f} \\ &\quad \times \int dn_f \, \mathrm{Tr}_{j_{f+} \otimes j_{f-}} (u g_f (g'_f)^{-1} v^{-1} |j_{f+}, n_f\rangle \otimes |j_{f-}, n_f\rangle \\ &\quad \times \langle j_{f+}, n_f | \otimes \langle j_{f-}, n_f |), \end{aligned} \tag{2.27}$$

where

$$\alpha_j = d_{j_+} d_{j_-}, \qquad \beta_j = d_{j_+ + j_-}. \tag{2.28}$$

Lemma 2.1 The operator C is Hermitian.

[2]Moreover, in the case $\gamma > 1$ the coherent states to be used below are the ones in their "antiparallel" version $|j, n\rangle \otimes \overline{|j, n\rangle}$ [9].

Proof We have

$$\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(ug_f(g'_f)^{-1}v^{-1}T^{\gamma}_{j_f}) = \beta_{j_f}\int dn\,\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(u\epsilon^T\epsilon g_f\epsilon^T\epsilon(g'_f)^{-1}\epsilon^T\epsilon v^{-1}n_f|j_{f+}j_{f-}\rangle\langle j_{f+}j_{f-}|n_f^{\dagger}), \tag{2.29}$$

where we have inserted the product $\epsilon^T\epsilon = 1$ with $\epsilon \in SU(2)$ defined by (A7). We arrive at

$$\begin{aligned}\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(ug_f(g'_f)^{-1}v^{-1}T^{\gamma}_{j_f}) &= \beta_{j_f}\int dn\,\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(u\epsilon^T\bar{g}_f(g'_f)^T\epsilon v^{-1}n_f|j_{f+}j_{f-}\rangle\langle j_{f+}j_{f-}|n_f^{\dagger})\\ &= \beta_{j_f}\int dn\,\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(\epsilon^T\bar{u}\bar{g}_f(g'_f)^Tv^T\epsilon n_f|j_{f+}j_{f-}\rangle\langle j_{f+}j_{f-}|n_f^{\dagger})\\ &= \beta_{j_f}\int dn\,\mathrm{Tr}_{j_{f+}\otimes j_{f-}}(\epsilon\bar{n}_f|j_{f+}j_{f-}\rangle\langle j_{f+}j_{f-}|n_f^T\epsilon^Tvg'_fg_f^{-1}u^{-1}) = \mathrm{Tr}_{j_{f+}\otimes j_{f-}}(vg'_fg_f^{-1}u^{-1}T^{\gamma}_{j_f}),\end{aligned} \tag{2.30}$$

which implies the lemma. Q.E.D.

Since the propagator is Hermitian, Feynman amplitudes are again independent of the orientations of faces and propagators.

Lemma 2.2 $\mathbb{T}^\gamma$ is a projector, namely $(\mathbb{T}^\gamma)^2 = \mathbb{T}^\gamma$.

Proof In the coherent states basis, it is easier to check that $(\mathbb{T}^\gamma)^3 = (\mathbb{T}^\gamma)^2$, which adding Hermiticity of $\mathbb{T}^\gamma$ implies the lemma. The equation $(\mathbb{T}^\gamma)^3 = (\mathbb{T}^\gamma)^2$ follows from the same equation on each strand, since

$$\begin{aligned}&\beta_j^3\int dndn'dn''|j_+,n\rangle\otimes|j_-,n\rangle\langle j_++j_-,n|j_++j_-,n'\rangle\\ &\quad\times\langle j_++j_-,n'|j_++j_-,n''\rangle\langle j_+,n''|\otimes\langle j_-,n''|\\ &=\beta_j^2\int dndn''|j_+,n\rangle\otimes|j_-,n\rangle\langle j_++j_-,n|j_+\\ &\quad+j_-,n''\rangle\langle j_+,n''|\otimes j_-,n''|,\end{aligned} \tag{2.31}$$

where we have used that $\mathbf{1}_{j_++j_-} = \beta_j\int dn'|j_++j_-,n'\rangle\times\langle j_++j_-,n'|$. Q.E.D.

Let us also notice that the lemma is easily proven in the angular momentum basis, where, from (2.25) it easily follows that $(T^\gamma_j)^2 = T^\gamma_j$.

Since $\mathbb{T}^\gamma$ and $\mathbb{P}$ do not commute, the propagator C can have nontrivial *spectrum* (with eigenvalues between 0 and 1). Slicing the eigenvalues should allow a renormalization group analysis. This is why we would like to call these kinds of theories *dynamic* GFTs.

Remark that since $\mathbb{T}^\gamma$ is a projector, the propagator C of the EPRL/FK theory is bounded in norm by the propagator of the BF theory, and that Feynman amplitudes for the EPRL/FK theory are therefore *bounded* by those of the BF theory; in particular, we expect milder ultraspin (large j) divergences in EPRL/FK.

1. Amplitudes

Combining the propagator and the vertex expressions, the integrations over all g, g' group variables can be performed explicitly, leading to the amplitude of any graph $\mathcal{G}$. This amplitude is given by an integral of a product over all faces of the graph as in (2.17), but the amplitudes for faces are different.

To compute these face amplitudes we distinguish between closed faces (no external edges) and open faces (which end on external edges).

Using the same numbering of the p edges and vertices along a closed face, its amplitude is given by

$$\begin{aligned}\mathcal{A}_f = \int\prod_{a=1}^{p}(dg_{\ell_a}dg'_{\ell_a})\sum_{j_{\ell_a}}\alpha_{j_{\ell_a}}\\ \times\mathrm{Tr}_{j_{\ell_a}+\otimes j_{\ell_a}-}((u_{\ell_a}g_{\ell_a}(g'_{\ell_a})^{-1}v_{\ell_a}^{-1})^{\eta_{\ell_af}}T^{\gamma}_{j_{\ell_a}})\prod_v V_v,\end{aligned} \tag{2.32}$$

where the constraint on j_+, j_- is implicitly understood from now on. We can perform the g integrals using (A5) or (A6) and we arrive at

$$\mathcal{A}_f = \sum_{j_f}\alpha_{j_f}\,\mathrm{Tr}_{j_{f+}\otimes j_{f-}}\vec{\prod}_{a=1}^{p}(h^{\eta_{\ell_af}}_{\ell_a,v_a}h^{\eta_{\ell_{a+1}f}}_{\ell_{a+1},v_a}T^{\gamma}_{j_f}), \tag{2.33}$$

with h_{ℓ_a,v_a} defined in (2.19) and $\ell_{p+1} = \ell_1$. Note that we use (A5) or (A6) to take into account the fact that $\eta_{\ell f}$ can change when we follow a face f. We find

$$\begin{aligned}\mathcal{A}_f = \sum_{j_f}\alpha_{j_f}\int\prod_{a=1}^{p}\beta_{j_f}dn_{\ell_a,f}\langle j_{f+}n_{\ell_a,f}|h^{\eta_{\ell_af}}_{\ell_a,v_a,+}h^{\eta_{\ell_{a+1}f}}_{\ell_{a+1},v_a,+}|\\ \times j_{f+}n_{\ell_{a+1},f}\rangle\langle j_{f-}n_{\ell_a,f}|h^{\eta_{\ell_af}}_{\ell_a,v_a,-}h^{\eta_{\ell_{a+1}f}}_{\ell_{a+1},v_a,-}|j_{f-}n_{\ell_{a+1},f}\rangle.\end{aligned} \tag{2.34}$$

2. BF limit

Let us see how we recover the $SU(2)$ BF model in the limit $\gamma = 1$. In this limit $j_- = 0$, hence $j_+ = j_+ + j_-$. Thus we are left with

$$\begin{aligned}\mathcal{A}_f|_{\gamma=1} = \sum_{j_f}d_{j_f}^{p+1}\int\prod_{a=1}^{p}dn_{\ell_a,f}\langle j_f,n_{\ell_a,f}\\ \times|h^{\eta_{\ell_af}}_{\ell_a,v_a,+}h^{\eta_{\ell_{a+1}f}}_{\ell_{a+1},v_a,+}|j_f,n_{\ell_{a+1},f}\rangle,\end{aligned} \tag{2.35}$$

where only one $SU(2)$ copy appears. We can use the completeness relation for coherent states (2.14), and cyclicity of the trace to reorder the product according to lines instead of vertices and we obtain

$$\mathcal{A}_f|_{\gamma=1} = \sum_j d_j^2 \int dn \langle jn| \prod_{a=1}^{p} h_{\ell_a, v_a, +}^{\eta_{\ell_a f}} h_{\ell_a, v_{a+1}, +}^{\eta_{\ell_a f}} |jn\rangle. \quad (2.36)$$

Redefining $t_{\ell_a} = u_{\ell_a,+}^{-1} v_{\ell_a,+}$ we finally obtain

$$\mathcal{A}_f|_{\gamma=1} = \sum_j d_j^2 \int dn \langle jn| \overrightarrow{\prod_{a\in f}} t_{\ell_a}^{\eta_{\ell_a f}} |jn\rangle = \delta\Big(\overrightarrow{\prod_{\ell\in f}} t_\ell^{\eta_{\ell f}}\Big), \quad (2.37)$$

consistently with (2.8).

3. Amplitudes with external edges

For a face with external edges, the expression is slightly modified as there is no integration on the external data.

Let us call G and $\tilde{G}$ the group labels of the incoming and outgoing external strands, respectively. We omit the edge index ℓ in the following. Moreover we indicate with $u_{\rm in}$, $v_{\rm in}$, $u_{\rm out}$, $v_{\rm out}$ the gauge transformations on the incoming, outgoing edges, respectively. Let q be the number of internal strands. The expression of the resulting face amplitude is similar to (2.32) except for the fact that we do not integrate on the external labels. On choosing

$$\eta_{{\rm in}f} = \eta_{{\rm out}f} = 1 \quad (2.38)$$

that is, the incoming and outgoing strand oriented according to the face, we find using (A5) and (A6)

$$\mathcal{A}_{\rm ext} = \sum_j \alpha_j \times {\rm Tr}\Big[u_{\rm in} G \tilde{G}^{-1} v_{\rm out}^{-1} T_j^\gamma u_{\rm out} u_{\ell_1} T_j^\gamma \times \Big(\prod_{a=2}^{q} u_{\ell_a} v_{\ell_{a+1}} T_j^\gamma \Big) v_{\ell_q}^{-1} v_{\rm in}^{-1} T_j^\gamma \Big] \quad (2.39)$$

where, to simplify the notation, we have chosen all propagators oriented according to the face. It is immediately verified that it reduces to (2.33) with $p = q + 2$ if we glue together the external edges with the insertion of a delta function $\delta(G\tilde{G}^{-1})\delta_{\sigma\sigma'}$.

III. STATIONARY PHASE FOR BF AND EPRL/FK MODELS

Let $\mathcal{G}$ be a graph in a GFT corresponding to the BF or EPRL/FK models, made of V vertices, L edges and F faces, usually labeled by letters v, ℓ and f. In the coherent state basis, its amplitude can in general be written as

$$\mathcal{A}_{\mathcal{G}} = \sum_{j_f \le \Lambda} \mathcal{N} \int \prod dh \prod dn \exp\Big\{ \sum_f j_f S_f[h, n] \Big\}, \quad (3.1)$$

where $\mathcal{N}$ is a normalization factor which is a rational function of the spins. As explained in the previous sections, the precise form of the face action and of the number of group variables $h \in SU(2)$ and unit vectors $n \in S^2$ depends on the choice of the model. Note that the sums over the spins j_f may lead to divergences, so that we introduce an ultraspin cutoff Λ that restricts the summation to spins below Λ. To derive the superficial power counting, we set $j_f = jk_f$ with $k_f \in [0, 1]$ and use the stationary phase method to derive the large j behavior of

$$\int \prod dh \prod dn \exp\Big\{ j \sum_f k_f S_f[h, n] \Big\}. \quad (3.2)$$

If the action is complex but has a negative real part, the contribution to this integral is quadratic fluctuations around zeroes of the real part of S which are stationary points of its imaginary part, otherwise the integral is exponentially suppressed as $j \to \infty$.

