Multiobjective Non–Linear Random Vibration analysis for performance–based Earthquake Engineering

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ABSTRACT: A new multiobjective linearization method for nonlinear random vibration analysis is presented. The strategy employs the Tail–Equivalent Linearization Method (TELM) which is a non–parametric linearization algorithm for multi–DOFs nonlinear systems. Due to the definition of conditioned probability, the joint tail probability of a multi–response structural system can be written as the product of a first marginal probability and some lower order conditioned tail probabilities. The algorithm decomposes the joint probability into a conveient form so that the conditioned probabilities are arranged consequently. In this case, each probability but the first is conditioned to previously computed responses. Then, TELM is recursively applied in order to define a set of interconnected linearized systems, each one defined in terms of its impulse response function. The definition of the base excitation by its cross–covariance or its power spectral density leads to the first marginal tail probability and to the power spectral density of the corresponding response. Afterwards, the interconnected linearized system is used to compute the tail probability and the power spectral density of each response in function of the previously analyzed responses' statistics. The computed joint probability can be used in random vibration analysis in order to get various statistics of the nonlinear response, such as the mean level–crossing rate and the joint first–passage probability. This work analyzes a series system, however, the procedure can be easily extended to the general case. Also, numerical applications illustrate the features of the method and comparison with results obtained by Monte Carlo simulations demonstrate its accuracy, in particular for high response thresholds.

1 INTRODUCTION

In performance–based earthquake engineering, it is important to properly consider non–linearities since failure usually occurs in the non–linear range of structural behavior. Equivalent linearization is one of the most powerful techniques in Random Vibrations because of its versatility in application to MDOF Finite Element Models. Several criteria have been proposed in years, in particular, one efficient non–parametric algorithm is the Tail–Equivalent Linearization Method (TELM) (Fujimura and Der Kiureghian 2007). It is able to predict non–Gaussian response distributions; also, non–stationary analysis can be performed (Der Kiureghian and Fujimura 2009), and it can even account for asymmetries providing non–symmetric response PDFs (Sessa and Der Kiureghian 2009).

However, dependence on a specific response affetcs equivalent linearization. TELM defines probability distributions of a single specific response only, thus, it is a single–objective analysis. Common structures usually require a multi–objective analysis: buildings' structural safety conditions often require that *each* inter–storey drift remains below a safety threshold. Thus, structural RV analysis can be considered as the investigation of the global reliability of a stochastic system which may be defined by a set of mutually related components both in series (such as in the inter–storey drift case) and parallel subsets.

Efficient algorithm have been developed for this purpose. Joint First–Passage Probability of interconnected systems (Song and Der Kiureghian 2006) leads to reasonably accurate estimates, however, the joint–PDF of the responses of interest is required. Matrix–based System Reliability (MSR) and Linear Programming (LP) (Song and Der Kiureghian 2003) efficiently compute reliability for any general system; the procedure is not dependent on the number of components and it is able to define the narrowest possible bounds for the available informations. However, because of its wide capability, Linear Programming can be demanding because it must be able to address any possible system configuration.

A procedure for multi–objective random vibration analysis is presented in this work. Its purpose is to define a discretized form of a non–Gaussian responses joint-PDF. This formulation is very fruitful because, eventually, the computed joint–PDF can be employed either for stationary or non–stationary analysis applying classical stochastic methods regardless of dimensions and parameters of the structural system. The underlying philosophy consists in defining the statistics of a "multi–objective" linearized system, whose equivalence condition is defined in therms of joint–tail probability of the system components.

Briefly, the joint–tail probability of a structural system is written as the product of conditioned probabilities of each response. Then, an equivalent linear system (TELS) is computed by TELM for each response. The TELS are defined in terms of a collection of Impulse Response Functions (IRF), each corresponding to a specified threshold.

Note that the IRF is defined as the evolution in time of a chosen response, given by a generical impulsive action. Thus, the proposed strategy will compute two different kinds of IRFs: a ground motion – response relationship, if only the marginal distribution of the specified response is needed, or a response–i – response–j relationship, if the needed distribution of the response j is conditioned on the response i.

Once the input excitation process statistics have been defined, it is possible to evaluate all the marginal distributions of the non–conditioned responses and also their power spectral densities which will be employed in order to get the statistics of the conditioned responses.

