Comparison Principle for Elliptic Equations with Mixed Singular Nonlinearities

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Abstract

We deal with existence and uniqueness of positive solutions of an elliptic boundary value problem modeled by

$$\begin{cases} -\Delta_p u = \dfrac{f}{u^\gamma} + g u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the usual p-Laplacian operator, $\gamma \geq 0$ and $0 \leq q \leq p-1$; f and g are nonnegative functions belonging to suitable Lebesgue spaces.

1 Introduction

In this paper we deal with an elliptic problem which simplest model is

$$\begin{cases} -\Delta_p u = \dfrac{f}{u^\gamma} + g u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is an open bounded subset of $\mathbb{R}^N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator ($1 < p < N$), $\gamma, q \geq 0$ are such that $q < p-1$ or $q = p-1$, which correspond to the *sublinear* and to the *linear* behaviour in case $p = 2$; here f, g are nonnegative functions belonging to suitable Lebesgue spaces. Clearly the Dirichlet problem (1.1) is *singular* since the request that the solution is zero on the boundary of the set implies that the right hand side blows up. For (1.1) we are mainly interested to existence and uniqueness of possibly unbounded solutions with *finite energy* (i.e. $u \in W_0^{1,p}(\Omega)$).

Let us briefly recall the mathematical framework concerning problem (1.1); we start with the non-singular case, namely $f \equiv 0$.

The main idea of this paper comes from the seminal paper [9] where the authors show existence and uniqueness of a solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ to (1.1) in case $p = 2$, $f \equiv 0$, $q < 1$ and g as a bounded nonnegative function. Let us also mention that classical arguments apply once that u is bounded in order to get a C^1-solution, at least when the set Ω is smooth enough. Later, in [6], in presence of a possibly unbounded g and if $q < p-1$, the existence of a solution is proven through an approximation process; here, even in the nonvariational case, it is proven existence of a solution with infinite energy (i.e. $u \notin W_0^{1,p}(\Omega)$) for rough data g.

Let us briefly underline that, when $p = 2$, problem (1.1) with $f \equiv 0$ is strongly related to the porous media equation in the following way: if u is a solution to (1.1) then for some positive constant $c, \tau > 0$

$$v(x,t) = c u(x)^q (t+\tau)^{\frac{-q}{1-q}},$$

is a solution to

$$g(x) u_t - \Delta v^{\frac{1}{q}} = 0.$$

On the other side there is a huge literature concerning the purely singular equations, namely $g \equiv 0$. In presence of regular f (say a positive $f \in C^\eta(\Omega)$), (1.1) was first treated in these pioneering works [17, 29, 38]; here the authors obtain existence and uniqueness of a classical solution (i.e. $u \in C^2(\Omega) \cap C(\overline{\Omega})$). Moreover, among other things, one has that: $u \in C^{2,\eta}(\Omega)$, $u \notin C^1(\overline{\Omega})$ if $\gamma > 1$ and $u \notin H_0^1(\Omega)$ if $\gamma \geq 3$. Furthermore we refer to [27] for more interesting results regarding the regularity of u.

For what concerns the weak theory of the purely singular case, existence of a distributional solution to (1.1) when the f is only a nonnegative function in $L^m(\Omega)$ ($m \geq 1$) is established in [7]. This solution, if $\gamma \leq 1$ (i.e. the *mild singular* case), attains the boundary datum in the classical sense of Sobolev traces; otherwise, when $\gamma > 1$ (i.e. the *strong singular* case), only a power of the solution has zero Sobolev trace and the solution is shown to be locally in the same space. Later, in [18, 19, 33], existence of solutions to (1.1) is given when the right hand side is of the general form $h(s)f$, with h as a nonnegative and not necessarily monotone function such that $h(s) \leq s^{-\gamma}$ near zero and just bounded at infinity. For the nonhomogeneous case in which $q = 0$ and $g \not\equiv 0$ we mention [32].

Dealing with uniqueness is more tricky; in [5] the authors show that the solution is unique in the class of $H_0^1(\Omega)$ and this kind of result has been extended to general nonincreasing nonlinearities and nonlinear operators in [31] for solutions in $W_0^{1,p}(\Omega)$. In [8], when $p = 2$, the authors show that there is at most one solution to (1.1) belonging to $W_0^{1,1}(\Omega)$.

In [33], uniqueness of a distributional solution belonging to $W_{\text{loc}}^{1,1}(\Omega)$ (with suitable boundary conditions) is shown for a general measure datum and a nonincreasing nonlinearity. Finally in presence of a very general nonlinear operator and a nonincreasing h it is shown in [19] the existence and uniqueness of a renormalized solution for a diffuse measure datum f. For further reading on singular problems we refer to [10–12, 21, 23, 24, 32]

As one should expect the literature concerning (1.1) in presence of both f and g not identically zero is less investigated. Already in [38] the author proves existence of a classical solution to (1.1) when both f and g are regular enough, $p = 2$ and $q < 1$. In the same direction we refer to [15] where it is also investigated the superlinear case, which is a completely different framework. The uniqueness of classical solutions to (1.1) is shown in [36] in presence of the Laplacian operator and $q < p - 1$; we also refer to [13] where, in case of regular f and g constant, it is proved existence and uniqueness of solutions to (1.1) if $q \leq 1$; here in the linear case it is proved existence under a smallness assumption on g and nonexistence otherwise. Then in [30], for $p > 1$, through a sub and supersolution argument it is shown existence of solutions to (1.1) when the right hand side is of the form $h(u) + k(u)$ and no monotonicity is assumed on h, k. In [14] it is investigated the existence of a solution to (1.1) in case $p = 2$ when f and g are functions in suitable Lebesgue spaces. Let us mention that in [35], for $p > 1$, the authors show existence and uniqueness of finite energy solutions to (1.1) under suitable assumptions on f, g. We finally refer to [22, 26] for more interesting results.