A. BF models in the coherent state representation

For BF models we have one group element $h_l \in SU(2)$. In dimension 4 one should work with $SU(2) \times SU(2)$ instead, which leads to two independent copies of the previous amplitude, so that we restrict ourselves to $SU(2)$ for simplification. The amplitude is given by (2.18). Including the k_f factor of (3.2) in the action and using

$$\langle n, j| g |n', j\rangle = \langle n| g |n'\rangle^{2j}, \quad (3.3)$$

with $|n\rangle$ a shorthand for $|\frac{1}{2} n\rangle$, it can be written in the form (3.1) with

$$S_f[h, n] = 2k_f \log \langle n| \overrightarrow{\prod_{\ell\in\partial f}} h_\ell^{\eta_{\ell,f}} |n\rangle. \quad (3.4)$$

Note that $|n\rangle\langle n|$ is a projector

$$|n\rangle\langle n| = \frac{1}{2}(1 + \sigma \cdot n), \quad (3.5)$$

so that the action reads

$$S_f[h, n] = k_f \log {\rm Tr}\Big[\Big(\overrightarrow{\prod_{\ell\in\partial f}} h_\ell^{\eta_{\ell,f}} \Big) (1 + \sigma \cdot n) \Big]. \quad (3.6)$$

Since the action is the logarithm of the trace of the product of an unitary element and a projector, it is clear that its real part is negative (it is the logarithm of the modulus of the trace, obviously bounded by 1) and maximal when the unitary element is one. This is attained at $h_\ell = 1$, but other solutions may be possible. In particular, the BF amplitude is invariant under the gauge transformations g_v at any vertex

$$h_\ell \to g_v h_\ell g_{v'}^{-1} \quad (3.7)$$

for any edge from the vertex v to the vertex v'. Therefore gauge transformations of the trivial solution $h_\ell = 1$ yield other equivalent solutions. More generally, there is a continuum of solutions connected to the trivial one which will translate into flat directions in the saddle point approximation.

To perform the saddle point expansion, we expand the group element to second order as

$$h_\ell = 1 - \frac{A_\ell^2}{2} + \mathrm{i}A_\ell \cdot \sigma + O(A_\ell^3), \tag{3.8}$$

with $A \in su(2) \times su(2)$ a Lie algebra element. By the same token, we expand the unit vectors as

$$n_f = n_f^{(0)} + \xi_f - \frac{\xi_f^2}{2} n_f^{(0)} + O(\xi_f^3), \quad \text{with} \quad n_f^{(0)} \cdot \xi_f = 0. \tag{3.9}$$

This expansion is determined by the requirement that $n_f^2 = 1$ up to third order terms. To alleviate the notation, we drop the superscript (0) in the sequel. Let us consider a face with edges $\ell_1, \ldots, \ell_p$, then to second order

$$\overrightarrow{\prod_{\ell \in \partial f}} h_\ell^{\eta_{\ell,f}} = 1 - \frac{A_f^2}{2} + \mathrm{i}\sigma \cdot A_f - \mathrm{i}\sigma \cdot \Phi_f, \tag{3.10}$$

with

$$A_f = \sum_{1 \le a \le p} \eta_{\ell_a,f} A_{\ell_a} \quad \text{and} \quad \Phi_f = \sum_{1 \le a < b \le p} \eta_{\ell_a,f}\eta_{\ell_b,f} A_{\ell_a} \wedge A_{\ell_b}. \tag{3.11}$$

Expanding then to second order (3.6), we get

$$S_f[A_\ell, \xi_f] = 2k_f\left\{\mathrm{i}n_f \cdot A_f - \frac{A_f^2}{2} + \frac{(n_f \cdot A_f)^2}{2} + \mathrm{i}\xi_f \cdot A_f + \mathrm{i}n_f \cdot \Phi_f\right\}, \tag{3.12}$$

and we have to estimate

$$\int \prod_\ell dA_\ell \prod_f d\xi_f \exp 2j \sum_f k_f \times \left\{\mathrm{i}n_f \cdot A_f - \frac{A_f^2}{2} + \frac{(n_f \cdot A_f)^2}{2} + \mathrm{i}\xi_f \cdot A_f + \mathrm{i}n_f \cdot \Phi_f\right\}, \tag{3.13}$$

as $j \to \infty$. Note that we do not integrate over the vectors n_f; the latter have to be chosen so that they are extrema of the imaginary part of S. Because all terms except the first one $\sum_f k_f n_f \cdot A_f$ are of second order, the imaginary part is stationary if and only if

$$\sum_f \mathrm{i}k_f n_f \cdot A_f = \sum_{\ell,f} \mathrm{i}\eta_{\ell,f} k_f n_f \cdot A_\ell = 0 \quad \forall\, A_\ell \in R^3, \tag{3.14}$$

which amounts to the closure condition

$$\sum_f \eta_{\ell,f} k_f n_f = 0, \tag{3.15}$$

to be fulfilled for any edge ℓ. This is the well-known requirement that, in the semiclassical limit, the vectors $j_f n_f$ are the sides of a triangle (respectively, the area bivectors of a tetrahedron) that propagates along ℓ in dimension 3 (respectively, dimension 4). The solutions of the closure conditions range from nondegenerate to maximally degenerate. In three-dimensional (respectively, four-dimensional) BF theory, a solution is said to be nondegenerate if all the tetrahedra (respectively, four-simplices) corresponding to the vertices of the graph have maximal dimension. At the opposite end, we say that a solution is maximally degenerate if all the vectors n_f are proportional to a single one n_0,

$$n_f = \sigma_f n_0 \quad \text{with} \quad \sigma_f \in \{-1, +1\}. \tag{3.16}$$

1. Maximally degenerate case

Let us first concentrate on the maximally degenerate solutions and show that for simply connected graphs (i.e., every closed loop can be shrunk to a point by deforming it through the faces), the quadratic saddle point approximation yields an upper bound estimate

$$\mathcal{A}_{\mathcal{G}} \lesssim \Lambda^{3F-3r}, \tag{3.17}$$

with r the rank of the $L \times F$ incidence matrix $\eta_{\ell,f}$. This is in accordance with the general results for BF theory presented (see also [13]).

To derive this result, we proceed with the following five steps.

(1) For a maximally degenerate solution, the closure constraints amount to

$$\sum_f \eta_{\ell,f} x_f = 0, \tag{3.18}$$

with $x_f = k_f \sigma_f$. Since the rank of the matrix $\eta_{\ell,f}$ is r, the x_f live in a $F - r$ dimensional subspace. The signs σ_f have to be adjusted so that $k_f > 0$. We end up with a summation over $F - r$ independent spins in (3.1). Let us note that since the incidence matrix has integer coefficients, all the spins may always be chosen to be half-integers, after multiplication by a suitable integer.

(2) For BF theory in the coherent state representation, we have a factor of d_j^2 per face, so that the normalization behaves as

$$\mathcal{N} = (d_j)^{2F} \sim j^{2F}, \tag{3.19}$$

where we have discarded an inessential multiplicative constant as $j \to \infty$.

(3) The integration over ξ_f can be performed using the Fourier representation of the δ function

$$\int d\xi_f \exp\{\mathrm{i}j\xi_f \cdot A\} = \frac{1}{j^2}\delta_{n_0^\perp}(A_f) \tag{3.20}$$

with an inessential factor of $(2\pi)^2$ absorbed in the integration measure. Note that the vector ξ_f is constrained to lie in the plane orthogonal to n_f, so that it enforces the constraint $A_f = 0$ only in that plane. Since $A_f = \sum_\ell \eta_{\ell,f} A_\ell$, these constraints are not independent. The number of independent constraints is $2r$, since everything takes place in the plane orthogonal to a vector $n_f = \sigma_f n_0$ which does not depend on the face. Altogether, the integration over the ξ_f yield a factor of j^{-2r} and implement the constraints

$$\sum_\ell \eta_{\ell,f} A_\ell = 0 \quad \text{in the directions orthogonal to}\, n_0. \tag{3.21}$$

(4) Using the previous constraints, the real part of the action involving A only vanishes, $(A_f)^2 - (n_f \cdot A_f)^2 = 0$.

(5) Because the graph is simply connected, the constraints (3.21) imply the existence of vectors $C_v \in \mathbb{R}^3$ attached to the vertices and orthogonal to n_0 such that

$$A_\ell - (n_0 \cdot A_\ell) n_0 = C_{s(\ell)} - C_{t(\ell)}, \tag{3.22}$$

with $s(\ell)$ (respectively, $t(\ell)$) the source (respectively, the target) of the edge ℓ. Then, the phase associated to a face f reads

$$n_0 \cdot \Phi_f = \sum_{\ell \in \partial f} \eta_{\ell,f} n_0 \cdot (C_{s(\ell)} \wedge C_{t(\ell)}). \tag{3.23}$$

The total contribution of all the faces to the action vanishes since

$$\sum_f k_f n_f \cdot \Phi_f = \sum_{f,\ell} \eta_{\ell,f} k_f n_f \cdot (C_{s(\ell)} \wedge C_{t(\ell)}) = \sum_\ell \Big(\sum_f \eta_{\ell,f} k_f n_f \Big) \cdot (C_{s(\ell)} \wedge C_{t_\ell}) = 0, \tag{3.24}$$

using the closure condition (3.15).

(6) However, its is important to note that the components of A parallel to n_0 are not constrained by (3.22) and their contribution to the action vanishes identically in the quadratic approximation. This is the reason why we only get an upper bound in the maximally degenerate case.

Accordingly, the bound for the amplitude can be estimated as

$$\sum_{\substack{F-r \text{ independent spins} \\ \text{of order } j \leq \Lambda}} j^{2F} \times j^{-2r} \sim \Lambda^{3F-3r}, \tag{3.25}$$

which is the result obtained in [12].

It is interesting to note that for a simply connected graph, the rank r of the incidence matrix can be written as $r = F - (V - 1)$. Indeed, the system of Eq. (3.21), whose rank is $2r$ allows to write the $2L$ variables A_l in terms of $2(V-1)$ differences $C_v - C_{v'}$, all in the direction orthogonal to n_0. Therefore, one has $2L - 2r = 2V - 2$, so that $r = L - V + 1$ and the amplitude of a simply connected graph scales as

$$\mathcal{A}_{\mathcal{G}} \lesssim \Lambda^{3(\chi_{\mathcal{G}} - 1)}, \tag{3.26}$$

with $\chi_{\mathcal{G}} = F - L + V$ the Euler characteristics of the graph. This also reproduces the result of [13], since $\chi_{\mathcal{G}} = \dim H^2_{\mathcal{G}} - \dim H^1_{\mathcal{G}} + \dim H^0_{\mathcal{G}} = \dim H^2_{\mathcal{G}} + 1$ for a simply connected graph. This is also in accordance with the results of [12] for graphs with planar jacket. The faces F_{jacket} of the planar jacket obey $F_{\text{jacket}} - L + V = 2$, since the associated surface has the topology of a sphere, and the remaining faces are in bijections with the cycles followed by the $N = F - F_{\text{jacket}}$ strands in the middle, so that the degree of divergence reads $\omega_{\mathcal{G}} = 3(F_{\text{jacket}} - L + V - 1) + 3N = 3(N+1)$.

