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Psychological Nash equilibria under ambiguity☆

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ABSTRACT

Psychological games aim to represent situations in which players may have belief-dependent motivations. In this setting, utility functions are directly dependent on the entire hierarchy of beliefs of each player. On the other hand, the literature on *strategic ambiguity* in classical games highlights that players may have ambiguous (or imprecise) beliefs about opponents' strategy choices. In this paper, we look at the issue of ambiguity in the framework of simultaneous psychological games by taking into account ambiguous hierarchies of beliefs and study a natural generalization of the psychological Nash equilibrium concept to this framework. We give an existence result for this new concept of equilibrium and provide examples that show that even an infinitesimal amount of ambiguity may alter significantly the equilibria of the game or can work as an equilibrium selection device. Finally, we look at the problem of stability of psychological equilibria with respect to ambiguous trembles on the entire hierarchy of correct beliefs and we provide a limit result that gives conditions so that sequences of psychological equilibria under ambiguous perturbation converge to psychological equilibria of the unperturbed game.

1. Introduction

Psychological games have been introduced to understand how emotions, opinions and intentions of the decision makers can affect a game. In the pioneering paper by Geanakoplos et al. (1989), this goal is tackled by assuming that players may have belief-dependent motivations.[1] More precisely, in a psychological game, each player's payoff depends on his hierarchy of beliefs; that is, it depends not only on what every player does but also on what he thinks every player believes, on what he thinks every player believes the others believe, and so on. Geanakoplos et al. (1989) present an equilibrium concept for this class of games, based on the idea that the entire hierarchy of beliefs of each player must be correct in equilibrium; moreover, they provide an existence result for this notion of equilibrium.

There is another strand of literature that focuses on the issue *strategic ambiguity* in classical strategic form games as it is well known that players may have ambiguous (or imprecise) beliefs about opponents' strategy choices. The classical Nash equilibrium concept is based on two ideas: the first is that each player best responds to the beliefs he has about his opponents' strategy choices; the second is that his beliefs are correct, that is, every player believes with probability 1 that his opponents follow their equilibrium strategies. In the equilibrium concepts for games under strategic ambiguity already studied in the literature, each player best responds to the beliefs he has about his opponents' strategy choices but these beliefs are now ambiguous (or imprecise), that is, they can take the form of a capacity or of a set of probability distributions (see for instance Dow and Werlang (1994), Eichberger and Kelsey (2000), Lehrer (2012), Riedel and Sass (2013), Battigalli et al. (2015) and De Marco and Romaniello (2015) and references therein). In particular, in the concept of *partially specified equilibrium* (Lehrer, 2012), ambiguity stems from the actual strategies chosen and has a specific structure as Lehrer (2012) states: *"players do not have precise knowledge of the mixed strategy chosen by each of the other players. Rather, players know only the probability of some subsets of pure strategies, but do not know the precise subdivision of probabilities within those subsets. They might know also the expected value of*

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[1] The literature on psychological games has increased considerably in the past decades; we recall (Battigalli and Dufwenberg, 2009) for further theoretical findings, Rabin (1993), Battigalli and Dufwenberg (2007) and Attanasi et al. (2010) for some applications, just to quote a few, and Attanasi and Nagel (2008) and Battigalli and Dufwenberg (2020) for surveys on psychological games and references.

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0165-4896/

some variables that depend on the actions taken by others.[2]" From the mathematical point of view, this kind of ambiguous beliefs of a player takes the form of a *set-valued map* (or *correspondence*[3]) that associates with every given profile of opponents' strategies (that represents the one actually played), a set of profiles of opponents' strategies that are indistinguishable (to the player) from the given one. In fact, they provide the same expected value for every random variable in a given set, which, in turn, is exogenous and characterizes the information available to the player. So, the set-valued map describes the first-order beliefs that the player perceives to be consistent with the actual play of his opponents, given the information he has and whatever is the actual play. In De Marco and Romaniello (2013) this approach has been slightly generalized by taking into account arbitrary set-valued maps that allow to study other perturbations of correct beliefs. Finally, it is worth noting that the equilibrium notions under strategic ambiguity give back the Nash equilibrium concept in case of no ambiguity; moreover, some limit results provide conditions that guarantee the convergence of sequences of equilibria of ambiguous games to the equilibria of the unambiguous games when ambiguity converges to zero (see De Marco and Romaniello (2013) and references therein).

It is clear that beliefs about opponents' strategy choices can be regarded as first-order beliefs; from this perspective, the literature on strategic ambiguity substantially looks at games in which first-order beliefs are ambiguous. However, in case payoffs depend explicitly on higher-order beliefs, it is possible that players perceive ambiguity regarding the entire hierarchy of beliefs. For instance, it might be the case that partially specified probabilities (or any kind of partial knowledge) appear directly in the second (or higher) order beliefs; else, in case some player (say John) has partially specified first-order beliefs, then every John's opponent, having correct second-order beliefs about John's first-order beliefs, has a set-valued belief as a natural consequence. As strategic ambiguity has been shown to affect equilibria in classical games, the natural question is in which way ambiguous beliefs affect psychological Nash equilibria. This is the key motivation of the present work; we combine these two relevant aspects of strategic interactions: psychological payoffs and ambiguous beliefs. More precisely, we look at the issue of ambiguity in the framework of psychological games by taking into account ambiguous hierarchies of beliefs and adapting the model of psychological games of Geanakoplos et al. (1989) to the ambiguity framework. The idea is that beliefs might be ambiguous (or imprecise) in equilibrium. More precisely, the function that maps strategy profiles to the *correct* hierarchies of beliefs, that is used in the classical definition of psychological Nash equilibria, is now replaced by a set-valued map (called *ambiguous belief correspondence*), that maps strategy profiles to the subsets of those hierarchies of beliefs that players perceive to be consistent with the corresponding strategy profile. Ambiguous belief correspondences provide a general tool to handle ambiguous hierarchies of beliefs and can cover several specific cases such as partially specified probabilities or perturbations of the correct belief function. This will be shown by different examples.

Following the standard approach, agents are assumed to have a pessimistic attitude towards ambiguity as they are endowed with the classical *maxmin* preferences to compare ambiguous alternatives.[4] From the mathematical point of view, such *maxmin* preferences (see Gilboa and Schmeidler (1989)) correspond to the maximization (with respect to the strategy of the corresponding player) of a marginal function computed along the graph of the ambiguous belief correspondence whose values, in turn, depend on the entire strategy profile. The equilibrium concept we introduce here, called *psychological Nash equilibrium under ambiguity*, appears to be the natural generalization of the psychological Nash equilibrium notion in Geanakoplos et al. (1989). We give an existence result for this equilibrium notion that is naturally based on the continuity properties of the ambiguous belief correspondences. We provide also different examples in order to better illustrate this new concept of equilibrium: they will show that even a little (infinitesimal) amount of ambiguity may alter significantly the equilibria of the game. However, the way in which the set of equilibria changes is not unequivocally determined but depends on the specific model. In fact, a first example shows that the set of equilibria might remain unaltered after the introduction of ambiguity, while, in a second example, the set of psychological equilibria under ambiguity is disjoint from the set of classical psychological equilibria. In a further example, ambiguity produces an equilibrium selection, that is, the set of psychological Nash equilibria under ambiguity is a proper subset of the (classical) psychological Nash equilibrium set. In contrast, in the last example, the set of psychological equilibria enlarges when ambiguity (represented by partially specified probabilities) is introduced.

The issue of equilibrium selection that arises from the example previously mentioned relates this work with another relevant strand of literature that concerns the classical theory of refinements of Nash equilibria.[5] These equilibrium concepts are based on properties of stability with respect to some kind of perturbations: roughly speaking, an equilibrium is stable if a game nearby has an equilibrium nearby. In the seminal paper by Selten (1975), the *trembling hand perfect equilibrium* concept selects equilibria that are stable with respect to the possibility that players believe that their opponents can make (infinitesimal) mistakes playing their equilibrium strategies: each equilibrium strategy should be *close* to the best reply against perturbed expectations about opponents' behavior, if the perturbation is small enough. In Geanakoplos et al. (1989) it is considered a notion of trembling hand perfect psychological equilibrium, that is constructed by perturbing the strategies as in Selten (1975) and keeping the hierarchies of beliefs fixed along the perturbation and equal to those that are correct given the unperturbed strategies. In the perfect equilibrium concept considered in Battigalli and Dufwenberg (2009), strategies are perturbed in the same way but now hierarchies of beliefs are perturbed accordingly, as equilibrium beliefs are determined by the strategies via their consistency condition. In the present paper we look at the problem of stability of psychological equilibria from another perspective as perturbations concern the entire hierarchy of correct beliefs and, as the literature on strategic ambiguity suggests, they (can) take the form of sets of hierarchies. However, our approach has an underlying problem that concerns understanding in which way ambiguous beliefs should converge to correct beliefs so that sequences of psychological equilibria under perturbation converge to psychological equilibria of the unperturbed game. We give a general limit theorem that tackles this issue. Then, we show how to construct selection criteria for classical psychological equilibria based on *ambiguous trembles* of the correct belief function.

