



## Two inequalities for the first Robin eigenvalue of the Finsler Laplacian

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**Abstract.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded, connected, open set with Lipschitz boundary. Let  $F$  be a suitable norm in  $\mathbb{R}^n$  and let  $\Delta_F u = \operatorname{div}(F_\xi(\nabla u)F(\nabla u))$  be the so-called Finsler Laplacian, with  $u \in H^1(\Omega)$ . In this paper, we prove two inequalities for  $\lambda_F(\beta, \Omega)$ , the first eigenvalue of  $\Delta_F$  with Robin boundary conditions involving a positive function  $\beta(x)$ . As a consequence of our result, we obtain the asymptotic behavior of  $\lambda_F(\beta, \Omega)$  when  $\beta$  is a positive constant which goes to zero

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**1. Introduction.** Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded, connected, open set with Lipschitz boundary.

Let  $F: \mathbb{R}^n \rightarrow [0, +\infty[$  be a  $C^2(\mathbb{R}^n \setminus \{0\})$ , convex, and positively 1-homogeneous function such that

$$a|\xi| \leq F(\xi) \leq b|\xi|, \quad \xi \in \mathbb{R}^n, \quad (1.1)$$

for some positive constants  $a$  and  $b$ . Throughout the paper, we will assume that  $F(\xi)$  is strongly convex, that is

$$[F^2]_{\xi\xi}(\xi) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}. \quad (1.2)$$

In what follows, we assume that  $\beta: \partial\Omega \rightarrow ]0, +\infty[$  is a continuous function and we define

$$m := \int_{\partial\Omega} \beta(x)F(\nu) d\mathcal{H}^{n-1} > 0, \quad (1.3)$$

where  $\nu$  is the unit outer normal to the boundary and  $d\mathcal{H}^{n-1}$  denotes the  $(n-1)$ -dimensional Hausdorff measure. Let us consider the following Robin

eigenvalue problem

$$\begin{cases} -\Delta_F u = \lambda_F(\beta, \Omega)u & \text{in } \Omega, \\ F(\nabla u)F_\xi(\nabla u) \cdot \nu + \beta(x)uF(\nu) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where  $u \in H^1(\Omega)$  and

$$\Delta_F u = \operatorname{div}(F_\xi(\nabla u)F(\nabla u))$$

is the so-called Finsler Laplacian. When  $F = \mathcal{E}$  is the Euclidean norm,  $\Delta_F$  reduces to the classic Laplace operator. Nevertheless, it is in general a nonlinear operator and it has been studied in several papers (see for instance [1, 3, 7, 10, 11]).

In [14] (see also [8] for the case that  $\beta$  is a positive constant), it is proved that the first eigenvalue of (1.4) is positive, simple, and has the following variational characterization

$$\lambda_F(\beta, \Omega) = \min_{v \in H^1(\Omega) \setminus \{0\}} \frac{\int_\Omega F^2(\nabla v)dx + \int_{\partial\Omega} \beta(x)v^2 F(\nu)d\mathcal{H}^{n-1}}{\int_\Omega v^2 dx}. \tag{1.5}$$

On the other hand,  $\lambda_F(\beta, \Omega)$  verifies a Faber-Krahn type inequality for suitable functions  $\beta(x)$ . Finally, the authors prove some estimates for  $\lambda_F(\beta, \Omega)$  in terms of geometric quantities related to the domain  $\Omega$ , in particular, a weighted anisotropic Cheeger inequality.

The aim of this paper is to prove, for a positive and continuous function  $\beta$ , two inequalities involving  $\lambda_F(\beta, \Omega)$  in terms of the following quantities

$$\sigma_F(\beta, \Omega) := \inf_{\substack{v \in H^1(\Omega) \setminus \{0\} \\ \int_{\partial\Omega} \beta(x)vF(\nu) d\mathcal{H}^{n-1} = 0}} \frac{\int_\Omega F^2(\nabla v) dx}{\int_\Omega v^2 dx}, \tag{1.6}$$

and

$$q_F(\beta, \Omega) := \inf_{\substack{h \in H^1(\Omega) \setminus \{0\} \\ \Delta_F h = 0}} \frac{\int_{\partial\Omega} \beta(x)h^2 F(\nu)d\mathcal{H}^{n-1}}{\int_\Omega h^2 dx}. \tag{1.7}$$