2. Nondegenerate case

In the nondegenerate case, the situation is slightly more complicated. The integration over the variables ξ_f yields a system of equations analogous to (3.21), but now with a vector n_f that varies from face to face,

$$\sum_\ell \eta_{\ell,f} A_\ell = 0 \quad \text{in the direction orthogonal to}\, n_f. \tag{3.27}$$

Because for fixed ℓ, the three (respectively, four) vectors $\eta_{\ell,f} n_f$ in dimension 3 (dimension 4) span a space of dimension 2 (respectively, 3), all the components of A_ℓ appear in the system (3.27), contrary to the maximally degenerate case (3.21), which only involves the components of A_ℓ orthogonal to n_0. In the example treated in detail below (see Sect. III B), (3.27) turns out to be equivalent to $\sum_\ell \eta_{\ell,f} A_\ell = 0$, which has rank $3r$ and yields the same degree of divergence. In the general case, we expect nondegenerate configurations to have a less divergent behavior, since the degree of divergence $3F - 3r$ obtained in [12] in the Abelian case is expected to be an overestimate in the general case and is the correct asymptotic behavior at least for many graphs.

3. Two-dimensional case

To close this section, let us see how the saddle point method allows us to recover the results presented in [22] in the simplest case of BF theory in dimension 2. In this case, GFT graphs are ordinary ribbon graphs with trivalent vertices representing triangles. The closure condition reads $k_{f_1} n_{f_1} = k_{f_2} n_{f_2}$ for every edge that separates two different faces and is vacuous for edges that appear twice as we go along a face (if we restrict ourselves to triangulations of orientable surfaces). Thus, for a genus 0 graph there is a single spin j and a single unit vector n. Moreover, the graph

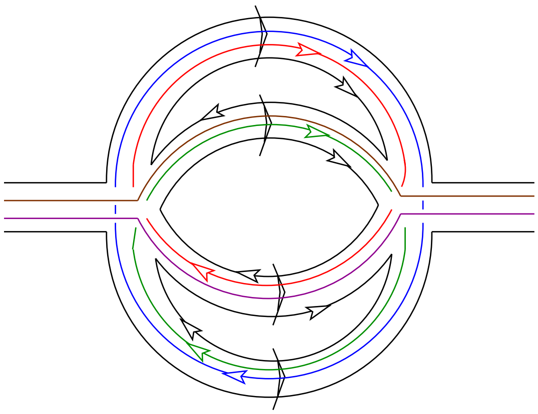

FIG. 4 (color online). The "self-energy" graph $\mathcal{G}_2$, quantum correction to the propagator.

is simply connected so the phase disappears. Choosing an arbitrary face, all other $F-1$ constraints are independent since the corresponding faces span a sphere with one hole that can be filled with the remaining face. Thus, $r=F-1$ and $\omega=3F-3(F-1)=3$, as expected from the relation (see [22])

$$\mathcal{A}_{\mathcal{G}} = \sum_{j\le\Lambda} j^{\chi_{\mathcal{G}}} \sim \Lambda^{\chi_{\mathcal{G}}+1}, \tag{3.28}$$

with $\chi_{\mathcal{G}}$ the Euler characteristic.

At higher genera the graph is no longer simply connected and the contribution of the antisymmetric part may be decisive. For instance, for the nonplanar double tadpole $\mathcal{G}_1$ (torus topology)

$$A_{\mathcal{G}_1} = \int dh_1 dh_2 \delta(h_1h_2(h_1)^{-1}(h_2)^{-1}), \tag{3.29}$$

an expansion to second order yields the action

$$S[n, A_1, A_2] = \mathrm{i} jn \cdot A_1 \wedge A_2. \tag{3.30}$$

The rank of this quadratic form is 4 and the Gaussian integration over A_1 and A_2 yields

$$\mathcal{A}_{\mathcal{G}_1} = \sum_{j\le\Lambda} j^2 \times j^{-4/2} \sim \Lambda, \tag{3.31}$$

in accordance with (3.28).

B. Divergence of the self-energy in the EPRL/FK model

The self-energy graph $\mathcal{G}_2$ of Fig. 4 has four open faces. It has six closed faces with two edges each. We label the internal propagators with an index a ranging from 1 to 4 and orient them in the same direction. We label the six closed faces with pairs of indices (a, b), $a<b$. Its amplitude reads

$$\mathcal{A}_{\mathcal{G}_2} = \prod_a du_a^{\pm} dv^{a\pm} \prod_{a<b} \mathcal{A}_{ab} \tag{3.32}$$

where, from (2.34) the face amplitude reads

$$\mathcal{A}_{ab} = \sum_j d_{j_+} d_{j_-} \beta_j^2 \int dn_{ab} dn'_{ab} \langle j_+ n_{ab} | u_{a+} u_{b+}^{-1} | j_+ n'_{ab}\rangle \times \langle j_+ n'_{ab} | v_{b+} v_{a+}^{-1} | j_+ n_{ab}\rangle \langle j_- n_{ab} | u_{a-} u_{b-}^{-1} | j_- n'_{ab}\rangle \times \langle j_- n'_{ab} | v_{b-} v_{a-}^{-1} | j_- n_{ab}\rangle. \tag{3.33}$$

Using (A16), we rewrite the amplitude above as

$$\mathcal{A}_{ab} = \sum_j d_{j_+} d_{j_-} \beta_j^2 \int dn dn' (\langle n | u_{a+} u_{b+}^{-1} | n'\rangle \times \langle n' | v_{b+} v_{a+}^{-1} | n\rangle)^{2j_+} (\langle n | u_{a-} u_{b-}^{-1} | n'\rangle \times \langle n' | v_{b-} v_{a-}^{-1} | n\rangle)^{2j_-}. \tag{3.34}$$

In order to perform a stationary phase analysis we rewrite the graph amplitude as

$$\mathcal{A}_{\mathcal{G}_2} = \sum_{j_f} \int \prod_a du_a^{\pm} \prod_a dv_a^{\pm} \prod_i dn_i \times \prod_f \{(d_{j_f})^2 d_{j_f^+} d_{j_f^-} \exp\{jS_f^+ + jS_f^-\}\}, \tag{3.35}$$

with $j_f^{\pm} = j\gamma^{\pm} k_f$, $k_f \in [0, 1]$ and j large. There is one coherent state per strand, which amounts here to label the coherent states by a couple of a face and an edge $i=(f, l)$ such that $\eta_{l,f} \neq 0$. The face action for $f=ab$ can be written as

$$S_f^{\pm} = 2\gamma^{\pm} k_f \log\{\langle n_{f,a} | u_a^{\pm} (u_b^{\pm})^{-1} | n_{f,b}\rangle \times \langle n_{f,b} | v_b^{\pm} (v_a^{\pm})^{-1} | n_{f,a}\rangle\}. \tag{3.36}$$

We employ the saddle point technique around the identity and develop the group elements as follows:

$$\begin{aligned} u_a^{\pm} &= 1 - \frac{(A_a^{\pm})^2}{2} + \mathrm{i}\sigma \cdot A_a^{\pm} + O(A_a^{\pm})^3, \\ v_a^{\pm} &= 1 - \frac{(B_a^{\pm})^2}{2} + \mathrm{i}\sigma \cdot B_a^{\pm} + O(B_a^{\pm})^3. \end{aligned} \tag{3.37}$$

Moreover, introducing the projector

$$|n_i\rangle\langle n_i| = \frac{1 + \mathrm{i}\sigma \cdot n_i}{2}, \tag{3.38}$$

the action at the identity for the face $f=ab$ reads

$$\begin{aligned} S_f^{\pm}[1, 1, n_i] &= \gamma^{\pm} k_{ab} \log \mathrm{Tr}\left\{\frac{1 + \mathrm{i}\sigma \cdot n_{f,a}}{2} \frac{1 + \mathrm{i}\sigma \cdot n_{f,b}}{2}\right\} \\ &= \gamma^{\pm} k_{ab} \log\left\{\frac{1 + n_{f,a} \cdot n_{f,b}}{2}\right\}, \end{aligned} \tag{3.39}$$

which is negative except for $n_{f,a} = n_{f,b} = n_f$. Therefore, we perform the expansion of the coherent state around an unit vector common to all the strands of the face

$$n_i = n_f + \xi_i - \frac{(\xi_i)^2}{2} n_f + O(\xi_i)^3, \quad \text{with} \quad n_f \cdot \xi_i = 0, \tag{3.40}$$

otherwise the integral is exponentially damped. To perform the expansion at second order of the action, it is convenient to rewrite this action as

$$\begin{aligned}S_f^{\pm} = {}& \gamma^{\pm} k_f \log \mathrm{Tr}\{|n_{f,a}\rangle\langle n_{f,a}|u_a^{\pm}(u_b^{\pm})^{-1}|n_{f,b}\rangle\langle n_{f,b}|\} + \gamma^{\pm} k_f \log \mathrm{Tr}\{|n_{f,b}\rangle\langle n_{f,b}|v_b^{\pm}(v_a^{\pm})^{-1}|n_{f,a}\rangle\langle n_{f,a}|\} \\ & - \gamma^{\pm} k_f \log \mathrm{Tr}\{|n_{f,a}\rangle\langle n_{f,a}||n_{f,b}\rangle\langle n_{f,b}|\}. \end{aligned} \tag{3.41}$$

Using the projector (3.38), the expansion to second order only involves traces of products of Pauli matrices and is straightforward but rather lengthy. A crucial intermediate result is the expansion to second order in A_1, A_2, ξ_1, ξ_2,

$$\begin{aligned}&\frac{1}{4}\mathrm{Tr}\bigg\{\bigg[1+\sigma\bigg(n+\xi_1-\frac{(\xi_1)^2}{2}n\bigg)\bigg]\bigg[1-\frac{(A_1-A_2)^2}{2}+\mathrm{i}\sigma\cdot(A_1-A_2+A_1\wedge A_2)\bigg]\bigg[1+\sigma\bigg(n+\xi_2-\frac{(\xi_2)^2}{2}n\bigg)\bigg]\bigg\} \\ &\quad = 1-\frac{(A_1-A_2)^2}{2}+\mathrm{i}n\cdot(A_1-A_2+A_1\wedge A_2)-\frac{(\xi_1-\xi_2)^2}{4}+\mathrm{i}(\xi_1+\xi_2)\frac{A_1-A_2}{2}+(\xi_1-\xi_2)\frac{n\wedge(A_1-A_2)}{2}.\end{aligned} \tag{3.42}$$

Gathering all terms together and taking the logarithm, we get

$$\begin{aligned}S_f^{\pm}[A^{\pm},B^{\pm},\xi_i] = {}& k_f\gamma^{\pm}\{-(A_a^{\pm}-A_b^{\pm})^2+[n_f\cdot(A_a^{\pm}-A_b^{\pm})]^2-(B_b^{\pm}-B_a^{\pm})^2+[n_f\cdot(B_b^{\pm}-B_a^{\pm})]^2+\mathrm{i}n_f\cdot(A_a^{\pm}-A_b^{\pm} \\ & +B_b^{\pm}-B_a^{\pm})+\mathrm{i}n_f\cdot(A_a^{\pm}\wedge A_b^{\pm}+B_b^{\pm}\wedge B_a^{\pm})+\mathrm{i}(\xi_{f,a}+\xi_{f,b})\cdot(A_a^{\pm}-A_b^{\pm}+B_b^{\pm}-B_a^{\pm}) \\ & -\frac{(\xi_{f,a}-\xi_{f,b})^2}{2}+(\xi_{f,a}-\xi_{f,b})\cdot[n_f\wedge(A_a^{\pm}-A_b^{\pm}-(B_b^{\pm}-B_a^{\pm}))]\}.\end{aligned} \tag{3.43}$$

To complete the computation, one has to perform a Gaussian integration with an action $S=\sum_f(S_f^+ + S_f^-)$. In order to disentangle this computation, it is convenient to perform the following change of variables:

$$A_a^{\pm} = A_a \pm \gamma^{\mp} X_a \quad \text{and} \quad B_a^{\pm} = B_a \pm \gamma^{\mp} Y_a. \tag{3.44}$$