The paper is organized as follows: Section 2 presents the model of psychological games under ambiguity and the equilibrium concept. The examples mentioned above are presented in

[2] Moreover, different players may obtain different specifications of the mixed strategies of the others.

[3] The terms *set-valued map* and *correspondence* are synonyms in the game theory literature and, in particular, in this paper.

[4] There are other types of preferences in case of ambiguity that could be potentially applicable to our framework. Our choice has the purpose to keep more simple the understanding of the novelties of our approach that mainly rely on the structure of beliefs.

[5] See, for example, Van Damme (1989) for an extensive survey and rich list of references.

Section 3. Section 4 focuses on the equilibrium existence theorem. The problem of stability of psychological Nash equilibria with respect to ambiguous trembles is studied in Section 5. Section 6 concludes.

2. Model and equilibria

We consider a finite set of players $I = \{1, \dots n\}$, and, for each player i, we denote with $A_i = \{a_i^1, \dots, a_i^{k(i)}\}$ the (finite) pure strategy set of player i. As usual, the set of strategy profiles A is the cartesian product of the strategy sets of each player, that is $A = A_1 \times \dots \times A_n = \prod_{i \in I} A_i$, and $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \times \dots \times A_n = \prod_{j \neq i} A_j$. Let Σ_i be the set of mixed strategies of player i, where each mixed strategy $\sigma_i \in \Sigma_i$ is a nonnegative vector $\sigma_i = (\sigma_i(a_i))_{a_i \in A_i} \in \mathbb{R}_+^{k(i)}$ such that $\sum_{a_i \in A_i} \sigma_i(a_i) = 1$. Denote also with $\Sigma = \prod_{i \in I} \Sigma_i$ and with $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$. We use (σ_i, σ_{-i}) with $\sigma_i \in \Sigma_i$ and $\sigma_{-i} \in \Sigma_{-i}$ to represent $\sigma \in \Sigma$.

2.1. Beliefs structure

Hierarchies of beliefs

Hierarchies of beliefs are constructed as in Geanakoplos et al. (1989). For any set S, $\Delta(S)$ denotes the set of probability measures on S. Then, for every player i, $B_i^1 := \Delta(\Sigma_{-i})$ is the set of the first-order beliefs of player i. Therefore, a first-order belief of player i, $b_i^1 \in B_i^1$, is a probability measure over the product of the other players' mixed strategy sets. The set B_i^1 is endowed with the weak topology and it is a separable and compact metric space because so it is Σ_{-i}.[6]

Higher-order beliefs are defined inductively as follows:

$$B_i^k := \Delta(\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}),$$
$$B_{-i}^k := \prod_{j \neq i} B_j^k, \quad B^k := \prod_{i \in I} B_i^k.$$

Moreover, for every k, B_i^k is compact and can be metrized as a separable metric space as done for B_i^1.[7]

Finally, the set of all *hierarchies of beliefs*[8] of player i is

$$B_i = \prod_{k=1}^{\infty} B_i^k.$$

The space B_i is a countable product of metric spaces so it is also metrizable in such a way that the topology induced by the corresponding metric is equivalent to the product topology. Moreover, under this topology, B_i is compact.

[6] This property is a consequence of the fact that $\Delta(S)$ can be metrized as a separable metric space if and only if S is a separable metric space. In particular, the metric is the Prokhorov distance (see Prokhorov (1956) or theorems 6.2 and 6.5 Chapter 2 in Parthasarathy (2005)). With this metric structure, the space is also compact (see theorem 6.4 Chapter 2 in Parthasarathy (2005)). Sometimes it can be useful to regard B_i^1 as a subset of the linear topological space of finite signed measures V_i^1, defined on the same σ-algebra. The space V_i^1 is endowed with the same weak topology and it is metrized as a separable metric space in the same way.

[7] More generally, the set of probability measures on a countable product of compact and separable metric spaces is still compact and separable (see pag. 46, 61 in Greever (1967)). Moreover, B_i^k can be regarded as compact subset of the linear topological space of finite signed measures V_i^k, endowed with the weak topology.

[8] The notion of hierarchy of beliefs can be found in several other papers (see for example Harsanyi (1967), Mertens and Zamir (1985), Brandenburger and Dekel (1993) and Battigalli and Siniscalchi (1999)).

Coherent beliefs

As pointed out in Geanakoplos et al. (1989), it is a common practice to restrict beliefs of each player i to the subset of *collectively coherent beliefs* $\overline{B}_i \subset B_i$, that is, the set of beliefs of player i in which he is sure that it is common knowledge that beliefs are coherent. Precisely,

Definition 2.1. A belief $b_i = (b_i^1, b_i^2, \dots) \in B_i$ is said to be *coherent* if, for every $k \in \mathbb{N}$, the marginal probability of b_i^{k+1} on $\Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}$ coincides with b_i^k, that is

$$\text{marg}(b_i^{k+1}, \Sigma_{-i} \times B_{-i}^1 \times B_{-i}^2 \times \dots \times B_{-i}^{k-1}) = b_i^k.$$

More precisely, the set of collectively coherent beliefs is defined as follows.[9]

Definition 2.2. Let $\hat{B}_i(0)$ be the set of coherent beliefs of player i. Inductively, for every $\alpha > 0$ let $\hat{B}_i(\alpha)$ be the set

$$\hat{B}_i(\alpha) := \{b_i \in \hat{B}_i(\alpha - 1) \mid \forall k \geq 1,\ b_i^{k+1}(\Sigma_{-i} \times X_{-i}^k(\alpha - 1)) = 1\},$$

where

$$X_j^k(\alpha) := \text{projection of } \hat{B}_j(\alpha) \text{ into } \prod_{l=1}^{k} B_j^l, \qquad X_{-i}^k(\alpha) := \prod_{j \neq i} X_j^k(\alpha).$$

Then, the set of *collectively coherent beliefs* $\overline{B}_i$ is defined by

$$\overline{B}_i = \bigcap_{\alpha > 0} \hat{B}_i(\alpha).$$

The set $\overline{B}_i$ is compact (see Battigalli and Dufwenberg (2009)). However, we give a self-contained proof below.

Lemma 2.3. *The set of collectively coherent beliefs $\overline{B}_i$ is a compact subset of B_i for every i.*

Proof. Given that the weak topology is Hausdorff and that intersection of compact sets in an Hausdorff space is compact, it is sufficient to prove that each $\hat{B}_i(\alpha)$ is compact, which results in proving that $\hat{B}_i(\alpha)$ is closed.[10]

We proceed by induction on α. Consider $\hat{B}_i(0)$ and let $\{b_{i,\nu}\}_{\nu \in \mathbb{N}} \subset \hat{B}_i(0)$ be a sequence converging in the product topology to a point $\tilde{b}_i$. Since B_i is compact then $\tilde{b}_i \in B_i$. Therefore, for every $k \geqslant 1$, the sequence $\{b_{i,\nu}^k\}_{\nu \in \mathbb{N}}$ weakly converges to $\tilde{b}_i^k \in B_i^k$. For every $k \geq 1$, we have to check that

$$\text{marg}(\tilde{b}_i^{k+1}, \Sigma_{-i} \times B_{-1}^1 \times \dots \times B_{-i}^{k-1}) = \tilde{b}_i^k. \tag{1}$$

Now, for every measurable $A \subset \Sigma_{-i} \times B_{-1}^1 \times \dots \times B_{-i}^{k-1}$, weak convergence implies that

$$\text{marg}(\tilde{b}_i^{k+1}, \Sigma_{-i} \times B_{-1}^1 \times \dots \times B_{-i}^{k-1})(A)$$
$$= \lim_{\nu \to \infty} \text{marg}(b_{i,\nu}^{k+1}, \Sigma_{-i} \times B_{-1}^1 \times \dots \times B_{-i}^{k-1})(A) = b_i^k(A).$$

Hence (1) holds and $\hat{B}_i(0)$ is compact in B_i.

