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STOCHASTIC RESPONSE OF FRACTIONAL VISCOELASTIC BEAMS

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Parole chiave: Viscoelasticità, Calcolo Frazionario, Modello frazionario di Kelvin-Voigt, Trave di Eulero-Bernoulli, Autofunzioni, Analisi nel domino della frequenza.

Abstract. *The aim of the present paper is the dynamic analysis of Euler-Bernoulli beam, characterized by fractional viscoelastic stress-strain relation, forced by stochastic load. Quasi-static viscoelastic behaviour of continuous Euler-Bernoulli beam has been investigated very recently, while the dynamic behaviour of fractional viscoelastic beam under stochastic loads is the new topic of study, and it is very useful for the vibration control in the real structures. The article provides an example of dynamic analysis in frequency domain of cantilever viscoelastic beam.*

Sommario. *Scopo del presente articolo è l'analisi dinamica della trave di Eulero-Bernoulli, avente legame tensione-deformazione viscoelastico frazionario, forzata da carico aleatorio. Il comportamento viscoelastico quasi-statico della trave continua di Eulero-Bernoulli è stato studiato recentemente, mentre il comportamento dinamico della trave viscoelastica frazionaria soggetta a carichi aleatori è argomento nuovo di ricerca, e tale studio risulta utile per il controllo delle vibrazioni nelle strutture reali. L'articolo fornisce un esempio di analisi dinamica nel dominio della frequenza di una trave a sbalzo viscoelastica.*

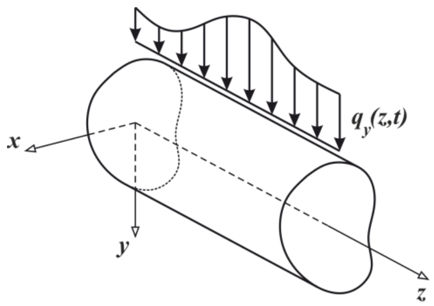
1 INTRODUCTION

The viscoelastic behaviour – typical of many materials i.e. rubber, glass, polymer, wood, ect. – has been investigated for more than two centuries. In the past the "classical" models as Maxwell and Kelvin-Voigt ones or more complex combinations of such units composed by springs and dashpots have been used to capture viscoelastic phenomena like relaxation and/or creep. However, these models, although very simple, show some inconsistencies. In the early years of twenty century Nutting [1] has shown that real experimental data of relaxation tests of viscoelastic material were well fitted by a power-law decay. The meaning of this experimental evidence is that the classical models are inconsistent because from them it is impossible to obtain power-law type relaxation functions. Starting from this observation

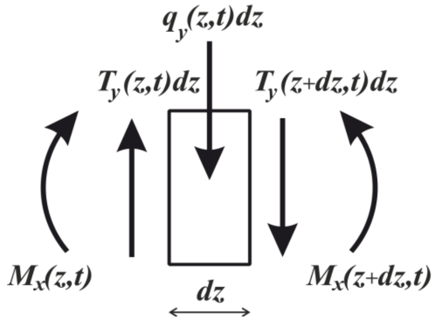
Scott-Blair et al. [2] introduced in the second part of the last century a new mathematical form to describe the viscoelastic behaviour. This new form of viscoelastic modelling agreed to the Nutting's data and it involved the fractional operators in the stress-strain relation. On the other hand, in the linear viscoelastic field the Boltzmann superposition principle is applicable, according to this principle the stress history is related to the strain history through a convolution integral, and if we introduce in the cited integral a power-law kernel we get a fractional differential or integral operator in the stress-strain relation. Such a model is called fractional viscoelastic model since fractional operators are involved, this fractional operators are the natural extension of classical differential/integral calculus [3,4].
In this paper the fractional operators are involved in a stress-strain relation of a continuous viscoelastic beam under the Euler-Bernoulli hypothesis. In particular we have considered the dynamic analysis of fractional viscoelastic beam forced by deterministic and stochastic loads and numerical examples have been provided.

2 FLEXURAL VIBRATION OF EULER-BERNOULLI BEAM MODELED USING FRACTIONAL KELVIN-VOIGT MODEL

Let us consider an isotropic homogeneous viscoelastic Euler-Bernoulli beam of length L, see Fig. 1, referred to the axes (x,y,z) with origin located at the centroid of the cross section, and (x,y) are principal axes of inertia of the cross section. All external spatially distributed loads, denoted as $q_y(z,t)$, are assumed to act in y-direction, thus orthogonally to the z-axis.



a) Layout of the beam



b) Free body diagram of the beam

Figure 1: Euler-Bernoulli beam.