The aim of this work is twofold. Firstly, we deal with uniqueness of finite energy solutions by employing the idea contained in [9]. More precisely we want to prove it for positive solutions to the Dirichlet problem associated to

$$- \Delta_p u = F(x, u), \tag{1.2}$$

where $p > 1$ and F is a Carathéodory function which is possibly unbounded both at the origin and at the infinity and such that

$$F(x, s)s^{1-p} \text{ decreases with respect to } s \text{ for a.e. } x \in \Omega. \tag{1.3}$$

Here the major difficult is dealing with a nonlinear operator when looking for comparison principles. Another issue which needs to be underlined is that the solutions are not required to be bounded; this implying the need of a suitable truncation arguments. It is also worth mentioning that (1.3) allows to deal with the case $q \leq p - 1$, at least for positive f if one considers the model case given by (1.1). This result is presented as the comparison principle given by Theorem 2.2 which, as a simple corollary, takes to uniqueness of finite energy solutions.

Other than uniqueness, we are interested to instances of finite energy solutions to (1.2); this is done both in the mild and in the strongly singular case by means of approximation arguments firstly if $q < p - 1$; then we also give an existence result in case $q = p - 1$. Summarizing, if $q < p - 1$, we provide existence of finite energy solutions to equations as in (1.1) if $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$, $\gamma \leq 1$ and $f \in L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)$, where we mean $L^1(\Omega)$ once that $\gamma = 1$.

Otherwise, we show that if $f \in L^m(\Omega)$ with $1 < \gamma < 2 - \frac{1}{m}$ then the existence is guaranteed under the same assumptions on g. Let us also highlight that, as remarked in Section 3.2, there are instances in which one could expect finite energy solutions up to $\gamma < 1 + \frac{p(m-1)}{(p-1)m}$.

Finally, once again if $f \in L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)$, we also show the existence of a solution in case $q = p - 1$ under a smallness assumption on g.

Let us mention that formally the change of variable $v = \dfrac{u^{\gamma+1}}{\gamma + 1}$ for $p = 2$ takes (1.1) to the following equation

$$- \Delta v + \frac{\gamma}{\gamma + 1} \frac{|\nabla v|^2}{v} = (\gamma + 1)^{\frac{\gamma+\theta}{\gamma+1}} g v^{\frac{\gamma+\theta}{\gamma+1}} + f, \tag{1.4}$$

which, for $g = 0$, was extensively studied in the past, see for instance [1–3, 20, 25]. The previous discussion could be formalized and the existence and uniqueness results given in the current paper could provide information regarding problem (1.4).

The plan of the paper is the following: in Section 2 we state and prove the comparison principle and the associated uniqueness result for problems as in (1.1) (Theorem 2.2 and Corollary 2.3). In Section 3 we give some existence results; precisely we investigate both the mild and the strongly singular case when $q < p - 1$ (Theorem 3.2 and Theorem 3.4); moreover we also treat a case in which $q = p - 1$ (Theorem 3.5).

1.1 Notation

In the entire paper Ω is an open and bounded subset of $\mathbb{R}^N$, with $N \geq 1$. We denote by ∂A the boundary and by $|A|$ the Lebesgue measure of a subset A of $\mathbb{R}^N$. By $C_c^k(\Omega)$, with $k \geq 1$, we mean the space of C^k functions with compact support in Ω.

For any $q > 1$, $q' := \frac{q}{q-1}$ is the Hölder conjugate exponent of q, while for any $1 \leq p < N$, $p^* = \frac{Np}{N-p}$ is the Sobolev conjugate exponent of p.

We denote by χ_E the characteristic function of $E \subset \Omega$, namely

$$\chi_E(x) = \begin{cases} 1 & x \in E, \\ 0 & x \in \Omega \setminus E, \end{cases}$$

and by $f^+ := \max(f, 0)$, $f^- := -\min(f, 0)$ the positive and the negative part of a function f. We will widely use the following function defined for a fixed $k > 0$ and $s \in \mathbb{R}$

$$T_k(s) = \max(-k, \min(s, k)), \tag{1.5}$$

and

$$V_\delta(s) = \begin{cases} 1 & s \leq \delta, \\ \dfrac{2\delta - s}{\delta} & \delta < s < 2\delta, \\ 0 & s \geq 2\delta. \end{cases} \tag{1.6}$$

If no otherwise specified, we will denote by C several constants whose value may change from line to line. These values will only depend on the data (for instance C may depend on Ω, N and p) but they will never depend on the indexes of the sequences we will often introduce.

2 Comparison Principle and Uniqueness

Let $1 < p < N$ and let us consider the following problem

$$\begin{cases} -\Delta_p u = F(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where the nonlinearity $F : \Omega \times (0, \infty) \to [0, \infty)$ is a general Carathéodory function. We start specifying the notion of weak solution to (2.1).

Definition 2.1 A positive function $u \in W_0^{1,p}(\Omega)$ is a weak solution to (2.1) if $F(x, u) \in L^1_{\text{loc}}(\Omega)$ and if

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \int_\Omega F(x, u)\varphi, \quad \forall \varphi \in C_c^1(\Omega). \tag{2.2}$$

In order to deal with uniqueness of solutions, we present a comparison principle for solutions to (2.1) provided the right hand side enjoys some monotonicity condition. In particular let us consider v_1, v_2 solutions to

$$\begin{cases} -\Delta_p v_i = G_i(x, v) & \text{in } \Omega, \\ v_i > 0 & \text{in } \Omega, \\ v_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where the nonlinearities $G_1, G_2 : \Omega \times (0, \infty) \to [0, \infty)$ are Carathéodory functions. We state the main result of this section.

Theorem 2.2 (Comparison Principle) *Let us assume G_1, G_2 are nonnegative functions such that either $G_1(x, s)s^{1-p}$ or $G_2(x, s)s^{1-p}$ is decreasing with respect to s and for almost every $x \in \Omega$ and*

$$G_1(x, s) \le G_2(x, s) \tag{2.4}$$

for almost every $x \in \Omega$ and for all $s \in (0, \infty)$. Let v_1 and v_2 be weak solutions to problem (2.3) with data, respectively, G_1, G_2 then $v_1 \le v_2$ almost everywhere in Ω.

