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# Chasing Non-diagonal Cycles in a Certain System of Algebras of Operations

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**Abstract** The mod 2 universal Steenrod algebra $Q$ is a non-locally finite homogeneous quadratic algebra closely related to the ordinary mod 2 Steenrod algebra and the Lambda algebra. The algebra $Q$ provides an example of a Koszul algebra which is a direct limit of a family of certain non-Koszul algebras $R_k$'s. In this paper we see how far the several $R_k$'s are to be Koszul by chasing in their cohomology non-trivial cocycles of minimal homological degree.

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## 1 Introduction

S. Priddy introduced the notion of Koszul algebra in [18] to construct resolutions for the Steenrod algebra and study the universal enveloping algebra of a Lie algebra. Ever since, the theory of Koszul algebras never stopped to attract the interest of people working in several areas of mathematics. Koszul algebras arise in fact in commutative and non-commutative algebraic geometry, representation theory, number theory, combinatorics and algebraic topology. The now nine years old [17] gives a beautiful account on the dramatic impact of Koszul algebras in the theory of quadratic algebras, and literature already offers several attempts to generalize the koszulness condition to non-quadratic algebras (see, for instance, [1] and [8]).

In this paper we deal with homogeneous graded augmented algebras $A$ isomorphic to a quotient of the form $T(V)/J(R)$, where $T(V) = \bigoplus_i T_i$ is the tensor algebra over a $\mathbb{K}$-vector space $V$ with basis $X = \{x_i \mid i \in \mathcal{I}\}$, $\mathcal{I}$ is a subset of $\mathbb{Z}$, $J(R)$ is the two-sided ideal of relations generated by some $R \subset T_2 = V \otimes V$, and the augmentation $\epsilon$ acts as follows

$$\epsilon : x_k \in T_1 \longmapsto 0 \in \mathbb{F}_2 \;\; \forall\, k \in \mathbb{Z}, \quad \text{and} \;\; \epsilon : 1 \in T_0 = \mathbb{K} \;\longmapsto 1.$$

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Once we assign to each monomial of type $x_{i_1} \cdots x_{i_k}$ length $k$ and *internal degree* $i_1 + \cdots + i_k$, and suppose $R$ homogeneous with respect to the internal degree, the tensor algebra $T(V)$ and the algebra $A \cong T(V)/J(R)$ become bigraded, and its cohomology

$$H^{i,j,k}(A) = \mathrm{Ext}_A^{i,j,k}(\mathbb{K}, \mathbb{K})$$

trigraded, the homological degree being $i$. The diagonal cohomology

$$D^{j,*}(A) = \oplus H^{j,j,*}(A)$$

is in general a subalgebra of $H(A)$. Among the several (and equivalent when $A$ is finitely generated) definition of koszulness we choose the so-called *diagonal purity*:

**Definition 1** A homogeneous quadratic algebra $A$ is said to be *Koszul* if

$$H(A) = D(A).$$

The mod 2 universal Steenrod algebra $Q$ is the algebra generated by $\{y_i \,|\, i \in \mathbb{Z}\}$ subject to the so-called generalized Adem relations:

$$y_{2k-1-n}\, y_k = \sum_j \binom{n-1-j}{j} y_{2k-1-j}\, y_{k+j-n} \quad (k \in \mathbb{Z},\ n \in \mathbb{N}_0). \tag{1.1}$$

The algebra $Q$ first appeared in [16], and it is isomorphic to the algebra of cohomology operations in the category of $H_\infty$-ring spectra (see [7], Ch. 3 and 8). Together with its odd $p$ analogue $Q(p)$, the universal Steenrod algebra $Q$ has been extensively studied, among others, by the authors ([2]-[6], [9]-[10], [12]-[14]). In particular, it has been proved that $Q$ and $Q(p)$ are Koszul algebras in [5] and [2], respectively.