Viscoelastic behaviour is described using a general model called fractional Kelvin-Voigt model, that is a springpot [4,5] in parallel with a spring (as shown in Fig. 2).

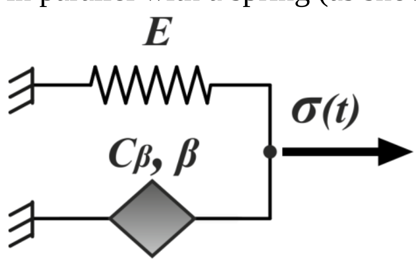


Figure 2: Fractional Kelvin-Voigt model.

Let $M_x(z,t)$ be the bending moment and $T_y(z,t)$ the shear in the section at abscissa z and at time *t*. The equilibrium equation for translations in the *y*-direction of the length dz of the beam is readily obtained by equating the inertia force to the sum of the forces exerted by the other parts of the beam and the external forces:

$$\frac{\partial^2 M_x(z,t)}{\partial z^2} = \rho(z)\frac{\partial^2 v(z,t)}{\partial t^2} - q_y(z,t) \tag{1}$$

being $\rho(z)$ the mass per unit length and $\partial M_x(z,t)/\partial z = T_y(z,t)$.

In virtue of the Euler- Bernoulli hypothesis, the cinematic and the mechanic relations read respectively:

$$\varepsilon(y,z,t) = -y\frac{\partial^2 v(z,t)}{\partial z^2} \tag{2}$$

$$\sigma(y,z,t) = \frac{M_x(z,t)}{I_x(z)}y \tag{3}$$

where $I_x(z)$ is the moment of inertia of the cross section with respect to the *x*-axis.

At this point, to capture the dynamic behaviour of a viscoelastic beam we need to introduce the appropriate constitutive law. For fractional Kelvin-Voigt model the law between the axial strain, $\varepsilon(y,z,t)$ and the stress $\sigma(y,z,t)$ is:

$$\sigma(y,z,t) = E(z)\varepsilon(y,z,t) + C_\beta\left({}_C D_{0^+}^{\beta}\varepsilon\right)(y,z,t) \tag{4}$$

Then introducing relation (4) into (2,3) to carry out the relation of bending moment $M_x(z,t)$ useful for Eq. (1), we obtain the flexural motion equation for this case:

$$\rho(z)\frac{\partial^2 v(z,t)}{\partial t^2} + \frac{\partial^2}{\partial z^2}\left\{E(z)I_x(z)\frac{\partial^2}{\partial z^2}\left[v(z,t)\right]\right\} + C_\beta\frac{\partial^2}{\partial z^2}\left\{I_x(z)\frac{\partial^2}{\partial z^2}\left[\left({}_C D_{0^+}^{\beta}v\right)(z,t)\right]\right\} = q_y(z,t) \tag{5}$$

Once it has been obtained the governing equation of motion, the flexural vibrations, solution of this differential equation, are, in any case, the linear combination of the eigenfunctions $\Phi_k(z)$ that are dependent on the constraints only (boundary conditions).

$$v(z,t) = \sum_{k=1}^{\infty} y_k(t)\Phi_k(z) \tag{6}$$

The coefficients of this linear combination, which are functions of time, are the modal coordinates $y_k(t)$ and depend on the initial conditions. In the next section numerical solution will be commented through a given example. We will use four eingenfunctions because results obtained with four eingenfunctions are perfectly identical than those obtained with forty eingenfunctions.

3 NUMERICAL SOLUTION AND REMARKS

Let us consider a fractional viscoelastic cantilever beam with span *L* of 5 meters subjected to an acceleration at the base: Gaussian white noise.

