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Length-Preserving Monomorphisms for Steenrod Algebras at odd primes

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Abstract Let p be an odd prime. In this paper we determine the group of length-preserving automorphisms for the ordinary Steenrod algebra $A(p)$ and for $\mathcal{B}(p)$, the algebra of cohomology operations for the cohomology of cocommutative $\mathbb{F}_p$-Hopf algebras. Contrarily to the $p = 2$ case, no length-preserving strict monomorphism turns out to exist.

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1 Introduction and Preliminaries

Let $\mathrm{Ext}_\Lambda(\mathbb{Z}_p, \mathbb{Z}_p)$ be the cohomology of a graded cocommutative Hopf algebra Λ over $\mathbb{F}_p$. When p is ad odd prime, the algebra $\mathcal{B}(p)$ of Steenrod operations on $\mathrm{Ext}_\Lambda(\mathbb{Z}_p, \mathbb{Z}_p)$ has been described in terms of generators and relations by Liulevicius in [20]. Namely, the generators, together with the unit 1, are the *p-th powers*

$$P^k : \mathrm{Ext}_\Lambda^{q,t}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathrm{Ext}_\Lambda^{q+2k(p-1),pt}(\mathbb{Z}_p, \mathbb{Z}_p) \quad (k, q, t \geq 0)$$

and the *Bockstein operator*

$$\beta : \mathrm{Ext}_\Lambda^{q,t}(\mathbb{Z}_p, \mathbb{Z}_p) \to \mathrm{Ext}_\Lambda^{q+1,pt}(\mathbb{Z}_p, \mathbb{Z}_p) \qquad (q, t \geq 0)$$

subject to the following relations:

$$\beta^2 = 0, \tag{1.1}$$

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$$P^aP^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} A(b,a,t)\; P^{a+b-t}P^t \quad \text{when } a < pb, \tag{1.2}$$

and

$$P^a\beta P^b = \sum_{t=0}^{\lfloor \frac{a}{p} \rfloor} B(b,a,t)\; \beta P^{a+b-t}P^t + \sum_{t=0}^{\lfloor \frac{a-1}{p} \rfloor} A(b,a-1,t)\; P^{a+b-t}\beta P^t \quad \text{when } a \le pb. \tag{1.3}$$

Coefficients in the several sums of (1.2) and (1.3) read as follows:

$$A(k,r,j) = (-1)^{r+j}\binom{(p-1)(k-j)-1}{r-pj},$$

and

$$B(k,r,j) = (-1)^{r+j}\binom{(p-1)(k-j)}{r-pj}.$$

The algebra of Liulevicius operations turned out to play an important role in stable homotopy computations (see for example [1], [2], [3], [20]). Out of some categorical peculiarities (see [4], [11], [23] and [27]), there are also significant connections with several other algebras of operations: for instance, the subalgebra $\mathcal{C}(p) \subset \mathcal{B}(p)$ generated by the set $\{P^i, \beta P^i \,|\, i \ge 0\} \cup \{1\}$ is a quotient of the Universal Steenrod algebra $\mathcal{Q}(p)$, introduced in [25] and broadly studied by the authors and other people ([5]-[8] and [10]-[18]).

Moreover, the ordinary Steenrod algebra $\mathcal{A}(p)$ and the algebra $\mathcal{A}(p)_L$ of cohomology operations for the cohomology of restricted Lie algebras over $\mathbb{F}_p$, are both quotients of $\mathcal{B}(p)$:

$$\mathcal{A}(p) \cong \frac{\mathcal{B}(p)}{(P^0-1)} \quad \text{and} \quad \mathcal{A}(p)_L \cong \frac{\mathcal{B}(p)}{(P^0)} \tag{1.4}$$

(for the latter, see Thm. 8.5 in [25]). When $p = 2$, Congruences (1.4) continue to hold once you replace the p-th power P^0 with the Steenrod square Sq^0.

Inspired by [19] and [26], where fractal structures inside $\mathcal{A}(p)$ – i.e. infinite descending chains of nested isomorphic subalgebras – led to find restrictions on the nilpotence of some elements in the Milnor basis, the authors investigated in the last few years the possible existence of fractal structures inside $\mathcal{Q}(p)$ (see [7], [4], [11], and [15]). In this paper we focus our attention on $\mathcal{B}(p)$.

Proposition 1.1 *The algebra $\mathcal{B}(2)$ admits a chain of nested subalgebras*

$$\mathcal{B}(2) = \mathcal{B}_0 \supset \mathcal{B}_1 \supset \cdots \supset \mathcal{B}_k \supset \cdots \tag{1.5}$$

all isomorphic to $\mathcal{B}(2)$.

