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CROSS-POWER SPECTRAL DENSITY AND CROSS-CORRELATION REPRESENTATION BY USING FRACTIONAL SPECTRAL MOMENTS

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Abstract. *In this paper the Cross-Power Spectral density function and the Cross-correlation function are reconstructed by the (complex) Fractional Spectral Moments. It will be shown that with the aid of Fractional spectral moments both Cross-Power Spectral Denstity and Cross-Correlation function may be represented in the whole domains of frequency (for Cross-Power Spectral Density) and time domain (for Cross-Correlation Function).*

Sommario. *Nel presente articolo la funzione Densità Spettrale di Potenza Incrociata e la Funzione di Correlazione Incrociata sono ricostruite attraverso i Momenti (complessi) Spettrali Frazionari. Sarà mostrato che con l'ausilio dei Momenti Spettrali Frazionari sia la Densità Spettrale di Potenza Incrociata che la Funzione di Correlazione Incrociata possono essere rappresentate nell'intero domino della frequenza (per la Densità Spettrale di Potenza Incrociata) e del tempo (per la Funzione di Correlazione Incrociata).*

1 INTRODUCTION

The spectral moments (SMs) introduced by Vanmarcke [1] are the moments of order $k \in \mathbb{N}$ of the one-sided Power Spectral Density (PSD). Such entities for k large may be divergent quantities [2] and then they are not useful quantities for reconstructing the PSD. Recently [2] the representation of the PSD and correlation function has been pursued by using fractional spectral moments. The latter are fractional moments of order $\gamma \in \mathbb{C}$. The appealing of such moments is related to the fact that $\Re(\gamma)$ remains constant and $\Im(\gamma)$ runs, then no diverge problems occur. The second important fact is that the Correlation function (CF)and the PSD are reconstructed in all the domain by means of such complex fractional moments. These important results have been obtained by using Mellin transform theorem and the achieved results, when discretization is performed along the imaginary axis, give rise to very accurate results for both CF and PSD.

In this paper the extension to the Cross-Correlation function (CCF) and the Cross-Power Spectral Density (CPSD) is presented. The CCF is not even nor odd and then as the first step the CCF is decomposed into an even and an odd function. Then the Mellin transform is applied for such functions. The Mellin transform is strictly related to the (complex) fractional moments of the even and the odd part of CCF. Inverse Mellin transform gives rise to the CCF as a generalized Taylor series of the type $\sum_{k=-m}^{m} c_k t^{-\gamma_k}$ where $\gamma_k = \rho + ik\Delta\eta$ and c_k are strictly related to Riesz fractional integrals of CCF in zero. It is also shown that such coefficients are related to the complex fractional spectral moments that are the moments of the one-sided CPSD. This is a very interesting result because in some cases the exact Fourier transform of some PSD is not known in analytical form while PSD is already known (see e.g. PSD of the type $s_0 t^{-\alpha}$ with $\alpha \in \mathbb{R}$) and then the fractional spectral moments may be evaluated either in time or in frequency domain without any difficulty.

2 CROSS-CORRELATION FUNCTION BY FRACTIONAL MOMENTS

Let $X_1(t)$ and $X_2(t)$ two stationary zero mean random processes, for which we can define the cross-correlation function $R_{X_1X_2}(\tau)$ or its Fourier transform named as cross-power spectral density function $S_{X_1X_2}(\omega)$. The cross-correlation function CCF is defined as

$$R_{X_1X_2}(\tau) = E\left[X_1(t)X_2\left(t+\tau\right)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{x_1x_2}(x_1(t), x_2(t+\tau))x_1x_2dx_1dx_2 \tag{1}$$

where $E[\cdot]$ means ensemble average. Usually, the CCF is neither even nor odd, then, for simplicity we decompose $R_{X_1X_2}(\tau)$ into an even function $u(\tau)$ and an odd function $v(\tau)$ as follows

$$R_{X_1X_2}(\tau) = \frac{1}{2}\left[R_{X_1X_2}(\tau) + R_{X_1X_2}(-\tau)\right] + \frac{1}{2}\left[R_{X_1X_2}(\tau) - R_{X_1X_2}(-\tau)\right] = u(\tau) + v(\tau). \tag{2}$$

