IL NUOVO CIMENTO **38 C** (2015) 165
DOI 10.1393/ncc/i2015-15165-3

Colloquia: VILASIFEST

Spectral geometry for quantum spacetime

Fedele Lizzi(*)

Dipartimento di Fisica, Università di Napoli Federico II - Napoli, Italy
INFN Sezione di Napoli - Napoli, Italy and
Departament de Estructura i Constituents de la Matèria, Institut de Ciéncies del Cosmos
Universitat de Barcelona - Catalonia, Spain

received 11 January 2016

Summary. — I make some considerations on the quantization of spacetime from a spectral point of view. The considerations range from the renormalization flow, to the standard model, to a new phase of spacetime.

1. – Introduction: quantum and classical

It is fitting that in a volume dedicated to Gaetano Vilasi I will discuss quantum gravity. Most of Gaetano research has been on classical mechanics, but if $\hbar$ is not very present in his papers, it has been indeed constantly present in our discussions. A notable exception is [1], where a classical dynamics point of view is taken to describe the Schrödinger equation as a dynamical system. Faithful to the many discussions we have had across the years, I will take the point of view that a geometric, or rather a noncommutative geometric, formalism is the most appropriate tool to describe classical and quantum mechanics, and go beyond to quantum field theory describing high energy interactions, and even further: the geometry of quantum spacetime. I will try to reproduce the spirit of our discussions, keeping the technicalities, even the formulas, at a minimum. References partly fill the gap. Unfortunately I cannot reproduce the food and the drinks...

To start with, one can pose the question: why quantum spacetime? The one we currently have seems to be doing an excellent job. It is based on Riemannian Geometry, and it serves well also quantum mechanics and field theory. Geometry is described in terms of points, lines, vector fields and the like. The same geometry, with some extra structures such as symplectic forms or Poisson brackets is used to describe the classical phase space. We know that the latter geometrical description does not survive quantization, in other words it does describe physical reality only in approximate sense,

(*) E-mail: `fedele.lizzi@na.infn.it`

macroscopically. The phase space of quantum mechanics is not described by points, but by operators which form an algebra.

There is a unified view of quantum and classical mechanics which may be useful for our considerations. Let us first consider the observables, the selfadjoint part of an algebra. For a point particle these can be functions of position and momenta, such as the x_1 coordinate, or angular momentum or energy. These functions form an algebra, *i.e.* it is possible to add and multiply them, multiply by a (complex) number and take the Hermitean (complex) conjugate. For technical reasons it is preferable to consider bounded continuos functions on the space, so that we can endow the algebra with a norm and define a normed $*$-algebra $\mathcal{A}$, *i.e.* for $a, b, \in \mathcal{A}$ and $\alpha, \beta \in \mathbb{C}$ and bar denoting usual complex conjugation, the following holds:

$$\begin{aligned} a^{**} &= a, \\ (ab)^* &= b^* a^*, \\ (\alpha a + \beta b)^* &= \bar{\alpha} a^* + \bar{\beta} b^*, \\ \|a + b\| &\leq \|a\| + \|b\|, \quad \|\alpha a\| = |\alpha|\, \|a\|, \\ \|ab\| &\leq \|a\|\, \|b\|, \\ \|a\| &\geq 0, \ \|a\| = 0 \iff a = 0. \end{aligned} \tag{1}$$

When the algebra is complete in the norm it is a C^*-algebra. The algebra is associative but not necessarily commutative. In the commutative case it can be proved (see for example [2, 3]) that a commutative C^*-algebra is *always* the algebra of continuous complex valued functions over a Hausdorff topological space.

Once we have the observables we need to define in which state our physical system is. In mathematics, a *state* on the C^*-algebra $\mathcal{A}$ is a linear functional, *i.e.* a map from the algebra to complex numbers which is positive and of unit norm:

$$\delta(a^* a) \geq 0, \qquad \forall\ a \in, \qquad \|\delta\| = \sup\{|\delta(a)| \mid \|a\| \leq 1\} = 1. \tag{2}$$

Any convex combination of states $\lambda\delta_1 + (1-\lambda)\delta_2$, with $0 \leq \lambda \leq 1$ is still a state. A pure state is a state which cannot be expressed as the convex sum of two other states. For an abelian algebra the pure states are in one to one correspondence with the points of the topological space. In this case any element of the algebra is a function, and a particular state associates to any point the value of the function at that particular point. Non pure states are density probabilities. To describe the evolution in time of the system we add some extra structure, a Hamiltonian and a Poisson bracket, and we have the evolution of the observables, in a classical Heisenberg picture. The transition to quantum mechanics is "simply" done considering the algebras of observables to be noncommutative. To quantify the noncommutativity it is necessary to introducee a quantity with the dimensions of an area in phase space, the dimensions of an action. The C^*-algebra can always be represented as bounded operators on a Hilbert space (GNS construction), and a unified description in terms of the momentum map can be given [4], just to make an example of geometrization of quantum mechanics. The key of the construction is the algebraic description, if an ordinary space is described by a commutative algebra, a *noncommutative space* will be described by a noncommutative algebra. Starting from this point, the programme of noncommutative geometry [5] is the translation of ordinary geometry in algebraic terms, so to allow a generalization to the noncommutative case. Physically the first noncommutative space is therefore the phase space of quantum mechanics, do we need more? And why? After all, the noncommutativity of quantum mechanics is relegated to position and momenta, coordinates commute among themselves, and the configuration space is an ordinary manifold to be described with the usual tool of differential geometry.