Proof We recall that the mod 2 universal Steenrod algebra $\mathcal{Q}(2)$ is the $\mathbb{F}_2$-algebra generated by x_k, $k \in \mathbb{Z}$, together with $1 \in \mathbb{F}_2$, subject to the so-called mod 2 *generalized Adem relations*:

$$R(k,n) = x_{2k-1-n}x_k + \sum_j \binom{n-1-j}{j} x_{2k-1-j}\, x_{k-n+j}, \quad \text{where } (k,n) \in \mathbb{Z} \times \mathbb{N}_0.$$

Following two different approaches (see [7] and [15]), the authors proved that the $\mathbb{F}_2$-linear map $\lambda : \mathcal{Q}(2) \longrightarrow \mathcal{Q}(2)$ acting as follows

$$1 \longmapsto 1 \quad \text{and} \quad x_{i_1} \ldots x_{i_m} \longmapsto x_{2i_1-1} \cdots x_{2i_m-1}$$

is well-defined and injective, hence it can be regarded as a length-preserving strict monomorphism of algebras.

We also recall that the algebra $\mathcal{B}(2)$ of Steenrod operations on the cohomology ring of any cocommutative Hopf $\mathbb{F}_2$-algebra Λ is generated by

$$Sq^k : \mathrm{Ext}_{\Lambda}^{q,t}(\mathbb{Z}_2, \mathbb{Z}_2) \longrightarrow \mathrm{Ext}_{\Lambda}^{q+k,2t}(\mathbb{Z}_2, \mathbb{Z}_2) \quad (k, q, t \geq 0)$$

subject to the following relations (see [25]):

$$Sq^a Sq^b = \sum_j \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j \quad (0 \leq a < 2b).$$

It is not hard to show that

$$\zeta : x_k \in \mathcal{Q}(2) \longmapsto \begin{cases} Sq^k & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

is an epimorphism of algebras. The map $\tilde{\lambda} : \mathcal{B}(2) \to \mathcal{B}(2)$ obtained by extending multiplicatively

$$1 \in \mathcal{B}(2) \longmapsto 1 \in \mathcal{B}(2) \quad \text{and} \quad Sq^k \in \mathcal{B}(2) \longmapsto Sq^{2k-1} \in \mathcal{B}(2)$$

is a map of algebras. In fact it makes commutative the following diagram

$$\begin{array}{ccc} \mathcal{Q}(2) & \xrightarrow{\lambda} & \mathcal{Q}(2) \\ \zeta \downarrow & & \downarrow \zeta \\ \mathcal{B}(2) & \xrightarrow[\tilde{\lambda}]{\cdots\cdots} & \mathcal{B}(2) \end{array}$$

Finally we set $\mathcal{B}_k = \mathrm{Im}\,\tilde{\lambda}^k$. □

Let p be an odd prime. A set of algebra generators for $\mathcal{B}(n)$ is given by $\mathcal{S} \cup \{1\}$, where

$$\mathcal{S} = \{\beta,\, P^k \mid k \geq 0\}.$$

Each monomial of $\mathcal{B}(n) \setminus \mathbb{F}_p$ has the form

$$m = c\,\beta^{\varepsilon_1} P^{i_1} \;\cdots\; \beta^{\varepsilon_n} P^{i_n} \beta^{\varepsilon_{n+1}}$$

where $c \in \mathbb{F}_p$, $i_h \geq 0$, $\varepsilon_j \in \{0, 1\}$, and β^0 has to be read as 1.

We assign a degree and a length to each monomial m of $\mathcal{B}(n)$ as follows. When $m = c\beta^{\varepsilon}$ with $(c, \varepsilon) \in \mathbb{F}_p \times \{0, 1\}$, we set $\deg(m) = \ell(m) = \varepsilon$.

If instead

$$m = c\,\beta^{\varepsilon_1} P^{i_1} \;\cdots\; \beta^{\varepsilon_n} P^{i_n} \beta^{\varepsilon_{n+1}} \neq 0$$

with $c \in \mathbb{F}_p^*$, we set

$$\deg(m) = \varepsilon_{n+1} + \sum_{j=1}^{n} \bigl(2(p-1)i_j + \varepsilon_j\bigr),$$

and

$$\ell(m) = n + \sum_{j=1}^{n+1} \varepsilon_j.$$

These notions are well-defined since Relations (1.2) and (1.3) are degree- and length-preserving.

Section 2 is devoted to prove the following Theorem.