In Section 2 a review of the classic formulation of TELM is presented. Afterwards, the decomposition of the multiobjective problem is shown in Section 3 and then, the conditioned tail probabilities are computed in Section 4. Finally, numerical applications and comparisons with the results of a Monte Carlo simulation are provided in Section 5.

2 TELM REVIEW

Let $F(t)$ be a random process defined as the response of a linear filter excited by a white noise $W(\tau)$:

$$F(t) = \int_0^t h_f(t-\tau) W(\tau)\, d\tau \tag{1}$$

where $h_f(\tau)$ is the Impulse Response Function (IRF) of the linear filter. The random process $F(t)$ can be expressed in discretized form as a random pulse train (Der Kiureghian 2000) as:

$$F(t) = \sum_{i=1}^{n} s_i(t)\, u_i = \mathbf{s}^T(t)\, \mathbf{u} \tag{2}$$

where $\mathbf{u}$ is a vector of standard normal random variables and $\mathbf{s}(t)$ is a deterministic vector depending on the base–excitation's covariance.

A multi–degrees–of–freedom (MDOF) nonlinear system can be defined by its equation of motion:

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{R}\left(\mathbf{U}\dot{\mathbf{U}}\right) = \mathbf{P}F(t) \tag{3}$$

where $\mathbf{U}$ denotes the displacements vector, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{P}$, respectively, the mass, damping and loads matrixes and $\mathbf{R}$ is the nonlinear restoring force. A generic response of interest $X(t)$ can be defined as nonlinear function of nodal displacements, velocities and accelerations. Our interest is in determining the tail probability $\Pr[x \leq X(t_n)]$ for a specified threshold x at time t_n. Defining a limit state function such as:

$$G(x, t_n, \mathbf{u}) = x - X(t) \tag{4}$$

TELM (Fujimura and Der Kiureghian 2007) is able to perform the First–Order Reliability Method (FORM) which evaluates the tail probability and the "design point" as the solution to the costrainted optimization problem:

$$\mathbf{u}^\star(x, t_n) = \arg\min\left[\|\mathbf{u}\| \,|G(x, t_n, \mathbf{u}) = 0\right]. \tag{5}$$

Note that $\mathbf{u}^\star$ is the closest point of the limit state surface 4 to the origin in the standard–normal space and corresponds to its maximum likelihood. Furthermore, FORM linearizes the limit state function and gets the first–order approximation of its tail–probability.

Now, the generic response of a linear system can be expressed in terms of its IRF:

$$X_L(t) = \int_0^t h(t-\tau) F(\tau)\, d\tau \tag{6}$$

or, in discretized form:

$$X_L(t) = \int_0^t h(t-\tau) \sum_{i=1}^{n} s_i(\tau)\, u_i\, d\tau = \mathbf{a}(t)\, \mathbf{u} \tag{7}$$

where the deterministic vector $\mathbf{a}(t)$ has elements:

$$a_i(t) = \int_0^t h(t-\tau)\, s_i(\tau)\, d\tau. \tag{8}$$

The limit state function 4 for the response 6 describes an hyperplane in the standard normal space, due to the linearity of the system. Geometrical considerations yield:

$$\mathbf{a}(t_n) = \frac{x}{\|\mathbf{u}^\star(x, t_n)\|} \frac{\mathbf{u}^\star(x, t_n)^T}{\|\mathbf{u}^\star(x, t_n)\|}. \tag{9}$$

The TELM equivalent condition requires that the tail probability of the nonlinear and linearized system must be the same, then, both systems must lead to the same design point. Once the nonlinear design point is computed by FORM, it is possible to get the linera IRF by the 9 and the 8. The Tail–equivalent Linearized System (TELS), then, is fully defined by a *collection* of IRFs, each one depending on a different threshold x_i.

3 MULTIOBJECTIVE RANDOM VIBRATION ANALYSIS

TELM is very powerful because it can be performed regardless of the number of DOFs and of the constitutive models (as long as they can be smoothed). Its only requirements are that the nonlinear system must reach stationarity before the considered time t_n and the Limit State Function's first derivative must be continuous in the standard normal space. While the first requirement can be easily achieved, the latter one could be harder to satisfy. In fact, most of the common problems require to evaluate the first–excursion probability with respect a safety threshold of either displacements or internal forces. While for some structures safety can be expressed in terms of a single response (such as for tanks, bridges and electrical substations), in many cases the global safety depends on a set of responses of interest. For example, the safety condition of a building requires that *each* inter–storey drift remains smaller than the safety threshold during the seismic motion. In that case, the global limit state function and the tail probability can be written as:

$$G(\mathbf{x}, t_n) = \min\left[x_1 - X_1(t_n), x_2 - X_2(t_n), \ldots, x_m - X_m(t_n)\right] \tag{10}$$

$$\Pr\left[G(\mathbf{x}, t_n) \le 0\right] = \Pr\left[x_1 \le X_1 \cup x_2 \le X_2 \cup \ldots \cup x_m \le X_m\right] = 1 - \Pr\left[\bigcap_{i=1}^{m} x_i \ge X_i\right]. \tag{11}$$