The first to note that that the union of quantum mechanics and general relativity cannot be compatible with a space in which points can be defined with infinite precision was Bronstein [6] shortly before being executed by Stalin's secret police. The reasoning has been later (independently) elaborated and refined by Doplicher and collaborators [7, 8]. In a simplified version goes as follows: in order to probe very short distances, necessary to localize a particle at a point, one has to use very energetic probes, with a very short wavelength, this is a quantum mechanical effect. But concentrating too much energy in a small volume leads to the creation of a black hole, a gravitational effect. Therefore the combination of the two theories makes it impossible (or at the very least problematic) to have perfect localization of events in configuration space. It is a "position-position" version of Heisenberg uncertainty principle and its explanation with the Heisenberg microscope. The role played by $\hbar$ in pause space is palyed by Planck's length $\ell_p = \sqrt{\hbar G/c^3}$ which provides a scale. One possibility to mimic this is to impose noncommutation rules for positions as well [7], something which is also suggested by string theory [9-11], a fact which rendered noncommutative geometry enormously popular some years ago.

A spacetime described by a noncommutative geometry has led to important studies of field theories on noncommmutative spaces (for a review see [12]), the simple transposition of the non commutativity of phase space to spacetime is not natural. There are problems with symmetries, which have to be considered quantum symmetries (see for example [13, 14]). Nevertheless, even if not of such a simple kind, some sort of quantum spacetime will be necessary, and the tools of noncommutative geometry seem appropriate.

2. – Let it flow

Let us add another element to the discussion. What we want to unify is not really gravity with point quantum mechanics, but rather with a quantum field theory. We have an enormously successful gauge quantum field theory which is able too explain the non gravitational interactions of particles. This is the standard model, supplemented by right handed neutrinos. The model has been tested in the Laboratory, most notably at the *Large Hadron Collider*, to energies to the order of the TeV's, and has proven to be extremely successful there. Applications of particle physics to astronomical and cosmological data can push this limit even higher, although in a less controlled and precise way.

Up to which energy should we believe in it? It is possible (and desirable) that new particles and new interactions are "behind the corner", and might be discovered in the near future. Supersymmetry [15] is possible candidate. Higher-dimensional terms in the Lagrangian could help [16]. While we do not know if, and which energy, these may be, it is not expected that the nature of field theory will change for several orders of magnitude. It is common belief that the fundamental structure of the theory will not change. But what will certainly change are the *numerical* values of the various "constants" of the model. I put quotation marks since these quantities are not constant at all, they *run* with energy. This is the principal tenet or the renormalization programme.

Let us look at the running of the coupling constants of the three interactions, the

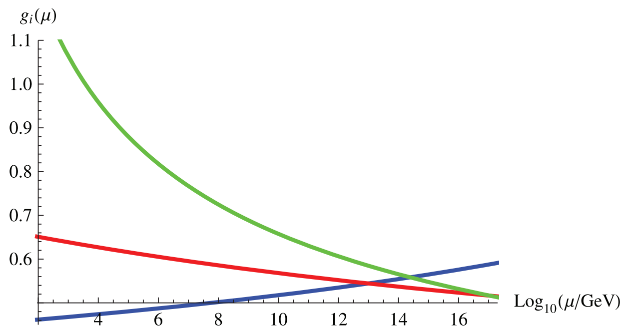

Fig. 1. – The running of the gauge interactions coupling constants.

result at one loop is shown in fig. 1, higher loops are not expected to change qualitatively the picture, little is known of nonperturbative effects. We see that two of the interaction decrease at high energy (asymptotic freedom), they are the ones corresponding to the nonabelian interactions, strong and weak, the third, corresponding to hyper charge, grows and would eventually reach a singularity (Landau pole) at an enormous energy. At an energy around 10^{18}–10^{19} GeV the onset of quantum gravity is expected. As we said this is the running in the absence of new physics, can we surmise something by looking at the picture? The main feature of it is the fact that the three constants almost meet at a single point. Until a few years ago data were compatible with the presence of a unification point, and this strongly motivated Grand Unified Theories (GUTs). It was taken as the indication that at the unification point there would be a larger, grand unified group. The theory, in its various incarnations, usually predicted proton decay at a rate which became excluded by experiments. The presence of supersymmetry could change the running, and cause the presence of an unification point, therefore supersymmetric grand unification is still viable.