The interest of this change of variables is that the terms linear in $A^{\pm}$ and $B^{\pm}$ now only involve A and B, while in the quadratic terms, the pair of variables A and B on one side and the pair X and Y on the other side decouple. We shall return in greater detail to this change of variable in Sec. III C in the case of a arbitrary graph, since it allows to separate the action, at the level of the quadratic approximation, into an $SU(2)$ BF action (variables A and B) and an ultralocal potential that only involves uncoupled variables attached to the vertices (variables X and Y). Turning back to the self-energy, we get

$$\begin{aligned}S_f[A,B,X,Y,\xi] &= S_f^+[A^+,B^+,\xi]+S_f^-[A^-,B^-,\xi] \\ &= k_f\{-(A_a-A_b)^2+[n_f\cdot(A_a-A_b)]^2-(B_b-B_a)^2+[n_f\cdot(B_b-B_a)]^2+\mathrm{i}n_f\cdot(A_a-A_b+B_b-B_a) \\ &\quad +\mathrm{i}n_f\cdot(A_a\wedge A_b+B_b\wedge B_a)+\mathrm{i}(\xi_{f,a}+\xi_{f,b})\cdot(A_a-A_b+B_b-B_a)-\frac{(\xi_{f,a}-\xi_{f,b})^2}{2} \\ &\quad +(\xi_{f,a}-\xi_{f,b})\cdot[n_f\wedge(A_a-A_b-(B_b-B_a))]\}+k_f\gamma^+\gamma^-\{-(X_a-X_b)^2+[n_f\cdot(X_a-X_b)]^2 \\ &\quad +\mathrm{i}n_f\cdot(X_a\wedge X_b)-(Y_b-Y_a)^2+[n_f\cdot(Y_b-Y_a)]^2+\mathrm{i}n_f\cdot(Y_b\wedge Y_a)\}.\end{aligned} \tag{3.45}$$

Performing the Gaussian integration over the two-dimensional vector $\chi_f=\xi_{f,a}-\xi_{f,b}$, one has

$$\int d\chi_f \exp jk_f\bigg\{-\frac{\chi_f^2}{2}+\chi_f\cdot[n_f\wedge(A_a-A_b-(B_b-B_a))]\bigg\} = \frac{2\pi}{j_f}\exp\frac{jk_f}{2}[n_f\wedge(A_a-A_b-(B_b-B_a))]^2. \tag{3.46}$$

Discarding an inessential constant in the limit $j\to\infty$ to alleviate the notations, the graph amplitude can therefore be written as

$$\mathcal{A}_{\mathcal{G}_2} = \sum_{j_f} j^{18}\bigg\{\int\prod_a dA_a\prod_a dB_a\prod_f d\xi_f \exp jS_{BF}(A,B,\xi)\int\prod_a dX_a \exp jQ(X)\int\prod_a dY_a\exp jQ(Y)\bigg\}, \tag{3.47}$$

with $\xi_f=\xi_{f,a}+\xi_{f,b}$. The BF-like action is

$$\begin{aligned}S_{BF}[A,B,\xi] = \sum_{a<b}k_{ab}\bigg\{&-\frac{1}{2}[n_f\wedge(A_a-A_b+B_b-B_a)]^2+\mathrm{i}n_{ab}\cdot(A_a-A_b+B_b-B_a)+\mathrm{i}n_{ab}\cdot(A_a\wedge A_b+B_b\wedge B_a) \\ &+\mathrm{i}\xi_{ab}\cdot(A_a-A_b+B_b-B_a)\bigg\},\end{aligned} \tag{3.48}$$

while the ultralocal terms are

$$Q[X] = \gamma^+\gamma^- \sum_{a<b} k_{ab}\{[n_{ab} \wedge (X_a - X_b)]^2 + \mathrm{i}n_{ab} \cdot (X_a \wedge X_b)\}. \quad (3.49)$$

The Gaussian integral over the variables A and B can be evaluated using the same techniques as in Sec. III A, devoted to BF theory. First, there are four closure conditions, one for each edge a,

$$\sum_{b=a+1}^{4} k_{ab}n_{ab} - \sum_{b=1}^{a-1} k_{ba}n_{ba} = 0 \quad (3.50)$$

or explicitly,

$$\begin{aligned} k_{12}n_{12} + k_{13}n_{13} + k_{14}n_{14} &= 0, \\ -k_{12}n_{12} + k_{23}n_{23} + k_{24}n_{24} &= 0, \\ -k_{13}n_{13} - k_{23}n_{23} + k_{34}n_{14} &= 0, \\ k_{14}n_{14} + k_{24}n_{24} + k_{34}n_{34} &= 0. \end{aligned} \quad (3.51)$$

Note that the first three equations are independent while the last one is their sum, so that the rank of the incidence matrix $\eta_{l,f}$ is 3. Let us investigate the case of nondegenerate configurations, which means that the six vectors $k_{ab}n_{ab}$ span a three-dimensional space. Geometrically, the solution of these closure constraints can be realized by constructing a tetrahedron (see Fig. 5) with faces labeled 1, 2, 3, 4 and assigning the vector $k_{ab}n_{ab}$ to an edge between faces a and b. Consequently, we sum over 6 independent spins in (3.35).

The Gaussian integration over the variables ξ_{ab} imposes the constraints

$$A_a - B_a = A_b - B_b \qquad \text{in the direction orthogonal to } n_{ab}. \quad (3.52)$$

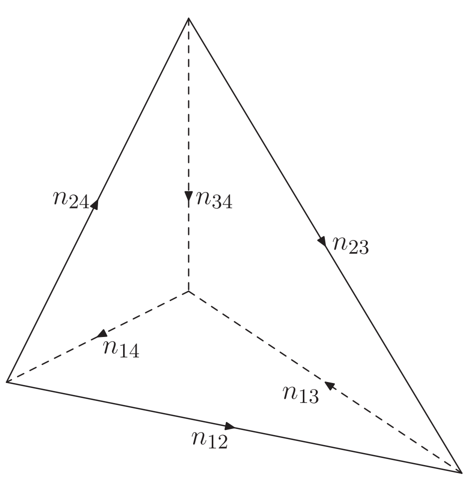

FIG. 5. The tetrahedron illustrating the closure condition.

However, out of the 12 relations in (3.52), only 9 of them are independent and they are equivalent to

$$A_a - B_a = C, \quad (3.53)$$

with $C \in \mathbb{R}^3$. First of all, it is clear that any solution of (3.53) is a solution of (3.52). Let us show that the converse also holds. Let consider all the equations involving edge 1,

$$A_1 - B_1 = A_a - B_a \quad \text{in the direction orthogonal to } n_{1a}, \quad a \in \{2, 3, 4\}. \quad (3.54)$$

Because of the closure constraint $\sum_{a>1} k_{1a}n_{1a} = 0$, the relation $A_1 - B_1 = A_a - B_a$ holds in the one dimensional space orthogonal to all three vectors $k_{1a}n_{1a}$. Thus the vectors $A_a - B_a$ are all equal along this direction. We then repeat the same reasoning for the other edges and conclude that the vectors $A_a - B_a$ are all equal along all directions using the nondegeneracy condition. As conclusion, the rank of (3.52) is 9 since it reduces the 12 degrees of freedom of the 4 vectors $A_a - B_a$ to a single vector and the Gaussian integration over ξ_{ab} enforcing this constraint yields a factor of j^{-9}.

Using this constraint, the real part of the quadratic action obviously vanishes. The imaginary part can be dealt with using the techniques of Sec. III A. Using first the constraint, we write

$$\begin{aligned} 0 &= (A_a + B_b) \wedge (B_a + A_b) \\ &= A_a \wedge B_a + B_b \wedge A_b + A_a \wedge A_b + B_b \wedge B_a. \end{aligned} \quad (3.55)$$

After summation over all faces, the net contribution of the phases to the amplitude vanishes

$$\sum_{a<b} \mathrm{i}k_{ab}n_{ab} \cdot [A_a \wedge A_b + B_b \wedge B_a] = -\sum_{a<b} \mathrm{i}k_{ab}n_{ab} \cdot [A_a \wedge B_a + B_b \wedge A_b], \quad (3.56)$$

where we have use the closure constraints. Altogether, the integration over the variables A, B and ξ yields

$$\int \prod_a dA_a \prod_a dB_a \prod_f d\xi_f \exp jS_{BF}(A, B, \xi) \sim j^{-9}, \quad (3.57)$$

as $j \to \infty$. Note that this, together with a j^{12} arising from the coherent state representation of the δ function (j^2 per face) and a summation over 6 independent spins, reproduces

$$\sum_{6 \text{ independent spins } \sim j < \Lambda} j^{12} \times j^{-9} \sim \Lambda^9, \quad (3.58)$$

which is the known result for $SU(2)$ BF theory. Since the rank r of the incidence matrix $\eta_{l,f}$ is 3, this reproduces with nondegenerate configurations the results of [12], with a degree of divergence $3F - 3r$.

Let us now consider the Gaussian integral over the independent variables X_a and Y_a,

$$\int \prod_a dX_a \exp jQ(X) \sim j^{-(\mathrm{rank}(Q)/2)}, \tag{3.59}$$

which simply amounts to compute the rank of the quadratic form

$$Q[X] = \gamma^+\gamma^- \sum_{a<b} k_{ab}\{[n_{ab} \wedge (X_a - X_b)]^2 + \mathrm{i}n_{ab} \cdot (X_a \wedge X_b)\}. \tag{3.60}$$

This quadratic form is associated with a symmetric bilinear form

$$B[X,Z] = \frac{1}{4}(Q(X+Z,X+Z) - Q(X-Z,X-Z)), \tag{3.61}$$

and its kernel is defined as the subspace of the variables X such that $B[X,Z] = 0$ for all Z. First, notice that $B[X,Z]$ is complex but the variables X and Z are real. Therefore, the orthogonality condition $B[X,Z] = 0$ has to be fulfilled for the real and the imaginary part separately. Since the real part is positive (but not definite positive), X has to obey

$$\sum_{a<b} k_{ab}[n_{ab} \wedge (X_a - X_b)]^2 = 0, \tag{3.62}$$

or equivalently $X_a - X_b = 0$ in the plane orthogonal n_{ab}. Using the nondegeneracy of the configuration, an analysis identical to that of the constraints (3.52) leads to $X_a = C$ with $C \in \mathbb{R}^3$ that do not depend on the edge. Then, the imaginary part of the relation $B(X,Z) = 0$ reads

$$\sum_{a<b}\{k_{ab}n_{ab} \cdot (C \wedge Z_b) + k_{ab}n_{ab} \cdot (Z_a \wedge C)\} = 0, \tag{3.63}$$

which is identically fulfilled for any Z_b because of the closure condition (3.50). Finally, the rank of Q is the dimension of the orthogonal of its kernel. Since the latter has dimension 3 and we have 4 vector variables X_a, we obtain $\mathrm{rank}(Q) = 12 - 3 = 9$, so that the Gaussian integral over X yields a power of $j^{-9/2}$. Obviously, the same holds for the integration over Y.

Therefore, we obtain the power counting for the self-energy with nondegenerate configurations as follows

$$\sum_{6\,\text{independent spins} \sim j < \Lambda} j^{24} \times j^{-6} \times j^{-9} \times (j^{-9/2})^2 \sim \Lambda^6, \tag{3.64}$$

with the factor j^{24} arising from a $d_{j^+}d_{j^-} \sim j^2$ for each of the six faces and a factor $d_j \sim j$ for each of the two strand in each face. The factor j^{-6} results form the Gaussian integration over the six variables $\chi_f = (\xi_{f,a} - \xi_{f,b})$ in (3.46) and the j^{-9} from the integration over A and B in (3.57). This reproduces the result of [11], with nondegenerate configurations. Note that this is an asymptotic behavior and not a mere bound as we had before, since all the zero modes of the quadratic approximation correspond to gauge degrees of freedom (3.53).

It is also of interest to notice that this result should also hold with finite nonzero spins on the external faces. Indeed, since the latter remain finite, the contribution of the external faces to the action can be neglected as $j \to \infty$.