The koszulness of $Q$ has not been worked out by using tools described in [11] and [18], since they are not suitable for non-locally finite algebras like $Q$. We rather handled a system of locally finite quadratic algebras $\{R_k, \phi_k\}$ proving that

$$Q \cong \varinjlim \{\, R_k, \phi_k \,\}, \quad \text{and} \quad H(Q) \cong \varprojlim \{H(R_k), \phi_k^*\}. \tag{1.2}$$

In fact, as it turned out, $\{R_k, \phi_k\}$ satisfies the Mittag-Leffler condition, and the non-zero elements in $H^{s,t,*}(R_k)$ with $s \neq t$ do not give any contribution to $H(Q)$. In this paper, we show that the algebras $R_k$'s are not Koszul for $k \geq 3$. Hence the algebra $Q$ turns out to be an example of a Koszul algebra which is a direct limit of non-Koszul algebras. Such curious phenomenon makes worthy to search for non-diagonal classes in $H^{s,t,*}(R_k)$ of minimal homological degree, and see how many steps they can be pushed backward in the inverse system (1.2). In a sense that we'll make precise in the next section, such classes appears as soon as possible. In fact we shall prove the following Theorem.

**Theorem 1** *For all $k \geq 3$, the graded vector $\mathbb{F}_2$-space $\mathrm{Ext}_{R_k}^{3,4,*}(\mathbb{F}_2, \mathbb{F}_2)$ has positive dimension.*

The material is organized as follows. In Section 2, we include some classical results on quadratic algebras, recall the definition of the algebras $R_k$'s and study the properties of a certain $\mathbb{F}_2$-basis picked in each of them. In Section 3, we give the proof of Theorem 1 split in several Lemmas and Propositions. Finally, Section 4 collects some open problems.

Despite the theoretical importance of ordinary and higher Massey products to generate non-diagonal cohomology classes (see [15]), such machinery has not been used here, since in our context it is quite hard to see whether a fixed Massey product is trivial or not.

## 2 PBW bases and the algebras $R_k$'s

Let $A$ be a graded augmented homogeneous algebra as in Section 1, and $p : T(V) \longrightarrow A$ be the quotient map (of augmented algebras). If

$$X = \{x_i \,|\, i \in \mathcal{I}\} \quad \text{with } \mathcal{I} \subseteq \mathbb{Z}$$

is a set of generators of $V$, then obviously the elements $a_i = p(x_i)$ generate $A_+$, the kernel of the augmentation $\epsilon : A \to \mathbb{K}$. For multi-indexes $I = (i_1, \ldots, i_n) \in \mathcal{I}^n$ and $J = (j_1, \ldots, j_m) \in \mathcal{I}^m$, we write $(I, J)$ for the multi-index $(i_1, \ldots, i_n, j_1, \ldots, j_m)$, and denote by $a_I$ the monomial $a_{i_1} \cdots a_{i_n}$, the one corresponding to the empty multi-index being 1. Any subset of $\cup \mathcal{I}^k$ is totally ordered by length first, and then by lexicographical order.

To a fixed basis of monomials $\mathcal{B}$ for $A$, we associate the following set of multi-indexes

$$S_{\mathcal{B}} = \{I \,|\, a_I \in \mathcal{B}\}.$$

**Definition 2** A basis of monomials $\mathcal{B}$ for $A$ is a *Poincaré-Birkhoff-Witt (PBW) basis* if the following conditions hold.
1. For any $I$ and $J$ in $S_{\mathcal{B}}$ such that $a_I a_J \neq 0$ and $(I, J) \notin \mathcal{S}_{\mathcal{B}}$, the multi-index of each monomial appearing in the expression of $a_I a_J$ in terms of elements of $\mathcal{B}$ is greater than $(I, J)$.
2. For any $k > 2$ and $(i_1, \ldots, i_k) \in S_{\mathcal{B}}$, the multi-indexes $(i_1, \ldots, i_j)$ and $(i_{j+1}, \ldots, i_k)$ are in $S_{\mathcal{B}}$ for each $j \in \{1, \ldots, k-1\}$.
3. If $(i_1, \ldots, i_j)$ and $(i_j, \ldots, i_k)$ are in $S_{\mathcal{B}}$, then $(i_1, \ldots, i_k)$ is in $S_{\mathcal{B}}$ as well.