Note that in the 11 the dependence on t_n of the X_i has been omitted for simplicity. TELM is not able to get the 11 because the first derivative of the 10 is usually non–continuous in the standard normal space. Furthermore, the LSF itself cannot be easily approximated by an hyperplane.

This drawback can be overcome by decomposing the joint–probability into the product of lower–order conditioned probabilities:

$$\Pr\left[G(\mathbf{x}, t_n) \ge 0\right] = \Pr\left[x_1 \ge X_1\right] \Pr\left[x_2 \ge X_2 | X_1\right] \ldots \Pr\left[x_m \ge X_m | X_1 \ldots X_{m-1}\right]. \tag{12}$$

The proposed stategy consists in computing the conditioned probability of the 16 by TELM and then to get the joint–tail probability in order to perform nonlinear random vibration analysis. Note that the Equations 10, 11 and 16 refer to a series system, *i.e.* the global chrisis occurs if any of the responses crosses its safety threshold. In case of parallel or hybrid systems, the Equation 16 can be easily modified by computing the complementary–conditioned tail probabilities.

4 THE CONDITIONED PROBABILITIES

Each term of the right side of the 16 can be computed separately. The first one is a marginal probability which can be computed by the usual formulation of TELM. In particular, let's consider a set of threshold of interest $\mathbf{x} = [x_1\, x_2\, \ldots x_m]^T$, for each one of them, TELM evaluates the correponding IRF in discretized form: $h_k^{1,l}$, where the indexes 1, k and l designate respectively the response, the time step and the threshold. The Fourier transform of the IRF is the Frequency Response Function (FRF) $H^{1,l}(\omega)$, corresponding to the lth threshold, which leads to the variance of the response:

$$\sigma_{1,l}^2 = 2\int_0^{\infty} \left\| H^{1,l}(\omega) \right\|^2 \Phi_{FF}(\omega)\, d\omega \tag{13}$$

where Φ_{FF} is the power spectral density of the base excitation. Then, the tail probability is:

$$\Pr\left[x_1^l \geq X_1\right] = \Phi\left[\frac{x_1^l}{\sigma_{1,l}}\right] \tag{14}$$

where $\Phi[\cdot]$ denotes the Standard Normal Cumulative Distribution Function. Also, the power spectral density of the displacements and of the acceleration is:

$$\Phi_{X_1X_1}^l(\omega) = \left\| H^{1,l}(\omega) \right\|^2 \Phi_{FF}(\omega) \tag{15}$$

note that the tail probability and the power spectral density are referred only to the threshold x^l. The power spectral density of the derivatives of X_1 can be easily obtained from the 15 as shown in (Lutes and Sarkani 2004). Performing a similar procedure, also the conditioned tail probabilities can be computed:

$$\Pr\left[x_2^l \geq X_2 | x_1^k = X_1\right] = \Phi\left[\frac{x_2^l}{\sigma_{2|1,l|k}}\right] \tag{16}$$

where:

$$\sigma_{2|1,l|k}^2 = 2\int_0^{\infty} \left\| H^{2|1,l}(\omega) \right\|^2 \Phi_{\ddot{X}_1\ddot{X}_1}^k(\omega)\, d\omega. \tag{17}$$

The frequency response function $H^{2|1,l}(\omega)$ it the response X_2 for a steady–state action in X_1. Its evaluation will be described in the following. Note that the 17 can be written only if the response X_2 given by an excitation in X_1 is uncoupled with the base excitation $F(t)$. This hypothesis is usually right for common buildings as long as the responses of interest are either the interstorey drifts or the floor displacements. In fact, if the first floor displacement is assigned, then the second floor drift can be considered independent of the base excitation. Thus, TELM can be performed in order to get the IRF $h_k^{2|1,l|k}$ where the index l denotes the lth threshold for X_2 and k denotes the kth threshold of X_1. For each threshold value of X_1, then, TELM defines a set of IRFs each corresponding to a threshold value of X_2. This procedure can be applied recursively to each response of interest in order to get all the conditioned tail probabilities; its generalized formulation is:

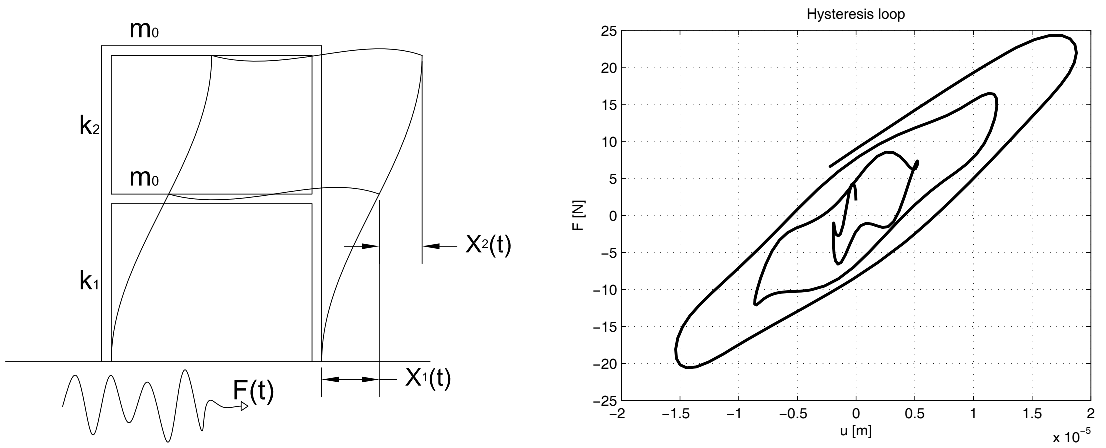
$$\Pr\left[x_j^l \geq X_j | x_i^k = X_i\right] = \Phi\left[\frac{x_j^l}{\sigma_{j|i,l|k}}\right]; \quad \sigma_{j|i,l|k}^2 = 2\int_0^{\infty} \left\| H^{j|i,l}(\omega) \right\|^2 \Phi_{\ddot{X}_i\ddot{X}_i}^k(\omega)\, d\omega. \tag{18}$$

The conditioned IRFs and FRFs $h_k^{j|i,l}$ and $H^{j|i,l}(\omega)$ can be easily computed by TELM, by using the jth limit state function and an excitation at the ith response. In case of interstorey drift responses, the ith excitation would be a storey acceleration.

Note that the input excitation of TELM should be defined with respect the power spectral density $\Phi_{\ddot{X}_i\ddot{X}_i}(\omega)$: first, the auto–covariance $\Gamma_{\ddot{X}_i\ddot{X}_i}$ is computed by taking the Fourier Transform of the PSD, then, the covariance matrix $\mathbf{G}$ can be built as discretized form of the auto–covariance (where each row–column corresponds to a specific time–step) which leads to the matrix $\mathbf{S}$:

$$\mathbf{S}\mathbf{S}^T = \mathbf{G}; \quad \mathbf{S} = [\mathbf{s}(t_1)\, \mathbf{s}(t_2) \ldots \mathbf{s}(t_n)] \tag{19}$$

which defines the excitation process by the Equation 2. However, this procedure could be demanding because the linearized system would depend on each specific threshold of the previous responses and on the base excitation. It has been shown in (Fujimura and Der Kiureghian 2007) that the linearized system does not depend on the excitation scale and that in common practice it is possible to evaluate the TELS with respect a White Noise excitation regardless of its real power spectral density. The so–called "White Noise approximation" leads to reasonably approximated linearized systems. Thus, the set of IRFs of each response of interest will be computed for each threshold of the response itself and regardless of the statistics of the other responses.