I personally do not find appealing the idea that the three interaction run on their own, meeting in pairs roughly when their values are approximatively equal, and then keep going. The weak force stronger than the strong, both marching towards freedom, the hypercharge, by now strongest of the three, slowly edging towards the Landau Pole... when suddenly quantum gravity arrives! To do what?

3. – Let us get some action

Let me go back to geometry. I said that the topological information is encoded in the algebra, which can always be represented as operators on a Hilbert space. But there is much more to geometry! The key ingredient is D, a generalization of the Dirac operator of quantum field theory. I will still call it Dirac operator tout-court. On an ordinary spin manifold the operator D can be just the usual $\partial\!\!\!/$. It is a square-root of the Laplacian, and it "knows" a lot about the space. The information is in the spectrum of the operator. The suggestive metaphore is to *hear the shape of a drum*. Since the ear listens to harmonics of the vibrating membrane, what counts are the eigenvalues of the Laplacian, by Helmotz equation this means solving for the eigenvalues of the Laplacian (with Dirichlet boundary conditions). It turns out that one cannot hear the shape of drum, not even with a fermionic ear (*i.e.* using the Dirac operator), but nearly so. Isospectral manifolds are quite rare and their construction contrived. Nevertheless the definition of the Dirac operator permits to describe in algebraic terms the geometry of spaces. For example, since points are pure states, if we have a distance among states (pure or otherwise) we implicitly have a distance among points. This can be done with the help of Dirac operatore. Fiven two states ψ and ϕ we define their distance as

$$d(\phi, \psi) = \sup_{\|[D,a]\| \leq 1} |\psi(a) - \phi(a)| \tag{3}$$

It is possible to prove that when the states are pure, and therefore correspond to points, the distance is the same as the geodesic one obtained with the metric defined by the anticommutator of the γ matrices which are present in D. Vector fields are projective modules of the algebras, and over the years a dictionary is being built. The set $\mathcal{A}, \mathcal{H}$ and D constitute what is called a *spectral triple*. Two more operators are fundamental, the chirality Γ, which for an ordinary four-dimensional manifod would be the usual γ_5, and J, charge conjugation. With this five elements set it is possible [17] to give a completely algebraic characterization of manifolds, and therefore to give meaning to the notion of *Noncommutative Manifolds*. The original Dirac operator was introduced originally to describe the motion of fermions via the Dirac equation. A free fermions is described by the Lagrangian([1])

$$S_F = \langle \Psi, D\Psi \rangle = \int \mathrm{d}x \, \sqrt{g} \Psi^\dagger D \Psi. \tag{4}$$

In the presence of a background electromagnetic field we have have to substitute the covariant derivative D_A and we may add a mass term. If we have different flavours and gauge interactions the fermions will have extra indices, and the D operator will be a matrix. We will in any case express the action in the form (4), with the proper interpretation of D. Classically this action is invariant under the following scale transformation:

$$\begin{aligned} |\Psi\rangle &\to \mathrm{e}^{\frac{1}{2}\phi} |\Psi\rangle, \\ D &\to \mathrm{e}^{-\frac{1}{2}\phi} D \mathrm{e}^{-\frac{1}{2}\phi}. \end{aligned} \tag{5}$$

Recalling the presence of $\sqrt{\det g}$ in (4) is easy to see that this transformation is related to Weyl rescaling where the coordinates are left unchanged, and the rescaling is on the metric as $g^{\mu\nu} \to \mathrm{e}^{2\phi} g^{\mu\nu}$

Let us now follow the procedure of [19-21]. This symmetry does not however survive quantization. Consider the partition function

$$Z(D) = \int [\mathrm{d}\Psi][\mathrm{d}\Psi^\dagger] e^{-S_F}. \tag{6}$$

This expression is formally a determinant, but first we need a scale to render D dimensionless. For this we introduce a normalization dimensional quantity μ

$$Z(D, \mu) = \int [\mathrm{d}\psi][\mathrm{d}\bar{\psi}] e^{-S_\psi} = \det \left(\frac{D}{\mu} \right). \tag{7}$$

([1]) In the course of this discussion, for technical reasons I will always consider a Euclidean signature. At the end a Wick rotation will be understood. This is a nontrivial procedure in this case [18], but I will skip the details.

Since D in unbounded the determinant still diverges, we need to introduce a regulator. The regularization can be done in several ways. In the spirit of noncommutative geometry, based on operators nd their spectrum, the most natural one consists in enforcing a cutoff by truncating the spectrum of the Dirac operator. This was considered long ago (and independently of noncommutative geometry), in [22-24].