As a simple corollary of the previous result, one has that uniqueness holds for weak solutions to (2.1).

Corollary 2.3 (Uniqueness) *Let us assume that F is a nonnegative function such that $F(x, s)s^{1-p}$ is decreasing with respect to s and for almost every $x \in \Omega$. Then there exists at most one weak solution to problem (2.1).*

Remark 2.4 Just to give an idea, Corollary 2.3 gives uniqueness of solutions to (2.1) when F is modelled by

$$F(x, s) = \frac{f(x)}{s^\gamma} + g(x)s^q, \text{ with } f + g > 0 \text{ a.e. in } \Omega,$$

or by

$$F(x, s) = \frac{f(x)}{s^\gamma} + g(x)s^{p-1}, \text{ with } f > 0 \text{ a.e. in } \Omega,$$

where f, g are nonnegative functions defined almost everywhere, $\gamma \ge 0$ and $0 \le q < p-1$.

2.1 Proof of the Comparison Principle

In this section we prove the comparison principle for weak solutions to problem (2.1) and, as a consequence, we deduce the uniqueness result, namely Corollary 2.3.

Proof of Theorem 2.2 First of all we need to show that for any weak solution u to (2.1), the formulation (2.2) can be extended for $W^{1,p}$-test functions. We consider a nonnegative $\varphi \in W_0^{1,p}(\Omega)$ and a sequence of nonnegative functions $\varphi_{\eta,n} \in C_c^1(\Omega)$ such that

$$\begin{cases} \varphi_{\eta,n} \overset{\eta \to 0}{\to} \varphi_n \overset{n \to \infty}{\to} \varphi & \text{in } W_0^{1,p}(\Omega) \\ \operatorname{supp}\varphi_n \subset\subset \Omega : 0 \le \varphi_n \le \varphi & \text{for all } n \in \mathbb{N}. \end{cases}$$

An example of such $\varphi_{\eta,n}$ is $\rho_\eta * (\varphi \wedge \phi_n)$ ($\varphi \wedge \phi_n := \inf(\varphi, \phi_n)$) where ρ_η is a smooth mollifier and ϕ_n is a sequence of nonnegative functions in $C_c^1(\Omega)$ which converges to φ in $W_0^{1,p}(\Omega)$.

Hence let us take $\varphi_{\eta,n}$ as a test function in (2.2), yielding to

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi_{\eta,n} = \int_\Omega F(x,u)\varphi_{\eta,n}.$$

We want to pass first η to zero and then n to infinity in the previous.

Since $u \in W_0^{1,p}(\Omega)$ one can pass to the limit the first term recalling that $\varphi_{\eta,n}$ converges to φ_n in $W_0^{1,p}(\Omega)$. For the right hand side one has that $F(x,u) \in L_{\text{loc}}^1(\Omega)$ that gives that we can pass $\eta \to 0$ since $\varphi_{\eta,n}$ converges $*$-weakly in $L^\infty(\Omega)$ to φ_n which has compact support in Ω. Hence we deduce

$$\int_\Omega |\nabla u|^{p-2}\nabla u . \nabla\varphi_n = \int_\Omega F(x,u)\varphi_n. \tag{2.5}$$

Now let observe that by the Young inequality

$$\int_\Omega F(x,u)\varphi_n \le \int_\Omega |\nabla u|^p + \int_\Omega |\nabla\varphi_n|^p,$$

and by the Fatou Lemma with respect to n, one gets

$$\int_\Omega F(x,u)\varphi \le C. \tag{2.6}$$

Now we take $n \to \infty$ in (2.5). For the term on the left hand side we can reason as already done when $\eta \to 0$. For the right hand side of (2.5) one can easily apply the Lebesgue Theorem since

$$F(x,u)\varphi_n \le F(x,u)\varphi \overset{(2.6)}{\in} L^1(\Omega),$$

which gives

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int_\Omega F(x,u)\varphi, \tag{2.7}$$

for every $\varphi \in W_0^{1,p}(\Omega)$.

Since v_1 and v_2 are weak solutions to problem (2.3) with data G_1, G_2 then, recalling (2.7), one can test both equations with $W_0^{1,p}$-functions. From here we suppose that $G_1(x,s)s^{1-p}$ is decreasing with respect to s for almost every $x \in \Omega$; if one is in the other case, then slight modifications will be needed.

Let us fix $\varepsilon > 0$ and $k \in \mathbb{N}$ and let us define

$$A_{k,\varepsilon} := \left\{ x \in \Omega : 0 \le (v_1(x) + \varepsilon)^p - (v_2(x) + \varepsilon)^p \le k \right\}, \quad A_{k,\varepsilon}^c = \Omega \setminus A_{k,\varepsilon},$$

and

$$A_k = \left\{ x \in \Omega : 0 \le v_1(x)^p - v_2(x)^p \le k \right\}, \quad A_k^c = \Omega \setminus A_k.$$

We consider the following two functions:

$$\psi_1 = \frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_1 + \varepsilon)^{p-1}}, \quad \psi_2 = \frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_2 + \varepsilon)^{p-1}}, \quad (2.8)$$

where T_k is defined by (1.5). Let us also underline that $\psi_1, \psi_2 \in W_0^{1,p}(\Omega)$ (see Remark 2.5 below). One has

$$\nabla\psi_1 = \left(\nabla v_1 - p\left(\frac{v_2 + \varepsilon}{v_1 + \varepsilon}\right)^{p-1} \nabla v_2 + (p-1)\left(\frac{v_2 + \varepsilon}{v_1 + \varepsilon}\right)^p \nabla v_1\right)\chi_{A_{k,\varepsilon}}$$
$$-(p-1)\frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_1 + \varepsilon)^p}\nabla v_1 \chi_{A_{k,\varepsilon}^c},$$

and

$$\nabla\psi_2 = -\left(\nabla v_2 - p\left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon}\right)^{p-1} \nabla v_1 + (p-1)\left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon}\right)^p \nabla v_2\right)\chi_{A_{k,\varepsilon}}$$
$$-(p-1)\frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_2 + \varepsilon)^p}\nabla v_2 \chi_{A_{k,\varepsilon}^c}.$$