(a) Structural Model

(b) Bouc–Wen hysteresis loop

Figure 1:

Table 1: Mechanical parameters of the structural models

Model	$k_1\left[\frac{N}{m}\right]$	$k_1\left[\frac{N}{m}\right]$	$m\,[kg]$
2DOF–1	$7.5\cdot10^7$	$7.0\cdot10^7$	$3.0\cdot10^4$
2DOF–2	$4.0\cdot10^7$	$2.0\cdot10^7$	$3.0\cdot10^4$

5 NUMERICAL APPLICATION

In order to test the proposed procedure, a numerical application has been developed. The chosen structural models are two degrees–of–freedom nonlinear buildings, whose geometric scheme is shown in Figure 5; also, mechanical parameters are summarized in Table 1. Note that the first–floor displacement has been set as the first response of interest $X_1\left(t\right)$ and the drift between the first and the second floot has been set as the second response $X_2\left(t\right)$. Nonlinearities are modeled by Bouc–Wen constitutive models (Bouc 1963), (Wen 1976), (Wen 1980) with $\alpha=0.05$, $A_0=n=1$ and $\beta=\gamma=35$. A sample hysteretic loop of the Bouc–Wen model is shown in Figure 5. The material's stiffness shown in Table 1 is the tangent of the Bouc–Wen loop at zero displacement.

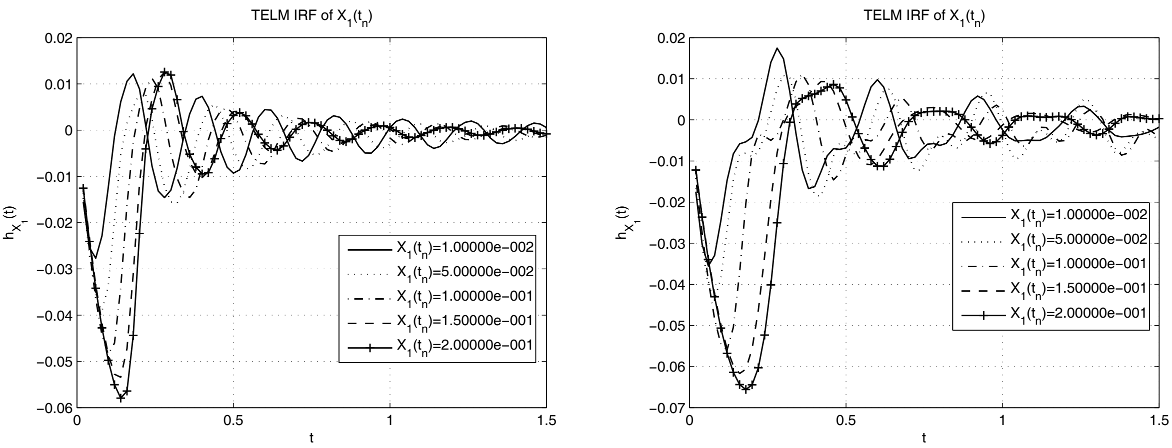
The first performance of TELM computes the IRFs of $X_1\left(t_n\right)$ for a set of 20 thresholds with step $\Delta x=1$cm; some of the computed IRFs are shown in Figure 2. Then, a random vibration analysis is performed in order to get the statistics and the power spectral density of the displacement X_1. The base excitation has been defined as an unfiltered white noise. The analysis has been performed for the white noise cutoff standard deviation $\sigma_{WN}=0.50g$. The displacement power spectral densities are shown in Figure 3.

Then, TELM has been performed again; the input excitation is the first displacement X_1 and the Limit State Function has been defined in terms of X_2. The computed IRFs are shown in Figure 4; note that each IRF is the evolution in time of the interstorey drift given by an impulsive action of the first floor displacement, as explained in the previous section.

The computed IRFs lead to the evaluation of the conditioned tail probabilities of X_2 given X_1. The conditioned response statistics are easily computed by evaluating the FRFs as the Fourier transform of the IRFs, and then to combine them with the power spectral densities of the displacement X_1. The conditioned PDFs of X_2 given X_1 are shown in Figure 5.

Finally, it is possible to employ the 16 in order to get the joint distribution of the responses. In Figure 6 the complementary–joint–PDF are shown, *i.e.* the probability that at least one of the responses is greater than the corresponding threshold.