By induction, suppose that $\hat{B}_i(\alpha)$ is compact. Consider a sequence $\{b_{i,\nu}\}_{\nu \in \mathbb{N}} \subset \hat{B}_i(\alpha + 1)$ converging in the product topology to $\tilde{b}_i$. Since $\hat{B}_i(\alpha+1) \subset \hat{B}_i(\alpha)$ and $\hat{B}_i(\alpha)$ is compact, then $\tilde{b}_i \in \hat{B}_i(\alpha)$. Moreover, by weak convergence we have

$$\tilde{b}_i^{k+1}(\Sigma_{-i} \times X_{-i}^k(\alpha)) = \lim_{\nu \to \infty} b_{i,\nu}^{k+1}(\Sigma_{-i} \times X_{-i}^k(\alpha)) = 1.$$

Therefore, $\tilde{b}_i \in \hat{B}_i(\alpha + 1)$ and $\hat{B}_i(\alpha + 1)$ is compact. □

In the remainder of the paper, with an abuse of notation we will denote with $\overline{B}_i$ the set of collectively coherent beliefs or any of its compact subsets.

[9] You can find this construction also in Geanakoplos et al. (1989).

[10] In fact $\hat{B}_i(\alpha)$ is a subset of the compact space B_i for every $\alpha \geq 0$.

Ambiguous hierarchies

Differently from Geanakoplos et al. (1989), where the beliefs of a player i are given by the elements $b_i \in \overline{B}_i$, we generalize the model and allow beliefs to be compact subsets[11] $K_i \subseteq \overline{B}_i$. We denote with $\mathscr{K}_i$ the set of all compact subsets of $\overline{B}_i$. This choice enables to consider the ambiguity players come up against during the game, due to the uncertainty about other players' actions and beliefs. The interpretation is similar to the classical one of games under strategic ambiguity: the agent does not have a precise belief b_i but knows that the belief can be any $b_i \in K_i$. Trivially, if K_i is a singleton, then the belief is not ambiguous, leading the theory back to the standard case.

Remark 2.4. We introduced ambiguity at the end of the process, representing ambiguous beliefs as compact subsets of the product space $\overline{B}_i$, but there is another natural approach to represent ambiguity on hierarchies of beliefs as shown in Ahn (2007).[12] Roughly speaking, Ahn's approach is to introduce ambiguity at each level of the hierarchy of beliefs, and then to take the product. However, Ahn himself proved the universality of the construction,[13] i.e. our approach is actually equivalent to Ahn's one when coherency of beliefs is common knowledge.

2.2. Game and equilibria

Following the model in Geanakoplos et al. (1989), each agent i is endowed with an utility function

$$u_i : \overline{B}_i \times \Sigma \to \mathbb{R}, \tag{2}$$

depending not only on the mixed strategy profile but also on agent's beliefs: $u_i(b_i, \sigma)$ represents the payoff to player i if he believed b_i and the strategy profile σ is actually played. Indeed, fixed b_i, $u_i(b_i, \cdot)$ can be (but not necessarily) the classical expected utility function as it is assumed in Geanakoplos et al. (1989). As agents face set-valued beliefs $K_i \in \mathscr{K}_i$, they have a set-valued payoff $\{u_i(b_i, \sigma)\}_{b_i \in K_i}$ for every given belief $K_i \in \mathscr{K}_i$ and strategy profile $\sigma \in \Sigma$. There are several ways in which agents' ambiguity might be solved depending on the agents' attitudes towards ambiguity.[14] In this paper we focus on the so called *maxmin* preferences (see Gilboa and Schmeidler, 1989): each agent i has an utility function $U_i : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined by

$$U_i\,(K_i, \sigma) = \inf_{b_i \in K_i} u_i(b_i, \sigma) \quad \forall (K_i, \sigma) \in \mathscr{K}_i \times \Sigma. \tag{3}$$

Remark 2.5. In formula (3), we are implicitly assuming that the definition of U_i is well posed. This is obviously satisfied if the function u_i is continuous; in that case it immediately follows that $\inf u_i(b_i, \sigma) = \min u_i(b_i, \sigma)$.

Now, it is possible to define the game.

Definition 2.6. A *normal form psychological game under ambiguity* is defined by

$$G = \{A_1, \ldots, A_n, U_1, \ldots, U_n\}$$

where the utility functions U_i are defined as in formula (3) for every $i \in N$.

In the classical models of strategic ambiguity, players have vague beliefs about their opponents' behavior and these beliefs might depend on the actual strategy; for instance, this is the case when players have partial knowledge of the strategies played by their opponents. In particular, when ambiguity is expressed by multiple probability distributions, each agent's beliefs take the form of a set-valued map (or correspondence) from the strategy profiles set to the set of probability distributions over opponents' strategies (see Lehrer (2012), Battigalli et al. (2015) and De Marco and Romaniello (2012)). In this paper we generalize this approach to hierarchies of beliefs: we assume that each agent i is endowed with a set-valued map $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$, (that we call *belief correspondence* of player i), where each image $\gamma_i(\sigma)$ is a not empty and compact set, i.e.:

$$\emptyset \neq \gamma_i(\sigma) \in \mathscr{K}_i \quad \forall \sigma \in \Sigma.$$

Each subset $\gamma_i(\sigma) \subseteq \overline{B}_i$ provides the set of hierarchies of beliefs that player i perceives to be consistent given the strategy profile σ.

Ambiguous belief correspondences provide a general tool to handle ambiguous hierarchies of beliefs as they include several specific models. For instance, Section 2.2.2 shows the embedding of games under partially specified probabilities into our framework. Differently, in some examples in Section 3, the specific belief correspondences taken into account represent a set-valued perturbation of the correct belief function used in psychological games without ambiguity (see Section 2.2.1 for more details on correct beliefs).

The equilibrium concept for psychological games under ambiguity is given below:

Definition 2.7. A *psychological Nash equilibrium under ambiguity* of the game G with belief correspondences $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a pair (K^*, σ^*), where $K^* = (K_1^*, \ldots, K_n^*)$ with $K_i^* \subseteq \overline{B}_i$ and $\sigma^* \in \Sigma$ such that, for every player i:

(i) $K_i^* = \gamma_i(\sigma^*)$;
(ii) $U_i(K_i^*, \sigma^*) \geqslant U_i(K_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

In this case, we can also say that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity.

We point out that the definition above captures, in a natural way, the main features of the classical equilibrium notion since condition (ii) requires that the equilibrium strategy of each player is optimal given his beliefs and condition (i) requires that beliefs must be consistent with the equilibrium strategy profile.

Similarly to Geanakoplos et al. (1989), psychological equilibria under ambiguity have a characterization as Nash equilibria. Let $w_i : \Sigma \times \Sigma \to \mathbb{R}$ be the *summary utility function* defined by

$$w_i(\sigma, \tau) = U_i(\gamma_i(\sigma), \tau) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau) \quad \forall (\sigma, \tau) \in \Sigma \times \Sigma. \tag{4}$$

Then the *summary form* of the game G is $\hat{G} := (A_1, \ldots, A_n, w_1, \ldots, w_n)$. Now, it immediately follows from the definition that

Lemma 2.8. *The profile $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity if and only if, for every player i,*

$$w_i(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geqslant w_i(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i. \tag{5}$$

Remark 2.9. Condition (5) above means that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium if and only if σ^* is a mixed strategy equilibrium of a classical strategic form game where utility functions are specified by the strategy profile σ^*, that is, the utility functions are $\sigma \in \Sigma \to w_i(\sigma^*, \sigma) \in \mathbb{R}$, for every player i.

[11] The assumption of compactness of beliefs is rather standard (see for instance Ahn (2007)) as it keeps the problem much more tractable from the mathematical point of view. Nevertheless it seems that non-compact beliefs might be realistic in some specific situation.

[12] Similar results about the universality of unambiguous hierarchies of beliefs can be found in Mariotti et al. (2005).

[13] Details are rather technical, we refer to Ahn's paper and in particular to Proposition 4 and the diagram in Figure 1 therein.

[14] See Gilboa and Marinacci (2013) for a survey and many references.