Choosing the more general viscoelastic constitutive law as the fractional Kelvin-Voigt model, the starting general equation is given in the form (5), which assumes the following form if, the stiffness $E(z)$, the density $\rho(z)$, and the moment of inertia $I_x(z)$, are all constant quantities:

$$\frac{\partial^2 v(z,t)}{\partial t^2}+\frac{C_\beta I}{\rho}\frac{\partial^4}{\partial z^4}\left[\left({}_C D_{0^+}^{\beta} v\right)(z,t)\right]+\frac{E\,I}{\rho}\frac{\partial^4 v(z,t)}{\partial z^4}=-\frac{\rho \ddot{z}_g(t)}{\rho} \tag{7}$$

where the coefficient C_β is selected as:

$$C_\beta = E\left(\frac{\mu}{E}\right)^{\beta} \tag{8}$$

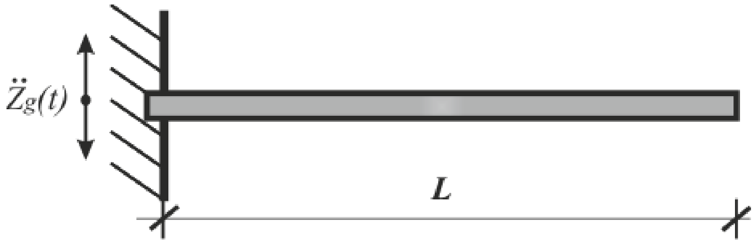


Figure 3: Cantilever viscoelastic beam

As aforementioned, flexural vibrations are expressed through, the linear combination of the eigenfunctions $\Phi_k(z)$ that for a cantilever beam are given as:

$$\begin{aligned}\Phi_k(z)=C_k\big[&(\sin\lambda_k z-\sinh\lambda_k z)(\sin\lambda_k L-\sinh\lambda_k L)+\\&+(\cos\lambda_k z-\cosh\lambda_k z)(\cos\lambda_k L+\cosh\lambda_k L)\big]\end{aligned} \tag{9}$$

with $k=1,2,\dots,\infty$.

Indeed, the general form of eingenfunctions is:

$$\Phi_k(z)=A\sin\lambda_k z+B\cos\lambda_k z+C\sinh\lambda_k z+D\cosh\lambda_k z \tag{10}$$

introducing appropriate boundary conditions for this case:

$$\begin{aligned}&\Phi_k(0)=0 \qquad &&\Phi'_k(0)=0\\&\Phi''_k(L)=0 \qquad &&\Phi'''_k(L)=0\end{aligned} \tag{11}$$

we obtain:

$$B+D=0 \qquad A+C=0$$

$$\frac{A}{B}=\frac{\sin\lambda_k L-\sinh\lambda_k L}{\cos\lambda_k L+\cosh\lambda_k L} \qquad \begin{vmatrix}(\sin\lambda_k L+\sinh\lambda_k L) & (\cos\lambda_k L+\cosh\lambda_k L)\\ (\cos\lambda_k L+\cosh\lambda_k L) & (-\sin\lambda_k L+\sinh\lambda_k L)\end{vmatrix}=0 \tag{12}$$

Solving this transcendental equation we determine $\lambda_k L$ and so the k-eigenfunctions. The trend of the first four eigenfunctions is shown in Fig. 4. Regarding the evaluation of the modal coordinates $y_k(t)$, introduce Eq. (6) into the motion equation (7) we get

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{\partial^2 y_k(t)}{\partial t^2}\Phi_k(z)dz+\frac{C_\beta I}{\rho}\sum_{k=1}^{\infty}\left[\left({}_C D_{0^+}^{\beta} y_k\right)(t)\right]\frac{\partial^4\Phi_k(z)}{\partial z^4}dz+\\&+\frac{EI}{\rho}\sum_{k=1}^{\infty}y_k(t)\frac{\partial^4\Phi_k(z)}{\partial z^4}dz=-\ddot{z}_g(t)\end{aligned}\tag{13}$$

then, multiplying both sides by $\Phi_j(z)$ and integrating from 0 to L, we obtain:

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{\partial^2 y_k(t)}{\partial t^2}\int_0^L\Phi_k(z)\Phi_j(z)dz+\frac{C_\beta I}{\rho}\sum_{k=1}^{\infty}\left[\left({}_C D_{0^+}^{\beta} y_k\right)(t)\right]\int_0^L\Phi_j(z)\frac{\partial^4\Phi_k(z)}{\partial z^4}dz\\&+\frac{EI}{\rho}\sum_{k=1}^{\infty}y_k(t)\int_0^L\Phi_j(z)\frac{\partial^4\Phi_k(z)}{\partial z^4}dz=-\int_0^L\Phi_j(z)\,\ddot{z}_g(t)dz\end{aligned}\tag{14}$$