Theorem 1.2 *Let p be an odd prime. All length-preserving injective algebra endomorphisms for $\mathcal{B}(p)$ are isomorphisms.*

As a consequence, the odd p-counterpart of (1.5) would possibly come from an $\mathbb{F}_p$-algebra monomorphism $\xi : \mathcal{B}(p) \longrightarrow \mathcal{B}(p)$, such that $\xi(\mathcal{S})$ does not belong to the graded F_p-linear span of $\mathcal{S}$. The existence of such monomorphism remains dubious.

Let A be an algebra graded by length (i.e. relations in A preserve the length of each monomial). As in [11] we denote by $\underline{\mathrm{Aut}}(A)$ the group of length-preserving algebra (from now on: *LPA-automorphisms*) with respect to ordinary composition.

A key feature to prove Theorem 1.2 is the fact that, for any $\phi \in \underline{\mathrm{Aut}}(\mathcal{B}(p))$, the elements in $\mathcal{S}$ are all ϕ-eigenvectors. By closely examining such LPA-automorphisms we shall be able to prove the following Theorem.

Theorem 1.3 *Let p be an odd prime. The groups $\underline{\mathrm{Aut}}(\mathcal{B}(p))$ is isomorphic to $\mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1)$.*

We end this section by recalling a result surely known by the experts.

Proposition 1.4 *As an $\mathbb{F}_p$-vector space, $\mathcal{B}(p)$ has a basis $\mathcal{A}dm$ made by all monic admissible monomials, i.e. elements in $\mathcal{S} \cup \{1\}$ together with monomials*

$$\beta P^k, \quad P^k \beta, \quad \beta P^k \beta \quad \textit{and} \quad \beta^{\varepsilon_0} P^{t_1} \beta^{\varepsilon_1} P^{t_2} \beta^{\varepsilon_{s-1}} P^{t_s} \beta^{\varepsilon_s},$$

where $k \geq 0$, $\varepsilon_i \in \{0, 1\}$, and $t_j \geq p t_{j+1} + \varepsilon_j \geq 0$ for $1 \leq j < s$.

Proof Follow the proof of Proposition 9 in [9], and make the suitable straightforward changes. □

Example 1.5 Let k be any positive integer. The elements of $\mathcal{B}(p)$

$$\bigl(P^0\bigr)^k, \quad \beta\bigl(P^0\bigr)^k, \quad \text{and} \quad \beta\bigl(P^0\bigr)^k\beta$$

all belong to $\mathcal{A}dm$.

Elements in $\mathcal{A}dm$ can be totally ordered by degree first, and then by lexicographic order.

2 Proof of Theorem 1.2

Throughout the rest of the paper p will denote an odd prime. The chosen line of attack will resemble as closely as possible [11, Section 2].

We start by collecting some consequences of Relations (1.2) and (1.3).

Lemma 2.1 *Let $\mathcal{S} = \{\beta,\, P^k \mid k \geq 0\} \subset \mathcal{B}(n)$. The following statements hold.*
i) P^0 commutes with all elements of type $\beta^\varepsilon P^k$. Furthermore $\beta^\varepsilon P^k P^0$ is admissible.
ii) $P^1 P^b = 0$ if and only if $b \equiv -1 \pmod p$.
iii) Suppose a and b are two distinct non-negative integers. $P^a P^b = P^b P^a$ if and only if $ab = 0$.

Proof Part i) is immediate. Since $P^1 P^0$ is admissible, to prove Part ii) we can assume $b \geq 1$. Relation (1.2) for $a = 1$ says that

$$P^1 P^b = (b+1)\, P^{b+1} P^0.$$

To prove Part iii), fix two integers a and b such that $0 \leq a < b$. The element $P^a P^b$ is involved on the left side of a relation of type (1.2). There, the binomial coefficient $A(b, a, a)$ is 0 unless $a = 0$. □

Proposition 2.2 *Let $\phi : \mathcal{B}(p) \longrightarrow \mathcal{B}(p)$ be any LPA-endomorphism. The element P^0 is a ϕ-eigenvector.*

Proof $P^0 P^0$ is admissible and $P^1 P^1$ is non-zero by Lemma 2.1 ii).

It follows that $\phi(P^0)$ and $\phi(P^1)$ are not proportional to β.

Setting $\phi(P^0) = d_0 P^{L(0)}$ and $\phi(P^1) = d_1 P^{L(1)}$, we deduce that $P^{L(0)}$ and $P^{L(1)}$ commutes from Lemma 2.1 i). By Lemma 2.1 iii), this can only happen if $L(0)L(1) = 0$.