2.2.1. Links with psychological games without ambiguity

Definition 2.7 is a natural generalization of the concept of psychological Nash equilibrium defined in Geanakoplos et al. (1989). Indeed, recall that in their setting a *normal form psychological game* is determined by

$$G^{GPS} = \{A_1, \dots, A_n, u_1, \dots, u_n\},$$

where the utility functions u_i are defined as in (2) for every $i \in I$. Their notion of psychological Nash equilibrium makes use of the concept of correct beliefs that are provided by the functions $\beta_i : \Sigma \to \overline{B}_i$ for every player i. For every $\sigma \in \Sigma$, $\beta_i(\sigma) = (b_i^1, b_i^2, \dots, b_i^k, \dots)$ is the hierarchy of beliefs in which player i believes (with probability 1) that his opponents follow σ_{-i}, that each opponent $j \neq i$ believes that his opponents follow σ_{-j}, and so on. More precisely, denote with $\delta_{\{x\}}$ the Dirac measure at the point x. Then, it follows that:

$$b_i^1 = \delta_{\{\sigma_{-i}\}}$$
$$marg(b_i^2, B_{-i}^1) = \delta_{\left\{(b_j^1)_{j\neq i}\right\}}$$
$$marg(b_i^3, B_{-i}^2) = \delta_{\left\{(b_j^2)_{j\neq i}\right\}}$$
$$\dots\dots\dots$$
$$marg(b_i^{k+1}, B_{-i}^k) = \delta_{\left\{(b_j^k)_{j\neq i}\right\}}$$
$$\dots\dots\dots$$

Therefore, a *psychological Nash equilibrium* of the game G^{GPS} is defined as a pair (b^*, σ^*) where $b^* = (b_1^*, \dots, b_n^*)$ with $b_i^* \in \overline{B}_i$ and $\sigma^* \in \Sigma$ such that, for every player i,

(i) $b_i^* = \beta_i(\sigma^*)$;
(ii) $u_i(b_i^*, \sigma^*) \geqslant u_i(b_i^*, (\sigma_i, \sigma_{-i}^*))$ for every $\sigma_i \in \Sigma_i$.

We can also say that $(\beta(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium of the game G^{GPS}.

The GPS summary utility function of player i, that here we denote with w_i^{GPS}, takes the form

$$w_i^{GPS}(\sigma, \tau) = u_i(\beta_i(\sigma), \tau) \quad \forall(\sigma, \tau) \in \Sigma \times \Sigma. \tag{6}$$

Hence, $(\beta(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium if and only if

$$w_i^{GPS}(\sigma^*, (\sigma_i^*, \sigma_{-i}^*)) \geqslant w_i^{GPS}(\sigma^*, (y_i, \sigma_{-i}^*)) \quad \forall y_i \in \Sigma_i, \forall i \in N. \tag{7}$$

Consequently, if we replace, in Definition 2.7, the set-valued map γ_i with the (single-valued) map β_i as defined in Geanakoplos et al. (1989), then we get the classical definition of psychological Nash equilibrium.

2.2.2. Links with strategic ambiguity

With an abuse of notation, we can roughly summarize a game under strategic ambiguity as a game in which the generic player i has an utility function u_i that depends only on his own strategy σ_i and on his first-order belief b_i^1, that is

$$u_i : B_i^1 \times \Sigma_i \to \mathbb{R}.$$

Moreover, player i is endowed with a set-valued map $S_i : \Sigma_{-i} \rightsquigarrow B_i^1$ where $S_i(\sigma_{-i})$ is the set of first-order beliefs that player i perceives to be consistent with σ_{-i}. Assuming that the player has maxmin preferences, his utility becomes

$$U_i : \Sigma \to \mathbb{R}, \text{ where } U_i(\sigma) = \min_{b_i^1 \in S_i(\sigma_{-i})} u_i(b_i^1, \sigma_i).$$

Then, the equilibria under strategic ambiguity can be found applying the standard Nash equilibrium concept to the game with utilities $U_1, \dots, U_n$.

In the particular case of *partially specified probabilities* (see Lehrer (2012)), the model is defined as follows: for every player i and for every $j \neq i$, the set $\mathcal{Y}_{ij}$ represents a set of random variables, defined over the set of j's pure strategies A_j, whose expectations are known to player i, i.e., given a mixed strategy σ_j, player i does not know σ_j in its entirety but knows only the expectations $E_{\sigma_j}[Y]$ (with respect to σ_j) of each $Y \in \mathcal{Y}_{ij}$. Therefore, player i's beliefs about player j's choice are given by all the strategies τ_j that satisfy

$$E_{\tau_j}[Y] = E_{\sigma_j}[Y], \ \forall Y \in \mathcal{Y}_{ij}.$$

Finally, Lehrer's partially specified probabilities are defined by the set-valued maps $K_i : \Sigma_{-i} \rightsquigarrow \Sigma_{-i}$ such that

$$K_i(\sigma_{-i}) = \left\{\tau_{-i} \in \Sigma_{-i} \;\middle|\; E_{\tau_j}[Y] = E_{\sigma_j}[Y] \quad \text{for all } Y \in \mathcal{Y}_{ij} \text{ and for all } j \neq i\right\}. \tag{8}$$

If we consider the set-valued maps $S_i : \Sigma_{-i} \rightsquigarrow B_i^1$ defined as

$$S_i(\sigma_{-i}) = \left\{\delta_{\{\tau_{-i}\}} \mid \tau_{-i} \in K_i(\sigma_{-i})\right\}, \tag{9}$$

with $\delta_{\{x\}}$ denoting the Dirac measure at the point x, we obtain the embedding of the Lehrer's model into the GPS' model.[15] Consequently, it could be possible to obtain a Nash equilibrium under strategic ambiguity described by set-valued maps $S_1, \dots, S_n$ as a psychological Nash equilibrium under ambiguity, considering the beliefs correspondence $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$, for all i, defined by

$$\gamma_i(\sigma) = \left\{b_i \in \overline{B}_i \mid b_i^1 \in S_i(\sigma_{-i})\right\}. \tag{10}$$

It is worth noting that in the γ_i defined above there are no constraints on higher-order beliefs since in the classical problems of strategic ambiguity these beliefs play no role. On the contrary, in psychological game theory higher-order beliefs enter the domain of the utility functions and may be affected by strategic ambiguity. The question is which consistency conditions should be imposed on higher-order beliefs in this case. In absence of further sources of ambiguity, one can consider the case in which higher-order beliefs are correct, conditionally on first-order beliefs. For example, if utility function depend just on first-order and second-order beliefs and strategic ambiguity is described by the set-valued maps $S_1, \dots, S_n$ in (9), then correct beliefs should be given by

$$\gamma_i(\sigma) = \left\{b_i \in \overline{B}_i \mid b_i^1 \in S_i(\sigma_{-i}),\ marg(b_i^2, B_{-i}^1) = \delta_{\left\{(b_j^1)_{j\neq i}\right\}} \text{ with } b_j^1 \in S_j(\sigma_{-j}) \text{ for all } j \neq i\right\}. \tag{11}$$

In an iterative way it follows the general case in which utilities depend on higher-order beliefs.

3. Examples

In this section we provide different examples that show some possible effects of ambiguity on psychological equilibria. In the first four examples the idea is to consider ambiguity as a perturbation. More precisely, we look at belief correspondences that provide a neighborhood of the correct beliefs $\beta(\sigma)$ for every σ and study the corresponding psychological equilibria. The last example looks at a different kind of ambiguous beliefs arising from *partially specified probabilities*.

In Example 3.1 we look at the original *Bravery Game* presented in Geanakoplos et al. (1989) and show that when beliefs are

[15] In Lehrer (2012), first-order beliefs are intended as probability distributions over pure strategies, while in GPS' approach, which we aim to generalize, first-order beliefs are given by probability distributions over mixed strategies. The two approaches can be reconciled considering the Dirac probability measures concentrated to the single strategy profiles.

perturbed by a small amount of ambiguity, the two equilibria in pure strategies (that we find without ambiguity) survive to ambiguity, while the unique equilibrium in completely mixed strategies is slightly perturbed but the perturbation converges to zero as ambiguity converges to zero. As ambiguity increases, the set of equilibria shrinks progressively to two equilibria and then to one equilibrium when ambiguity is sufficiently large.