In this way we define the Mellin transform [3, 4, 5] of the even function $u(\tau)$, in particular for the positive half-plane of τ in the form

$$\mathrm{M}_{u^+}(\gamma-1) = \mathcal{M}\{u(\tau)U(\tau),\, \gamma\} = \int_0^{\infty} u^+(\tau)\tau^{\gamma-1}d\tau \tag{3}$$

where $U(t)$ is the unit step function and $\gamma \in \mathbb{C}$ with $\gamma = \rho + i\eta$, while for the negative half-plane we have

$$\mathrm{M}_{u^-}(\gamma-1) = \mathcal{M}\{u(\tau)U(-\tau),\, \gamma\} = \int_{-\infty}^{0} u^-(\tau)(-\tau)^{\gamma-1}d\tau \tag{4}$$

the terms $\mathrm{M}_{u^+}(\gamma-1)$ and $\mathrm{M}_{u^-}(\gamma-1)$ may be interpreted as the *fractional moments* of half-function $u^+(\tau)$ and $u^-(\tau)$ respectively.

From Eqs. (3) and (4) it may be observed that $\mathrm{M}_{u^+}(\gamma-1) = \mathrm{M}_{u^-}(\gamma-1)$, because the $u(\tau)$ is even, whereby the $u(\tau)$ may be obtained as the inverse Mellin transform as:

$$u(\tau) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathrm{M}_{u^+}(\gamma-1)|\tau|^{-\gamma}d\gamma; \qquad \tau \in \mathbb{R}. \tag{5}$$

Moreover we define the Mellin transform of odd part of CCF $v(\tau)$, for which we have for the positive half-plane of τ

$$\mathrm{M}_{v^+}(\gamma-1) = \mathcal{M}\{v(\tau)U(\tau),\,\gamma\} = \int_0^\infty v^+(\tau)\tau^{\gamma-1}d\tau \tag{6}$$

and for negative value of τ we get

$$\mathrm{M}_{v^-}(\gamma-1) = \mathcal{M}\{v(\tau)U(-\tau),\,\gamma\} = \int_{-\infty}^0 v^-(\tau)(-\tau)^{\gamma-1}d\tau. \tag{7}$$

From Eqs. (6) and (7) it may be observed that $\mathrm{M}_{v^+}(\gamma-1) = -\mathrm{M}_{v^-}(\gamma-1)$ because $v(\tau)$ is an odd function. By using the inverse Mellin transform theorem we can restore the given function $v(\tau)$ by its fractional moments, that is

$$v(\tau) = \frac{\operatorname{sgn}(\tau)}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} \mathrm{M}_{v^+}(\gamma-1)|\tau|^{-\gamma}d\gamma; \qquad \tau\in\mathbb{R}. \tag{8}$$

Based on the previous results and remembering Eq. (2) the CCF can be represented in the whole domain by using the fractional moments $\mathrm{M}_{u^+}(\gamma-1)$ and $\mathcal{M}_{v^+}(\gamma-1)$ in the following form

$$\begin{aligned} R_{X_1X_2}(\tau) &= \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty}\left[\mathrm{M}_{u^+}(\gamma-1) + \operatorname{sgn}(\tau)\mathrm{M}_{v^+}(\gamma-1)\right]|\tau|^{-\gamma}d\gamma \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\mathrm{M}_{u^+}(\gamma-1) + \operatorname{sgn}(\tau)\mathrm{M}_{v^+}(\gamma-1)\right]|\tau|^{-\gamma}d\eta \end{aligned} \tag{9}$$

in Eq. (9) we take into account that the integral in the inverse Mellin transform is performed along the imaginary axis while $\rho = \Re\{\gamma\}$ remains fixed for which we have that $d\gamma = id\eta$. The representation of the CCF is valid provided ρ belongs to the so called *fundamental strip* of Mellin transform, since both $\mathrm{M}_{u^+}(\gamma-1)$ and $\mathrm{M}_{v^+}(\gamma-1)$ are holomorphic in the fundamental strip [3, 4, 5].