We choose ψ_1 and ψ_2 as test functions in equations solved by, respectively, v_1 and v_2 and we subtract them yielding to

$$\int_{A_{k,\varepsilon}} \left(|\nabla v_1|^p - \left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon}\right)^p |\nabla v_2|^p - p\left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon}\right)^{p-1}|\nabla v_2|^{p-2}\nabla v_2 \cdot \left(\nabla v_1 - \left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon}\right)\nabla v_2\right)\right)$$
$$+ \int_{A_{k,\varepsilon}} \left(|\nabla v_2|^p - \left(\frac{v_2 + \varepsilon}{v_1 + \varepsilon}\right)^p |\nabla v_1|^p - p\left(\frac{v_2 + \varepsilon}{v_1 + \varepsilon}\right)^{p-1}|\nabla v_1|^{p-2}\nabla v_1 \cdot \left(\nabla v_2 - \left(\frac{v_2 + \varepsilon}{v_1 + \varepsilon}\right)\nabla v_1\right)\right)$$
$$+ (p-1)\int_{A_{k,\varepsilon}^c} \left(\frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_2 + \varepsilon)^p}|\nabla v_2|^p - \frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_1 + \varepsilon)^p}|\nabla v_1|^p\right)$$
$$\leq \int_\Omega \left(\frac{G_1(x, v_1)}{(v_1 + \varepsilon)^{p-1}} - \frac{G_2(x, v_2)}{(v_2 + \varepsilon)^{p-1}}\right)T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+).$$

Now using the following classical estimate due to the convexity of the power function (recall that $p > 1$)

$$|\xi|^p - |\eta|^p - p|\eta|^{p-2}\eta \cdot (\xi - \eta) \geq 0, \quad \forall \xi, \eta \in \mathbb{R}^N,$$

one has

$$(p-1)\int_{A_{k,\varepsilon}^c} \frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_2 + \varepsilon)^p}|\nabla v_2|^p$$
$$\leq (p-1)\int_{A_{k,\varepsilon}^c} \frac{T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+)}{(v_1 + \varepsilon)^p}|\nabla v_1|^p \quad (2.9)$$
$$+ \int_\Omega \left(\frac{G_1(x, v_1)}{(v_1 + \varepsilon)^{p-1}} - \frac{G_2(x, v_2)}{(v_2 + \varepsilon)^{p-1}}\right)T_k(((v_1 + \varepsilon)^p - (v_2 + \varepsilon)^p)^+).$$

Noting that the first term of (2.9) is nonnegative, we have

$$
\begin{aligned}
0 \le (p-1) & \int_{A_{k,\varepsilon}^c} \frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^p} |\nabla v_1|^p \\
& + \int_\Omega \left(\frac{G_1(x,v_1)}{(v_1+\varepsilon)^{p-1}} - \frac{G_2(x,v_2)}{(v_2+\varepsilon)^{p-1}} \right) T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+) \\
& \overset{2.4}{\le} (p-1) \int_{A_{k,\varepsilon}^c} \frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^p} |\nabla v_1|^p \\
& + \int_\Omega \left(\frac{G_1(x,v_1)}{(v_1+\varepsilon)^{p-1}} - \frac{G_1(x,v_2)}{(v_2+\varepsilon)^{p-1}} \right) T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+).
\end{aligned}
\tag{2.10}
$$

Denoting $r_{k,\varepsilon}, \tilde{r}_{k,\varepsilon}$ as follows

$$
\begin{aligned}
r_{k,\varepsilon} = (p-1) & \frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^p} |\nabla v_1|^p \chi_{A_{k,\varepsilon}^c} \\
& + \left(\frac{G_1(x,v_1)}{(v_1+\varepsilon)^{p-1}} - \frac{G_1(x,v_2)}{(v_2+\varepsilon)^{p-1}} \right) T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+),
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{r}_{k,\varepsilon} = (p-1) & \frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^p} |\nabla v_1|^p \chi_{A_{k,\varepsilon}^c} \\
& + \frac{G_1(x,v_1)}{(v_1+\varepsilon)^{p-1}} T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+),
\end{aligned}
$$

then one has

$$
0 \le r_{k,\varepsilon}^+ \le \tilde{r}_{k,\varepsilon}. \tag{2.11}
$$

Since v_1, v_2 are positive then one has that $r_{k,\varepsilon}^+$ ($r_{k,\varepsilon}^-$) converges to r_k^+ (r_k^- resp.) and $\tilde{r}_{k,\varepsilon}$ converges to $\tilde{r}_k$ almost everywhere in Ω, where

$$
r_k = (p-1) \frac{T_k((v_1^p - v_2^p)^+)}{v_1^p} |\nabla v_1|^p \chi_{A_k^c} + \left(\frac{G_1(x,v_1)}{v_1^{p-1}} - \frac{G_1(x,v_2)}{v_2^{p-1}} \right) T_k((v_1^p - v_2^p)^+),
$$

and

$$
\tilde{r}_k = (p-1) \frac{T_k((v_1^p - v_2^p)^+)}{v_1^p} |\nabla v_1|^p \chi_{A_k^c} + \frac{G_1(x,v_1)}{v_1^{p-1}} T_k((v_1^p - v_2^p)^+).
$$