We observe that if  $\beta(x) = \beta$  is a positive parameter, then

$$q_F(\beta, \Omega) = \beta q_F(\Omega), \tag{1.8}$$

where

$$q_F(\Omega) := \inf_{\substack{h \in H^1(\Omega) \setminus \{0\} \\ \Delta_F h = 0}} \frac{\int_{\partial\Omega} h^2 F(\nu)d\mathcal{H}^{n-1}}{\int_\Omega h^2 dx} \tag{1.9}$$

while  $\sigma_F(\beta, \Omega)$  does not depend on  $\beta$  and then, in this case, we denote it by  $\sigma_F(\Omega)$ . On the other hand, in the Euclidean case, when  $\Omega$  has two axes of symmetry,  $\sigma_\mathcal{E}(\Omega)$  coincides with the first nontrivial Neumann eigenvalue of the Laplace operator  $\mu(\Omega)$  (see for instance [16–18]).

Furthermore, under certain assumptions,  $q_{\mathcal{E}}(\Omega)$  coincides with the first non-trivial Steklov eigenvalue  $q$  related to the biharmonic Laplacian

$$\begin{cases} \Delta^2 v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \Delta v = q \frac{\partial v}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \quad (1.10)$$

This is shown in [5] by means of a generalized Fichera duality principle, provided  $\Omega$  satisfies a uniform outer ball condition. We recall that problem (1.10) was first considered by Kuttler and Sigillito in [17, 18] where among other things they studied the isoperimetric properties related to the first eigenvalue. In the last years this kind of problems have been intensively studied in the literature, we refer the reader for instance to [4, 6, 12, 20], and the references therein for further studies.

In particular, one can find some physical interpretation of the Steklov boundary conditions in [6] where the authors state also several Navier-Robin problems for the biharmonic operator. Finally, when  $\beta$  is not a positive constant, in [15], the authors prove that if  $\partial\Omega \in C^2$ , then  $q_{\mathcal{E}}(\beta, \Omega)$  coincides with the first nontrivial eigenvalue of the following “weighted” Steklov type problem

$$\begin{cases} \Delta^2 v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \\ \Delta v = q_{\mathcal{E}}(\beta, \Omega) \frac{1}{\beta(x)} \frac{\partial v}{\partial \nu} & \text{on } \partial\Omega. \end{cases} \quad (1.11)$$

Our main result is the following

**Theorem 1.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a bounded, connected, open set with Lipschitz boundary, then*

$$\frac{1}{\lambda_F(\beta, \Omega)} \leq \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \quad (1.12)$$

and

$$\frac{1}{\lambda_F(\beta, \Omega)} \leq \frac{1}{\lambda_F(\Omega)} + \frac{1}{q_F(\beta, \Omega)}, \quad (1.13)$$

where  $\beta: \partial\Omega \rightarrow ]0, +\infty[$  is a continuous function,  $m$  is given by (1.3), and  $\lambda_F(\Omega)$  is the first Dirichlet eigenvalue of the Finsler Laplacian.

In the Euclidean case, when  $\beta(x) = \beta > 0$  is constant, the inequalities (1.12) and (1.13) were proved in [21] for  $\lambda_{\mathcal{E}}(\beta, \Omega)$ , the first Robin eigenvalue of the Laplacian by using the P-function method (see also [22]).

Successively in [16], Kuttler proves the same result with a simpler proof whose key ingredient is an algebraic inequality between geometric and arithmetic means. Our result, in this order of idea, allows to extend the results of [21] to the case when  $\beta$  is not constant and to a larger class of elliptic operators. Our proof follows the idea contained in [16].

We prove inequalities (1.12) and (1.13) in Sections 2 and 3, respectively.