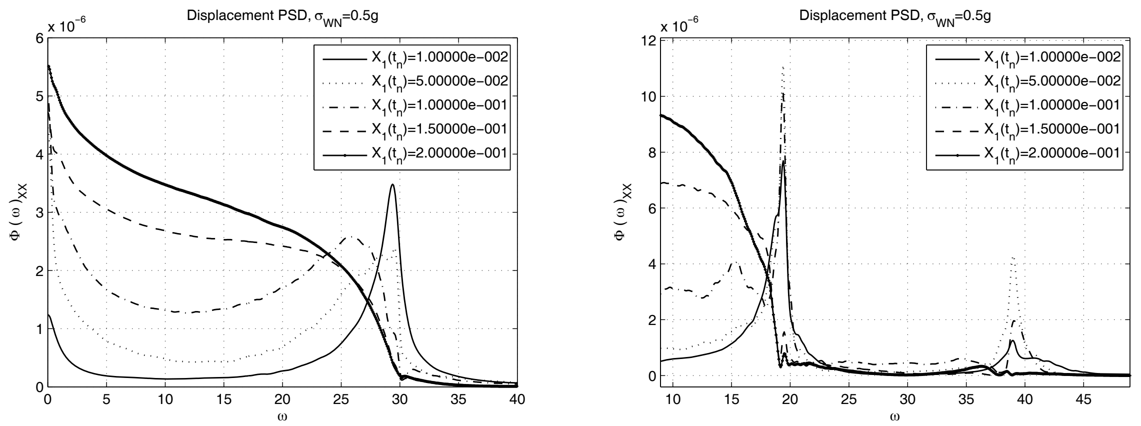
In order to check the approximation of the proposed strategy, a Monte Carlo simulation has been performed. The response statistics obtained with 10,000 samplings have been compared with the results of TELM. In particular, Figure 7 shows the difference of the complementary–joint–PDFs with respect the thresholds x_1 and x_2, specifically, in both cases, the computed error is not negligible for small thresholds. This expected behavior is due a drawback of TELM: the linearized system depends on the dimensions of the hysteresis loops. For this reason, the nonlinear relationship between forces and displacements is close to be linear at small thresholds, then, the response statistics are quite close to be Gaussian. However, it has been shown in (Fujimura 2006) that this drawback is overcomed by the evaluation of the extreme–response statistics: the error of the response statistics at t_n does not affects the first–passage probability. Also, in common practice, the structural safety is usually computed for higher values of the threshold, where the error between TELM and the Monte Carlo



(a) Model 2DOF–1

(b) Model 2DOF–2

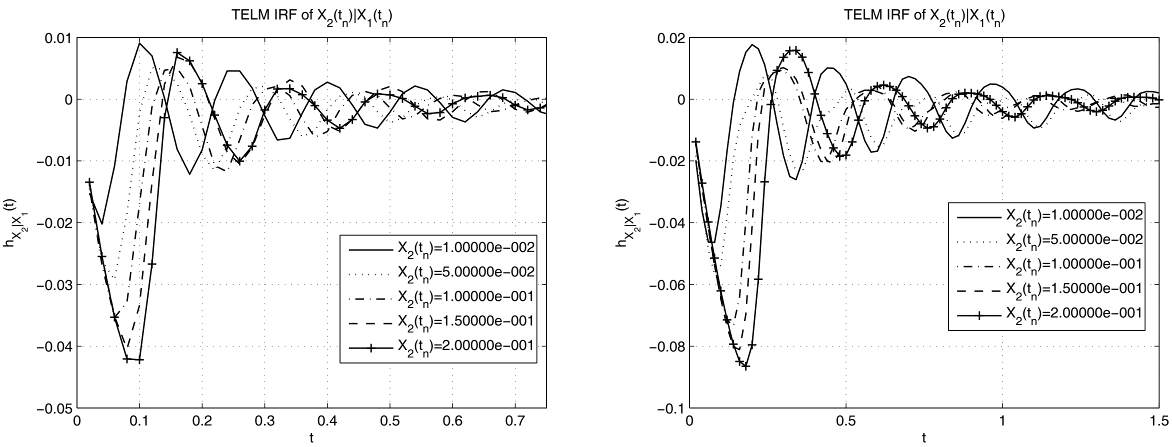
Figure 2: Impulse Response Functions – $X_1\left(t\right)$



(a) Model 2DOF–1

(b) Model 2DOF–2

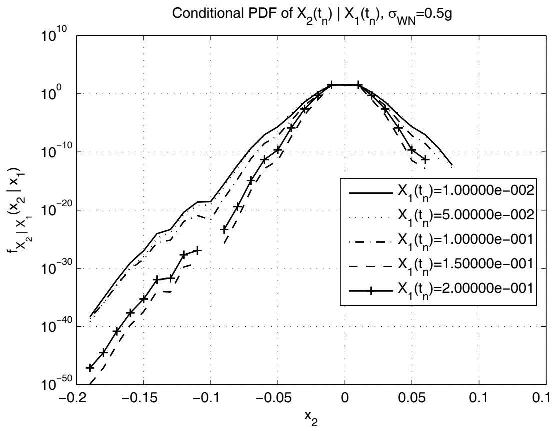
Figure 3: Power spectral densities of X_1 – $\sigma_{WN} = 0.50g$