Moreover, using that $T_k(s) \le s$ for $s \ge 0$, we deduce that

$$
\frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^p} \chi_{A_{k,\varepsilon}^c} \le 1, \tag{2.12}
$$

and

$$
\frac{T_k(((v_1+\varepsilon)^p - (v_2+\varepsilon)^p)^+)}{(v_1+\varepsilon)^{p-1}} \le \frac{(v_1+\varepsilon)^p - \varepsilon^p}{(v_1+\varepsilon)^{p-1}} \le p v_1, \tag{2.13}
$$

where the last inequality holds by means of the Langrange Theorem.
It follows that

$$
\tilde{r}_{k,\varepsilon} \overset{(2.12),(2.13)}{\le} (p-1)|\nabla v_1|^p + p G_1(x,v_1)v_1. \tag{2.14}
$$

Since $v_1 \in W_0^{1,p}(\Omega)$ and from (2.6) one has that the right hand side of (2.14) belongs to $L^1(\Omega)$. This implies, applying the Lebesgue Theorem, that $\tilde{r}_{k,\varepsilon}$ strongly converges to $\tilde{r}_k$ in $L^1(\Omega)$. Now starting from (2.11) and applying the Vitali Theorem, we obtain that

$$r_{k,\varepsilon}^+ \to r_k^+ \text{ strongly in } L^1(\Omega). \tag{2.15}$$

As regards $r_{k,\varepsilon}^-$, applying the Fatou Lemma, we have

$$\limsup_{\varepsilon \to 0} \int_\Omega -r_{k,\varepsilon}^- \le \int_\Omega -r_k^-. \tag{2.16}$$

Hence we deduce that

$$0 \stackrel{(2.10)}{\le} \limsup_{\varepsilon \to 0} \int_\Omega r_{k,\varepsilon} = \limsup_{\varepsilon \to 0} \int_\Omega (r_{k,\varepsilon}^+ - r_{k,\varepsilon}^-) \stackrel{(2.15),(2.16)}{\le} \int_\Omega (r_k^+ - r_k^-) = \int_\Omega r_k.$$

Thus, until now, we have shown that

$$0 \le \int_\Omega \left((p-1)\frac{T_k((v_1^p - v_2^p)^+)}{v_1^p}|\nabla v_1|^p \chi_{A_k^c} + \left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) T_k((v_1^p - v_2^p)^+) \right). \tag{2.17}$$

Now we pass to the limit in (2.17) as k tends to infinity. We note that $\chi_{A_k^c}$ tends to 0 as k tends to infinity. Moreover, using (2.12) with $\varepsilon = 0$, we have

$$\frac{T_k((v_1^p - v_2^p)^+)}{v_1^p}|\nabla v_1|^p \chi_{A_k^c} \le |\nabla v_1|^p \in L^1(\Omega),$$

since $v_1 \in W_0^{1,p}(\Omega)$. This implies, applying the Lebesgue Theorem, that

$$\frac{T_k((v_1^p - v_2^p)^+)}{v_1^p}|\nabla v_1|^p \chi_{A_k^c} \to 0 \text{ strongly in } L^1(\Omega). \tag{2.18}$$

As regards the second term in the right hand side of (2.17), from $G_1(x,s)s^{1-p}$ decreasing with respect to s, one has that

$$0 \le -\left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) T_k((v_1^p - v_2^p)^+), \tag{2.19}$$

where the right hand side of (2.19) is increasing in k. Applying Beppo Levi's Theorem, we obtain that

$$\lim_{k \to \infty} \int_\Omega \left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) T_k((v_1^p - v_2^p)^+) = \int_\Omega \left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) (v_1^p - v_2^p)^+. \tag{2.20}$$

By passing to the limit as k tends to infinity in (2.17), using (2.18) and (2.20), we have

$$0 \le \int_\Omega \left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) (v_1^p - v_2^p)^+. \tag{2.21}$$

Furthermore from the fact that $G_1(x,s)s^{1-p}$ is decreasing with respect to s, one yields to

$$\left(\frac{G_1(x, v_1)}{v_1^{p-1}} - \frac{G_1(x, v_2)}{v_2^{p-1}} \right) (v_1^p - v_2^p)^+ \le 0 \quad \text{a.e. in } \Omega,$$

which, gathered with (2.21), gives that $(v_1^p - v_2^p)^+ \equiv 0$, that is $v_1 \le v_2$ almost everywhere in Ω. □

Remark 2.5 Here we show that ψ_1, ψ_2 defined by (2.8) belong to $W_0^{1,p}(\Omega)$. We focus on ψ_2. As a consequence of Lemma 1.1 contained in [37] and the fact the v_1, v_2 have finite energy, we have that the function ψ_h defined as

$$\psi_h = \frac{T_k(((v_1 + \varepsilon)^p - (T_h(v_2 + \varepsilon))^p)^+)}{(v_2 + \varepsilon)^{p-1}}$$

belongs to $W_0^{1,p}(\Omega)$ for every $h \geq 0$. Moreover, by computing its gradient, we get

$$\nabla \psi_h = -p \nabla v_2 \chi_{\{v_2 + \varepsilon \leq h\} \cap A_{k,\varepsilon,h} \cap B_h} + p \left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon} \right)^{p-1} \nabla v_1 \chi_{A_{k,\varepsilon,h} \cap B_h}$$

$$-(p-1) \frac{T_k(((v_1 + \varepsilon)^p - (T_h(v_2 + \varepsilon))^p)^+)}{(v_2 + \varepsilon)^p} \nabla v_2,$$

where

$$A_{k,\varepsilon,h} = \left\{ x \in \Omega : 0 \leq (v_1(x) + \varepsilon)^p - (T_h(v_2(x) + \varepsilon))^p \leq k \right\}$$

and

$$B_h = \{ x \in \Omega : v_1(x) + \varepsilon \geq T_h(v_2(x) + \varepsilon) \}.$$

It follows from the definition of $A_{k,\varepsilon,h}$ that

$$\left(\frac{v_1 + \varepsilon}{v_2 + \varepsilon} \right)^{p-1} \leq \left(\frac{k}{\varepsilon^p} + 1 \right)^{\frac{p-1}{p}}.$$

This implies that

$$|\nabla \psi_h|^p \leq C(p, k, \varepsilon) \left(|\nabla v_2|^p + |\nabla v_1|^p \right),$$

with $C(p, k, \varepsilon)$ a positive constant dependent only on p, k, ε. Hence, using $v_1, v_2 \in W_0^{1,p}(\Omega)$, we deduce that $\{\psi_h\}$ is bounded in $W_0^{1,p}(\Omega)$ uniformly in h. Moreover ψ_h converges to ψ_2 almost everywhere in Ω. So that ψ_h converges to ψ_2 weakly in $W_0^{1,p}(\Omega)$ and ψ_2 belongs to $W_0^{1,p}(\Omega)$. As regards φ, in a similar way it is possible to prove that ψ_1 belongs to $W_0^{1,p}(\Omega)$.