**2. Proof of the inequality (1.12).** First of all, in order to verify that inequality (1.12) is well posed, we show that  $\sigma_F(\beta, \Omega)$  is positive. To see this, we first prove that  $\sigma_F(\beta, \Omega)$  is a minimum. Let  $v_k \in H^1(\Omega) \setminus \{0\}$  be a minimizing sequence such that  $\int_{\partial\Omega} \beta(x)v_k F(\nu) d\mathcal{H}^{n-1} = 0$ ,  $\|v_k\|_{L^2(\Omega)} = 1$ , and

$$\lim_k \int_{\Omega} F^2(\nabla v_k) dx = \sigma_F(\beta, \Omega). \tag{2.1}$$

Then  $v_k$  is bounded in  $H^1(\Omega)$  and there exists a subsequence, still denoted by  $v_k$ , such that  $v_k$  converges in  $L^2(\Omega)$  to a function  $v \in H^1(\Omega)$  with  $\|v\|_{L^2(\Omega)} = 1$ . Furthermore, by the classical trace embedding theorem,  $v_k$  converges to  $v$  also in  $L^2(\partial\Omega)$  and then  $\int_{\partial\Omega} \beta(x)v F(\nu) d\mathcal{H}^{n-1} = 0$ . Taking  $v$  as test function in (1.6) and using Fatou’s lemma, we finally get

$$\sigma_F(\beta, \Omega) \leq \int_{\Omega} F^2(\nabla v) dx \leq \liminf_k \int_{\Omega} F^2(\nabla v_k) dx = \sigma_F(\beta, \Omega).$$

So  $v$  is a minimizer for  $\sigma_F(\beta, \Omega)$  and we can show that  $\sigma_F(\beta, \Omega) > 0$  arguing by contradiction. Indeed, if  $\sigma_F(\beta, \Omega) = 0$ , then there exists  $v \in H^1(\Omega) \setminus \{0\}$  such that  $\int_{\partial\Omega} \beta(x)v F(\nu) d\mathcal{H}^{n-1} = 0$ ,  $\|v\|_{L^2(\Omega)} = 1$ , and  $\int_{\Omega} F^2(\nabla v) dx = 0$  and hence  $v = C$  almost everywhere in  $\bar{\Omega}$ , with  $C \in \mathbb{R}$ , and  $C \neq 0$ . Since  $m > 0$ , this is in contradiction with

$$C \int_{\partial\Omega} \beta(x) F(\nu) d\mathcal{H}^{n-1} = 0.$$

Now we prove inequality (1.12). For the reader’s convenience, from now on, we will use the following notation

$$E_F(w) := \int_{\Omega} F^2(\nabla w) dx \tag{2.2}$$

for any  $w \in H^1(\Omega)$ . Let  $u$  be a positive eigenfunction corresponding to  $\lambda_F(\beta, \Omega)$  and

$$c = \frac{1}{m} \int_{\partial\Omega} \beta(x)u F(\nu) d\mathcal{H}^{n-1}, \tag{2.3}$$

with  $m$  defined in (1.3). By the Minkowski inequality and, recalling the definition of  $\sigma_F(\beta, \Omega)$  in (1.6), we have

$$\sqrt{\int_{\Omega} u^2 dx} \leq \sqrt{\int_{\Omega} (u - c)^2 dx} + \sqrt{c^2|\Omega|} \leq \sqrt{\frac{E_F(u)}{\sigma_F(\beta, \Omega)}} + \sqrt{c^2|\Omega|}.$$

Squaring and using the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
 \int_{\Omega} u^2 dx &\leq \frac{E_F(u)}{\sigma_F(\beta, \Omega)} + c^2 |\Omega| + 2\sqrt{\frac{E_F(u)c^2|\Omega|}{\sigma_F(\beta, \Omega)}} \\
 &\leq \frac{E_F(u)}{\sigma_F(\beta, \Omega)} + c^2 |\Omega| + \frac{E_F(u)|\Omega|}{m} + \frac{c^2 m}{\sigma_F(\beta, \Omega)} \\
 &= E_F(u) \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) + c^2 m \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) \\
 &= \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) (E_F(u) + c^2 m).
 \end{aligned} \tag{2.4}$$