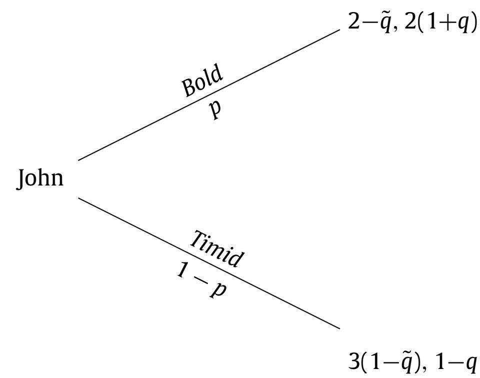
Similar results are obtained in Example 3.2 where the Bravery Game is slightly modified in the payoffs but the main behavioral assumptions on players remain substantially unaltered. In this case, all the three equilibria that we get in case of no ambiguity survive to ambiguity when the perturbation is sufficiently small. Also in this case, the set of equilibria shrinks progressively to two equilibria and then to one equilibrium when ambiguity increases.

A further variation of the Bravery Game is given in Example 3.3. Here the behavioral assumption on players is rather changed (namely in John's preferences) so that the story behind the game is different. We get a unique psychological equilibrium in the unambiguous case. In case of ambiguity, we get a unique equilibrium that differs from the unique equilibrium in the unambiguous case; however, when ambiguity converges to zero, the equilibrium of the ambiguous game converges to the equilibrium of the unambiguous one.

These examples show that the set of equilibria in presence of ambiguity can be disjoint from the set of equilibria in the unambiguous case, or, ambiguity plays the role of equilibrium selector. This seems a bit surprising as, in classical games with material payoffs, the presence of strategic ambiguity often enlarges the set of equilibria. On the other hand, in the examples previously quoted, the set of equilibria is refined only when the amount of ambiguity is large enough. In contrast, Example 3.4 shows that even just an infinitesimal amount of ambiguity may work as an equilibrium selector. More precisely, we consider a variation of the *Confidence Game* presented in Geanakoplos et al. (1989) and it turns out that the psychological game without ambiguity has two equilibria while an infinitesimal amount of ambiguity destroys one (and only one) of them.

The last Example 3.5 shows a completely different effect of ambiguity as the set of equilibria is enlarged by an entire interval when ambiguity is introduced. Moreover, the structure of ambiguity is different as it is built upon partially specified probabilities at the level of first-order beliefs, while, higher-order beliefs are correct conditionally on first-order ones.

Example 3.1. We consider the *Bravery Game* presented in figure 2 in Geanakoplos et al. (1989) in order to illustrate the effects of (an infinitesimal amount of) ambiguity on the set of equilibria. For the sake of completeness we recall the story as presented originally: Player 1 (John) must publicly take a decision, and is concerned about what friend Player 2 (Anne) will think about him. He can take a bold decision, which exposes him to the possibility of danger, or a timid decision, so John's pure strategy space is $A_1 = \{Bold, Timid\}$. Anne is inactive (that is, Anne does not choose any action in the game). John chooses *Bold* with probability p and *Timid* with probability $1-p$. His payoff depends not only on what he does but also on what he believes Anne thinks of his temper (that is, on what he believes she thinks he will do). In this game, it is considered the case in which John cares only about the expectation $\tilde{q}$ of his belief about the expectation q of Anne's first-order belief. Moreover, John prefers to be timid rather than bold, unless he thinks that Anne expects him to be bold, in which case he prefers not to disappoint her. Anne prefers to think of her friend as bold; in addition, it is good for her if he is bold. The game and payoffs are described below:



Note that, as Anne is a non-active player, the mixed strategy profile is identified just by John's mixed strategy p. With an abuse of notation, in this example the correct belief functions simply map the strategy p to the expectations of correct beliefs: more precisely, $\beta_2(p) = p$ tells that the expectation of Anne's first-order correct beliefs about John's strategy p must be equal to p and $\beta_1(p) = p$ tells that the expectation of John's correct second-order beliefs about Anne's expectation of correct first-order belief about John's strategy p must be equal to p as well.

Now, the expected utility of John depends only on $\tilde{q}$ and p and it takes the following form:

$$u_1(\tilde{q}, p) = p(2 - \tilde{q}) + 3(1 - p)(1 - \tilde{q}) = p(2\tilde{q} - 1) + 3(1 - \tilde{q}).$$

Firstly, let us look at psychological Nash equilibria (without ambiguity). If we replace in u_1 the generic $\tilde{q}$ with the correct belief $\beta_1(p)$, we get the function w_1^{GPS}. More precisely, for every pair of John's mixed strategies (p, y) we get

$$\begin{aligned} w_1^{GPS}(p, y) &= u_1(\beta_1(p), y) \\ &= y(2p - 1) + 3(1 - p) \quad \forall p \in [0, 1] \text{ and } \forall y \in [0, 1]. \end{aligned}$$

Recall that p gives a psychological Nash equilibrium if and only if

$$w_1^{GPS}(p, p) \geqslant w_1^{GPS}(p, y) \quad \forall y \in [0, 1].$$

Now, we immediately get that

$$\begin{aligned} w_1^{GPS}(p, 0) > w_1^{GPS}(p, y) \ \forall y \in]0, 1], \ \text{if}\, p < 1/2; \\ w_1^{GPS}(p, 1) > w_1^{GPS}(p, y) \ \forall y \in [0, 1[, \ \text{if}\, p > 1/2; \\ w_1^{GPS}(p, y) = 3/2 \ \forall y \in]0, 1], \ \text{if}\, p = 1/2. \end{aligned}$$

Consequently, this game has three psychological Nash equilibria:

- $p = 1 = \tilde{q} = q$: John chooses to be *Bold*;
- $p = 0 = \tilde{q} = q$: John chooses to be *Timid*;
- $p = 1/2 = \tilde{q} = q$: John randomizes with probability $p = 1/2$.

Now, we introduce a specific form of ambiguous beliefs in the game. Suppose that John's belief is not a singleton anymore, but it is an interval: $\gamma_1^{\varepsilon}(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$ with $\varepsilon > 0$ is the set-valued map that describes John's (second-order) beliefs.

In order to compute John's summary utility function, we firstly compute, for every pair of John's mixed strategies (p, y) the following:

$$\arg\min_{\tilde{q} \in \gamma_1^{\varepsilon}(p)} u_1(\tilde{q}, y) = \left\{ \tilde{q}' \in [0, 1] \;\middle|\; u_1(\tilde{q}', y) = \min_{\tilde{q} \in \gamma_1^{\varepsilon}(p)} u_1(\tilde{q}, y) \right\}.$$

We get

$$\begin{aligned} \arg\min_{\tilde{q} \in \gamma_1^{\varepsilon}(p)} u_1(\tilde{q}, y) &= \arg\min_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}(2y - 3) + 3 - y] \\ &= \min\{p + \varepsilon, 1\}, \quad \forall y \in [0, 1]. \end{aligned}$$

Denote with $p^+ = \min\{p+\varepsilon, 1\}$. Therefore, for every pair of John's mixed strategies (p, y),

$$w_1(p, y) = U_1(\gamma_1^\varepsilon(p), y) = \min_{\tilde{q}\in\gamma_1^\varepsilon(p)} [\tilde{q}(2y-3)+3-y]$$
$$= p^+(2y-3)+3-y = y(2p^+-1)+3(1-p^+).$$

Recall that p gives a psychological Nash equilibrium under ambiguity if and only if

$$w_1(p, p) \geqslant w_1(p, y) \quad \forall y \in [0, 1].$$

Now, we get that

$$w_1(p, 0) > w_1(p, y)\ \forall y \in]0, 1],\ \text{if}\ p^+ < 1/2;$$
$$w_1(p, 1) > w_1(p, y)\ \forall y \in [0, 1[,\ \text{if}\ p^+ > 1/2;$$
$$w_1(p, y) = 3/2\ \forall y \in]0, 1],\ \text{if}\ p^+ = 1/2.$$

Since $p^+ = 1/2 \iff p = (1-2\varepsilon)/2$ then

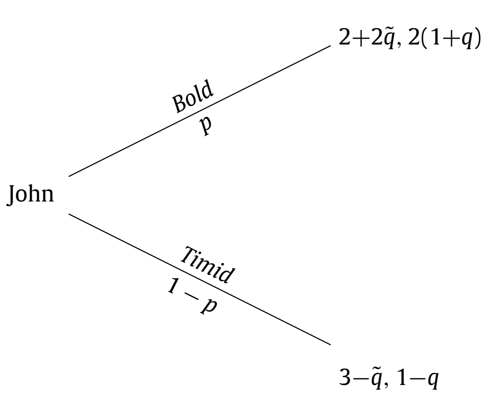
$$w_1(p, 0) > w_1(p, y)\ \forall y \in]0, 1],\ \text{if}\ p < (1-2\varepsilon)/2; \tag{12}$$
$$w_1(p, 1) > w_1(p, y)\ \forall y \in [0, 1[,\ \text{if}\ p > (1-2\varepsilon)/2; \tag{13}$$
$$w_1(p, y) = 3/2\ \forall y \in]0, 1],\ \text{if}\ p = (1-2\varepsilon)/2. \tag{14}$$

Now, (12) can give only the equilibrium corresponding to $p = 0$, (13) can give only the equilibrium corresponding to $p = 1$ and finally (14) can give only the equilibrium corresponding to $p = (1-2\varepsilon)/2$. The existence of these equilibria depends on ε.