3 Existence Results in Some Model Equations

In this section we give existence results to (2.1) for some explicit nonlinearities F of the following form

$$F(x, s) = f(x)h(s) + g(x)k(s), \tag{3.1}$$

where f, g are nonnegative functions belonging to suitable Lebesgue space, with $f \not\equiv 0$, and $h, k : (0, \infty) \to [0, \infty)$ are continuous nonnegative functions such that

$$\exists \, \gamma \geq 0, \underline{C} > 0 : h(s) \leq \frac{\underline{C}}{s^{\gamma}} \quad \forall s \in (0, \infty), \tag{3.2}$$

and

$$\exists \, q \geq 0, \overline{C} > 0 : k(s) \leq \overline{C} s^q \quad \forall s \in (0, \infty). \tag{3.3}$$

Remark 3.1 Let us observe that (3.3) implies that k can be extended by continuity at 0 defining $k(0) = 0$.

We underline that we are not assuming any kind of monotonicity on the functions h, k but just some control from the above. Moreover, the case of continuous and bounded h, k are well contained in our existence result.

For the sake of clarity we reformulate the problem under the assumption (3.1):

$$\begin{cases} -\Delta_p u = f(x)h(u) + g(x)k(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

At first we state an existence result in case $\gamma \leq 1$ and $q < p - 1$, which we recall that corresponds to the sublinear case when $p = 2$; let us explicitly note that in the sequel we define $\left(\frac{p^*}{1-\gamma}\right)' := 1$ if $\gamma = 1$.

In particular one has the following result.

Theorem 3.2 *Let* $f \in L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)$ *be a nonnegative function not identically zero and let* $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$ *be a nonnegative function. Let h and k be nonnegative continuous functions satisfying (3.2) with $\gamma \leq 1$ and (3.3) with $q < p - 1$ respectively. Then there exists at least one weak solution to problem (3.4).*

Remark 3.3 In the case $f \equiv 0$, if k is an increasing function satisfying (3.3), the existence of a weak solution to (3.4) is contained in [6].

Next we deal with the more difficult case of a strong singularity; here, in order to deduce an existence result, we need some regularity on the Ω.

Theorem 3.4 *Let Ω satisfy the interior ball condition and let $f \in L^m(\Omega)$ with $m > 1$ be a nonnegative function and let $g \in L^{\left(\frac{p^*}{1+q}\right)'}(\Omega)$ be a nonnegative function. Let h and k be nonnegative continuous functions satisfying (3.2) with $1 < \gamma < 2 - \frac{1}{m}$ and (3.3) with $q < p - 1$ respectively. Then there exists at least one weak solution to problem (3.4).*

Finally we also dealt with $q = p - 1$. In the next result we denote by C_p the best constant for the Poincaré inequality in Ω; we also recall that $\overline{C}$ is the one defined by (3.3).

Theorem 3.5 *Let $f \in L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)$ be a nonnegative function not identically zero and let g such that $||g||_{L^\infty(\Omega)} < (\overline{C}C_p^p)^{-1}$. Let h and k be nonnegative continuous functions satisfying (3.2) with $\gamma \leq 1$ and (3.3) with $q = p - 1$ respectively. Then there exists at least one weak solution to problem (3.4).*

Remark 3.6 Collecting the existence results contained in Theorems 3.2, 3.4 and 3.5 with the uniqueness result contained in Corollary 2.3 we obtain that there exists a unique solution $u \in W_0^{1,p}(\Omega)$ to

$$-\Delta_p u = F(x, u),$$

under the assumptions of Theorems 3.2 and 3.4 in case $(h(s)+k(s))s^{1-p}$ is decreasing with respect to s and requiring that $f + g$ is almost everywhere positive in Ω.

Moreover under the assumptions of Theorem 3.5 one has a unique solution if $h(s)s^{1-p}$ is decreasing with respect to s and f is almost everywhere positive in Ω.

3.1 Proof of the Existence Results

Let us introduce the following scheme of approximation

$$\begin{cases} -\Delta_p u_n = f_n h_n(u_n) + g_n k_n(u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where $f_n = T_n(f)$ and $g_n = T_n(g)$. Moreover, defining $h(0) := \lim_{s \to 0} h(s)$, we set

$$h_n(s) = \begin{cases} T_n(h(s)) & \text{for } s > 0, \\ \min(n, h(0)) & \text{otherwise,} \end{cases} \quad \text{and} \quad k_n(s) = \begin{cases} T_n(k(s)) & \text{for } s > 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of a weak solution $u_n \in W_0^{1,p}(\Omega)$ is guaranteed by [28]. Moreover, by Theorem 4.2 of [37], we get that u_n is bounded and, since the right hand side of (3.5) is nonnegative, that u_n is nonnegative.

Remark 3.7 Under the assumptions of Remark 3.6 one has that the approximating sequence $\{u_n\}$ is increasing w.r.t. n. Indeed defining $F_n(x, s) = f_n(x)h_n(s) + g_n(x)k_n(s)$ one deduces that for every n in $\mathbb{N}$

$$F_n(x, s) \le F_{n+1}(x, s) \quad \forall s \in (0, \infty) \text{ and for a.e. } x \in \Omega.$$

This allows to apply Theorem 2.2, yielding to

$$u_n \le u_{n+1} \qquad \forall n \in \mathbb{N}.$$

Proof of Theorem 3.2 We divide the proof in two steps. In the first one, we show a priori estimates on u_n, solutions to (3.5). In the second one we pass to the limit our approximation in order to deduce the existence of a weak solution to (3.4).