By (2.3), Hölder's inequality, and (1.5), we see that (2.4) implies

$$\begin{aligned}
 \int_{\Omega} u^2 dx &\leq \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) \left( E_F(u) \right. \\
 &\quad \left. + \frac{(\int_{\partial\Omega} \beta(x)F(\nu)d\mathcal{H}^{n-1}) (\int_{\partial\Omega} \beta(x)u^2 F(\nu)d\mathcal{H}^{n-1})}{m} \right) \\
 &= \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) \left( E_F(u) + \int_{\partial\Omega} \beta(x)u^2 F(\nu)d\mathcal{H}^{n-1} \right) \\
 &= \left( \frac{1}{\sigma_F(\beta, \Omega)} + \frac{|\Omega|}{m} \right) \left( \lambda_F(\beta, \Omega) \int_{\Omega} u^2 dx \right)
 \end{aligned} \tag{2.5}$$

which gives (1.12).

**Remark 2.1.** Let  $\Omega$  be an open set of  $\mathbb{R}^n$  with Lipschitz boundary. We denote by  $P_F(\Omega)$  the so-called anisotropic perimeter defined as follows (see for instance [2])

$$P_F(\Omega) = \int_{\partial\Omega} F(\nu) d\mathcal{H}^{n-1},$$

where  $\nu$  denotes the unit outer normal to  $\partial\Omega$ . We stress that when  $\beta(x) = \beta$  is a positive constant, the inequality (1.12) gives the following asymptotic behavior of  $\lambda_F(\beta, \Omega)$ , when  $\beta$  goes to zero:

$$\lim_{\beta \rightarrow 0} \frac{\lambda_F(\beta, \Omega)}{\beta} = \frac{P_F(\Omega)}{|\Omega|}. \tag{2.6}$$

Indeed if  $\beta$  is a positive constant, then  $m = \beta P_F(\Omega)$  and we have

$$\frac{P_F(\Omega)}{|\Omega|} \geq \frac{\lambda_F(\beta, \Omega)}{\beta} \geq \frac{P_F(\Omega)\sigma_F(\Omega)}{P_F(\Omega)\beta + |\Omega|\sigma_F(\Omega)}, \tag{2.7}$$

where the first inequality follows by using a constant as test function in (1.5) and the second by using (1.12). Taking in (2.7) the limit for  $\beta$  which goes to zero, one gets (2.6).

**Remark 2.2.** Let  $\mu_F(\Omega)$  be the first nontrivial Neumann eigenvalue of the Finsler Laplacian (see for instance [9]), it holds

$$\sigma_F(\beta, \Omega) \leq \mu_F(\Omega). \quad (2.8)$$

Indeed if  $u$  is an eigenfunction corresponding to  $\mu_F(\Omega)$ , then  $\int_{\Omega} u \, dx = 0$  and

$$\mu_F(\Omega) = \frac{\int_{\Omega} F^2(\nabla u) \, dx}{\int_{\Omega} u^2 \, dx}.$$

Inequality (2.8) follows by taking as test in (1.6) the function  $v(x) = u(x) - c$ , where  $c$  is as in (2.3).

**3. Proof of the inequality (1.13).** First of all, we observe that the trace embedding theorem ensures that  $q_F(\beta, \Omega)$  is positive and then inequality (1.13) is well posed.

Let  $u$  be a positive eigenfunction corresponding to  $\lambda_F(\beta, \Omega)$  and let us consider the functions  $v$  and  $h$  which solve the following problems respectively

$$\begin{cases} \Delta_F v = \Delta_F u & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.1)$$

and

$$\begin{cases} \Delta_F h = 0 & \text{in } \Omega, \\ h = u & \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

The maximum principle assures that  $u \leq v + h$ . Moreover, by using the same notation of Section 2, it holds

$$E_F(u) \geq E_F(v). \quad (3.3)$$

Indeed, the convexity of  $F^2$  and the homogeneity of  $F$  imply

$$\begin{aligned} \int_{\Omega} F^2(\nabla v) \, dx &\geq - \int_{\Omega} F^2(\nabla u) \, dx + 2 \int_{\Omega} F(\nabla v) F_{\xi}(\nabla v) \cdot \nabla v \, dx \\ &= - \int_{\Omega} F^2(\nabla u) \, dx + 2 \int_{\Omega} F^2(\nabla v) \, dx, \end{aligned}$$

where the last equality follows as  $v$  is the solution of (3.1).