It is useful to note that the relation between the fractional moments and fractional operators exist. In order to show this we introduce the Riesz fractional integral denoted as $\left(\mathbf{I}^\gamma R_{X_1X_2}\right)(\tau)$, defined as

$$\left(\mathbf{I}^\gamma R_{X_1X_2}\right)(\tau) = \frac{1}{2\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}\int_{-\infty}^{\infty}\left(R_{X_1X_2}\right)(\bar{\tau})|\tau-\bar{\tau}|^{\gamma-1}d\bar{\tau}, \qquad \rho>0,\,\rho\neq 1,3,\ldots \tag{10}$$

and the complemetary Riesz fractional integral, denoted as $\left(\mathbf{H}^\gamma R_{X_1X_2}\right)(\tau)$, defined as

$$\left(\mathbf{H}^\gamma R_{X_1X_2}\right)(\tau) = \frac{1}{2\Gamma(\gamma)\sin\left(\frac{\gamma\pi}{2}\right)}\int_{-\infty}^{\infty}\frac{\left(R_{X_1X_2}\right)(\bar{\tau})\operatorname{sgn}(\tau-\bar{\tau})}{|\tau-\bar{\tau}|^{1-\gamma}}d\bar{\tau}, \qquad \rho>0,\,\rho\neq 1,3,\ldots \tag{11}$$

Based on the definition of Riemann-Liouville fractional integral, denoted with $\left(\mathrm{I}^{\gamma}_{\pm}R_{X_1X_2}\right)(\tau)$ [4, 5] we get an useful relationship between Riesz and Riemann-Liouville fractional operators, that is

$$\left(\mathbf{I}^{\gamma}R_{X_1X_2}\right)(\tau)=\frac{\left(\mathrm{I}^{\gamma}_{0^+}R_{X_1X_2}\right)(\tau)+\left(\mathrm{I}^{\gamma}_{0^-}R_{X_1X_2}\right)(\tau)}{2\cos\left(\frac{\gamma\pi}{2}\right)};\quad \left(\mathbf{H}^{\gamma}R_{X_1X_2}\right)(\tau)=\frac{\left(\mathrm{I}^{\gamma}_{0^+}R_{X_1X_2}\right)(\tau)-\left(\mathrm{I}^{\gamma}_{0^-}R_{X_1X_2}\right)(\tau)}{2\sin\left(\frac{\gamma\pi}{2}\right)}. \tag{12}$$

From Eqs. (12) it is easily to demonstrate the relation between fractional operators of CCF at the origin and fractional moments, indeed we have

$$\left(\mathbf{I}^{\gamma}R_{X_1X_2}\right)(0)=\frac{\mathrm{M}_{R^+_{X_1X_2}}(\gamma-1)+\mathrm{M}_{R^-_{X_1X_2}}(\gamma-1)}{2\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}=\frac{\mathrm{M}_{u^+}(\gamma-1)}{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)} \tag{13a}$$

$$\left(\mathbf{H}^{\gamma}R_{X_1X_2}\right)(0)=\frac{\mathrm{M}_{R^+_{X_1X_2}}(\gamma-1)-\mathrm{M}_{R^-_{X_1X_2}}(\gamma-1)}{2\Gamma(\gamma)\sin\left(\frac{\gamma\pi}{2}\right)}=\frac{\mathrm{M}_{v^+}(\gamma-1)}{\Gamma(\gamma)\sin\left(\frac{\gamma\pi}{2}\right)}. \tag{13b}$$