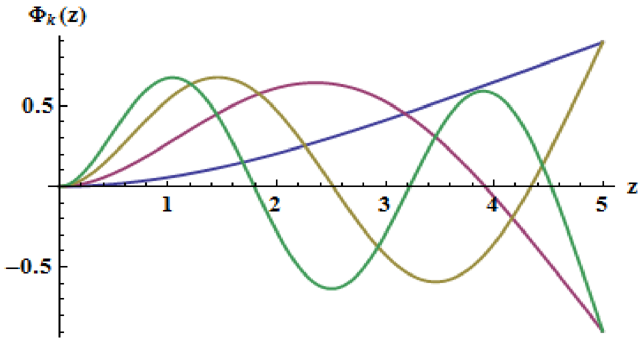


Figure 4: Eingenfunctions for cantilever beam

farther, using the relations of orthogonality

$$\begin{aligned}&\int_0^L\Phi_j(z)\Phi_k(z)dz=\delta_{jk}\\&\int_0^L\Phi_j(z)\frac{d^4\Phi_k(z)}{dz^4}dz=\delta_{jk}R_k\end{aligned}\tag{15}$$

The governing differential equation in terms of $y_k(t)$ is derived as

$$\frac{\partial^2 y_k(t)}{\partial t^2}+\frac{C_\beta I}{\rho}R_k\left({}_C D_{0^+}^{\beta} y_k\right)(t)+\frac{EI}{\rho}R_k\,y_k(t)=-P_k\,\ddot{z}_g(t)\tag{16}$$

Where $P_k=\int_0^L\Phi_j(z)\,dz$ represents the modal coefficient of participation. In time domain, the equation (16) is solved starting from the initial conditions of quiet and discretizing the operator of fractional derivative according to the binomial coefficients of the formulation of Grunwald-Letnikov [3,4]. It is possible to operate in frequencies domain obtaining the same results in terms of statistics. Considering Eq. (16), and using Fourier transform, we obtain:

$$-\omega^2 Y_k(\omega)+\frac{C_\beta I}{\rho}R_k\ (I\omega)^\beta Y_k(\omega)+\frac{EI}{\rho}R_k\, Y_k(\omega)=-P_k\ \ddot{Z}_g(\omega) \tag{17}$$

And then:

$$Y_k(\omega)=H_k(\omega)(-P_k)\ddot{Z}_g(\omega) \tag{18}$$

Where:

$$H_k(\omega)=\left(-\omega^2+\frac{C_\beta I}{\rho}R_k\ (I\omega)^\beta+\frac{EI}{\rho}R_k\right)^{-1} \tag{19}$$

is the transfer function of the viscoelastic considered system. The next step is to determine the power spectral density matrix as regards the modal responses $Y_k(\omega)$. As known it is an hermitian matrix whose diagonal elements are:

$$S_{y_{jj}}=\lim_{T\to\infty}\frac{1}{2\pi T}E\left[Y_j(\omega)Y_j^*(\omega)\right]=H_j(\omega)H_j^*(\omega)P_j^2 S_0 \tag{20}$$

And in a similar manner the elements outside the main diagonal appear to be:

$$S_{y_{jk}}=H_j(\omega)H_k^*(\omega)P_j P_k S_0 \tag{21}$$

Where:

$$S_0=\lim_{T\to\infty}\frac{1}{2\pi T}E\left[\ddot{Z}_g(\omega)\ddot{Z}_g^*(\omega)\right] \tag{22}$$

is the power spectral density of Gaussian white noise. In particular the parameters have been selected as: $E=210000\cdot10^5\ daN/m^2$, $I=8.226\cdot10^{-6}\ m^4$, $\rho=7850\ daN/m^3$, $\mu=9.56\cdot10^8$, $\alpha=0.3$, $S_0=1\,N\ s^2$.

Considering these parameters, we obtain the numerical co-spectrum, i.e. the real part of the power spectral density matrix, and the numerical quad-spectrum, i.e. the imaginary part of the power spectral density matrix.