Finally, let us mention that we have derived this power counting with nondegenerate configurations. In the next section, we shall discuss maximally degenerate configurations.

C. A bound for maximally degenerate configurations

Consider a general graph $\mathcal{G}$ in the EPRL/FK model with F faces f. Since we are going to take the limit $j_f \to \infty$ for the internal spins, the contribution of the external faces can be neglected, as long as their spins remain finite. Recall that the graph amplitude can be written as

$$\mathcal{A}_{\mathcal{G}} = \int \prod dh \prod \mathcal{A}_f, \tag{3.65}$$

with the face amplitude given by (2.34). The graph amplitude may be rewritten as in (3.1)

$$\mathcal{A}_f = \sum_{j_f}\left\{d_{j_f^+}d_{j_f^-}(d_j)^p \int \prod dn \prod dh \exp j \sum_f \{S_f^+ + S_f^-\}\right\}, \tag{3.66}$$

with

$$S_f^{\pm}[n,h] = 2k_f\gamma^{\pm} \sum_{1\le q\le p} \log\Big\{\langle n_{f,l_q} | (h^+_{v_q,l_q})^{\epsilon_{v_q,l_q}\eta_{l_q,f}} (h^+_{v_q,l_{q+1}})^{\epsilon_{v_q,l_{q+1}}\eta_{l_{q+1},f}} | n_{f,l_q}\rangle\Big\}. \tag{3.67}$$

In the limit $j \to \infty$, we expect that the main contribution to this integral arises from the neighborhood of the identity for the group elements. At the identity, the action reads

$$S_f^{\pm}[n,1] = 2k_f\gamma^{\pm} \log \mathrm{Tr}\left[\overrightarrow{\prod_q} \frac{1}{2}(1 + \mathrm{i}n_{f,l_q} \cdot \sigma)\right]. \tag{3.68}$$

This is the trace of a product of rank one projectors; its real part is always negative and vanishes when all the projectors are equal. Therefore, we expand the unit vectors n_{f,l_q} around a unit vector n_f common to all edges of the face,

$$n_{f,l_q} = n_f + \xi_{f,l_q} - \frac{(\xi_{f,l_q})^2}{2} n_f + O(\xi_{f,l_q})^3, \quad \text{with} \quad n_f \cdot \xi_{f,l_q} = 0, \tag{3.69}$$

together with an expansion of the group elements around the identity

$$h_{v,l} = 1 - \frac{(A_{v,l})^2}{2} + \mathrm{i}\sigma \cdot A_{v,l} + O(A_{v,l})^3. \tag{3.70}$$

The expansion of the action to second order follows the same steps as Sec. III B. It is convenient to introduce

$$D^{\pm}_{f.v_q} = \epsilon_{v_q,l_q}\eta_{l_q,f}A^{\pm}_{v_q,l_q} + \epsilon_{v_q,l_{q+1}}\eta_{l_{q+1},f}A^{\pm}_{v_q,l_{q+1}} \quad (3.71)$$

and

$$\Phi^{\pm}_{f,v_q} = \eta_{l_q,f}\eta_{l_{q+1},f}A^{\pm}_{v_q,l_q} \wedge A^{\pm}_{v_q,l_{q+1}}. \quad (3.72)$$

After some algebra, the second order expansion of the action reads

$$\begin{aligned} S^{\pm}_f[A_{v,l},\xi_{f,l}] = k_f\gamma^{\pm}\sum_q\Big\{&(n_f\wedge D^{\pm}_{f,l_q})^2 + 2\mathrm{i}n_f\cdot D^{\pm}_{f,v_q} \\ &+ 2\mathrm{i}n_f\cdot\Phi^{\pm}_{f,v_q} - \frac{1}{2}(\xi_{f,l_q})^2 + \frac{1}{2}\xi_{f,l_q}\cdot\xi_{f,l_{q+1}} \\ &+ \frac{\mathrm{i}}{2}n_f\cdot(\xi_{f,l_q}\wedge\xi_{f,l_{q+1}}) + \mathrm{i}\xi_{f,l_q}\cdot(D^{\pm}_{f,v_{q-1}} \\ &+ D^{\pm}_{f,v_q}) + \xi_{f,l_q}\cdot[n_f\wedge(D^{\pm}_{f,v_q} - D^{\pm}_{f,v_{q-1}})]\Big\}. \end{aligned} \quad (3.73)$$

In order to simplify the analysis, we perform a change of variable similar to (3.44),

$$A^{\pm}_{v,l} = A_{v,l} \pm \gamma^{\mp}X_{v,l}. \quad (3.74)$$

Terms linear in $A^{\pm}_{v,l}$ are all of the form

$$\gamma^{+}L_{v,l}\cdot A^{+}_{v,l} + \gamma^{-}L_{v,l}\cdot A^{-}_{v,l} = L_{v,l}A_{v,l}, \quad \text{with} \quad L_{v,l}\in\mathbb{R}^3, \quad (3.75)$$

so that they do not involve the variables $X_{v,l}$. Terms quadratic in $A^{\pm}_{v,l}$ are all of the form

$$\gamma^{+}B[A^{+}_{v,l},A^{+}_{v',l'}] + \gamma^{-}B[A^{-}_{v,l},A^{-}_{v',l'}], \quad (3.76)$$

where the bilinear form $B[A^{\pm}_{v,l},A^{\pm}_{v',l'}]$ is either a scalar product $A^{\pm}_{v,l}\cdot A^{\pm}_{v',l'}$ or a wedge product $n_f\cdot(A^{\pm}_{v,l}\wedge A^{\pm}_{v',l'})$. Substituting $A^{\pm}_{v,l}$ and $A^{\pm}_{v',l'}$, it is easily seen that

$$\begin{aligned} &\gamma^{+}B[A^{+}_{v,l},A^{+}_{v',l'}] + \gamma^{-}B[A^{-}_{v,l},A^{-}_{v',l'}] \\ &\quad = B[A_{v,l},A_{v',l'}] + \gamma^{+}\gamma^{-}B[X_{v,l},X_{v',l'}]. \end{aligned} \quad (3.77)$$

Then, we can express the total action $S_f = S^{+}_f + S^{-}_f$ as a sum of a BF-type action

$$\begin{aligned} S^{BF}_f[A,\xi] = k_f\sum_q\Big\{&(n_f\wedge D_{f,l_q})^2 + 2\mathrm{i}n_f\cdot D_{f,v_q} \\ &+ 2\mathrm{i}n_f\cdot\Phi_{f,v_q} - \frac{1}{2}(\xi_{f,l_q})^2 + \frac{1}{2}\xi_{f,l_q}\cdot\xi_{f,l_{q+1}} \\ &+ \frac{\mathrm{i}}{2}n_f\cdot(\xi_{f,l_q}\wedge\xi_{f,l_{q+1}}) + \mathrm{i}\xi_{f,l_q}\cdot(D_{f,v_{q-1}} + D_{f,v_q}) \\ &+ \xi_{f,l_q}\cdot[n_f\wedge(D_{f,v_q} - D_{f,v_{q-1}})]\Big\}, \end{aligned} \quad (3.78)$$

with $D^{\pm}_{f,v}$ and $\Phi_{f,v}$ as in (3.71) and (3.72) but with $A_{v,l}$ instead of $A^{\pm}_{v,l}$, and an ultralocal potential

$$Q_f[X] = \gamma^{+}\gamma^{-}k_f\sum_q\{(n_f\wedge D_{f,l_q})^2 + 2\mathrm{i}n_f\cdot\Phi_{f,v_q}\}. \quad (3.79)$$

To relate the BF face action to the more conventional one we encountered in Sec. III A, let us perform the integration over the variables $\xi_{f.v_q}$, starting with ξ_{f,v_p},

$$\begin{aligned} &\int d\xi_{f,v_p}\exp jk_f\Big\{\frac{1}{2}(\xi_{f,l_p})^2 + \frac{1}{2}\xi_{f,l_p}\cdot(\xi_{f,l_{p-1}} + \xi_{f,l_1}) + \frac{\mathrm{i}}{2}\xi_{f,l_p}\cdot[n_f\wedge(\xi_{f,l_{p-1}} - \xi_{f,l_1})]\mathrm{i}\xi_{f,l_p}\cdot(D_{f,v_{p-1}} + D_{f,v_p}) \\ &\quad + \xi_{f,l_p}\cdot[n_f\wedge(D_{f,v_p} - D_{f,v_{p-1}})]\Big\} \\ &= \frac{1}{j^2}\exp\frac{jk_f}{2}\Big\{\frac{1}{2}(\xi_{f,l_{p-1}} + \xi_{f,l_1}) + \frac{\mathrm{i}}{2}[n_f\wedge(\xi_{f,l_{p-1}} - \xi_{f,l_1})] + \mathrm{i}[D_{f,v_{p-1}} + D_{f,v_p} - n_f(n_f\cdot(D_{f,v_{p-1}} + D_{f,v_p}))] \\ &\quad + n_f\wedge(D_{f,v_p} - D_{f,v_{p-1}})\Big\}^2 \\ &= \frac{1}{j^2}\exp jk_f\Big\{\frac{1}{2}\xi_{f,l_{p-1}}\cdot\xi_{f,l_1} + \frac{\mathrm{i}}{2}n_f\cdot(\xi_{f,l_{p-1}}\wedge\xi_{f,l_1}) - 2\mathrm{i}n_f\cdot(D_{f,v_{p-1}}\wedge D_{f,v_1}) - 2(n_f\wedge D_{f,v_{p-1}})(n_f\wedge D_{f,v_{p-1}}) \\ &\quad + \xi_{f,l_{p-1}}\cdot(\mathrm{i}D_{f,v_p} + n_f\wedge D_{f,v_p}) + \xi_{f,l_1}\cdot(\mathrm{i}D_{f,v_{p-1}} + n_f\wedge D_{f,v_{p-1}})\Big\}. \end{aligned} \quad (3.80)$$

Note that ξ_{f,v_p} is orthogonal to n_f so that it couples only to the projection of $D_{f,v_{p-1}} + D_{f,v_p}$ onto the subspace orthogonal to n_f. Gathering all the terms in the action pertaining to the edges l_{q-1} and l_1, we get

$$\begin{aligned} &-\frac{1}{2}(\xi_{f,l_{q-1}})^2 - \frac{1}{2}(\xi_{f,l_1})^2 + \frac{1}{2}\xi_{f,l_{q-1}}\cdot\xi_{f,l_1} + \frac{\mathrm{i}}{2}n_f\cdot(\xi_{f,l_{q-1}}\wedge\xi_{f,l_1})\mathrm{i}\xi_{f,l_{p-1}}\cdot(D_{f,v_{p-2}} + D_{f,v_{p-1}} + D_{f,v_p}) \\ &\quad + \xi_{f,l_{p-1}}\cdot[n_f\wedge(D_{f,v_{p-1}} - D_{f,v_{p-2}} + D_{f,v_p})]\mathrm{i}\xi_{f,l_1}\cdot(D_{f,v_p} + D_{f,v_1} + D_{f,v_{p-1}}) + \xi_{f,l_1}\cdot[n_f\wedge(D_{f,v_1} - D_{f,v_{p-1}} - D_{f,v_p})] \\ &\quad - (D_{f,v_p}\wedge n_f)^2 - (D_{f,v_{p-1}}\wedge n_f)^2 - 2(D_{f,v_p}\wedge n_f)\cdot(D_{f,v_{p-1}}\wedge n_f) + 2\mathrm{i}n_f\cdot(\Phi_{v_q} - D_{f,v_{p-1}}\wedge D_{f,v_1}). \end{aligned} \quad (3.81)$$

The integration over the variable ξ_{f,l_p} has a simple graphical interpretation. We have contracted the line l_q and merged the vertex v_q (associated with D_{f,V_q}) with the vertex v_{q-1} (associated with $D_{f,v_{q-1}}$) into a new vertex (still called v_{q-1}), associated with $D_{f,v_{q-1}} + D_{f,v_q}$ and $\Phi_{f,v_q} - D_{f,v_{p-1}} \wedge D_{f,v_1}$. Therefore, we may pursue this procedure till we obtain a face with only two edges. Then, we proceed as in Sec. III B for the self-energy and integrate over $\xi_{f,1} - \xi_{f,2}$. The remaining variable $\xi_{f,1} + \xi_{f,2}$ is a Lagrange multiplier for the constraint $\sum_q D_{f,v_q} = 0$, or explicitly

$$\sum_{1\leq q\leq p} \epsilon_{v_q,l_q}\eta_{l_q,f}A_{v_q,l_q} + \epsilon_{v_q,l_{q+1}}\eta_{l_{q+1},f}A_{v_q,l_{q+1}} = 0, \quad (3.82)$$

which is nothing but the constraint (3.21) written in terms of the variables $\epsilon_{v_{q-1},l_q}A_{v_{q-1},l_q} + \epsilon_{v_q,l_q}A_{v_q,l_q}$. Then, the rest of the discussion follows the same path as in Sec. III A. There are $2r$ independent constraints in the maximally degenerate case. The real part of the action vanishes identically as well as the imaginary part for a simply connected graph, once we have used these constraints and the closure constraints. Accordingly, we have

$$\int \prod_{v,l} dA_{v,l} \prod_{f,l} dn_{f,l} \exp j\Big\{\sum_f S_f^{BF}[A,n]\Big\} \lesssim j^{\left(\sum_f L_f - 1\right)} \times j^{-2r}, \quad (3.83)$$

with L_f the number of edges in the face f.