The cutoff is enforced considering only the first N eigenvalues of D. To this extent consider the projector

$$P_N = \sum_{n=0}^{N} |\lambda_n\rangle \langle\lambda_n| \tag{8}$$

with λ_n and $|\lambda_n\rangle$ eigenvalues and eigenvectors of D respectively. The integer N is defined as

$$N = \max n, \text{ such that } \lambda_n \leq \Lambda \tag{9}$$

with Λ the energy cutoff, enforced considering only the space spanned by the eigenvectors corresponding to eigenvalues smaller than Λ. The choice of a sharp cutoff could be changed in favour of a cutoff function χ, for the moment we are considering as function the characteristic function of the unit interval, sometimes it may be useful to consider an exponential decay.

Given the pivotal role played by the Dirac operator it is interesting to study geometries for which the Dirac operator or the Laplacian have a discrete finite spectrum, such as the fuzzy sphere or disc [25-28], and one can study the geometric properties of a pace with a cutoff [29]. The cutoff may be an artifact necessary to make sense of divergences, but it may be also a physical meaningful quantity signaling, for example, a phase transition, or in any case a scale at which the theory is profoundly changes and may not be described anymore by the same quantities.

Define therefore the regularized partition function:

$$\begin{aligned} Z(D,\mu) &= \prod_{n=1}^{N} \frac{\lambda_n}{\mu} \\ &= \det\left(\mathbb{I} - P_N + P_N \frac{D}{\mu} P_N\right) \\ &= \det\left(\mathbb{I} - P_N + P_N \frac{D}{\Lambda} P_N\right) \det\left(\mathbb{I} - P_N + \frac{\Lambda}{\mu} P_N\right) \\ &= Z_\Lambda(D,\Lambda) \det\left(\mathbb{I} - P_N + \frac{\Lambda}{\mu} P_N\right). \end{aligned} \tag{10}$$

Note that while the action is invariant by the transformation (5), the measure of the functional integral is not. We have an *anomaly*. If we wish to retain the symmetry we can add a term to the measure, and writing this as an exponential this is tantamount to adding a term to the action. Or we may consider an effective theory, for which the symmetry is broken by a physical scale. The first point of view was taken in [19], here we will instead follow [21] and [30] and take Λ to be a physically meaningful scale.

Let us now consider the change of the partition function under the renormalization flow, *i.e.* let us see how it transforms under $\mu \to \gamma\mu$. The flow is

$$Z(D,\mu) \to Z(D,\mu) e^{-\log\gamma \,\mathrm{tr}\, P_N}. \tag{11}$$

On the other side we have that

$$\mathrm{tr}\, P_N = N = \mathrm{tr}\,\chi\left(\frac{D}{\Lambda}\right) = S_B(\Lambda, D) \tag{12}$$

for the choice of χ the characteristic function on the interval, a consequence of our sharp cutoff on the eigenvalues.

The quantity S_B has been introduced by Chamseddine and Connes [31] and is called the *Spectral Action*. Here we provided a derivation of it. It can also be derived using the zeta function [32].

Technically the bosonic spectral action is a sum of residues and can be expanded in a power series in terms of Λ^{-1}, and then evaluated using standard heath kernel techniques [33]. The result, for Dirac operators square toor of a generalizaed Laplacian, is of a field theory coupled in the correct way to gravity (given by the metric of the γ^μ), and in the presence of connection, the appropriate term given by the square of the curvature. The spectral action is a most natural quantity in the noncommutative geometry framework, being dependent on the spectrum of the Dirac operator. In this sense it is a purely geometric action, and it can be extended to the noncommutative case. In fact a rather simple noncommutative geometry, an almost commutative geometry, gives spectacular results when applied to the standard model, which is what we were discussing earlier.

4. – Standard

The standard model of particle interaction can be expresses as an almost commutative geometry, namely we take the ingredients of the spectral triple introduced in sect. **3**, and we take the product of two algebras, one infinite-dimensional and commutative (continuous functions on spacetime), the other noncommutative, but finite-dimensional, *i.e.* a matrix algebra. The algebra acts on a Hilbert space, in turn the product of spinor times a finite-dimensional space, and the Dirac operator will have a similar split

$$\begin{aligned} \mathcal{A} &= C(\mathcal{R}^4) \otimes \mathcal{A}_F, \\ \mathcal{H} &= \mathrm{Sp}(\mathcal{R}^4) \otimes \mathcal{H}_F, \\ D &= \not{\partial} + \not{\omega} \otimes \mathbb{I} + \gamma_5 \otimes D_F, \end{aligned} \tag{13}$$

where ω_μ is the Levi-Civita connection corresponding to the metric defined by the anticommutator of the γ's, in this sense we are in a curved spacetime, and therefore coupled to a background gravitational field. To describe the standard model we take

$$\mathcal{A}_F = \mathrm{Mat}(\mathbb{C})_3 \oplus \mathbb{H} \oplus \mathbb{C} \tag{14}$$

with $\mathbb{H}$ the quaternions, which we represent as 2×2 matrices. The unitaries of the algebra correspond to the symmetries of the standard model: $SU(3) \oplus SU(2) \oplus U(1)$, a unimodularity condition takes care of the extra $U(1)$. The grading, given by $\Gamma = \gamma_5 \times \gamma$, splits it into a left and right subspace: $\mathcal{H}_L \oplus \mathcal{H}_R$. The J operator basically exchange the two chiralities and conjugates, thus effectively making the algebra act form the right. For $\mathcal{H}_F$ we take the zoo of known fermions: in total there are 96 degrees of freedom per generation, including right handed neutrinos([2]). The finite-dimensional operator D_F contains all fermion masses, or rather Yukawa couplings.