More precisely, if $0 < \varepsilon < 1/2$, we get three equilibria. The equilibrium corresponding to $p = 0$ and the one corresponding to $p = 1$ are those surviving to the presence of ambiguity. The third equilibrium, corresponding to $p = (1-2\varepsilon)/2$, converges to $p = 1/2$ as ambiguity (that is ε) converges to 0.

If $\varepsilon = 1/2$, we get only the equilibria $p = 0$ and $p = 1$. For $\varepsilon > 1/2$, there is only one equilibrium corresponding to $p = 1$. Summarizing, when ambiguity is sufficiently small, the unique effect is a small perturbation of the equilibrium in purely mixed strategies. As ambiguity increases, the set of equilibria shrinks to a unique equilibrium $p = 1$.

Example 3.2. We consider a slight variation of the Bravery game presented above. In this new example only the payoffs of John are (slightly) modified but, as in the previous example, John prefers to be timid rather than bold, unless he thinks that Anne expects him to be bold. The game and payoffs are described below:



The expected utility of John[16] takes the following form for every belief $\tilde{q}$ and mixed strategy p:

$$u_1(\tilde{q}, p) = p(2+2\tilde{q}) + (1-p)(3-\tilde{q}) = \tilde{q}(3p-1)+3-p.$$

[16] Again, Anne's expected utility does not play any role in equilibrium so it is superfluous.

The correct belief functions are defined as in the previous example, that is $\beta_1(p) = p$ and $\beta_2(p) = p$. Therefore John's GPS summary utility function w_1^{GPS} is given, for every pair of John's mixed strategies (p, y), by

$$w_1^{GPS}(p, y) = u_1(\beta_1(p), y) = p(3y-1)+3-y$$
$$= y(3p-1)+3-p \quad \forall p \in [0, 1] \text{ and } \forall y \in [0, 1].$$

Recall that p gives a psychological Nash equilibrium if and only if

$$w_1^{GPS}(p, p) \geqslant w_1^{GPS}(p, y) \quad \forall y \in [0, 1].$$

Now, we immediately get that

$$w_1^{GPS}(p, 0) > w_1^{GPS}(p, y)\ \forall y \in]0, 1],\ \text{if}\ p < 1/3;$$
$$w_1^{GPS}(p, 1) > w_1^{GPS}(p, y)\ \forall y \in [0, 1[,\ \text{if}\ p > 1/3;$$
$$w_1^{GPS}(p, y) = 8/3\ \forall y \in]0, 1],\ \text{if}\ p = 1/3.$$

This game has three psychological Nash equilibria:

- $p = 1 = \tilde{q} = q$: John chooses to be *Bold*;
- $p = 0 = \tilde{q} = q$: John chooses to be *Timid*;
- $p = 1/3 = \tilde{q} = q$: John randomizes with probability $p = 1/3$.

Now, we consider the same ambiguity of the previous example: $\gamma_1^\varepsilon(p) = [p-\varepsilon, p+\varepsilon] \cap [0, 1]$ with $\varepsilon > 0$ is the set-valued function that describes John's (second-order) beliefs. We get

$$\arg\min_{\tilde{q}\in\gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \left\{\tilde{q}' \in [0, 1] \;\middle|\; u_1(\tilde{q}', y) = \min_{\tilde{q}\in\gamma_1^\varepsilon(p)} u_1(\tilde{q}, y)\right\}.$$

Then

$$\arg\min_{\tilde{q}\in\gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg\min_{\tilde{q}\in[p-\varepsilon,p+\varepsilon]\cap[0,1]} [\tilde{q}(3y-1)+3-y]$$
$$= \begin{cases} \min\{p+\varepsilon, 1\} & \text{if } y \in [0, 1/3[, \\ \gamma_1^\varepsilon(p) & \text{if } y = 1/3, \\ \max\{p-\varepsilon, 0\} & \text{if } y \in]1/3, 1]. \end{cases}$$

Denote with $p^- = \max\{p-\varepsilon, 0\}$ and $p^+ = \min\{p+\varepsilon, 1\}$. Therefore, for every pair of John's mixed strategies (p, y), we have:

$$w_1(p, y) = U_1(\gamma_1^\varepsilon(p), y) = \min_{\tilde{q}\in\gamma_1^\varepsilon(p)} [\tilde{q}(3y-1)+3-y] =$$
$$\begin{cases} p^+(3y-1)+3-y = y(3p^+-1)+3-p^+ & \text{if } y \in [0, 1/3[, \\ 3-y = \frac{8}{3} & \text{if } y = 1/3, \\ p^-(3y-1)+3-y = y(3p^--1)+3-p^- & \text{if } y \in]1/3, 1]. \end{cases}$$

Recall that p gives a psychological Nash equilibrium under ambiguity if and only if

$$w_1(p, p) \geqslant w_1(p, y) \quad \forall y \in [0, 1].$$

Now, three cases are possible:

(i) If p is such that $1/3 < p^- < p^+$, (that is $p > 1/3+\varepsilon$), then $w_1(p, y)$ is strictly increasing in $[0, 1]$ and attains its maximum at $y = 1$. So, in this case, there is only one equilibrium corresponding to $p = 1$.

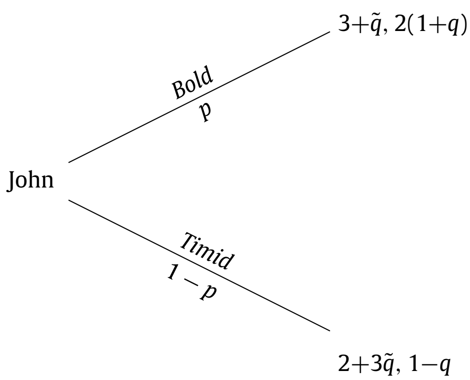
(ii) If p is such that $p^- < p^+ < 1/3$, (that is $p < 1/3-\varepsilon$), then $w_1(p, y)$ is strictly decreasing in $[0, 1]$ and attains its maximum at $y = 0$. So, in this case, the unique equilibrium corresponds to $p = 0$.

(iii) If p is such that $p^- \leqslant 1/3 \leqslant p^+$, (that is $p \in [1/3-\varepsilon, 1/3+\varepsilon]$), then $w_1(p, y)$ is strictly increasing in $[0, 1/3]$ and strictly decreasing in $[1/3, 1]$. Therefore $y = 1/3$ is the maximum point and there is only one equilibrium corresponding to $p = 1/3$.

So, if $0 < \varepsilon < 1/3$, we get again three equilibria. More precisely, all the three psychological equilibria corresponding to $p = 0$, $p = 1$ and $p = 1/3$ survive to the presence of ambiguity. If $1/3 \leqslant \varepsilon < 2/3$ only $p = 1$ and $p = 1/3$ survive while $p = 0$ is destroyed by the presence of ambiguity. If $\varepsilon \geqslant 2/3$, then only the equilibrium corresponding to $p = 1/3$ survives to ambiguity. Note in particular that for $\varepsilon \geqslant 1$, the set-valued map that describes John's (second-order) beliefs is $\gamma_1^\varepsilon(p) = [0, 1]$ for every p, representing an extreme form of ambiguity that can be interpreted as full ignorance.

Summarizing, when ambiguity is sufficiently small, there are no effects on equilibria. As shown by the previous example, as ambiguity increases, the set of equilibria shrinks progressively to two equilibria and then to one equilibrium when ambiguity is sufficiently large. The unique equilibrium in this case is in completely mixed strategies.