Step 1. Let us choose u_n as a test function in the weak formulation of (3.5) and from the Hölder inequality and from EqS. 3.2, 3.3, one gets

$$\int_\Omega |\nabla u_n|^p = \int_\Omega (f_n h_n(u_n) u_n + g_n k_n(u_n) u_n)$$

$$\le \underline{C} \int_\Omega f_n u_n^{1-\gamma} + \overline{C} \int_\Omega g_n u_n^{1+q}$$

$$\le \underline{C} \|f\|_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} \|u_n\|_{L^{p^*}(\Omega)}^{1-\gamma} + \overline{C} \|g\|_{L^{\left(\frac{p^*}{q+1}\right)'}(\Omega)} \|u_n\|_{L^{p^*}(\Omega)}^{q+1}. \tag{3.6}$$

If $\|u_n\|_{L^{p^*}(\Omega)} \le 1$, we deduce that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ uniformly in n. Otherwise, recalling that $0 \le 1 - \gamma < q + 1 < p$, we obtain, applying the Sobolev embedding Theorem on the left-hand side of (3.6), that

$$\|u_n\|_{L^{p^*}(\Omega)}^p \le C \left(\|f\|_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} + \|g\|_{L^{\left(\frac{p^*}{q+1}\right)'}(\Omega)} \right) \|u_n\|_{L^{p^*}(\Omega)}^{q+1}. \tag{3.7}$$

This implies, dividing by $\|u_n\|_{L^{p^*}(\Omega)}^{q+1}$ both members of (3.7), that $\{u_n\}$ is bounded in $L^{p^*}(\Omega)$ uniformly in n. It follows from (3.6) that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$ with respect to n. This implies that there exists a nonnegative function u in $W_0^{1,p}(\Omega)$ such that $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and almost everywhere in Ω. Let

us take $0 \leq \varphi \in W_0^{1,p}(\Omega)$ as test function in the weak formulation of (3.5); one obtains, using the Young inequality, that

$$\int_\Omega (f_n h_n(u_n) + g_n k_n(u_n))\, \varphi = \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \leq \frac{1}{p'} \int_\Omega |\nabla u_n|^p + \frac{1}{p} \int_\Omega |\nabla \varphi|^p \leq C. \tag{3.8}$$

Hence $\{f_n h_n(u_n) + g_n k_n(u_n)\}$ is bounded in $L^1_{\mathrm{loc}}(\Omega)$ and, applying Theorem 2.1 of [4], that ∇u_n converges almost everywhere in Ω to ∇u.

Step 2. In this second step we prove that u obtained in the first step is a weak solution to (3.4).

First of all we apply the Fatou Lemma in (3.8) in order to deduce that

$$\int_\Omega (f h(u) + g k(u))\varphi \leq \liminf_{n \to \infty} \int_\Omega (f_n h_n(u_n) + g_n k_n(u_n))\, \varphi \leq C,$$

hence $(f h(u) + g k(u))\varphi \in L^1(\Omega)$ for any nonnegative $\varphi \in W_0^{1,p}(\Omega)$. As a consequence, if $h(s)$ is unbounded as s tends to 0, we deduce that

$$\{u = 0\} \subset \{f = 0\}, \tag{3.9}$$

up to a set of zero Lebesgue measure.

From now on, we assume that $h(s)$ is unbounded as s tends to 0. Let φ be a nonnegative function in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Choosing it as test function in the weak formulation of (3.5) we have

$$\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi = \int_\Omega (f_n h_n(u_n) + g_n k_n(u_n))\varphi. \tag{3.10}$$

We want to pass to the limit in (3.10) as n tends to infinity. We fix $\delta > 0$ and we decompose the right hand side in the following way:

$$\int_\Omega (f_n h_n(u_n) + g_n k_n(u_n))\varphi = \int_{\{u_n \leq \delta\}} (f_n h_n(u_n) + g_n k_n(u_n))\varphi$$
$$+ \int_{\{u_n > \delta\}} (f_n h_n(u_n) + g_n k_n(u_n))\varphi. \tag{3.11}$$

Therefore we have, thanks to Lemma 1.1 contained in [37], that $V_\delta(u_n)\varphi$ belongs to $W_0^{1,p}(\Omega)$, where $V_\delta(s)$ is defined by (1.6). So we take it as test function in the weak formulation of (3.5) and we obtain

$$\int_{\{u_n \leq \delta\}} (f_n h_n(u_n) + g_n k_n(u_n))\varphi \overset{(1.6)}{\leq} \int_\Omega (f_n h_n(u_n) + g_n k_n(u_n)) V_\delta(u_n)\varphi$$
$$= \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi\, V_\delta(u_n) - \frac{1}{\delta} \int_{\{\delta < u_n < 2\delta\}} |\nabla u_n|^p \varphi$$
$$\leq \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi\, V_\delta(u_n)$$

Using that V_δ is bounded we deduce that $|\nabla u_n|^{p-2} \nabla u_n V_\delta(u_n)$ converges to $|\nabla u|^{p-2} \nabla u V_\delta(u)$ weakly in $L^{p'}(\Omega)^N$ as n tends to infinity. This implies that

$$\lim_{n \to \infty} \int_{\{u_n \leq \delta\}} (f_n h_n(u_n) + g_n k_n(u_n))\varphi \leq \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\, V_\delta(u). \tag{3.12}$$