In this way, the CCF may be expressed in the form:

$$R_{X_1X_2}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(\gamma)\left[\cos\left(\frac{\gamma\pi}{2}\right)\left(\mathbf{I}^{\gamma}R_{X_1X_2}\right)(0)+\mathrm{sgn}(\tau)\sin\left(\frac{\gamma\pi}{2}\right)\left(\mathbf{H}^{\gamma}R_{X_1X_2}\right)(0)\right]|\tau|^{-\gamma}d\eta. \tag{14}$$

The integrals in Eq. (9) and Eq. (14) may be discretized by using the trapezoidal rule in order to obtain the approximate form of given function $R_{X_1X_2}(\tau)$, namely

$$\begin{aligned}R_{X_1X_2}(\tau)&\approx\frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m}\left[\mathrm{M}_{u^+}(\gamma_k-1)+\mathrm{sgn}(\tau)\mathrm{M}_{v^+}(\gamma_k-1)\right]|\tau|^{-\gamma_k}\\&=\frac{\Delta\eta|\tau|^{-\rho}}{2\pi}\sum_{k=-m}^{m}\Gamma(\gamma_k)\left[\cos\left(\frac{\gamma_k\pi}{2}\right)\left(\mathbf{I}^{\gamma_k}R_{X_1X_2}\right)(0)+\mathrm{sgn}(\tau)\sin\left(\frac{\gamma_k\pi}{2}\right)\left(\mathbf{H}^{\gamma_k}R_{X_1X_2}\right)(0)\right]|\tau|^{-ik\Delta\eta}\end{aligned} \tag{15}$$

where the exponent γ is discretized in the form $\gamma_k=\rho+ik\Delta\eta$, $\Delta\eta$ is the discretization step of the imaginary axis, and m is the truncation number of the summation, that is chosen in such a way that any term $n>m$ in the summation has a negligible contribution. Notice that the Eq. (15) is a not-divergent summation, because ρ remains fixed, and this is a very important aspect if we want to restore the given function in a large domain of τ.

From Eq. (14) we recognize that Eq. (14) is a sort of a Taylor series since by knowing the (fractional) operators in zero of the given function then the function may be reconstructed. The appealing of the expansion in Eq. (15) does not diverge for $\tau\to\infty$, since the real part of the exponent γ remains fixed and only the imaginary part runs.

3 CROSS-POWER SPECTRAL DENSITY FUNCTION BY FRACTIONAL SPECTRAL MOMENTS

In this section the introduced representation by fractional moments will be applied to restore the cross-power spectral density function.

Let $S_{X_1X_2}(\omega)$ be a Fourier transform of CCF that is so called Cross-Power Spectral Density (CPSD), that is

$$\begin{aligned} S_{X_1X_2}(\omega) &= \mathcal{F}\left\{R_{X_1X_2}(\tau);\,\omega\right\} = \int_{-\infty}^{\infty} R_{X_1X_2}(\tau)e^{i\omega\tau}d\tau = \int_{-\infty}^{\infty} R_{X_1X_2}(\tau)\left[\cos(\omega\tau)+i\sin(\omega\tau)\right]d\tau \\ &= 2\left[\int_{0}^{\infty} u(\tau)\cos(\omega\tau)d\tau + i\int_{0}^{\infty} v(\tau)\sin(\omega\tau)d\tau\right] = \hat{\mathrm{U}}(\omega)+i\hat{\mathrm{V}}(\omega) \end{aligned} \tag{16}$$

where $\hat{U}(\omega)$ and $\hat{V}(\omega)$ are the Fourier transform of even and odd part of CCF, respectively, and they represent the real and the imaginary part of CPSD.