The algebra is represented on this Hilbert space (in a reducible way) in such a way that that several resctionons imposed by noncommutative geometry are satisfied. Those restrictions have to do with the representations of forms, consistency of chirality, Dirac operator and charge conjugation and other mathematical consistency requirements, for details see [36]. The restrictions are such that the scheme does not work for all gauge theories. For example only representations of the algebra (not simply the group) are allowed. This means that only the fundamental and trivial representations are possible. This is true for the standard model, but not for the grand unified theory $SU(5)$. The gauge bosons come from the covariant D.

At this point all that is necessary is to crank the machine [31, 36] and calculate the various terms in the spectral and sermonic actions. The calculations, although straightforward are complicated, but the reward is great: *the full Lagrangian of the standard model coupled with background gravity.* There are some important features to note:

- The Higgs field is a product of the action, it does not have to be inserted by hand, it comes out with its quartic potential naturally from the heath kernel expansion. In this sense it is on a par with the other vector bosons, it corresponds to fluctuations in the internal space, and is therefore a scalar.
- The three gauge group coupling constants come out to be equal (apart form the usual 3/5 normalization for the hypercharge coupling).
- Yukawa couplings (masses) and mixings are taken as inputs, but not the Higgs parameter (in particular the coefficient of the quartic coupling λ), which come out to be fixed function of the other parameters, which are dominated by the top Yukawa coupling.

Since the relations among the couplings are not preserved by the renormalization flow, we have to choose a scale at which we write this action. The natural scale is one at which the couplings would unify, and one can consider a range of values. Then one has a prediction for the Higgs mass, once the vev v of the potential, which is experimentally known, is inserted in the relation $m_H^2 = 2\lambda v$. This prediction was made, in the most recent version of the basic model, in [36]. It is

$$M_H = 170\,\mathrm{GeV} \tag{15}$$

and is wrong!

It now depends how one chooses to consider this theory. if you take it as a mature fully formed theory then the result is wrong. Period. If one takes it (as I do) as a tool to investigate the standard model starting from first principles, then I think it is remarkable

([2]) Note the the full Hilbert space is the tensor product of this finite-dimensional spacetimes the usual spinorial degrees of freedom. So the states are overcounted. This is called fermion doubling [34], a necessary feature, at first perceived as a problem [35], we will see later that it may be an asset.

that a theory based on pure mathematical result gets reasonable numbers, I think that the measurement of the Higgs as a reason to understand in which direction one has to improve on the theory.

5. – Not so standard

Let us try to understand in this framework the origin the standard model algebra, and see if it may shed light on the mass of the Higgs. I mentioned earlier that is possible to define in purely algebraic terms when a commutative spectral triple describes a manifold [17]. Since the conditions are algebraic one can apply them to the finite-dimensional case. In this case one finds that the finite-dimensional algebra must be of the kind

$$\mathcal{A}_{\mathcal{F}} = \mathbb{M}_a(\mathbb{H}) \oplus \mathbb{M}_{2a}(\mathbb{C}) \qquad a \in \mathbb{N}. \tag{16}$$

Acting on a finite Hilbert space of dimension $2(2a)^2$. The first nontrivial case is for a $a = 2$, a finite Hilbert space of dimension 32, the number of degrees of freedom for one generation. Compatibility with the chirality gives as algebra

$$\mathcal{A}_{LR} = \mathbb{H}_L \oplus \mathbb{H}_R \oplus \mathbb{M}_4(\mathbb{C}). \tag{17}$$

This is the algebra corresponding to a Pati-Salam Grand Unified Theory [37], a rare case of GUT for which the fermions come in the fundamental representation of the gauge groups. This kind of theory requires a field for the breaking to the standard model. We will call this field σ. It should appear in the Dirac operator in the position corresponding to the neutrino Majorana mass. But unfortunately putting a nonzero entry in that position, and cracking of the machine, does not produce a field. Hence one has to include it by hand [38]. Which is quite unpleasant. Nevertheless, running the physical quantitates with this field does change the Higgs mass, making it compatible with the experimental value, at the price of a partial loss of predictivity since a new parameter should be added. Physics is therefore telling us that into his framework right handed neutrinos, and Majorana masses are fundamental.

A few possibilities to improve the model have been devised. In [39] (which predates the measurement of the Higgs mass) C. Stephan noticed that enlarging the Hilbert space could generate this field. In [40] one of the mathematical conditions is violated, which again is in some sense unpleasant.