As proved by the second author in [12], The algebra $Q$ has a PBW linear basis made by 1 and the *admissible monomials*

$$\{y_{i_1} \cdots y_{i_h} \mid i_j \geq 2i_{j+1} \text{ for each } j = 1, \ldots, h-1\}. \tag{2.1}$$

In [5], the authors introduced a system of locally finite quadratic algebras $\{R_k, \phi_k\}$ defined as follows. The algebra $R_k$ is generated by

$$y_k, y_{k-1}, y_{k-2}, \cdots,$$

and its ideal of relations is generated by

$$P_{2i-1-n,i} = y_{2i-1-n} y_i - \sum_j \binom{n-1-j}{j} y_{2i-1-j} y_{i+j-n} \tag{2.2}$$

with $\max\{i, 2i-1\} \leq k$. The map $\phi_k : R_k \longrightarrow R_{k+1}$ of the system maps $y_I \in R_k$ onto the homonymous monomial in $R_{k+1}$. Such maps are not all inclusions. In fact it can be proved that the dimension of $\operatorname{Ker} \phi_{2h}$ is positive for $h > 0$, for example $\operatorname{Ker} \phi_2$ contains $y_2 y_2 y_1$ which is non-zero in $R_2$. On the contrary the maps $\phi_{2h+1}$ are all monomorphisms. For $h > 0$ this comes as a consequence of Lemma 2 below.

For the rest of the paper we shall adopt a nomenclature resembling the ordinary Steenrod algebra: each $P_{a,b}$ in (2.2) will be called an *Adem relation*, while a monomial $y_{i_1} \cdots y_{i_k}$ (and its multi-index as well) will be said to be *admissible* if $i_h \geq 2i_{h+1}$.

To recall the recursive procedure explained in [5] to define a basis $\mathcal{B}_k$ for $R_k$, and reproduced almost verbatim below, we gives the following Definition.

**Definition 3** A non-admissible monomial $y_{i_1} y_{i_2} \cdots y_{i_\ell}$ in $R_k$ is said to be *exceptional* if $i_{h+1} > (k+1)/2$ for any non-admissible couple $(i_h, i_{h+1})$.

By Definition 3, it immediately follows that $R_k$ contains exceptional monomials if and only if $k \geq 2$.

In $\mathcal{B}_k$ we first include 1 and the set $\mathcal{C}_k$ of admissible monomials. Fixed a length $l$ and an internal degree $m$ such that $R_k^{l,m} \setminus \operatorname{Span}(\mathcal{C}_k^{l,m})$ is not empty, we pick in $\mathcal{B}_k$ an exceptional monomial $c_1$ with maximal label in $R_k^{l,m}$. This is possible since, by Lemma 3.1 in [5], the number of monomials in $R_k^{l,m}$ is finite. We now iterate the procedure. If

$$R_k^{l,m} \setminus \operatorname{Span}(\mathcal{C}_k^{l,m} \cup \{c_1, ..., c_{t-1}\})$$

is not empty, then we pick the monomial $c_t$ of maximal label out, and put it in $\mathcal{B}_k^{l,m}$. Monomials in $\mathcal{C}_k^{l,m} \cup \{c_1, ..., c_t\}$ are independent by construction, hence we get a basis for $R_k^{l,m}$ in a finite number of steps. Finally $\mathcal{B}_k = \cup \mathcal{B}_k^{l,m}$.

Since the main tool in the proof of Theorem 1 consists in finding suitable exceptional monomials not belonging to $\mathcal{B}_k$, it is worthy to examine some properties of such basis.

**Proposition 1** *Let $k > 2$. For any exceptional monomial $y_i y_j y_\ell$ in $R_k$ not belonging to $\mathcal{B}_k$, the monomial $y_i y_j$ is exceptional, while $y_j y_\ell$ is admissible and involved in at least one Adem relation.*

*Proof* A priori, the only other possibility for an exceptional $y_i y_j y_\ell$ to be written as sum of monomials of greater label is that $y_i y_j$ is admissible and involved in at least one Adem relation, and $y_j y_\ell$ is exceptional. This case, however, cannot occur, since in any non-trivial relation

$$y_i y_j y_\ell = \sum y_{i_h} y_{j_h} y_{\ell_h},$$

all the $\ell_h$'s should be necessarily equal to $\ell$, and there exists at least one summand with $y_{i_h} y_{j_h}$ neither admissible or exceptional.

**Lemma 1** *Let $k > 1$. In each relation in $R_k$ involving both admissible monomials and exceptional monomials, the minimal label is exceptional.*

*Proof* Consider in $R_k^{l,m}$ a relation

$$y_{I_1} + \cdots + y_{I_s} + y_{J_1} + \cdots + y_{J_t} = 0,$$

where the $y_I$'s are admissible, the $y_J$'s are exceptional, and none of them can be cancelled out.