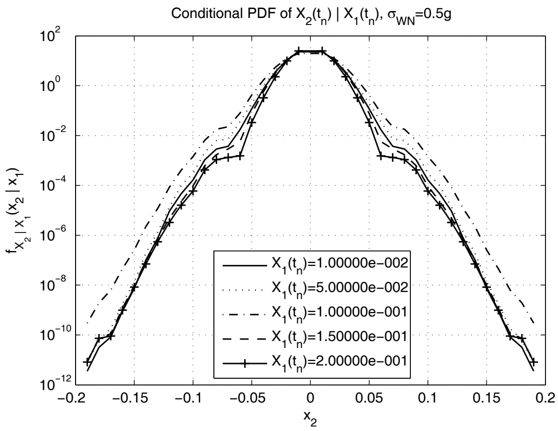
(a) Model 2DOF–1

(b) Model 2DOF–2

Figure 4: Impulse Response Functions – $X_2\left(t\right)$

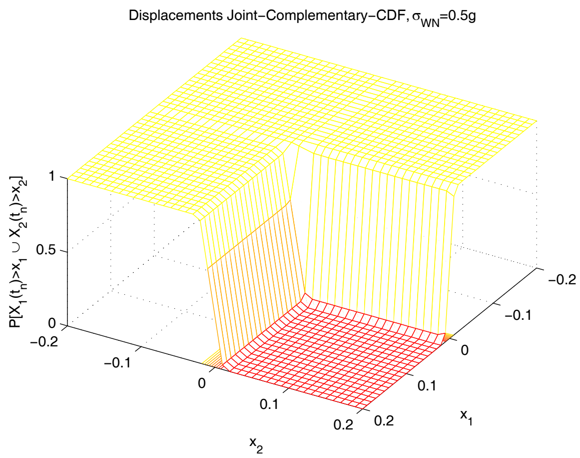


(a) Model 2DOF–1

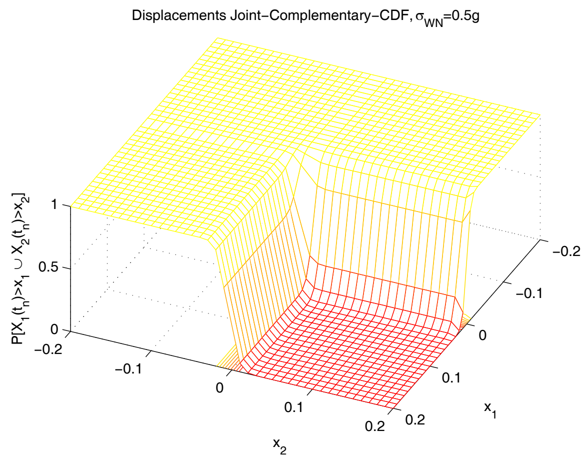


(b) Model 2DOF–2

Figure 5: PDF of $X_2\left(t\right)$ given $X_1 - \sigma_{WN} = 0.5g$



(a) Model 2DOF–1



(b) Model 2DOF–2

Figure 6: Complementary Joint–CDF – $\sigma_{WN} = 0.50g$

simulation is negligible.