Let us finally analyze the ultralocal terms given by the quadratic form Q_f defined in (3.79). Gathering the contributions of all faces, we get

$$Q[X] = \sum_v \Big\{\sum_f \gamma^+\gamma^- k_f[n_f \wedge (A_{v,\overleftarrow{l}_{f,v}} - A_{v,\vec{l}_{f,v}})]^2 + 2in_f \cdot (A_{v,\overleftarrow{l}_{f,v}} \wedge A_{v,\vec{l}_{f,v}})\Big\}, \quad (3.84)$$

with $\overleftarrow{l}_{f,v}$ (respectively, $\vec{l}_{f,v}$) the edge entering (respectively, leaving) the vertex v along the face f. First we notice that this is a sum over all vertices of quadratic forms defined at each vertex involving only variables attached to that vertex. This is the reason why we called such a term "ultralocal."

We then proceed as we did for the self-energy. The quadratic form has a real and an imaginary part, but its arguments are real. Therefore, the kernel of the associated bilinear form is the intersection of the kernel of the real part and of the imaginary part. Because the real part is a sum of squares, at each vertex and for each face we have

$$A_{v,\overleftarrow{l}_{f,v}} = A_{v,\vec{l}_{f,v}} \quad \text{in the direction orthogonal to } n_f. \quad (3.85)$$

Since in the maximally degenerate case all the n_f are proportional to n_0, this simply implies that all vectors $A_{v,l} = C_v$ in the plane orthogonal to n_0, while the components collinear to n_0 are left unconstrained. Then, as in Sec. III B, the closure constraints imply that $A_{v,l} = C_v$ also lies in the kernel of the imaginary part. If we denote by d_v the valence of vertex v (d_v may be lower than 5 since the external faces carrying spin 0 have to be removed), we get a rank of $2d_v - 2$, (there are $3d_v$ variables and $2 + d_v$ solutions), so that

$$\int \prod dX \exp jQ(X) \lesssim j^{-\left(\sum_v (2d_v-2)/2\right)}. \quad (3.86)$$

Taking all the terms together, we get

$$A_{\mathcal{G}} \lesssim \Lambda^{3F-3r+F+V-\sum_v d_v}. \quad (3.87)$$

The first term is the power counting of the graph $\mathcal{G}$ in BF theory with group $SU(2)$, while the second one results from a difference of normalization between EPRL/FK and BF theories. The last one is minus the half of sum of the ranks of the ultra local quadratic forms at each vertex. Since the latter are less important for nondegenerate configurations, we expect the maximally degenerate configuration to give a larger contribution, as long as the external spin remain finite. In particular, for the self-energy we have $d_v = 4$ so that the maximally degenerate configurations are bounded by Λ^9.

When applied to the self-energy with $d_v = 4$ because we set the external spins to 0, we get a bound in Λ^9, which suggests that degenerate configurations dominate in the EPRL model. However, this is only an upper bound since the zero modes of the degenerate configurations are not all gauge degrees of freedom, in particular, the component of A and B along n_0 do not contribute to the action in the quadratic approximation. These modes require a more thorough study involving higher order terms. Nevertheless, using the asymptotic behavior of $6j$ symbols and fusion coefficients (see [11]), we show in Appendix C that degenerate configurations indeed dominate this correction to the self-energy, but with an asymptotic behavior in Λ^7 instead of Λ^9. This is not in contradiction with the results of [11], since the latter use the relation $(6j)^2 \sim \frac{1}{V}$, which implicitly assumes that the configuration is nondegenerate. Therefore, in the sum over spins we have to identify a partial sum made of spins obeying a relation such that maximally degenerate configurations exist. This partial sum behaves like Λ^7, while the remaining terms containing the nondegenerate configurations are in Λ^6.

IV. CONCLUDING REMARK: HINT ON A PHASE TRANSITION

By parity, the ϕ^5 4-dimensional GFT has no two-point function contribution to first order in the coupling constant. At second order beyond the self-energy graph $\mathcal{G}_2$, the only other graphs have tadpoles, hence they are absent in the colored model; in the noncolored model they have fewer faces, so we can expect the amplitude of $\mathcal{G}_2$ to provide the dominant correction to the effective propagator of the model.

Since that amplitude $\mathcal{A}_{\mathcal{G}_2}$ is positive, we can expect the whole self-energy correction Σ to be also positive. The corresponding geometric power series for the dressed or effective propagator

$$C_{\text{dressed}} = C + C\Sigma C + C\Sigma C\Sigma C + \ldots = C\left(\frac{1}{1 - \Sigma C}\right) \tag{4.1}$$

should therefore be singular when the spectrum of ΣC has eigenvalue 1. This should occur for λ large enough, depending on the ultraviolet cutoff Λ. This is usually the signal of a phase transition. For instance, in an ordinary ϕ^4 model a positive mass term corresponds to a double well potential which signals a breaking of the $\phi \to -\phi$ symmetry. In the vector ϕ^4 "Ginzburg-Landau" model, it leads to the famous continuous symmetry breaking with appearance of an associated Goldstone boson.

At a more speculative level, this hint of a phase transition supports a scenario in which ordinary macroscopic smooth space-time is an emergent phenomenon. Group field theory, in particular its perturbative phase, might be a more fundamental description and space-time might result from condensation through a phase transition. This scenario is a version of what has been called geometrogenesis.

In this scenario the relationship of group field theory to space-time, gravitons and general relativity would be somewhat similar to that between QCD and effective theories of nuclear forces.

ACKNOWLEDGMENTS

T. K. and P. V. thank CPHT École Polytechnique and LPT Orsay for the hospitality. A. T. acknowledges the CNCSIS Grant No. Idei 454/2009, ID-44, and the Grant No. PN 09 37 01 02. P. V. acknowledges the European Science Foundation Exchange Grant No. 2595 under the Research Networking Program "Quantum Geometry and Quantum Gravity" and thanks the Università di Napoli for partial financial support. We also acknowledge J. Ben Geloun, R. Gurau and S. Speziale for fruitful discussions.

APPENDIX A: HARMONIC ANALYSIS ON $SU(2)$ AND COHERENT STATES

We include in this appendix well-known formulas for self-completeness. We start with

$$\int dg R^{(j)m}_{k}(g) = \delta_{j0}\delta_{m0}\delta_{k0} \tag{A1}$$

$$R^{(j)m}_{k}(g) R^{(\tilde{j})\tilde{m}}_{\tilde{k}}(g) = \sum_{J=|j-\tilde{j}|}^{j+\tilde{j}} (j, m; \tilde{j}, \tilde{m}|J, M) \times (J, K|j, k; \tilde{j}, \tilde{k}) R^{(J)M}_{K}(g), \tag{A2}$$

where $R^{(j)m}_{k}(g)$ are unitary representations of $SU(2)$ and $(J, K|j, k; \tilde{j}, \tilde{k})$ are the Clebsch-Gordan coefficients. We use the normalizations of [4].

We have

$$\int dg \bar{R}^{(j)}{}_{n}{}^{m}(g) R^{(j')p}{}_{q}(g) = \frac{1}{d_j}\delta(j, j')\delta^m_q \delta^p_n \tag{A3}$$

$$\int dg R^{(j)m}{}_{n}(g) R^{(j')p}{}_{q}(g) = \frac{1}{d_j}\delta(j, j')\epsilon^{mp}\epsilon_{nq} \tag{A4}$$

with $\bar{R}^{(j)}{}_{n}{}^{m}(g) = R^{(j)m}{}_{n}(g^{-1})$ which imply

$$\int dg \, \mathrm{Tr}_j Ag \, \mathrm{Tr}_{j'} Bg^{-1} = \frac{1}{d_j}\delta(j, j') \, \mathrm{Tr}_j AB \tag{A5}$$

$$\int dg \, \mathrm{Tr}_j Ag \, \mathrm{Tr}_{j'} Bg = \frac{1}{d_j}\delta(j, j') \, \mathrm{Tr}_j A\epsilon B^T \epsilon^T \tag{A6}$$

with $\epsilon \in SU(2)$,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A7}$$

We have $\epsilon^T \epsilon = 1$ and $\epsilon g \epsilon^T = \bar{g}$.

1. nj symbols

We have the $3j$ symbols

$$\iota^{m_1 m_2 m_3} = \frac{(-1)^{j_1 - j_2 + m_3}}{\sqrt{d_{j_3}}}(j_3, -m_3|j_1, m_1; j_2, m_2), \tag{A8}$$

the $6j$ symbols

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{pmatrix} = \sum_{m_1..m_6} \iota^{m_4 m_3 m_5} \iota_{m_5 m_1 m_6} \iota_{m_3}{}^{m_2 m_1} \iota_{m_2 m_4}{}^{m_6}, \tag{A9}$$

and the $9j$ symbols

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix} = \sum_{m_1..m_6} \iota^{m_1 m_2 m_3} \iota^{m_4 m_5 m_6} \iota^{m_7 m_8 m_9} \iota_{m_1 m_4 m_7} \iota_{m_2 m_5 m_8} \iota_{m_3 m_6 m_9}. \tag{A10}$$

For the $15j$ symbols see [4]. The indices are raised and lowered with the tensor ϵ.