By the Minkowski inequality, by the definition of  $q_F(\beta, \Omega)$  given in (1.7), and recalling the following variational characterization of  $\lambda_F(\Omega)$  (see for instance [3])

$$\lambda_F(\Omega) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{E_F(u)}{\int_{\Omega} u^2 \, dx}, \quad (3.4)$$

we get

$$\begin{aligned}
 \left( \int_{\Omega} u^2 dx \right)^{\frac{1}{2}} &\leq \left( \int_{\Omega} v^2 dx \right)^{\frac{1}{2}} + \left( \int_{\Omega} h^2 dx \right)^{\frac{1}{2}} \\
 &\leq \left( \frac{E_F(v)}{\lambda_F(\Omega)} \right)^{\frac{1}{2}} + \left( \frac{\int_{\partial\Omega} \beta(x) h^2 F(\nu) d\mathcal{H}^{n-1}}{q_F(\beta, \Omega)} \right)^{\frac{1}{2}} \\
 &\leq \left( \frac{E_F(u)}{\lambda_F(\Omega)} \right)^{\frac{1}{2}} + \left( \frac{\int_{\partial\Omega} \beta(x) u^2 F(\nu) d\mathcal{H}^{n-1}}{q_F(\beta, \Omega)} \right)^{\frac{1}{2}}.
 \end{aligned}$$

Squaring and using the arithmetic-geometric mean inequality, we have

$$\begin{aligned}
 \int_{\Omega} u^2 dx &\leq \frac{E_F(u)}{\lambda_F(\Omega)} + \frac{\int_{\partial\Omega} \beta(x) u^2 F(\nu) d\mathcal{H}^{n-1}}{q_F(\beta, \Omega)} \\
 &\quad + 2 \left( \frac{E_F(u) \int_{\partial\Omega} \beta(x) u^2 F(\nu) d\mathcal{H}^{n-1}}{\lambda_F(\Omega) q_F(\beta, \Omega)} \right)^{\frac{1}{2}} \\
 &\leq \left( \frac{1}{\lambda_F(\Omega)} + \frac{1}{q_F(\beta, \Omega)} \right) E_F(u) + \int_{\partial\Omega} \beta(x) u^2 F(\nu) d\mathcal{H}^{n-1} \\
 &\quad \times \left( \frac{1}{q_F(\beta, \Omega)} + \frac{1}{\lambda_F(\Omega)} \right) \\
 &\leq \left( \frac{1}{\lambda_F(\Omega)} + \frac{1}{q_F(\beta, \Omega)} \right) \left( E_F(u) + \int_{\partial\Omega} \beta(x) u^2 F(\nu) d\mathcal{H}^{n-1} \right),
 \end{aligned}$$

which gives (1.13).

**Remark 3.1.** We stress that inequality (1.13) can also be written as follows

$$0 \leq \frac{1}{\lambda_F(\beta, \Omega)} - \frac{1}{\lambda_F(\Omega)} \leq \frac{1}{q_F(\beta, \Omega)}, \quad (3.5)$$

this inequality gives an upper bound of the distance between the first Dirichlet and Robin eigenvalue of the Finsler Laplacian in terms of  $q_F(\beta, \Omega)$ . In particular, if  $\beta(x) = \beta$  is constant, (3.5) can be rewritten by (1.8) as

$$0 \leq \frac{1}{\lambda_F(\beta, \Omega)} - \frac{1}{\lambda_F(\Omega)} \leq \frac{1}{\beta q_F(\Omega)}, \quad (3.6)$$

with  $q_F(\Omega)$  defined by (1.9). Inequality (3.6) implies that if  $\beta \rightarrow +\infty$ , then  $\lambda_F(\beta, \Omega) \rightarrow \lambda_F(\Omega)$  as is well known.

**Remark 3.2.** We observe that (1.13) gives a geometric inequality involving  $q_F(\beta, \Omega)$ . Indeed, if we consider  $h \equiv 1$  as test function in (1.7), we have

$$q_F(\beta, \Omega) \leq \frac{m}{|\Omega|}, \quad (3.7)$$

that reads in the constant case by (1.8) as

$$q_F(\Omega) \leq \frac{P_F(\Omega)}{|\Omega|},$$

where  $q_F(\Omega)$  is defined in (1.9).