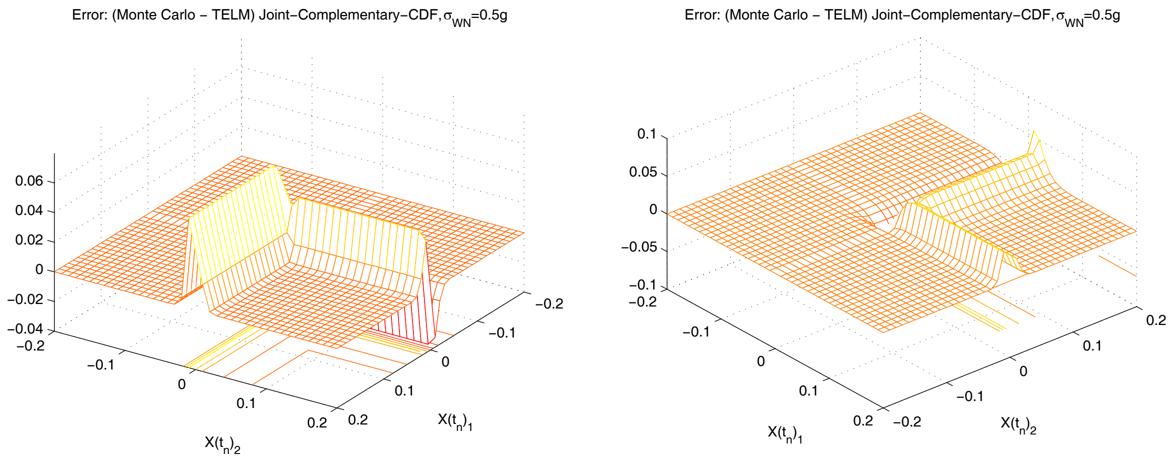
6 CONCLUSIONS

An extension of the Tail Equivalent Linearization Method is presented. Its goal is to compute the joint tail probability of structural systems whose reliability depends on two or more responses of interest. The proposed procedure can be performed regardless of the system dimensions, the only requirements are the existence and uniqueness of TELS. In this paper, just the series system case has been investigated, however, the algorithm can be easily extended to the general case of parallel and hybrid systems. Furthermore, numerical applications have been provided; in order to be able to show the joint probabilities, only the two–degrees of freedom has been presented. In any case, a generalized procedure can be applied also to multi–dimensional systems.

The proposed algorithm leads to reasonably accurate results which can be employed in order to get the level–crossing rate and the Joint First–Passage Probability of interconnected systems (Song and Der Kiureghian 2006); it appears particularly convenient for buildings. Also, the TELM algorithm has been implemented in OpenSees, a framework for structural analysis provided by the Pacific Earthquake Engineering Research Center and it is available for scientific purposes.

However, further work can be developed. In fact, the *white noise approximation* affects the computation of the conditioned Impulse Response Functions (IRFs); the error is negligible for the provided applications, however, it could be higher for complex systems and narrow band excitations. A further development would define the conditioned IRFs with respect conveniently filtered white noises in order to get the structural filtering.

Also, a further development is the implementation of multi–support excitations: the provided structure presents two responses whose the second one can be expressed in function of the first response and regard-



(a) Model 2DOF–1 (b) Model 2DOF–2

Figure 7: Monte Carlo - TELM error – $\sigma_{WN} = 0.50g$

less of the base excitation. If the structure has coupled responses, then TELM should take into account of the multiple dependence by using multi–support excitations.

REFERENCES

Bouc, R. (1963). Forced vibrations of mechanical systems with hysteresis. In A. Publishing (Ed.), *Proceedings of the 4th conference on nonlinear oscillation*, Prague, pp. 315. Academia Publishing.

Der Kiureghian, A. (2000). The geometry of random vibrations and solutions by form and sorm. *Probabilistic Engineering Mechanics 15*(1), 81 – 90.

Der Kiureghian, A. & K. Fujimura (2009). Nonlinear stochastic dynamic analysis for performance-based earthquake engineering. *Earthquake Engineering & Structural Dynamics 38*(5), 719–738.

Fujimura, K. (2006). *Tail Equivalent Linearization Method for Nonlinear Random Vibration*. PhD dissertation, University of California – Berkeley, Department of Civil and Environmental Engineering.

Fujimura, K. & A. Der Kiureghian (2007). Tail-equivalent linearization method for nonlinear random vibration. *Probabilistic Engineering Mechanics 22*(1), 63 – 76.

Lutes, L. D. & S. Sarkani (2004). *Random Vibrations – Analysis of Structural and Mechanical Systems* (2004 ed.). Oxford, UK: Elsevier.

Sessa, S. & A. Der Kiureghian (2009, September). Application of the tail–equivalent linearization method for stochastic dynamic analysis with asymmetric hysteresis. In *Proceedings of the 10th International Conference on Structural Safety And Reliability – Osaka*.

Song, J. & A. Der Kiureghian (2003). Bounds on system reliability by linear programming. *Journal of Engineering Mechanics 129*(6), 627–636.

Song, J. & A. Der Kiureghian (2006). Joint first-passage probability and reliability of systems under stochastic excitation. *Journal of Engineering Mechanics 132*(1), 65–77.

Wen, Y. (1976). Method for random vibration of hysteretic systems. *Journal of Engineering Mechanics Division, ASCE 102*(2), 249 – 163.

Wen, Y. (1980). Equivalent linearization for hysteretic systems under random excitations. *Journal of Applied Mechanics ASME 47*(3), 150 – 154.