With Devastato and Martinetti we proposed [41-43] a *Grand Symmetry* based on

$$\mathbb{M}(\mathbb{H})_4 \oplus \mathbb{M}(\mathbb{C})_8 \tag{18}$$

Recall that a finite "manifold" in this context is an algebra: $M_a(\mathbb{H}) \oplus M_{2a}(\mathbb{C})$ acting on a $2(2a)^2$-dimensional Hilbert space. So far we had $a = 2$, corresponding to dimension 32 (96 for three generation). But these 96 states, are multiplied by spinors, another 4 degrees of freedom. There is an over counting, since for example the four degrees of freedom of the electron (electron and positron of right and left chiralities) are counted in the 32, and in the degrees of freedom of the spinor. Some states can be projected out without problems, but other (the ones of mixed chirality), cannot. Hence really the Hilbert space is of dimension 128 (384) for three generations([3]),

For $a = 4$ and the finite *Grand Algebra*:

$$M_4(\mathbb{H}) \oplus M_8(\mathbb{C}) \tag{19}$$

a 128-dimensional space is required. This is exactly the dimension of the Hilbert space if we take the fermion doubling into account. Let us look at the Hilbert space with a different splitting:

$$\mathcal{H} = sp(L^2(\mathbb{R}^4)) \otimes \mathcal{H}_F = L^2(\mathbb{R}^4) \otimes \mathsf{H}_F. \tag{20}$$

It is possible to represent the Grand Algebra on this Hilbert space, but the representation is highly nontrivial. In particular it does not act diagonally on the spinor indices, it mixes them. I refer to [41-43] for details. The key point is that in the process spacetime indices, related to the Euclidean symmetries, mix with internal, gauge indices. The suggestion is to consider this algebra to be some high energy description, so that the standard model is some sort of effective low energy theory, coming after the breaking due to the Dirac operator.

There are two things that the Grand symmetry achieves. On the one hand this scheme the σ field comes out (at the price of the addition of a new parameter) on a par with the Higgs. Moreover it is possible to see that the braking to the standard model algebra is indeed dynamical. But the spacetime describe by this algebra does not have the usual (low energy) son structure, it is in some sense in a "pregeometric state". This begs the question, which geometry is there at the scale Λ and above?

6. – Looking up

We are working under the hypothesis is that Λ has a physical meaning, a scale indicating a phase transition. We can try to infer some properties of this regime studying the high energy limit of the action with the cutoff. Usually we use probes that are bosons, hence consider the expansion of the spectral action in the high momentum limit [44]. To this extent we can use the results of Barvisnky and Vilkovisky [45] who were able to sum all derivatives for the action (12) for the cutoff function χ a decreasing exponential. The expression (for details see [44]) is a series expansion in Λ, different form the heath kernel we mentioned earlier, but involving the same elements: Ricci tensor and scalar, Laplacian, curvature tensor.

For a Dirac operator of the kind

$$D = i\gamma^\mu \nabla_\mu + \gamma_5 \phi = i\gamma^\mu(\partial_\mu + \omega_\mu + iA_\mu) + \gamma_5\phi \tag{21}$$

with $g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$, we get, to leading order,

$$S_B \simeq \frac{\Lambda^4}{(4\pi)^2} \int \mathrm{d}^4x \left[-\frac{3}{2} h_{\mu\nu} h_{\mu\nu} + 8\phi \frac{1}{-\partial^2} \phi + 8F_{\mu\nu} \frac{1}{(-\partial^2)^2} F_{\mu\nu} \right]. \tag{22}$$

([3]) It may be worth to comment at this point that we have nothing to offer was to why there are three generations in this context.

To understand the meaning of this action recall how we obtain propagation of waves and correlation of points in the usual quantum field theory with the action:

$$S[J,\varphi] = \int \mathrm{d}^4x \left[\varphi(x)\left(\partial^2 + m^2\right)\varphi(x) - J(x)\varphi(x)\right] \tag{23}$$

and equation of motion

$$\left(\partial^2 + m^2\right)\varphi(y) = J(y). \tag{24}$$

It is the Green's function $G(x-y)$ which "propagates" the source:

$$\varphi_J(x) = \int \mathrm{d}^4y J(y)G(x-y). \tag{25}$$

In momentum representation we have

$$\begin{aligned} \varphi(x) &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\ e^{ikx}\ \hat{\varphi}(k), \\ J(x) &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\ e^{ikx}\ \hat{J}(k), \\ G(x-y) &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4k\ e^{ik(x-y)}\ \hat{G}(k). \end{aligned} \tag{26}$$

The propagator is

$$G(k) = \frac{1}{(k^2+m^2)} \tag{27}$$

The field at a point depends on the value of the field in nearby points, and the points "talk" to each other exchanging virtual particles. Let us consider the case of a scalar, like the Higgs, and that of a vector, like a photon or a graviton:

$$S[J,\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\varphi(x)F(\partial^2)\varphi(x) - J(x)\varphi(x)\right). \tag{28}$$