Another expression of $S_{X_1X_2}(\omega)$ may be obtained starting from Eq. (9) and by performing the Fourier transform of it, obtaining the following relationship

$$S_{X_1X_2}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(1-\gamma)\left[\sin\left(\frac{\gamma\pi}{2}\right)\mathrm{M}_{u^+}(\gamma-1) + i\,\mathrm{sgn}(\omega)\cos\left(\frac{\gamma\pi}{2}\right)\mathrm{M}_{v^+}(\gamma-1)\right]|\omega|^{\gamma-1}d\eta \tag{17}$$

which can be discretized, obtaining

$$S_{X_1X_2}(\omega) \approx \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m}\Gamma(1-\gamma_k)\left[\sin\left(\frac{\gamma_k\pi}{2}\right)\mathrm{M}_{u^+}(\gamma_k-1) + i\,\mathrm{sgn}(\omega)\cos\left(\frac{\gamma_k\pi}{2}\right)\mathrm{M}_{v^+}(\gamma_k-1)\right]|\omega|^{\gamma_k-1} \tag{18}$$

Eq. (15) and (17) are the extension of the previous results [2] in terms of autocorrelation and power spectral density to the case of cross-correlation and cross-power spectral density.

Another way to represent the CPSD by using *fractional spectral moments* (FSMs). The spectral moments have been introduced by Vanmarcke [1], and the generalization of these quantities with fractional exponent, just call fractional spectral moments, has been performed by Cottone & Di Paola in [2]. The FSMs for the CPSD are defined as

$$\Lambda_{u^+}(-\gamma) = \int_{0}^{\infty}\Re\left\{S_{X_1X_2}(\omega)\right\}\omega^{-\gamma}d\omega = \int_{0}^{\infty}\hat{\mathrm{U}}(\omega)\omega^{-\gamma}d\omega \tag{19a}$$

$$\Lambda_{v^+}(-\gamma) = \int_{0}^{\infty}\Im\left\{S_{X_1X_2}(\omega)\right\}\omega^{-\gamma}d\omega = \int_{0}^{\infty}\hat{\mathrm{V}}(\omega)\omega^{-\gamma}d\omega. \tag{19b}$$

By using the definitions of FMs, in Eqs. (3) and (6), and using some properties of Fourier transform of fractional operators (see Appendix B), it may be easily demonstrated that the following identities

$$\mathrm{M}_{u^+}(\gamma-1) = \frac{\Gamma(\gamma)\cos(\gamma\pi/2)}{\pi}\Lambda_{u^+}(-\gamma); \qquad \mathrm{M}_{v^+}(\gamma-1) = \frac{\Gamma(\gamma)\sin(\gamma\pi/2)}{\pi}\Lambda_{v^+}(-\gamma). \tag{20}$$

hold true. This is a very useful result since in many cases of engineering interest the stochastic process like for wind or wave actions is defined by the spectral properties in frequency domain rather than by the correlation or cross correlation in time domain. As a consequence the fractional moments, in virtue of Eq. (20), may be easier calculated by Eq. (19).

By using the Eqs. (20) we can obtain the following expression, that shown another exact representation of CPSD

$$S_{X_1X_2}(\omega) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\left[\Lambda_{u^+}(-\gamma) + i\,\text{sgn}(\omega)\Lambda_{v^+}(-\gamma)\right]\cos\left(\frac{\gamma\pi}{2}\right)\sin\left(\frac{\gamma\pi}{2}\right)\Gamma(\gamma)\Gamma(1-\gamma)|\omega|^{\gamma-1}d\eta \tag{21}$$

Eq. (21), taking into account that $\cos(\gamma\pi/2)\sin(\gamma\pi/2)\,\Gamma(\gamma)\Gamma(1-\gamma) = \pi/2$ may be rewritten as

$$S_{X_1X_2}(\omega) = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\Lambda_{u^+}(-\gamma) + i\,\text{sgn}(\omega)\Lambda_{v^+}(-\gamma)\right]|\omega|^{\gamma-1}d\eta \tag{22}$$

or in discretized form

$$\begin{aligned} S_{X_1X_2}(\omega) &\approx \frac{\Delta\eta}{4\pi}\sum_{k=-m}^{m}\left[\Lambda_{u^+}(-\gamma_k) + i\,\text{sgn}(\omega)\Lambda_{v^+}(-\gamma_k)\right]|\omega|^{\gamma_k-1} \\ &= \frac{\Delta\eta|\omega|^{\rho-1}}{4\pi}\sum_{k=-m}^{m}\left[\Lambda_{u^+}(-\gamma_k) + i\,\text{sgn}(\omega)\Lambda_{v^+}(-\gamma_k)\right]|\omega|^{i\Delta\eta}. \end{aligned} \tag{23}$$