If the integer $\bar{k}$ is sufficiently large, we may assume that in $R_{\bar{k}}$ there are all the generalized Adem relations required to write each

$$\phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{J_h})$$

as a sum $c_1^h + \cdots + c_{t_h}^h$, where $c_j^h \in \mathcal{C}_{\bar{k}}$ for any $j = 1, \ldots, t_h$. Monomials in this sum have a label greater than $J_h$, since in every generalized Adem relation the minimal label belongs to the non-admissible monomial. By definition $\mathcal{C}_{\bar{k}} \subseteq \mathcal{B}_{\bar{k}}$, hence every $\phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{I_i})$ in the equality

$$\sum_{i=1}^{s} \phi_{\bar{k}-1} \circ \cdots \circ \phi_k(y_{I_i}) + \sum_{h=1}^{t} (c_1^h + \cdots + c_{t_h}^h) = 0$$

has to be cancelled out. This is only possible if, for each $i$, there exists a label $J_h$ less than $I_i$.

**Proposition 2** *The basis $\mathcal{B}_k$ satisfies Condition 1 and 2 of Definition 2.*

*Proof* Consider two monomials $y_I = y_{i_1} \cdots y_{i_s}$ and $y_J = y_{j_1} \cdots y_{j_t}$ in $\mathcal{B}_k$ such that $y_I y_J$ is not zero and not belonging to $\mathcal{B}_k$.

If $y_I y_J$ is exceptional, then it is a sum of admissible monomials and exceptional monomials of greater label by construction of $\mathcal{B}_k$. Otherwise $y_I$ and $y_J$ are both admissible and the Adem relation $P_{i_s, j_1}$ is available in $R_k$. In this case we can argue as in the proof of Theorem 3.1 in [19], ch. 1. We start to apply a generalized Adem relation to $y_{i_s} y_{j_1}$. Thus we get $y_I y_J$ as a sum of monomials of greater label and smaller *moment*. Applying again a suitable generalized Adem relation to every non-admissible pair which comes out, we eventually succeed in writing $y_I y_J$ as sum of admissibles and possibly exceptionals of greater label.

Condition 2 is immediately verified for admissible monomials. When a monomial is instead exceptional, validity of Condition 2 comes quite easily from Lemma 1.

**Proposition 3** *For $k > 2$ the algebra $R_k$ does not admit any PBW basis.*

*Proof* Let $\mathcal{B}$ be any basis of $R_k$ containing only monomials. By definition of $R_k$, it follows that $\mathcal{B}$ contains every monomial of length 2 which is admissible or exceptional. Let now $r$ be the integral part of $(k+1)/2$. If $r$ is even, consider in $R_k$ the monomial $y_{2r-1} y_{r+1}$ and $y_{r+1} y_{r/2}$. Both are in $\mathcal{B}$, since the former is exceptional, and the latter is admissible; but $y_{2r-1} y_{r+1} y_{r/2}$ does not belong to $\mathcal{B}$, in fact

$$y_{2r-1}(y_{r+1} y_{r/2}) = y_{2r-1}(y_r y_{1+r/2}) = 0 \cdot y_{1+r/2} = 0.$$

Thus $\mathcal{B}$ does not satisfy Condition 3 of Definition 2. When $r$ is odd, $\mathcal{B}$ is not PBW for the same reason: one could check that

$$y_{2r-1} y_{r+2} y_{(r-1)/2} = y_{2r-1} y_r y_{(r+3)/2} = 0.$$

**Lemma 2** *For $\ell > 0$, the algebra $R_{2\ell+2}$ is isomorphic to the free product $R_{2\ell+1} \sqcup \mathbb{F}_2[y_{2\ell+2}]$, i.e. the algebra freely generated by $R_{2\ell+1}$ and $\mathbb{F}_2[y_{2\ell+2}]$.*

*Proof* Since the multi-index $(2\ell{+}2, 2\ell{+}2)$ is exceptional in $R_{2\ell+2}$, and $\dim R^{s,s(2\ell+2)}_{2\ell+2}$ is 1, necessarily $y^s_{2\ell+2}$ belongs to $\mathcal{B}_{2\ell+2}$ for every $s \geq 0$. Hence $\mathbb{F}_2[y_{2\ell+2}]$ is a subalgebra of $R_{2\ell+2}$. The statement now follows from the fact that no Adem relation available in $R_{2\ell+2}$ involves $y_{2\ell+2}$.