2. Coherent states

Let us first consider the $SU(2)$ case. We introduce the following parametrization for coherent states in the spin-$1/2$ fundamental representation

$$\left|\frac{1}{2}, n\right\rangle = e^{i\theta \hat{m}\cdot(\sigma/2)} \left|\frac{1}{2}, \frac{1}{2}\right\rangle, \tag{A11}$$

with

$$\hat{m} = (\sin\phi, -\cos\phi, 0), \tag{A12}$$

and σ_i the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \text{(A13)}$$

This represents a rotation g_n of the vector $n_0 = (0, 0, 1)$ of an angle θ around the $\hat{m}$ direction

$$n_0 \to n = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta), \quad \text{(A14)}$$

with $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$. The coherent state $|j, n>$ is obtained in the same way, replacing the generators $i\sigma_i/2$ with the appropriate operators J_i in the $2j + 1$ dimensional representation. With this parametrization, the scalar product of coherent states reads

$$\langle j, n|j, \tilde{n}\rangle = \left(\cos\frac{\theta}{2} \cos\frac{\tilde{\theta}}{2} + \sin\frac{\theta}{2} \sin\frac{\tilde{\theta}}{2} e^{-i(\phi-\tilde{\phi})}\right)^{2j}, \quad \text{(A15)}$$

which implies

$$\langle j, n|j, \tilde{n}\rangle = \left(\left\langle\frac{1}{2}, n \middle| \frac{1}{2}, \tilde{n}\right\rangle\right)^{2j}. \quad \text{(A16)}$$

In the representation space V^j of dimension $d_j \equiv 2j+1$, we have

$$\mathbf{1}_j = \sum_m |j, m\rangle\langle j, m|, \quad \text{(A17)}$$

where $|j, m\rangle$, $m \in [-j, j]$ is the usual orthonormal basis in V^j. We have

$$R^{(j')m}{}_{m'}(g) \equiv \langle j, m|g|j, m'\rangle. \quad \text{(A18)}$$

Hence

$$\delta^m_{m'} = d_j \int_{SU(2)} dg R_j^{(j)m}(g) \bar{R}_{m'}^{(j)j}(g). \quad \text{(A19)}$$

APPENDIX B: SELF-ENERGY: COMPARISON WITH [11] AND NORMALIZATION CONVENTIONS

We return to the "self-energy" graph $\mathcal{G}_2$ of Fig. 4. We first rewrite the propagator in a slightly different way, using the gauge invariance. We perform an $SU(2)$ gauge transformation. We multiply the $u^\pm$, $v^\pm$ variables simultaneously by $SU(2)$ elements h and $\tilde{h}$, which are the same for the left and right components

$$u^\pm \to h^{-1}u^\pm, \qquad v^\pm \to \tilde{h}^{-1}v^\pm, \quad \text{(B1)}$$

and we integrate over h, $\tilde{h}$ so that (2.27) becomes

$$\begin{aligned} C(g, g') = &\int dh d\tilde{h} du dv \prod_f \sum_{j_f} \alpha_f \beta_f \int dn_f \\ &\times \sum_{m,\tilde{m},k,\tilde{k}} (g'_{f+} v_+^{-1}|j_+ m\rangle\langle j_+ \tilde{m}|u_+ g_{f+} R_{j_+}^{(j_+)m} \\ &\times (\tilde{h}n_f) R_{\tilde{m}}^{(j_+)j_+}((hn_f)^\dagger)) \otimes (g'_{f-} v_-^{-1}|j_- k\rangle \\ &\times \langle j_- \tilde{k}|u_- g_{f-} R_{j_-}^{(j_-)k}(\tilde{h}n_f) R_{\tilde{k}}^{(j_-)j_-}((hn_f)^\dagger)). \quad \text{(B2)} \end{aligned}$$

Note that we have also used (2.13). Considering the tensor product of representations (A2), we get

$$\begin{aligned} C(g, g') = &\int dh d\tilde{h} du dv \prod_f \sum_{j_f} \alpha_f \beta_f \int dn_f \sum_{m,\tilde{m},k,\tilde{k}} (j_+ + j_-, M|j_+, m; j_-, k)(j_+, \tilde{m}; j_-, \tilde{k}|j_+ + j_-, \tilde{M}) R^{(j_+ + j_-)M}{}_{j_+ + j_-}(\tilde{h}n_f) \\ &\times R^{(j_+ + j_-)j_+ + j_-}{}_{\tilde{M}}((hn_f)^\dagger)(g'_{f+} v_+^{-1}|j_+ m\rangle\langle j_+ \tilde{m}|u_+ g_{f+}) \otimes (g'_{f-} v_-^{-1}|j_- k\rangle\langle j_- \tilde{k}|u_- g_{f-}). \quad \text{(B3)} \end{aligned}$$

The integration over n_f through (A4) finally yields

$$C(g, g') = \int dH du dv \prod_{f=1}^4 \sum_{j_f} \alpha_{j_f} \mathrm{Tr}(u g_f (g'_f)^{-1} v^{-1} T_{j_f}^\gamma(H)), \quad \text{(B4)}$$

with $H = \tilde{h}h^\dagger$, and

$$\begin{aligned} T_{j_f}^\gamma(H) = &\beta_{j_f} \sum_{\substack{m\tilde{m} \\ k\tilde{k}}} |j_{f+} m_f\rangle\langle j_{f+} \tilde{m}_f| \otimes |j_{f-} k_f\rangle\langle j_{f-} \tilde{k}_f| \\ &\times \iota_{m_f k_f - M_f} \iota^{\tilde{m}_f \tilde{k}_f - \tilde{M}_f} R^{(j_{f+} + j_{f-})M_f}{}_{\tilde{M}_f}(H), \quad \text{(B5)} \end{aligned}$$

where we have used (A8). Using this expression and the amplitude expression, it is checked below that this corresponds to the normalizations of [11], with $k = 2$.

Note that this choice $k = 2$ is *not* the one advocated in [23].

To prove this statement we rewrite the graph amplitude for the self-energy, inserting the new expression of the propagator (B4). We can neglect the open faces, since the external legs have vanishing spin. Hence, we get

$$\mathcal{A}(\mathcal{G}_2) = \int dH du dv \prod_{a<b} \mathcal{A}_{ab}(u, v, H), \quad \text{(B6)}$$

with the face amplitude

$$\begin{aligned} \mathcal{A}_{ab} = &\sum_{j_a, j_b} \alpha_{j_a} \alpha_{j_b} \int dg_{ab} d\tilde{g}_{ab} \mathrm{Tr}_{j_{a+} \otimes j_{a-}} \\ &\times (u_a g_{ab} \tilde{g}_{ab}^{-1} v_a^{-1} T_{j_a}^\gamma(H_a)) \\ &\times \mathrm{Tr}_{j_{b+} \otimes j_{b-}}(u_b g_{ab} \tilde{g}_{ab}^{-1} v_b^{-1} T_{j_b}^\gamma(H_b)). \quad \text{(B7)} \end{aligned}$$

The amplitude for this self-energy graph is written in [11] as

$$\mathcal{A}(\mathcal{G}_2) = \sum_{j_{ab} \iota_a} \prod_{a<b} d(j_{ab})(6j(j_{ab}^+)6j(j_{ab}^-))^2 \left(\prod_a f_a\right)^2, \quad \text{(B8)}$$

where in [11], $d(j_{ab}) \simeq j_{ab}^k$ in the ultraspin regime. One further has

$$6j(j_{ab}) = \begin{pmatrix} j_{12} & j_{13} & j_{14} \\ j_{23} & j_{24} & j_{34} \end{pmatrix} \tag{B9}$$

defined as in Eq. (A9), while

$$f_1 = \sqrt{d_{j_{12}} d_{j_{13}} d_{j_{14}}} \begin{Bmatrix} j_{12}^+ & j_{14}^+ & j_{13}^+ \\ j_{12}^- & j_{14}^- & j_{13}^- \\ j_{12} & j_{14} & j_{13} \end{Bmatrix}, \tag{B10}$$

and cyclically for f_2, f_3, and f_4. Both expressions (B6) and (B8) are calculated for external $j's$ put to zero, that is, the contributions of faces with external legs are put to 1.

To compare our expression to (B8), we perform the integration on the variables u and v, and we rewrite the integrand of face amplitudes in the form

$$(g_{ab}^{\pm} \tilde{g}_{ab}^{\pm -1})^{m_{ab}}_{k_{ab}} (v_a^{\pm -1})^{k_{ab}}_{p_{ab}} (T_{j_a}^{\gamma})^{p_{ab}}_{q_{ab}} (H_a)(u_a^{\pm})^{q_{ab}}_{m_{ab}}, \tag{B11}$$

where we use the shorthand notation

$$(g)^m_n = R_n^{(j)m}(g). \tag{B12}$$

We need to perform integrals of the form

$$\int du_a^{\pm} \prod_{b \neq a} (u_a^{\pm})^{q_{ab}}_{m_{ab}} \tag{B13}$$

$$\int dv_a^{\pm} \prod_{b \neq a} (v_a^{\pm -1})^{k_{ab}}_{p_{ab}}, \tag{B14}$$

with $a, b = 1, \ldots, 4$ (we have 16 integrals in total). Using (A2) and (A3), we obtain for $a = 1$

$$\int du_1^{\pm} (u_1^{\pm})^{q_{12}}_{m_{12}} (u_1^{\pm})^{q_{13}}_{m_{13}} (u_1^{\pm})^{q_{14}}_{m_{14}} = \iota_{\pm}^{q_{12}q_{13}q_{14}} \iota^{\pm}_{m_{12}m_{13}m_{14}} \tag{B15}$$

$$\int dv_1^{\pm} (v_1^{\pm -1})^{k_{12}}_{p_{12}} (v_1^{\pm -1})^{k_{13}}_{p_{13}} (v_1^{\pm -1})^{k_{14}}_{p_{14}} = \iota_{\pm}^{k_{12}k_{13}k_{14}} \iota^{\pm}_{p_{12}p_{13}p_{14}} \tag{B16}$$

and similar results for $a = 2, 3, 4$. All indices are double for $+$ and $-$ variables, that is, they should carry an extra superscript (e.g. $m_{ab} \to m_{ab}^{\pm}$). We now perform the integration on the variables g, $\tilde{g}$. For each face $\mathcal{A}_{ab}$ they appear twice, once attached to the propagator containing the a variables, once to the propagator containing the b ones. This explains the switch in the indices below. We have 6 integrals to perform for each $SU(2)$ copy. By means of (A3) we find

$$\int dg_{ab}^{\pm} d\tilde{g}_{ab}^{\pm} (g_{ab}^{\pm} \tilde{g}_{ab}^{-1})^{m_{ab}}_{k_{ab}} (g_{ab}^{\pm} \tilde{g}_{ab}^{-1})^{m_{ba}}_{k_{ba}} = \frac{1}{d_{j_{ab}^+} d_{j_{ab}^-}} \epsilon^{m_{ab}m_{ba}} \epsilon_{k_{ab}k_{ba}}. \tag{B17}$$

To compare to the expression of [11] we still have to perform the integration over H_a appearing in the T_j^{γ}. From (B5) we have

$$(T_{j_a}^{\gamma})^{p_{ab}}_{q_{ab}} (H_a) = \beta_a \iota^{p_{ab}^+ p_{ab}^-}{}_{M_{ab}} \iota_{q_{ab}^+ q_{ab}^-}{}^{\tilde{M}_{ab}} (H_a)^{M_{ab}}_{\tilde{M}_{ab}}. \tag{B18}$$

Therefore we have 4 integrals of the form

$$\beta_a \prod_{b \neq a} (\iota^{p_{ab}^+ p_{ab}^-}{}_{M_{ab}} \iota_{q_{ab}^+ q_{ab}^-}{}^{\tilde{M}_{ab}}) \int dH_a \prod_{b \neq a} (H_a)^{M_{ab}}_{\tilde{M}_{ab}}. \tag{B19}$$

We obtain

$$\int dH_1 (H_1)^{M_{12}}_{\tilde{M}_{12}} (H_1)^{M_{13}}_{\tilde{M}_{13}} (H_1)^{M_{14}}_{\tilde{M}_{14}} = \iota_{M_{12}M_{13}M_{14}} \iota_{\tilde{M}_{12}\tilde{M}_{13}\tilde{M}_{14}} \tag{B20}$$

and similarly for the others. We replace this result into (B19) for each $a \in \{1, .., 4\}$. We replace then all integration results (B15)–(B17) and (B19), into the expression for the graph amplitude (B6) and we obtain

$$\mathcal{A}(\mathcal{G}_2) = \left(\prod_{a<b} d_{j_{ab}^+} d_{j_{ab}^-}\right) (6j_{ab}^+ 6j_{ab}^-)^2 \left(\prod_{a=1}^{4} f_a\right)^2. \tag{B21}$$

As already stated above, this reproduces (B8) with $k = 2$.