Example 3.3. We consider another variation of the Bravery Game. Everything is unaltered except for John's payoff. Now, we consider the case in which John prefers to be bold rather than timid, unless he believes that is more likely that Anne expects him to be bold.



Again, Anne's expected utility does not play any role. For every belief $\tilde{q}$ and mixed strategy p, the expected utility of John is:

$$u_1(\tilde{q}, p) = p(3 + \tilde{q}) + (1 - p)(2 + 3\tilde{q}) = p(1 - 2\tilde{q}) + 3\tilde{q} + 2.$$

The correct belief functions are defined as in the previous examples, that is $\beta_1(p) = p$ and $\beta_2(p) = p$. Firstly, let us look at psychological Nash equilibria (without ambiguity). Recall that p gives a psychological Nash equilibrium if and only if

$$w_1^{GPS}(p, p) \geqslant w_1^{GPS}(p, y) \quad \forall y \in [0, 1],$$

where

$$w_1^{GPS}(p, y) = u_1(\beta_1(p), y) = y(1 - 2p) + 3p + 2.$$

Now, we immediately get that

$$w_1^{GPS}(p, 1) > w_1^{GPS}(p, p) \ \forall p < 1/2,$$
$$w_1^{GPS}(p, 0) > w_1^{GPS}(p, p) \ \forall p > 1/2,$$
$$w_1^{GPS}(p, y) = 7/2 \quad \forall y \in [0, 1] \text{ if } p = 1/2.$$

Consequently, the unique psychological Nash equilibrium corresponds to $p = 1/2$.

We introduce ambiguity as done in the previous examples: $\gamma_1^\varepsilon(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$, with $\varepsilon > 0$, is the set-valued map that describes John's (second-order) beliefs. For every pair of John's mixed strategies (p, y) it follows that

$$\arg\min_{\tilde{q}\in\gamma_1^\varepsilon(p)} u_1(\tilde{q}, y) = \arg\min_{\tilde{q}\in[p-\varepsilon,p+\varepsilon]\cap[0,1]} [\tilde{q}(3 - 2y) + y + 2] = \max\{p - \varepsilon, 0\}.$$

Denote again with $p^- = \max\{p - \varepsilon, 0\}$. Therefore, for every pair of John's mixed strategies (p, y) we have that

$$w_1(p, y) = U(\gamma_1^\varepsilon(p), y) = \min_{\tilde{q}\in\gamma_1^\varepsilon(p)} [\tilde{q}(3 - 2y) + y + 2] = p^-(3 - 2y) + y + 2 = y(1 - 2p^-) + 3p^- + 2.$$

Now, three cases are possible:

(i) If p is such that $p^- < 1/2$ (that is $p < 1/2+\varepsilon$), then $w_1(p, y)$ is strictly increasing in $[0, 1]$ and attains its maximum at $y = 1$.

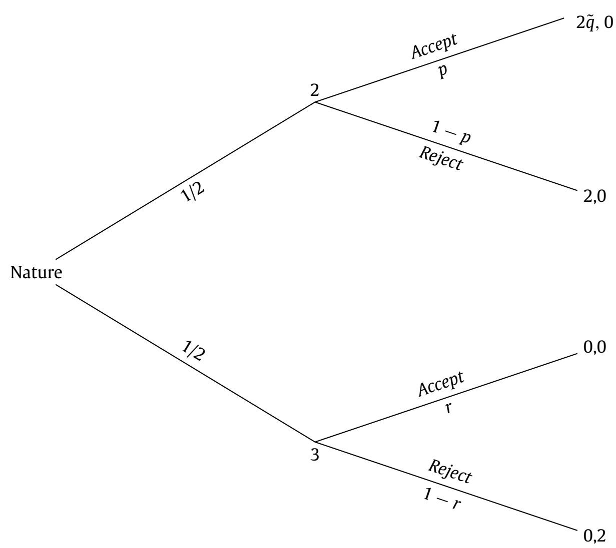
(ii) If p is such that $p^- > 1/2$ (that is $p > 1/2+\varepsilon$), then $w_1(p, y)$ is strictly decreasing in $[0, 1]$ and attains its maximum at $y = 0$.

(iii) If p is such that $p^- = 1/2$, (that is $p = 1/2 + \varepsilon$), then $w_1(p, y) = 3p^- + 2 \ \forall y \in [0, 1]$.

Now, in case $0 < \varepsilon \leqslant 1/2$, condition (iii) gives that $p = 1/2+\varepsilon$ is an equilibrium, while condition *(i)*, *(ii)* imply that there are no other equilibria. In case $\varepsilon > \frac{1}{2}$, it follows that $p^- < 1/2$ for every $p \in [0, 1]$, then condition (i) gives that $p = 1$ is the unique equilibrium.

Summarizing, the example shows that ambiguity produces a different equilibrium with respect to the non-ambiguous case. When ambiguity is sufficiently small, the equilibrium under ambiguity is close to the non-ambiguous one. As ambiguity increases, the equilibrium under ambiguity converges to a strategy profile that is not an equilibrium of the game without ambiguity.

Example 3.4. In the games presented above, the presence of ambiguity perturbs the set of equilibria and possibly refines it in case the amount of ambiguity is sufficiently large. The example presented below shows instead that even an infinitesimal amount of ambiguity may alter significantly the set of equilibria and, in particular, work as an equilibrium selector. In order to achieve this goal we consider a slight variation (in few payoffs) of the *Confidence Game* in figure 3 of Geanakoplos et al. (1989). For the sake of completeness, we recall the story behind this game (explaining the few differences from the original version): John has invited a woman (Anne) for a date, but he is not sure she will accept. He cannot tell whether Anne is Player 2, who likes him, or Player 3, who does not (nature chooses the woman's identity, with equal probabilities). Player 3 will not accept in any case. Even if Player 2 likes him, it is not certain that she will accept. Player 2 will go out with him only if she thinks he is fully confident of himself, that is, if she believes that John's probability assessment of being accepted is equal to 1. Here there is the main difference with the original Confidence Game in Geanakoplos et al. (1989) where Player 2 will go out with him if she thinks John is *sufficiently* confident of himself, meaning that she believes that his probability assessment is greater than a given threshold level smaller than 1. The pure strategy set of Player 2 is $A_2 = \{Accept, Reject\}$ and the pure strategy set of Player 3 is $A_3 = \{Accept, Reject\}$. We denote with p the mixed strategy of Player 2, where, with an abuse of notation, p is the probability of *Accept* and $1 - p$ is the probability of *Reject*. Similarly r is the mixed strategy of Player 3; again, with an abuse of notation, r is the probability of *Accept* and $1 - r$ is the probability of *Reject*. It is assumed that Player 3's utility does not depend on beliefs while Player 2's utility depends on her second-order beliefs. Moreover, as done in the previous examples, it is considered the case in which only the expectations of beliefs play a role in Player 2's utility function. We denote with $q \in [0, 1]$ the expectation of John's first-order beliefs about Player 2's mixed strategy p and $\tilde{q} \in [0, 1]$ the expectation of Player 2's second-order beliefs about the expectation q of John's first-order beliefs. The game is represented below.



Differently from the original Confidence Game we assume that the psychological utility of Player 2 is not affected by what Player 2 believes John believes Player 3 will play.[17] Moreover, for the sake of simplicity we do not report John's utility as he is inactive. So, we can focus on the following normal form game between Player 2 and Player 3:

Player 2	Player 3	
	Accept	*Reject*
Accept	$\tilde{q}, 0$	$\tilde{q}, 1$
Reject	1,0	1,1

A mixed strategy profile is identified by the pair (p, r). Also in this example, the correct belief functions simply map the strategy profiles (p, r) to correct expectation of beliefs; more precisely, $\beta_1(p, r) = p$ tells that the expectation of John's correct first-order beliefs about Player 2's strategy p must be equal to p and $\beta_2(p, r) = p$ tells that the expectation of Player 2's correct second-order beliefs about John's first-order belief about Player 2's strategy p must be equal to p as well.