The FSMs can be also used to represent the CCF, obtaining the following expression

$$R_{X_1X_2}(\tau) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\Gamma(\gamma)\left[\cos\left(\frac{\gamma\pi}{2}\right)\Lambda_{u^+}(-\gamma) + \text{sgn}(\tau)\sin\left(\frac{\gamma\pi}{2}\right)\Lambda_{v^+}(-\gamma)\right]|\tau|^{-\gamma}d\eta \tag{24}$$

or in discretization form

$$R_{X_1X_2}(\tau) \approx \frac{\Delta\eta}{2\pi^2}\sum_{k=-m}^{m}\Gamma(\gamma_k)\left[\cos\left(\frac{\gamma_k\pi}{2}\right)\Lambda_{u^+}(-\gamma_k) + \text{sgn}(\tau)\sin\left(\frac{\gamma_k\pi}{2}\right)\Lambda_{v^+}(-\gamma_k)\right]|\tau|^{-\gamma_k}. \tag{25}$$

4 NUMERICAL EXAMPLE

Let us consider two linear oscillators forced by a white noise process $W(t)$

$$\begin{cases} \ddot{X}_1(t) + 2\zeta_1\omega_1\dot{X}_1(t) + \omega_1^2X_1(t) = p_1W(t) \\ \ddot{X}_2(t) + 2\zeta_2\omega_2\dot{X}_1(t) + \omega_2^2X_2(t) = p_2W(t) \end{cases} \tag{26}$$

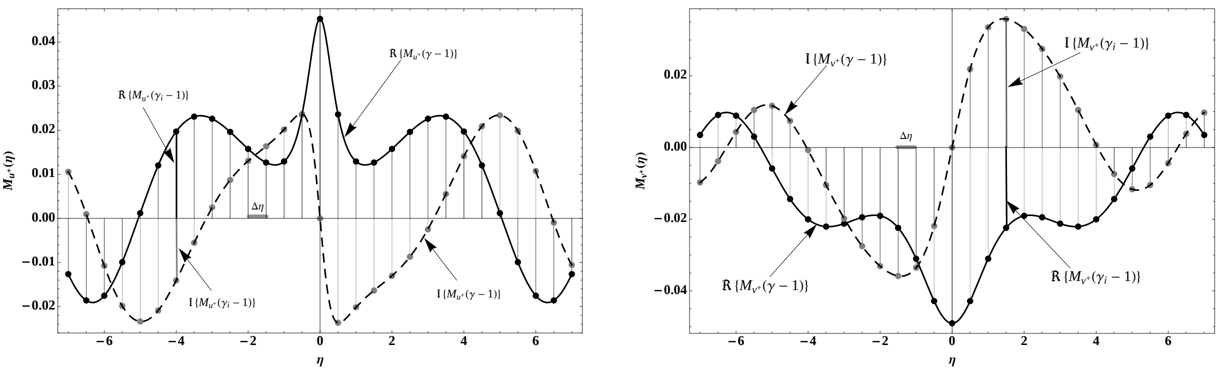
The cross-power spectral density $S_{X_1X_2}(\omega)$ is defined as

$$S_{X_1X_2}(\omega) = \frac{p_1p_2S_0}{\left[\left(\omega_1^2-\omega^2\right) - 2i\zeta_1\omega_1\omega\right]\left[\left(\omega_2^2-\omega^2\right) + 2i\zeta_2\omega_2\omega\right]} \tag{27}$$

where the S_0 is the PSD of the white noise $W(t)$. The cross-correlation function $R_{X_1X_2}(\tau)$ is evaluated by making the inverse Fourier transform of the cross PSD, that is

$$R_{X_1X_2}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{p_1p_2S_0e^{-i\omega\tau}d\omega}{\left[\left(\omega_1^2-\omega^2\right) - 2i\zeta_1\omega_1\omega\right]\left[\left(\omega_2^2-\omega^2\right) + 2i\zeta_2\omega_2\omega\right]} \tag{28}$$