**Proposition 4** *The algebras $R_k$'s are all Koszul for $k \leq 2$.*

*Proof* For $k \leq 1$, there are no exceptional monomials, hence $\mathcal{B}_k$ is made only by admissibles. This in particular implies that $\mathcal{B}_k$ is a PBW basis, and the existence of a PBW basis is a sufficient condition for a locally finite algebra to be Koszul. In fact, the argument along the proof of Theorem 5.2 in [18] can be adapted to any locally finite algebra. (see also Theorem 3.1 in [17]). Let now $k = 2$. By Lemma 2, $R_2$ is isomorphic to the free product of two Koszul algebras, hence it is itself Koszul (see, for instance Proposition 1.1 in [17], ch. 3).

Our chase of non-diagonal trivial classes in $H^{i,i+j}(R_k)$ for $k > 2$ and $j > 0$ starts with the following Proposition.

**Proposition 5** $H^{1,1+j}(R_k) = 0$ *and* $H^{2,2+j}(R_k) = 0$ *for all $j > 0$ and for all $k$.*

*Proof* This is essentially Corollary 5.3 in [17], ch. 1. Note that the algebras $R_k$ are all quadratic and 1-generated in the sense of [17], p. 6.

Proposition 5 implies that the minimal possible homological degree for non-diagonal cohomology classes of $R_k$ is 3, and Theorem 1 precisely says that for $k \geq 3$ such minimum is actually achieved. In other words neither of the algebras $R_k$'s for $k \geq 3$ is 4-Koszul in the sense of [17], p. 29.

We end this section by summarizing some algebraic features of the Adem relations repeatedly used in the next section. The omitted proof just relies on the arithmetics of the indices and the mod 2 binomial coefficients in (2.2).

**Proposition 6** *i) No Adem relation* (2.2) *contains monomials of type $y_{2s}y_s$ among its summands.*
*ii)* $P_{2s-1,s} = y_{2s-1}y_s$.
*iii)* $P_{2s-1-2^n,s} = y_{2s-1-2^n}y_s - y_{2s-1}y_{s+2^n}$ *for all $n \geq 0$.*

Obviously, the Adem relations in Parts ii) and iii) of Proposition 6 are available in $R_k$ when $k \geq \max\{s, 2s-1\}$.

## 3 Proof of Theorem 1

For every Adem relation $P_{a,b}$ available in $R_k$, we shall often denote by $F_{(a,b)}$ its admissible part. In other words,

$$P_{a,b} = y_a y_b - F_{(a,b)}.$$

The element in the cobar construction corresponding to $y_I \in \mathcal{B}_k \setminus \{1\}$ will be denoted by $\alpha_I$.

**Proposition 7** *For every $h \geq 0$ the graded $\mathbb{F}_2$-vector space $H^{3,4,*}(R_{8h+3})$ is non-zero.*

*Proof* Consider the cochain $\beta(8h+3)$ equal to the sum

$$\alpha_{8h+3}\,\alpha_{(4h+3,2h+1)}\,\alpha_{h+1} + \alpha_{(8h+3,4h+3)}\,\alpha_{2h+1}\,\alpha_{h+1} + \alpha_{8h+3}\,\alpha_{4h+2}\,\alpha_{(2h+2,h+1)}.$$

We have

$$\delta\alpha_{(4h+3,2h+1)} = \alpha_{4h+3}\,\alpha_{2h+1} + \alpha_{4h+2}\,\alpha_{2h+2}.$$

Furthermore $\delta\alpha_{(8h+3,4h+3)}$ is simply equal to $\alpha_{8h+3}\,\alpha_{4h+3}$, since $y_{8h+3}\,y_{4h+3}$ is exceptional, and

$$\delta\alpha_{(2h+2,h+1)} = \alpha_{2h+2}\,\alpha_{h+1} \quad \text{by Proposition 6, i).}$$

It immediately follows that $\beta(8h+3)$ is a cocycle. To see that $\beta(8h+3)$ is not a coboundary, note that for any cochain $\gamma$ of homological degree 2 and length 4, the monomial $\alpha_{8h+3}\,\alpha_{(4h+3,2h+1)}\,\alpha_{h+1}$ never appears as top term among the summands of $\delta\gamma$, since $y_{8h+3}\,y_{4h+3}\,y_{2h+1} = 0$ (see the proof of Proposition 3), and $y_{4h+3} \cdot (y_{2h+1}\,y_{h+1})$ is null by Proposition 6, ii).