APPENDIX C: ASYMPTOTICS OF 6j AND FUSION COEFFICIENTS IN THE DEGENERATE CASE

In this appendix, we investigate the asymptotic behavior of the self-energy correction using sum over spins. We first derive the asymptotics of the 6j symbols which yields the power counting in the $SU(2)$ BF theory and then fusion coefficients f appearing in (B21) to obtain the power counting in the EPRL model.

1. Degenerate 6j and BF theory

In the general case, the 6j symbols can be written using Racah's single sum formula (see for instance [24])

$$\begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24} \end{Bmatrix} = \Delta(1,2,3)\Delta(1,2,4)\Delta(1,3,4)\Delta(2,3,4) \times \sum_k \frac{(-1)^k (k+1)!}{F(k)}, \tag{C1}$$

with

$$\Delta(a,b,c)=\left(\frac{(-j_{ab}+j_{bc}+j_{ac})!(j_{ab}-j_{bc}+j_{ac})!(j_{ab}+j_{bc}-j_{ac})!}{(j_{ab}+j_{bc}+j_{ac}+1)!}\right)^{1/2}, \tag{C2}$$

and

$$F(k)=(k-j_{12}-j_{23}-j_{13})!(k-j_{13}-j_{34}-j_{14})!(k-j_{23}-j_{34}-j_{24})!(k-j_{12}-j_{24}-j_{14})!(j_{12}+j_{23}+j_{34}+j_{14}-k)!(j_{12}+j_{13}+j_{34}+j_{24}-k)!(j_{23}+j_{13}+j_{24}+j_{14}-k)!. \tag{C3}$$

The sums runs over all integers k such that the arguments of the factorials are non-negative.

Consider a degenerate tetrahedron which is reduced to a single edge, whose vertices are labeled 1, 2, 3, 4 with 1 and 4 on the boundary of the edge. The associated spin (lengths of the edges of the tetrahedron) between vertices a and b ($a<b$) is j_{ab} and we have $j_{ac}=j_{ab}+j_{bc}$ if $a<b<c$. Therefore only the three spins j_{12}, j_{23}, j_{34} are independent and we have

$$F(k)=(k-2j_{12}-2j_{23})!(k-2j_{12}-2j_{23}-2j_{34})!(k-2j_{23}-2j_{34})!(k-2j_{12}-2j_{23}-2j_{34})!(2j_{12}+2j_{23}+2j_{34}-k)!(2j_{12}+2j_{23}+2j_{34}-k)!(2j_{12}+4j_{23}+2j_{34}-k)!. \tag{C4}$$

The sum over k is restricted to the single term $k=2(j_{12}+j_{23}+j_{34})$ and we have

$$F(k)=(2j_{12})!(2j_{34})!(2j_{23})!. \tag{C5}$$

There are also simplifications in the factors Δ,

$$\begin{aligned}\Delta(1,2,3)&=\left(\frac{(2j_{23})!(2j_{12})!}{(2j_{12}+2j_{23}+1)!}\right)^{1/2},\\ \Delta(1,2,4)&=\left(\frac{(2j_{12})!(2j_{23}+2j_{34})!}{(2j_{12}+2j_{23}+2j_{34}+1)!}\right)^{1/2},\\ \Delta(2,3,4)&=\left(\frac{(2j_{34})!(2j_{23})!}{(2j_{34}+2j_{23}+1)!}\right)^{1/2},\\ \Delta(1,3,4)&=\left(\frac{(2j_{34})!(2j_{12}+2j_{23})!}{(2j_{12}+2j_{23}+2j_{34}+1)!}\right)^{1/2}.\end{aligned} \tag{C6}$$

Taking all the terms together we get

$$\begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24}\end{Bmatrix}=(-1)^{2(j_{12}+j_{23}+j_{34})}\left(\frac{(2j_{23})!(2j_{12})!}{(2j_{12}+2j_{23}+1)!}\times\frac{(2j_{34})!(2j_{23})!}{(2j_{34}+2j_{23}+1)!}\times\frac{(2j_{34})!(2j_{12}+2j_{23})!}{(2j_{12}+2j_{23}+2j_{34}+1)!}\times\frac{(2j_{12})!(2j_{23}+2j_{34})!}{(2j_{12}+2j_{23}+2j_{34}+1)!}\right)^{1/2}\times\frac{(2j_{12}+2j_{23}+2j_{34}+1)!}{(2j_{12})!(2j_{23})!(2j_{34})!}, \tag{C7}$$

which simplifies into

$$\begin{Bmatrix} j_{12} & j_{23} & j_{13} \\ j_{34} & j_{14} & j_{24}\end{Bmatrix}=\frac{(-1)^{2(j_{12}+j_{23}+j_{34})}}{\sqrt{2j_{12}+2j_{23}+1}\sqrt{2j_{34}+2j_{23}+1}}. \tag{C8}$$

This yields an asymptotic behavior ($k_{ab}\in[0,1]$ fixed)

$$\begin{Bmatrix} jk_{12} & jk_{23} & jk_{13} \\ jk_{34} & jk_{14} & jk_{24}\end{Bmatrix}^2\sim_{j\to\infty}\frac{1}{j^2}. \tag{C9}$$

Consequently, the degree of divergence of the $SU(2)$ BF theory graph (double sunset without external legs) made of two vertices (each with one $6j$), three edges and six faces (each with one $dj=2j+1$) is

$$d_{\rm BF}^{\rm degenerate}=3+6-2=7<d_{\rm BF}^{\rm non\ degenerate}=9, \tag{C10}$$

where we sum over only three spins in the maximally degenerate case. Let us note that this is less than the degree of divergence of nondegenerate configurations, so that the latter are dominant in BF theory, at least for this graph.

2. Degenerate fusion coefficients and the EPRL model

Using the notations of [11] (Appendix B), the fusion coefficients can be expressed as a product of a $9j$ and a $3j$ coefficient,

$$\begin{aligned}f_i^{i^+i^-}(j_1,j_2,j_3,0)&=\delta_{i^+,j_3^+}\delta_{i^-,j_3^+}\delta_{i^-,j3}\sqrt{d_{j_1}d_{j_2}d_{j_3}}\\&\times\left(\frac{(2j_1^+)!(2j_1^-)!(2j_2^+)!(2j_2^-)!(j_1^++j_1^-+j_2^++j_2^--j_3)!(j_1^++j_1^-+j_2^++j_2^-+j_3+1)!}{(2j_1^++2j_1^-+1)!(2j_2^++2j_2^-+1)!(j_1^++j_2^+-j_3^+)!(j_1^++j_2^++j_3^++1)!(j_1^-+j_2^--j_3^-)!(j_1^-+j_2^-+j_3^-+1)!)}\right)^{1/2}\\&\times\left(\frac{(2j_3^+)!(2j_3^-)!(j_3^++j_3^--j_1^+-j_1^-+j_2^++j_2^-)!(j_3^++j_3^-+j_1^++j_1^--j_2^+-j_2^-)!}{(1+2j_3^++2j_3^-)!(j_3^+-j_1^++j_2^+)!(j_3^++j_1^+-j_2^+)!(j_3^--j_1^-+j_2^-)!(j_3^-+j_1^--j_2^-)!)}\right)^{1/2},\end{aligned} \tag{C11}$$

with as usual $j_a^{\pm} = \frac{1\pm\gamma}{2} j_a$. The second factor is the contribution from the $9j$ while the third one is that of the $3j$. Let us notice that if $\gamma = 1$, then $j_a^+ = j_a$ and $j_a^- = 0$, so that $f_i^{i^+ i^-}(j_1, j_2, j_3, 0) = \delta_{i^+, j_3^+} \delta_{i^-, j_3^-} \delta_{i, j_3}$. Thus, the theory reduces to an $SU(2)$ BF theory using (B21).

The result is symmetrical in the indices 1, 2, 3, so let us write the degeneracy condition on the triangle as $j_1 + j_2 = j_3$ to eliminate j_3. After some simplifications, we get

$$f_i^{i^+ i^-}(j_1, j_2, j_3, 0) = \delta_{i^+, j_3^+} \delta_{i^-, j_3^-} \delta_{i, j_3} \times \sqrt{\frac{2j_1 + 2j_2 + 1}{(2j_1^+ + 2j_2^+ + 1)(2j_1^- + 2j_2^- + 1)}}. \tag{C12}$$

Thus, the fusion coefficients scale as

$$f_i^{i^+ i^-}(jk_1, jk_2, jk_3) \sim_{j\to\infty} \frac{\delta_{jk^+, jk_3^+} \delta_{i^-, jk_3^-} \delta_{i, j_3}}{\sqrt{j}}. \tag{C13}$$

Accordingly, the power counting of the maximally degenerate configurations is (the summation over the intertwiners i is trivial thanks to the Kronecker symbols)

$$d_{\mathrm{EPRL}}^{\mathrm{degenerate}} = 3 + 12 + 2\left(2 \times (-1) + 4 \times \frac{-1}{2}\right) = 7 > d_{\mathrm{EPRL}}^{\mathrm{non\ degenerate}} = 6, \tag{C14}$$

the first term is the contribution of the $6j$ (two per vertices) and the second one the contribution from the fusion coefficients (four per vertices). Therefore, the degenerate configurations dominate in the EPRL model for this graph, in accordance with the quadratic approximation. Nevertheless, there is no reason to believe that this is a general feature of the model, since the quadratic approximation only yields an upper bound.

[1] D. V. Boulatov, Mod. Phys. Lett. A **7**, 1629 (1992); arXiv:hep-th/9202.H. Ooguri, Mod. Phys. Lett. A **7**, 2799 (1992); arXiv:hep-th/9205090.
[2] L. Freidel, Int. J. Theor. Phys. **44**, 1769 (2005).
[3] D. Oriti, *Approaches to Quantum Gravity* (Cambridge University Press, Cambridge, 2009).
[4] C. Rovelli, *Quantum Gravity* (Cambridge University Press, Cambridge, 2004).
[5] L. Freidel and D. Louapre, Phys. Rev. D **68**, 104004 (2003).
[6] J. Magnen, K. Noui, V. Rivasseau, and M. Smerlak, Classical Quantum Gravity **26**, 185012 (2009).
[7] J. Engle, R. Pereira, and C. Rovelli, Phys. Rev. Lett. **99**, 161301 (2007).
[8] E. R. Livine and S. Speziale, Phys. Rev. D **76**, 084028 (2007).
[9] L. Freidel and K. Krasnov, Classical Quantum Gravity **25**, 125018 (2008).
[10] J. Engle, E. Livine, R. Pereira, and C. Rovelli, Nucl. Phys. **B799**, 136 (2008).
[11] C. Perini, C. Rovelli, and S. Speziale, Phys. Lett. B **682**, 78 (2009).
[12] J. B. Geloun, T. Krajewski, J. Magnen, and V. Rivasseau, Classical Quantum Gravity **27**, 155012 (2010).
[13] V. Bonzom and M. Smerlak, Lett. Math. Phys. **93**, 295 (2010).
[14] L. Freidel, R. Gurau, and D. Oriti, Phys. Rev. D **80**, 044007 (2009).
[15] F. Conrady and L. Freidel, Phys. Rev. D **78**, 104023 (2008).
[16] R. Gurau, arXiv:0907.2582.
[17] A. Tanasa, Classical Quantum Gravity **27**, 095008 (2010).
[18] R. Gurau, Annales Henri Poincare **11**, 565 (2010).
[19] R. Gurau, arXiv:1006.0714.
[20] J. Ben Geloun, J. Magnen, and V. Rivasseau, arXiv:0911.1719.
[21] A. M. Perelomov, *Generalized Coherent States and Their Applications* (Springer, Berlin, 1986).
[22] J. C. Baez, Lect. Notes Phys. **543**, 25 (2000).
[23] E. Bianchi, D. Regoli, and C. Rovelli, Classical Quantum Gravity **27**, 185009 (2010).
[24] R. Gurau, Annales Henri Poincare **9**, 1413 (2008).