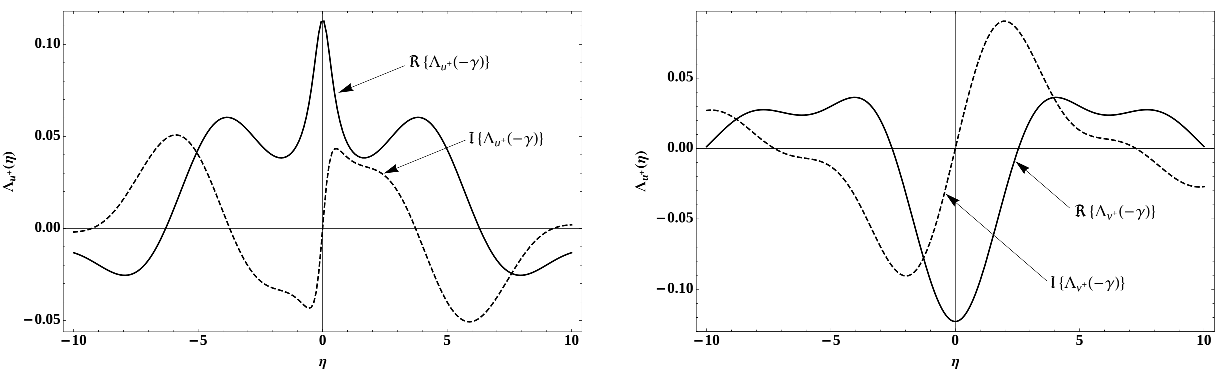
For the numerical application the following parameters have been selected $\omega_1 = 2\omega_2 = \pi$, $\zeta_1 = 2\zeta_2 = 1/2$ and $p_1 = 4p_2 = 2$. Starting from the knowledge of $u(t)$ and $v(t)$ we can define the fractional moments $\mathrm{M}_{u^+}(\gamma - 1)$ and $\mathrm{M}_{v^+}(\gamma - 1)$. These FMs are complex quantities and their real and imaginary parts, for fixed value of ρ, are shown in Figure 1(a) and 1(b) respectively. Moreover in these figures are shown the discretized fractional moments $\mathrm{M}_{u^+}(\gamma_k - 1)$ and $\mathrm{M}_{v^-}(\gamma_k - 1)$ obtained by discretization of that are used for approximated forms on CCF in Eq. (18) and CPSD in Eq. (18). While real part $\hat{\mathrm{U}}(\omega)$ and imaginary part $\hat{\mathrm{V}}(\omega)$ of CPSD are



(a) Fractional moments of $u(\tau)$ **(b)** Fractional moments of $v(\tau)$

Figure 1: Real and imaginary part of fractional moments of even and odd function for $\rho = 1/2$

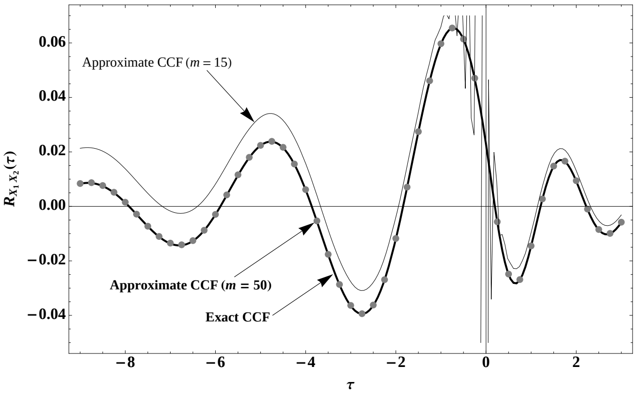
derived from Eq. (27). By knowing $\mathrm{U}(\omega)$ and $\mathrm{V}(\omega)$ the trend of fractional spectral moments $\Lambda_{u^+}(-\gamma)$ and $\Lambda_{v^+}(-\gamma)$ are determined and they are shown in Figure 2(a) and 2(b) respectively.