To prove the non-koszulness of the algebras $R_{8h+5}$ for $h \geq 0$, we need some Lemmas. Moreover, the case $k = 5$ has to be managed separately.

**Lemma 3** *The exceptional monomials of length 3 not belonging to $\mathcal{B}_5$ are*

$$y_s y_5 y_{2t-1} \qquad \text{for } t \leq 1 \text{ and } s \leq 4t+1,$$

*and*

$$y_s y_4 y_{2t-1} \qquad \text{for } t \leq 1 \text{ and } s \leq 4t-1.$$

*Proof* By Lemma 1, an exceptional monomial not in $\mathcal{B}_5$ has either the form $y_a y_5 y_b$ or $y_a y_4 y_b$.The only way to write them as sum of admissibles and exceptionals of greater label is to have them involved in the equality

$$F_{(s,c)} y_3 = y_s F_{(c,3)} \qquad \text{for } s \leq 2c-1. \tag{3.1}$$

Once we write each non-admissible monomials in (3.1) as sum of admissibles and exceptionals, it is not hard to show that, the monomial of lowest multi-index will be $y_s y_5 y_{c-2}$ if $c$ is odd, and $y_s y_4 y_{c-1}$ if $c$ is even.

**Proposition 8** *There exists a non-trivial class in $H^{3,4,*}(R_5)$.*

*Proof* Since $\delta\alpha_{(5,1)} = \alpha_5\alpha_1 + \alpha_3\alpha_3$, the element

$$\beta(5) = \alpha_5\alpha_{(5,1)}\alpha_3 + \alpha_{(5,5)}\alpha_1\alpha_3 + \alpha_5\alpha_3\alpha_{(5,1)} + \alpha_5\alpha_{(3,5)}\alpha_1$$

is a cocycle. By Lemma 3, $y_5y_5y_1$ does not belong to $\mathcal{B}_5$, and $y_5y_1y_3$ is neither admissible nor exceptional. This implies that $\beta(5)$ is not a coboundary.

**Lemma 4** *Let $h > 0$. The monomial $y_{8h}y_{4h+6}y_{2h}$ in $R_{8h+5}$ is not in $\mathcal{B}_{8h+5}$.*

*Proof* Consider the relation

$$F_{(8h,4h+2)}y_{2h+4} = y_{8h}F_{(4h+2,2h+4)}. \tag{3.2}$$

When we write the polynomial on the first side as sum of admissibles and exceptional, the lowest label is $J = (8h+2, 4h+4, 2h)$. On the right side, we find

$$y_{8h}y_{4h+7}y_{2h-1} + y_{8h}y_{4h+6}y_{2h} + y_{8h}y_{4h+5}y_{2h+1}.$$

The third monomial is also equal to

$$F_{(8h,4h+3)}y_{2h+3},$$

whose lowest multi-index is in any case bigger than $J$. Replacing $y_{8h}y_{4h+5}y_{2h+1}$ by $F_{(8h,4h+3)}y_{2h+3}$ in (3.2), we succeed in writing $y_{8h}y_{4h+6}y_{2h}$ as sum of admissibles and exceptionals of greater label.

**Lemma 5** *The only Adem relation* (2.2) *containing* $y_{4h+6}y_{2h+2}$ *among its summands is* $P_{4h+4,2h+4}$. *The monomial* $y_{4h+6}y_{2h+0}$ *is instead involved in just two Adem relations, namely* $P_{4h+2,2h+4}$ *and* $P_{4h,2h+6}$.

*Proof* Among the Adem relations $P_{i,j}$ of internal degree $6h+6$ that possibly involve $y_{4h+6}y_{2h+2}$, the one with maximal label $(i,j)$ is $P_{4h+4,2h+4}$, which actually contains $y_{4h+6}y_{2h+2}$ among its summands. Note now that

$$F_{(4h+4-t,2h+4+t)} = \sum_j \binom{3t+2-j}{j} y_{4h+7+2t-j}\, y_{2h+1-2t+j},$$

and the binomial coefficient corresponding to $j = 2t+1$ is non-zero only for $t = 0$. A similar argument proves the second part of the statement.