The proof will be over if we show that $L(1) \neq 0$. Suppose the contrary.

By Lemma 2.1 ii), we see that $P^1 P^{p-1} = 0$. By applying ϕ we would get

$$d_1 P^0 \phi(P^{p-1}) = 0. \tag{2.1}$$

Since $P^0 \beta$ is admissible, Lemma 2.1 i) forces $\phi(P^{p-1})$ in (2.1) to be zero, against the injectivity of the map ϕ. □

Proposition 2.3 *Let $\phi : \mathcal{B}(p) \longrightarrow \mathcal{B}(p)$ be any LPA-endomorphism. The element β is a ϕ-eigenvector.*

Proof By Proposition 2.2 we know that $\phi(P^0) = c_0 P^0$ for a certain $c_0 \in \mathbb{F}_p^*$. The elements βP^0 and $P^0 \beta$ hit different elements through the injective map ϕ. By Lemma 2.1 i) $\phi(\beta)$ cannot be proportional to a p-th power. □

Lemma 2.4 *Let $\phi : \mathcal{B}(p) \longrightarrow \mathcal{B}(p)$ be any LPA-endomorphism. If $\phi(P^1) = d_1 P^\ell$ for a certain $d_1 \in \mathbb{F}_p^*$, then*

$$\phi(P^k) = d_k P^{\ell k}$$

for each $k \geq 1$ and a suitable $d_k \in \mathbb{F}_p^*$.

Proof By Proposition 2.3, the element $\phi(P^k)$ is proportional to a pure p-th power for every k. We set

$$\phi(P^k) = d_k P^{L(k)}$$

for a suitable $d_k \in \mathbb{F}_p^*$.

Relation (1.3) for $a = 1$ gives

$$P^1 \beta P^k = k \cdot \beta P^{k+1} P^0 + P^{k+1} \beta P^0. \tag{2.2}$$

Once we apply the map ϕ to both sides of (2.2), for dimensional reasons we get

$$\ell + L(k) = L(k+1) \qquad \forall\, k \geq 1$$

or, equivalently,

$$L(k) = \ell\, k \qquad \forall\, k \geq 1$$

as claimed. □

Proposition 2.5 *Let $\phi : \mathcal{B}(p) \longrightarrow \mathcal{B}(p)$ be any LPA-endomorphism. For any $m \in \mathbb{N}$ the element P^m is a ϕ-eigenvector.*

Proof Again by Proposition 2.3, the element $\phi(P^1)$ is proportional to a pure p-th power. Suppose, by contradiction, that

$$\phi(P^1) = d_1 P^\ell$$

for a suitable $d_1 \in \mathbb{F}_p^*$ and $\ell > 1$. Relation (1.2) for $a = 2p-1$ and $b = 2$ gives

$$P^{2p-1} P^2 = 0.$$

We now apply the map ϕ to both sides. By Lemma 2.4 we get

$$P^{(2p-1)\ell} P^{2\ell} = 0$$

which is not true for $\ell > 1$. In fact, if this is the case,

$$\ell + 1 < 2\ell - \lfloor \frac{\ell}{p} \rfloor.$$

When we try to express $P^{(2p-1)\ell} P^{2\ell}$ as sum of admissibles, we find

$$P^{(2p-1)\ell} P^{2\ell} = \sum_{t=0}^{2\ell - \lfloor \frac{\ell}{p} \rfloor} A(2\ell, (2p-1)\ell, t)\; P^{(2p+1)\ell - t} P^t,$$

and

$$A(2\ell, (2p-1)\ell, \ell+1) = -1.$$

So far, we have proved that necessarily

$$\phi(P^1) = d_1 P^1.$$

Now the statement comes from Lemma 2.4.

Obviously, Propositions 2.2, 2.3, and 2.5 together prove Theorem 1.2.

3 Length-preserving automorphisms for $\mathcal{B}(n)$

Fix an element ϕ in $\underline{\mathrm{Aut}}(\mathcal{B}(p))$. Consistently with notations introduced in Section 2, we set

$$\phi(P^k) = d_k P^k \quad \text{and} \quad \phi(\beta) = c\,\beta.$$

Lemma 3.1 *For each $k \geq 0$ we have*

$$d_k = d_1 \cdot \left(\frac{d_1}{d_0}\right)^{k-1}.$$

Proof By applying the map ϕ to both sides of (2.2) we deduce that

$$d_1 d_k = d_{k+1} d_0 \qquad \forall k \geq 0,$$

and our statement comes from a straightforward inductive argument. □

We have now all the ingredients to prove Theorem 1.3.