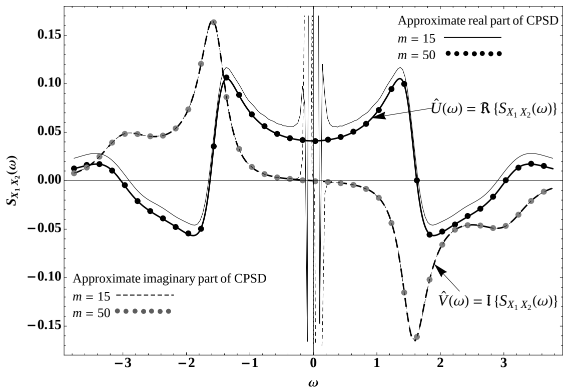
(a) Fractional spectral moments of $u(\tau)$ **(b)** Fractional spectral moments of $v(\tau)$

Figure 2: Real and imaginary part of fractional spectral moments of even and odd function for $\rho = 1/2$

By using approximate representation by FMs (Eqs. (15) and (18)) or by FSMs (Eqs. (25) and (23)) we may restore the CCF and CPSD, as well shown in Figure 3(a) and 3(b), respectively. In particular Figure 3(a) shows the comparison between the exact CCF and the approximation form obtained by FMs or FSMs, while in Figure 3(b) the overlap of exact and approximate representation, obtained by FMs or FSMs, of real and imaginary part of CPSD are shown. In both figures the approximate representation of CCF and CPSD are performed with fixed value of truncation length $\bar{\eta} = m\Delta\eta$ and different value of considered terms m. In the example the chosen truncation parameters are $m = 15, 50$ and $\bar{\eta} = 30$.



(a) Exact and approximate CCF



(b) Exact and approximate CPSD

Figure 3: Cross-correlation function and cross-power spectral density function

It is noted that the perfect coalescence between exact and approximate representation is obtained for $m = 50$ that corresponds to $\Delta\eta = 3/5$.

5 CONCLUSIONS

In this paper the usefulness of (complex) fractional spectral moments of the CPSD function or of the CCF has been highlighted. It has been shown that such a fractional spectral moments may be evaluated either by starting from the CCF or by the CPSD. Very exact simple relationships allow us to work in time or in frequency domain by using such a fractional spectral moments. The second obtained goal is that, by using Mellin transform theorem, the fractional spectral moments may be evaluated by the Riesz and the complementary Riesz integrals in zero. The third goal is that integration is performed along the imaginary axis and then no divergence problems occur for both $\tau \to \infty$ (in time domain) or for $\omega \to \infty$ (in frequency domain). Accuracy of the results is provided with the aid of the numerical example and the accuracy of the results is impressive in all time or frequency ranges.

REFERENCES

[1] Vanmarcke E., *Properties of spectral moments with applications to random vibrations*, Journal of Engineering Mechanical Division 98, No. 2, ASME 1972, 425-446.

[2] G. Cottone, M. Di Paola, *A new representation of power spectral density and correlation function by means of fractional spectral moments*, Probabilistic Engineering Mechanics 25 (2010), 348-353.

[3] R. B. Paris, D. Kaminski, *Asymptotics and Mellin-Barnes Integrals*, Cambridge University Press, New York, 2001.

[4] G. S. Samko, A. A. Kilbas, O. I. Marichev, *Fractional integrals and derivatives: theory and applications*, New York (NY): Gordon and Breach: 1993. Transl. from the russian.

[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.