Now, considering the Eq. (6) in frequencies domain, it is possible to determine the power spectral density of the transversal displacement of the beam, in fact:

$$S_v(\omega,z)=\lim_{T\to\infty}\frac{1}{2\pi T}E\left[v(\omega,t)v^*(\omega,t)\right]=\lim_{T\to\infty}\frac{1}{2\pi T}E\left[\sum_{k=1}^{\infty}\Phi_k(z)H_k(\omega)\right.$$
$$\left.\ddot{Z}_g(\omega)(-P_k)\sum_{j=1}^{\infty}\Phi_j(z)H_j^*(\omega)\ddot{Z}_g(\omega)(-P_j)\right]=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\Phi_k(z)\Phi_j(z)S_{y_{jk}} \tag{23}$$

It is a real function and below it shows his space-time evolution:

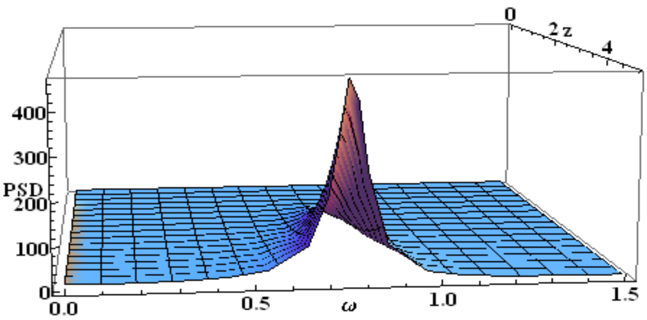


Figure 5: Power spectral density of the transversal displacement of the beam

Also we show the power spectral density of the transversal displacement of the beam for fixed cross section z, in particular at z = 1,2,3,4,5 meters:

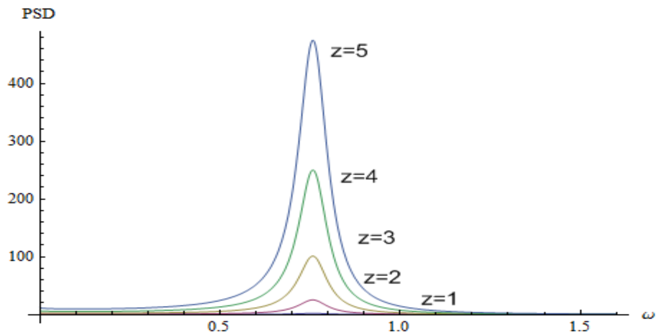
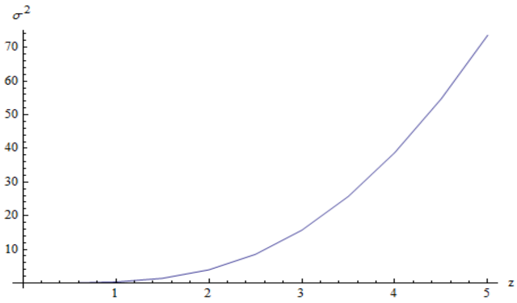


Figure 6: Power spectral density of the transversal displacement of the beam for fixed cross section z

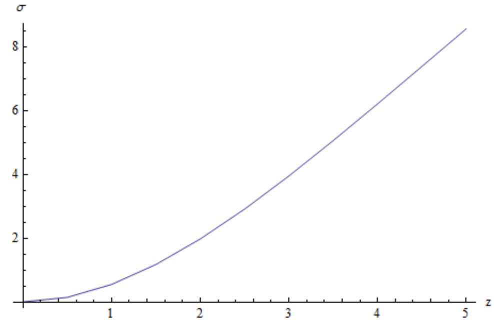
Finally, the variance $\sigma_v^2(z)$ of the transversal displacement of the beam is determined according to the relation:

$$\sigma_v^2(z)=\int_0^{\infty}S_v(z,\omega)\,d\omega \tag{24}$$

In conclusion we show the trend of statistics along the axis of the beam:



a) Variance



b) Standard Deviation

Figure 7: Statistics of the response (transversal displacement of the beam)

It is evident by observing these graphs that the variance of the transversal displacement of the beam is greater at the free end.

4 CONCLUSIONS

We propose the dynamic analysis of a viscoelastic continuous beam under stochastic loads. Viscoelastic behaviour has been taken into account by fractional Kelvin-Voigt model that is the proper model for capturing the viscoelasticity phenomena since it exhibits an intermediate behaviour between elastic and viscous. In particular this model is made by perfect spring in parallel with fractional springpot and it involves the fractional operators in stress-strain relation.

The present analysis regards a viscoelastic beam under the Euler-Bernoulli hypothesis forced by stochastic loads and the dynamical study is performed in the frequency domain.

Moreover dynamic analysis in frequency domain of a continuous cantilever beam under Gaussian white noise is shown in the numerical example.

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