Proof of Theorem 3.4 We take u_n as a test function in (3.5) yielding to

$$\int_\Omega |\nabla u_n|^p \leq \int_\Omega (f_n h_n(u_n) u_n + g_n k_n(u_n) u_n) \leq \underline{C} \int_\Omega f_n u_n^{1-\gamma} + \overline{C} \int_\Omega g_n u_n^{1+q}$$

$$\leq \underline{C} \int_\Omega f_n u_n^{1-\gamma} + \overline{C} ||g||_{L^{\left(\frac{p^*}{q+1}\right)'}(\Omega)} ||u_n||_{L^{p^*}(\Omega)}^{q+1}. \tag{3.17}$$

Hence, we just need an estimate on the first term of the right hand side of (3.17). First of all let us observe that there exists a nonincreasing and continuous function $\underline{h} : [0, \infty) \to [0, \infty)$ such that

$$\underline{h}(s) \leq h_n(s), \ \forall \, s > 0, \ n \in \mathbb{N}.$$

For the construction of such $\underline{h}$ we refer to [18]. Hence let us consider $v_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ solution to

$$\begin{cases} -\Delta_p v_n = \underline{h}(v_n) f_n & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Once again, reasoning as in [18, 19], one has that v_n is nondecreasing with respect to n and also that $u_n \geq v_n \geq v_1$. Moreover, it follows from the Hopf Lemma (see Lemma A.3 of [34]) that

$$v_1(x) \geq C\delta(x), \text{ for } x \in \Omega,$$

where $\delta(x)$ is the distance function from the boundary $\partial\Omega$.
Thanks to the previous we can finally estimate the term on the right hand side of (3.17) as follows:

$$\int_\Omega f_n u_n^{1-\gamma} \leq C^{1-\gamma} ||f||_{L^m(\Omega)} \left(\int_\Omega \frac{1}{\delta^{(\gamma-1)m'}} \right)^{\frac{1}{m'}},$$

which is finite since $\gamma < 2 - \frac{1}{m}$. This allows to have an estimate on u_n in $W_0^{1,p}(\Omega)$ which is independent on n. Hence one can reason as in Step 2 of Theorem 3.2 in order to deduce the existence of a weak solution. $\qquad\square$

Finally we prove Theorem 3.5.

Proof of Theorem 3.5 We choose u_n itself as a test function in the weak formulation of (3.5) and applying the Hölder inequality and the Poincaré inequality, we get

$$\int_\Omega |\nabla u_n|^p \overset{(32)}{\leq} \underline{C} \int_\Omega f u_n^{1-\gamma} + \overline{C} ||g||_{L^\infty(\Omega)} \int_\Omega u_n^p$$

$$\leq \underline{C} ||f||_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} ||u_n||_{L^{p^*}(\Omega)}^{1-\gamma} + \overline{C} ||g||_{L^\infty(\Omega)} C_P^p \int_\Omega |\nabla u_n|^p,$$

which, recalling $1 - \overline{C} ||g||_{L^\infty(\Omega)} C_P^p > 0$, implies that

$$\int_\Omega |\nabla u_n|^p \leq \frac{C}{1 - \overline{C} ||g||_{L^\infty(\Omega)} C_P^p} ||f||_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} ||u_n||_{L^{p^*}(\Omega)}^{1-\gamma}. \tag{3.18}$$

Applying the Sobolev embedding Theorem in the right hand side of (3.18), we have

$$||u_n||_{W_0^{1,p}(\Omega)}^p \leq \frac{C S^{1-\gamma}}{1 - \overline{C} ||g||_{L^\infty(\Omega)} C_P^p} ||f||_{L^{\left(\frac{p^*}{1-\gamma}\right)'}(\Omega)} ||u_n||_{W_0^{1,p}(\Omega)}^{1-\gamma},$$

where S is the constant of the embedding. Since $p > 1 - \gamma$ it follows that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. So, up to subsequence, we have $u_n \to u$ weakly in $W_0^{1,p}(\Omega)$ and almost

everywhere in Ω. Finally we can repeat the argument of Step 2 of Theorem 3.4 in order to conclude that u is a solution to (3.4). $\qquad\qquad\square$

3.2 A Concluding Remark

Here we underline that the result in Theorem 3.4 is not sharp, at least in the model case. Let $\Omega \subset \mathbb{R}^N$ be open and bounded with smooth boundary and let us consider the following problem

$$
\begin{cases}
-\Delta u = \dfrac{f}{u^\gamma} + gu^q & \text{in } \Omega, \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{3.19}
$$

where $\gamma > 1, q < 1, 0 < f \in L^1(\Omega)$ and $g \in L^\infty(\Omega)$ nonnegative. We recall the following result proven in [39].

Theorem 3.8 *Let $\gamma > 1$, $q < 1$ and let us suppose that there exists a function $u_0 \in H_0^1(\Omega)$ such that*

$$
\int_\Omega f u_0^{1-\gamma} < \infty.
\tag{3.20}
$$

Then there exists a solution $u \in H_0^1(\Omega)$ to (3.19).

Using the previous result we have the following existence theorem:

Theorem 3.9 *Let $f \in L^m(\Omega)$ with $m > 1$ be a nonnegative function and let $g \in L^\infty(\Omega)$ be a nonnegative function. Let $1 < \gamma < 3 - \frac{2}{m}$ and $q < 1$ then there exists a solution to problem (3.19).*

Proof In order to show the existence of a solution we employ (3.20) with $u_0 = \delta(x)^t$ for some $t > \frac{1}{2}$ and where $\delta(x)$ is the distance function from the boundary $\partial\Omega$. Indeed, one can show that an application of the Hölder inequality

$$
\int_\Omega f u_0^{1-\gamma} \le C \int_\Omega \delta^{t(1-\gamma)m'}
$$

and the last integral is finite thanks on the assumption $\gamma < 3 - \frac{2}{m}$. $\qquad\qquad\square$

We also remark that, in [16], Theorem 3.8 is extended for the case of the p-Laplacian operator with $p > 2$. In this case one can show that a similar result to Theorem 3.9 with $1 < \gamma < 1 + \frac{p(m-1)}{(p-1)m}$.

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