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### References

- [1] Alvino, A., Ferone, V., Lions, P.-L., Trombetti, G.: Convex symmetrization and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **14**, 275–293 (1997)
- [2] Amar, M., Bellettini, G.: A notion of total variation depending on a metric with discontinuous coefficients. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11**, 91–133 (1994)
- [3] Belloni, M., Ferone, V., Kawohl, B.: Isoperimetric inequalities, Wulff shape and related questions for strongly nonlinear elliptic operators. *Z. Angew. Math. Phys.* **54**, 771–783 (2003)
- [4] Berchio, E., Gazzola, F., Mitidieri, E.: Positivity preserving property for a class of biharmonic elliptic problems. *J. Differential Equations* **229**, 1–23 (2006)
- [5] Bucur, D., Ferrero, A., Gazzola, F.: On the first eigenvalue of a fourth order Steklov problem. *Calc. Var. Partial Differential Equations* **35**, 103–131 (2009)
- [6] Buoso, D., Kennedy, J.B.: The Bilaplacian with Robin boundary conditions. [arXiv:2105.11249](https://arxiv.org/abs/2105.11249) (2021)
- [7] Della Pietra, F., di Blasio, G., Gavitone, N.: Sharp estimates on the first Dirichlet eigenvalue of nonlinear elliptic operators via maximum principle. *Adv. Nonlinear Anal.* **9**, 278–291 (2020)
- [8] Della Pietra, F., Gavitone, N.: Faber–Krahn inequality for anisotropic eigenvalue problems with Robin boundary conditions. *Potential Anal.* **41**, 1147–1166 (2014)
- [9] Della Pietra, F., Gavitone, N., Piscitelli, G.: A sharp weighted anisotropic Poincaré inequality for convex domains. *C. R. Acad. Sci. Paris* **355**, 748–752 (2017)
- [10] Della Pietra, F., Gavitone, N.: Anisotropic elliptic problems involving Hardy-type potentials. *J. Math. Anal. Appl.* **397**, 800–813 (2013)
- [11] Ferone, V., Kawohl, B.: Remarks on a Finsler–Laplacian. *Proc. Amer. Math. Soc.* **137**(1), 247–253 (2009)
- [12] Ferrero, A., Gazzola, F., Weth, T.: On a fourth order Steklov eigenvalue problem. *Analysis* **25**, 315–332 (2005)
- [13] Fichera, G.: Su un principio di dualità per talune formule di maggiorazione relative alle equazioni differenziali. *Atti Accad. Naz. Lincei* **19**, 411–418 (1955)



- [14] Gavitone, N., Trani, L.: On the first Robin eigenvalue of a class of anisotropic operators. *Milan J. Math.* **86**, 201–223 (2018)
- [15] Hersch, J., Payne, L.E.: One-dimensional auxiliary problems and a priori bounds. *Abh. Math. Semin. Univ. Hamburg* **36**, 57–65 (1971)
- [16] Kuttler, J.R.: A note on a paper of Sperb. *Z. Angew. Math. Phys.* **24**, 431–434 (1973)
- [17] Kuttler, J.R., Sigillito, V.G.: Inequalities for membrane and Stekloff eigenvalues. *J. Math. Anal. Appl.* **23**, 148–160 (1968)
- [18] Kuttler, J.R., Sigillito, V.G.: An inequality for a Stekloff eigenvalue by the method of defect. *Proc. Amer. Math. Soc.* **20**, 357–360 (1969)
- [19] Payne, L.E.: Isoperimetric inequalities and their applications. *SIAM Rev.* **9**, 453–488 (1967)
- [20] Raulot, S., Savo, A.: Sharp bounds for the first eigenvalue of a fourth-order Steklov problem. *J. Geom. Anal.* **25**, 1602–1619 (2015)
- [21] Sperb, R.P.: Untere und obere Schranken für den tiefsten Eigenwert der elastisch gestützten Membran. *Z. Angew. Math. Phys.* **23**, 231–244 (1972)
- [22] Sperb, R.P.: *Maximum Principles and Applications*. Academic Press, Cambridge (1981)

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