**Proposition 9** *For* $h > 0$*, the* $\mathbb{F}_2$*-graded vector space* $H^{3,4,*}(R_{8h+5})$ *is non-zero.*

*Proof* We have seen by Lemma 4 that $y_{8h}y_{4h+6}y_{2h} \in R_{8h+5}$ is not in $\mathcal{B}_{8h+5}$, and the monomial $y_{4h+6}y_{2h}y_{h+2}$ is neither admissible nor exceptional. This implies that the cochain

$$\begin{aligned}\beta(8h+5) = {} & \alpha_{8h}\,\alpha_{(4h+6,2h)}\,\alpha_{h+2} + \alpha_{(8h,4h+6)}\,\alpha_{2h}\,\alpha_{h+2} \\ & +\alpha_{8h}\,\alpha_{4h+2}\,\alpha_{(2h+4,h+2)} + \alpha_{(8h,4h)}\,\alpha_{2h+6}\,\alpha_{h+2}\end{aligned}$$

is not a coboundary. To see that $\beta(8h+5)$ is a cocycle use the fact that

$$\delta\alpha_{(4h+6,2h)} = \alpha_{4h+6}\,\alpha_{2h} + \alpha_{4h+2}\,\alpha_{2h+4} + \alpha_{4h}\,\alpha_{2h+6}$$

that follows from Lemma 5.

**Proposition 10** *For every* $h \geq 0$ *the graded* $\mathbb{F}_2$*-vector space* $H^{3,4,*}(R_{8h+7})$ *is non-zero.*

*Proof* Note first that $y_{8h+4}\, y_{4h+6}\, y_{2h+2}$ does not belong to $\mathcal{B}_{8h+7}$. It is in fact the lowest monomial in the equality

$$y_{8h+4}F_{(4h+4,2h+4)} = F_{(8h+4,4h+4)}y_{2h+4}.$$

This proves that the cochain

$$\begin{aligned}\beta(8h+7) = {} & \alpha_{8h+4}\,\alpha_{(4h+6,2h+2)}\,\alpha_{h+2}\\ & +\alpha_{(8h+4,4h+6)}\,\alpha_{2h+2}\,\alpha_{h+2} + \alpha_{8h+4}\,\alpha_{4h+4}\,\alpha_{(2h+4,h+2)}\end{aligned}$$

is not a coboundary.

To prove that $\beta(8h+7)$ represents a non-trivial class in in $\mathrm{Ext}^{3,4}_{R_{8h+7}}(\mathbb{F}_2,\mathbb{F}_2)$ is essential to know that

$$\delta\alpha_{(4h+6,2h+2)} = \alpha_{4h+6}\alpha_{2h+2} + \alpha_{4h+4}\alpha_{2h+4}, \quad \text{and} \quad \delta\alpha_{(2h+4,h+2)} = \alpha_{2h+4}\,\alpha_{h+2}$$

which come from Lemma 5 and Proposition 6 i), respectively.

**Proposition 11** *For every $h > 0$, $H^{3,4,*}(R_{8h+1})$ is non-trivial.*

*Proof* In the relation

$$F_{(8h,4h+1)}y_{2h+2} = y_{8h}F_{(4h+1,2h+2)}$$

the monomial $y_{8h}y_{4h+3}y_{2h}$ can be written as sum of admissible and exceptional monomials of greater label, hence it is not in $\mathcal{B}_{8h+1}$. This in particular implies that the cochain

$$\begin{aligned}\beta(8h+1) = {} & \alpha_{8h}\,\alpha_{(4h+3,2h)}\,\alpha_{h+1} + \alpha_{(8h,4h+3)}\,\alpha_{2h}\,\alpha_{h+1}\\ & +\alpha_{8h}\,\alpha_{4h+1}\,\alpha_{(2h+2,h+1)} + \alpha_{(8h,4h)}\,\alpha_{2h+3}\,\alpha_{h+1},\end{aligned}$$

is not a coboundary. To see that $\delta\beta(8h+1)$ is zero, use the fact that

$$\delta\alpha_{(2s,s)} = \alpha_{2s}\,\alpha_s, \quad \text{and} \quad \delta\alpha_{(2s+3,s)} = \alpha_{2s+1}\,\alpha_{s+2} + \alpha_{2s}\,\alpha_{s+3} \tag{3.3}$$

for all $s \in \mathbb{Z}$; you also need that $\delta\alpha_{(8h,4h+3)} = \alpha_{8h}\,\alpha_{4h+3}$ since $y_{8h}\,y_{4h+3}$ is exceptional in $R_{8h+1}$.