In this case the equation of motion is $F(\partial^2)\phi(x) = J(x)$, giving

$$G = \frac{1}{F(\partial^2)}, \quad G(k) = \frac{1}{F(-k^2)} \tag{29}$$

and

$$\varphi_J(x) = \int \mathrm{d}^4y J(y)G(x-y) = \frac{1}{(2\pi)^4}\int \mathrm{d}^4k e^{ikx} J(k)\frac{1}{F(-k^2)}. \tag{30}$$

Let us analyze the short distance limit of this expression, it corresponds to $k \to \infty$. In this case the Green's function becomes

$$\varphi_J(x) \xrightarrow[K\to\infty]{} \begin{cases} \dfrac{1}{(2\pi)^4}\displaystyle\int \mathrm{d}^k e^{ikx} J(k)k^2 = (-\partial^2)J(x), & \text{for scalars and vectors,} \\ \dfrac{1}{(2\pi)^4}\displaystyle\int \mathrm{d}^k e^{ikx} J(k) = J(x), & \text{for gravitons,} \end{cases} \tag{31}$$

which in momentum space corresponds to

$$G(x-y) \propto \begin{cases} (-\partial^2)\delta\,(x-y)\,, & \text{for scalars and vectors,} \\ \delta\,(x-y)\,, & \text{for gravitons.} \end{cases} \tag{32}$$

The correlation vanishes for noncoinciding points, heuristically, nearby points "do not talk to each other". The spectral action is pointing to some sort of *nongeometrical phase of spacetime*, in which points loose their usual meaning. A behavior confirmed by asymptotic freedom studies [46].

7. – And then ...

I purposely avoided the word conclusion for this final section. There cannot be any conclusion to this note, as there cannot be conclusions to the endless discussions I shared with Gaetano. It is not clear how far we are from a consistent theory of quantum spacetime. It is clear that we do not yet have it, and we can only hope that we are moving along the right direction, although it must be said that we are moving, like a quantum-mechanical particle, in several directions at the same time. This is no warranty that at least one of them is correct, nevertheless it is fun to keep trying!

* * *

This article is based upon work from COST Action MP1405 QSPACE, supported by COST (European Cooperation in Science and Technology). FL is partially supported by CUR Generalitat de Catalunya under projects FPA2013-46570 and 2014 SGR 104, MDM-2014-0369 of ICCUB (Unidad de Excelencia *Maria de Maeztu*), and by UniNA and Compagnia di San Paolo under the grant *Programma STAR 2013*.

REFERENCES

[1] Marmo G. and Vilasi G., *Mod. Phys. Lett. B*, **10** (1996) 545, [hep-th/9605191].
[2] Landi G., *An Introduction to Noncommutative Spaces and their Geometries, Springer Lect. Notes Phys.*, Vol. **51** (Springer Verlag, Berlin, Heidelberg) 1997.
[3] Gracia-Bondia J. M., Varilly J. C. and Figueroa H., *Elements of noncommutative geometry* (Birkhauser, 2000).
[4] Carinena J. F., Clemente-Gallardo J. and Marmo G., *Theor. Math. Phys.*, **152** (2007) 894, [math-ph/0701053 [MATH-PH]].
[5] Connes A., *Noncommutative Geometry* (Academic Press) 1994.
[6] Bronstein M., Phys. Z. Sowjetunion, Band 9, Heft 2-3, pp. 140-157 (1936). *English translation, Gen. Relativ. Gravit.*, **44** (2012) 267.
[7] Doplicher S., Fredenhagen K. and Roberts J. E., *Commun. Math. Phys.*, **172** (1995) 187, [hep-th/0303037].