Fixed a triple

$$(c, d_0, d_1) \in F_p^* \times \mathbb{F}_p^* \times \mathbb{F}_p^* \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1),$$

by multiplicatively extending the $\mathbb{F}_p$-linear map Φ_{c,d_0,d_1} defined on the set $\mathcal{S}$ as follows

$$\Phi_{c,d_0,d_1}(\beta) = c\,\beta \quad \text{and} \quad \Phi_{c,d_0,d_1}(P^k) = d_1 \cdot \left(\frac{d_1}{d_0}\right)^{k-1} P^k,$$

we get a well-defined map in $\underline{\mathrm{Aut}}(\mathcal{B}(p))$. In fact all Relations (1.1)-(1.3) are preserved.

4 Some remarks on the ordinary Steenrod algebra

The Adem relations in the ordinary Steenrod algebra $\mathcal{A}(p) = \mathcal{B}(n)/(P^0 - 1)$ do not preserve the length of monomials. For instance, we have

$$P^1 P^1 = 2P^2.$$

Yet, we could define an LPA-endomorphism for $\mathcal{A}(p)$ being an injective algebra homorphism preserving the length of admissible monomials. Specializing a result of Section 2, we get the following Proposition.

Proposition 4.1 *In the ordinary Steenrod algebra $\mathcal{A}(p)$ we have*

$$P^1 P^b = (b+1) P^{b+1}.$$

Proposition 4.2 *Let ϕ be an LPA-endomorphism for $\mathcal{A}(p)$. For all $k \geq 0$, the elements $\phi(P^k)$ are proportional to p-th powers.*

Proof We argue by contradiction. Suppose that $\phi(P^{\bar{b}}) = d_{\bar{b}}\beta$. By applying the map ϕ to both sides of

$$P^1 P^{\bar{b}} = (\bar{b}+1)P^{\bar{b}+1}$$

we would get

$$d_1 d_{\bar{b}} P^{L(1)}\beta = (b+1)d_{b+1}P^{L(b+1)},$$

which is not justified by any Adem relation. □

Let $\phi : \mathcal{A}(p) \longrightarrow \mathcal{A}(p)$ be any LPA-endomorphism. It is easy to adapt the arguments along the proof of Lemma 2.4 and Proposition 2.5 to show that all p-th powers are ϕ-eigenvectors. Now injectivity of ϕ implies that $\phi(\beta) = c\,\beta$ for a suitable $c \in \mathbb{F}_p^*$. This proves the following Theorem.

Theorem 4.3 *Let p be an odd prime. All length-preserving injective algebra endomorphisms for $\mathcal{A}(p)$ are isomorphisms.*

We denote by $\underline{\mathrm{Aut}}(\mathcal{A}(p))$ the group of $\mathcal{A}(p)$-automorphisms preserving the length of admissible monomials. Fixed an element $\phi \in \underline{\mathrm{Aut}}(\mathcal{A}(p))$, we set

$$\phi(\beta) = c\,\beta \quad \text{and} \quad \phi(P^k) = d_k P^k \quad \forall\, k > 0.$$

Arguing as in the proof of Lemma 3.1, we get $d_k = d_1^k$, and we can infer that

$$\underline{\mathrm{Aut}}(\mathcal{A}(p)) \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/(p-1).$$

In fact, each $(c, d_1) \in \mathbb{F}_p^* \times \mathbb{F}_p^*$ determines an element Φ_{c,d_1} acting as follows on monic monomials of length 1:

$$\Phi_{c,d_1}\beta = c\,\beta \quad \text{and} \quad \Phi_{c,d_1}P^k = d_1^k\,P^k.$$

As a final remark we note that the quotient map

$$\pi : \mathcal{B}(n) \longrightarrow \mathcal{A}(p)$$

induces the monomorphism

$$\underline{\mathrm{Aut}}(\pi) : \Phi_{c,d_1} \in \underline{\mathrm{Aut}}(\mathcal{A}(p)) \longmapsto \Phi_{c,1,d_1} \in \underline{\mathrm{Aut}}(\mathcal{B}(p)).$$

Our analysis leaves $G = \underline{\mathrm{Aut}}(\mathcal{A}(p)_L)$ out. The relation $P^0 = 0$ makes (2.2) less restrictive in an $\mathcal{A}(p)_L$-context, hence the authors plan to figure out how G is related with $\underline{\mathrm{Aut}}(\mathcal{B}(p))$ and $\underline{\mathrm{Aut}}(\mathcal{A}(p))$ in a future paper.

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