So far, we have proved that $H(R_k)$ has at least a non-trivial cohomology class in homological degree 3 and length 4 for any odd $k \geq 3$. Therefore the proof of Theorem 1 ends with Proposition 12 below. Following [17], p. 25, in the next statement we denote by $A \sqcap B$ the direct sum of two graded algebras $A$ and $B$. Here and in Section 4 we write $\tilde{\beta}$ to denote the cohomology class represented by the cochain $\beta$.

**Proposition 12** *For any $\ell > 0$, $H(R_{2\ell})$ is isomorphic to*

$$H(R_{2\ell-1}) \sqcap \Lambda[\tilde{\alpha}_{2\ell}],$$

*where $\Lambda[\tilde{\alpha}_{2\ell}]$ is the exterior algebra on the class in $H^{1,1,2\ell}(R_{2\ell})$ represented by the dual of $y_{2\ell}$.*

*Proof* Immediate from Lemma 2 and Proposition 1.1 in [17], ch. 3.

## 4 Open problems

The cohomology of the algebras $R_k$'s for $k \geq 3$ is far from being fully understood. Recursion in defining $\mathcal{B}_k$ prevents us from stating results in the vein of Lemma 3 for bigger $k$'s and larger lengths, it also inhibits any effort to give a complete description of the cohomology rings, and leaves many questions open.

For instance, we could ask whether the non-diagonal part of $H(R_k)$ is nilpotent. The answer is surely negative for $k = 3$ and $k = 5$, since the cochains

$$\xi(3) = \alpha_{(1,3)}\alpha_0\alpha_1 + \alpha_1\alpha_{(3,0)}\alpha_1 + \alpha_1\alpha_1\alpha_{(2,1)}$$

and

$$\xi(5) = \alpha_{(1,5)}\alpha_1\alpha_1\alpha_1 + \alpha_1\alpha_{(5,1)}\alpha_1\alpha_1 + \alpha_1\alpha_3\alpha_{(3,1)}\alpha_1 + \alpha_1\alpha_3\alpha_1\alpha_2\alpha_{(2,1)}$$

represent torsion-free classes in $H^{3,4}(R_3)$ and in $H^{4,5}(R_5)$ respectively. If follows in particular, that $H^{3n,4n}(R_3)$ and $H^{4n,5n}(R_5)$ are non-zero for any $n \geq 0$. On the other hand, classes $\tilde{\beta}(k) \in H^{3,4}(R_k)$ have nilpotency degree 2. For instance, the cochain $\beta(3)^2$ represents 0 in $H^{6,8}(R_3)$, being hit by

$$\begin{array}{lll}
\alpha_3\alpha_{(3,1)}\alpha_{(1,3,3)}\alpha_1\alpha_1 & +\alpha_{(3,3)}\alpha_1\alpha_{(1,3,3)}\alpha_1\alpha_1 & +\alpha_3\alpha_{(3,1)}\alpha_{(1,3)}\alpha_{(3,1)}\alpha_1 \\
+\alpha_{(3,3)}\alpha_1\alpha_{(1,3)}\alpha_{(3,1)}\alpha_1 & +\alpha_3\alpha_{(3,1)}\alpha_{(1,3)}\alpha_2\alpha_{(2,1)} & +\alpha_{(3,3)}\alpha_1\alpha_{(1,3)}\alpha_2\alpha_{(2,1)} \\
+\alpha_3\alpha_2\alpha_{(2,1,3)}\alpha_{(3,1)}\alpha_1 & +\alpha_3\alpha_2\alpha_{(2,1,3,3)}\alpha_1\alpha_1 & +\alpha_3\alpha_2\alpha_{(2,1,3)}\alpha_2\alpha_{(2,1)}.
\end{array}$$

Another open question concerns the existence of a maximal homological degree for the generators of the cohomology algebras. In any case, for any $k > 2$ we could provide examples of classes in $H^{4,5}(R_k)$ that are not product of elements of lower homological degree.

A final question concerns how the “resistance” of each non-diagonal class in the inverse system (1.2) is related to its degrees and $k$. For example the class $\tilde{\beta}(8h+3)$ can be pulled back just once, i.e. it is not in $\mathrm{Im}(\phi^*_{8h+4} \circ \phi^*_{8h+3})$; $\tilde{\beta}(8h+1)$ and $\tilde{\beta}(8h+7)$ resist for 3 steps, while the cycle $\tilde{\beta}(8h+5)$ can be pulled back 5 times when $h > 0$.

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