[8] Doplicher S., *AIP Conf. Proc.*, **589** (2001) 204, [hep-th/0105251].
[9] Lizzi F. and Szabo R. J., *Chaos Solitons Fractals*, **10** (1999) 445, [hep-th/9712206].
[10] Landi G., Lizzi F. and Szabo R. J., *Commun. Math. Phys.*, **206** (1999) 603, [hep-th/9806099].
[11] Seiberg N. and Witten E., *JHEP*, **9909** (1999) 032, [hep-th/9908142].
[12] Szabo R. J., *Phys. Rep.*, **378** (2003) 207, [hep-th/0109162].
[13] Aschieri P., Dimitrijevic M., Kulish P., Lizzi F. and Wess J., *Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory, Springer Lect. Notes Phys.*, Vol. **774** (Springer Verlag, Berlin, Heidelberg) 2009.
[14] Aschieri P., Lizzi F. and Vitale P., *Phys. Rev. D*, **77** (2008) 025037, [arXiv:0708.3002 [hep-th]].
[15] Martin S. P., *Adv. Ser. Direct. High Energy Phys.*, **21** (2010) 1; (*Adv. Ser. Direct. High Energy Phys.*, **18** (1998) 1), [hep-ph/9709356].
[16] Devastato A., Lizzi F., Valcarcel Flores C. and Vassilevich D., *Int. J. Mod. Phys. A*, **30** (2015) 07; 1550033, [arXiv:1410.6624 [hep-ph]].
[17] Connes A., *J. Noncommut. Geom.*, **7** (2013) 1, [arXiv:0810.2088 [math.OA]].
[18] D'Andrea F., Kurkov M. and Lizzi F., "Clifford Structures in Noncommutative Geometry and New Scalar Fields in the Standard Model", to appear.
[19] Andrianov A. A. and Lizzi F., *JHEP*, **1005** (2010) 057, [arXiv:1001.2036 [hep-th]].
[20] Andrianov A. A., Kurkov M. A. and Lizzi F., *PoS CNCFG*, **2010** (2010) 024, [arXiv:1103.0478 [hep-th]].
[21] Andrianov A. A., Kurkov M. A. and Lizzi F., *JHEP*, **1110** (2011) 001, [arXiv:1106.3263 [hep-th]].
[22] Andrianov A. A. and Bonora L., *Nucl. Phys. B*, **233** (1984) 232.
[23] Andrianov A. A. and Bonora L., *Nucl. Phys. B*, **233** (1984) 247.
[24] Fujikawa K. and Suzuki H., *Path Integrals And Quantum Anomalies* (Oxford university Press) 2004.
[25] Hoppe J., "Quantum theory of a massless relativistic surface and a two-dimensional bound state problem", PhD Thesis, Massachusetts Institute of Technology, 1982, reprinted in *Soryushiron Kenkyu* Vol. **80** (1989), pp. 145–202.
[26] Madore J., *Class. Quant. Grav.*, **9** (1992) 69.
[27] Lizzi F., Vitale P. and Zampini A., *JHEP*, **0308** (2003) 057, [hep-th/0306247].
[28] Lizzi F., Vitale P. and Zampini A., *JHEP*, **0509** (2005) 080, [hep-th/0506008].
[29] D'Andrea F., Lizzi F. and Martinetti P., *J. Geom. Phys.*, **82** (2014) 18, [arXiv:1305.2605 [math-ph]].
[30] Kurkov M. A. and Lizzi F., *Mod. Phys. Lett. A*, **27** (2012) 1250203, [arXiv:1210.2663 [hep-th]].
[31] Chamseddine A. H. and Connes A., *Commun. Math. Phys.*, **186** (1997) 731, [hep-th/9606001].
[32] Kurkov M. A., Lizzi F., Sakellariadou M. and Watcharangkool A., *Phys. Rev. D*, **91** (2015) 6; 065013, [arXiv:1412.4669 [hep-th]].
[33] Vassilevich D. V., *Phys. Rep.*, **388** (2003) 279, [hep-th/0306138].
[34] Lizzi F., Mangano G., Miele G. and Sparano G., *Phys. Rev. D*, **55** (1997) 6357, [hep-th/9610035].
[35] Gracia-Bondia J. M., Iochum B. and Schucker T., *Phys. Lett. B*, **416** (1998) 123, [hep-th/9709145].
[36] Chamseddine A. H., Connes A. and Marcolli M., *Adv. Theor. Math. Phys.*, **11** (2007) 991, [hep-th/0610241].
[37] Pati J. C. and Salam A., *Phys. Rev. D*, **10** (1974) 275; (*Phys. Rev. D*, **11** (1975) 703).
[38] Chamseddine A. H. and Connes A., *JHEP*, **1209** (2012) 104, [arXiv:1208.1030 [hep-ph]].
[39] Stephan C. A., *Phys. Rev. D*, **79** (2009) 065013, [arXiv:0901.4676 [hep-th]].
[40] Chamseddine A. H., Connes A. and van Suijlekom W. D., *J. Geom. Phys.*, **73** (2013) 222, [arXiv:1304.7583 [math-ph]].
[41] Devastato A., Lizzi F. and Martinetti P., *JHEP*, **1401** (2014) 042, [arXiv:1304.0415 [hep-th]].

[42] Devastato A., Lizzi F. and Martinetti P., *Fortsch. Phys.*, **62** (2014) 863, [arXiv:1403.7567 [hep-th]].
[43] Devastato A. and Martinetti P., "Twisted spectral triple for the Standard Model and spontaneous breaking of the Grand Symmetry," arXiv:1411.1320 [hep-th].
[44] Kurkov M. A., Lizzi F. and Vassilevich D., *Phys. Lett. B*, **731** (2014) 311, [arXiv:1312.2235 [hep-th]].
[45] Barvinsky A. O. and Vilkovisky G. A., *Nucl. Phys. B*, **333** (1990) 471.
[46] Alkofer N., Saueressig F. and Zanusso O., *Phys. Rev. D*, **91** (2015) 2; 025025, [arXiv:1410.7999 [hep-th]].