A psychological Nash equilibrium is unequivocally determined by a mixed strategy profile (p^*, r^*) such that

$$w_2^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_2^{GPS}((p^*, r^*), (y, r^*)) \quad \forall y \in [0, 1],$$
$$w_3^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_3^{GPS}((p^*, r^*), (p^*, y)) \quad \forall y \in [0, 1].$$

Now, it is clear that strategy *Reject* is strictly dominant for Player 3. So, in equilibrium r^* must be equal to 0. Hence, we need only to find Player 2's best reply, given that $r^* = 0$. In this case, the expected utility for Player 2 playing y and having second-order belief $\tilde{q}$ is

$$u_2(\tilde{q}, y) = y(\tilde{q} - 1) + 1.$$

So, if $\tilde{q} < 1$ then Player 2's best reply is $y = 0$. If $\tilde{q} = 1$, then every $y \in [0, 1]$ is a best reply. It follows that

$$w_1^{GPS}((0, 0), (0, 0)) \geqslant w_1^{GPS}((0, 0), (y, 0)) \quad \forall y \in [0, 1],$$
$$w_1^{GPS}((1, 0), (1, 0)) = w_1^{GPS}((1, 0), (y, 0)) \quad \forall y \in [0, 1].$$

Therefore the strategy profiles $(p, r) = (0, 0)$ and $(p, r) = (1, 0)$ are psychological Nash equilibria. Note that there are no other Psychological Nash equilibria of the form $(p, 0)$ with $p \in]0, 1[$. In fact,

$$w_1^{GPS}((p, 0), (p, 0)) < w_1^{GPS}((p, 0), (0, 0)) \quad \forall p \in]0, 1[,$$

and so the strategy profiles $(p, 0)$ are not psychological Nash equilibria for every $p \in]0, 1[$.

Now, suppose that Player 2's second-order belief is given by the interval $\gamma_2^\varepsilon(p) = [p - \varepsilon, p + \varepsilon] \cap [0, 1]$ with $\varepsilon > 0$. For the sake of simplicity, we assume also that ε is small enough. Ambiguity does not affect Player 3's utility so that $r = 0$ is again a strictly dominant strategy for her. It follows again that every psychological equilibrium under ambiguity is given by a pair (p, r) with $r = 0$. So we only have to find Player 2's best reply to $r = 0$. Again, given that $r = 0$, the expected utility for Player 2 playing y and having second-order belief $\tilde{q}$ is

$$u_2(\tilde{q}, y) = y(\tilde{q} - 1) + 1.$$

For every pair of Player 2's mixed strategy p and y, we get

$$\arg\min_{\tilde{q} \in \gamma_2^\varepsilon(p)} u_2(\tilde{q}, y) = \arg\min_{\tilde{q} \in [p-\varepsilon, p+\varepsilon] \cap [0,1]} [\tilde{q}y + 1 - y] = \begin{cases} \gamma_2^\varepsilon(p) & \text{if } y = 0, \\ \max\{p - \varepsilon, 0\} & \text{if } y \in]0, 1]. \end{cases}$$

Denote with $p^- = \max\{p - \varepsilon, 0\}$. Therefore, given the two strategy profiles $(p, 0)$ and $(y, 0)$,

$$w_2((p, 0), (y, 0)) = U_2(\gamma_2^\varepsilon(p), y) = \min_{\tilde{q} \in \gamma_2^\varepsilon(p)} [\tilde{q}y + 1 - y] = \begin{cases} 1 & \text{if } y = 0, \\ p^- y + 1 - y = y(p^- - 1) + 1 & \text{if } y \in]0, 1]. \end{cases}$$

Now, since

$$p^- \leqslant 1 - \varepsilon \implies p^- - 1 \leqslant -\varepsilon < 0,$$

then $U_2(\gamma_2^\varepsilon(p), y)$ is strictly decreasing with respect to y in the interval $[0, 1]$. It follows that

$$w_2((0, 0), (0, 0)) \geqslant w_2((0, 0), (y, 0)) \quad \forall y \in [0, 1],$$

which implies that $p = 0$ is a best reply to $r = 0$ given that beliefs are consistent with $p = 0$. Hence $(p, r) = (0, 0)$ is a psychological Nash equilibrium under ambiguity. Moreover,

$$w_2((p, 0), (p, 0)) < w_2((p, 0), (0, 0)) \quad \forall p \in]0, 1],$$

therefore every $p > 0$ cannot be an equilibrium strategy when $r = 0$. Hence, the unique psychological Nash equilibrium under ambiguity is $(p, r) = (0, 0)$.

Summarizing, the example shows that even only an infinitesimal amount of ambiguity works as an equilibrium selector as it destroys the psychological equilibrium $(p, r) = (1, 0)$ and selects only the equilibrium $(p, r) = (0, 0)$.

Example 3.5. This example illustrates a two-player psychological game in which one of the players (John) has ambiguous first-order beliefs that are described by partially specified probabilities. As explained is Section 2.2.2, we consider the case in which the other player (Anne) has correct second order beliefs but then, as a natural consequence, Anne has set-valued second-order beliefs.

There are two players: the pure strategy set of Player 1 (say John) is $A_1 = \{U, D\}$ and the pure strategy set of Player 2 (say Anne) is $A_2 = \{L, C, R\}$. We denote with p the mixed strategy of John, where, with an abuse of notation, p is the probability of U and $1 - p$ is the probability of D. So we can identify the

[17] However, these beliefs do not affect equilibria even in the Confidence Game of Geanakoplos et al. (1989).

set of mixed strategies of John as $\Sigma_1 = \{p \mid p \in [0, 1]\}$. Similarly $r = (r_1, r_2, r_3)$ denotes the mixed strategy of Anne: r_1 is the probability of L, r_2 is the probability of C, r_3 is the probability of R and $r_1 + r_2 + r_3 = 1$. As $r_3 = 1 - r_1 - r_2$, with an abuse of notation we identify the set of mixed strategies of Anne as $\Sigma_2 = \{(r_1, r_2) \mid r_1, r_2 \geqslant 0,\ r_1 + r_2 \leqslant 1\}$.

John's utility is not affected by beliefs while Anne's utility depends on her second-order beliefs. Moreover, as done in the previous examples, it is considered the case in which only the expectations of the beliefs play a role. We denote with $q = (q_1, q_2, q_3)$ the expectations of John's first-order beliefs about Anne's mixed strategy $r = (r_1, r_2, r_3)$, that is q_1 is the expectation of r_1, q_2 is the expectation of r_2, q_3 is the expectation of r_3. Consequently $q_1 + q_2 + q_3 = 1$. Let $\tilde{q} = (\tilde{q}_1, \tilde{q}_2, \tilde{q}_3)$ denote the expectations of Anne's second-order beliefs about the expectation q of John's first-order beliefs, that is $\tilde{q}_1$ is the expectation of q_1, $\tilde{q}_2$ is the expectation of q_2, $\tilde{q}_3$ is the expectation of q_3 and consequently $\tilde{q}_1 + \tilde{q}_2 + \tilde{q}_3 = 1$. The normal form game is the following:

Player 1	Player 2		
	L	C	R
U	0,0	$1, 1 + \tilde{q}_2$	0,1
D	1,0	0,0	0,1

A mixed strategy profile is identified by the pair (p, r). Since only the expectations of the second component of Anne's mixed strategy play a role, then, with an abuse of notation, the (simplified) correct belief functions map the strategy profiles (p, r) to correct expectation of this component. More precisely, $\beta_1(p, r) = r_2$ tells that the expectation of John's correct first-order beliefs about Anne's component r_2 must be equal to r_2 and $\beta_2(p, r) = r_2$ tells that the expectation of Anne's correct second-order beliefs about John's correct first-order belief about Anne's strategy component r_2 must be equal to r_2 as well.

A psychological Nash equilibrium is unequivocally determined by a mixed strategy profile (p^*, r^*) such that

$$w_1^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_1^{GPS}((p^*, r^*), (y, r^*)) \quad \forall y \in \Sigma_1,$$
$$w_2^{GPS}((p^*, r^*), (p^*, r^*)) \geqslant w_2^{GPS}((p^*, r^*), (p^*, y)) \quad \forall y \in \Sigma_2.$$

Firstly, let us analyze Anne's utility. Given any mixed strategy p of John, the expected utility of Anne for every second-order belief $\tilde{q}_2$ and mixed strategy y, is

$$u_2(\tilde{q}_2, (p, (y_1, y_2))) = py_2(1 + \tilde{q}_2) + 1 - y_1 - y_2.$$

By definition we get

$$w_2^{GPS}((p, r), (p, r)) = u_2(r_2, (p, (r_1, r_2))) \text{ and}$$
$$w_2^{GPS}((p, r), (p, y)) = u_2(r_2, (p, (y_1, y_2))).$$

Anne's best reply correspondence for every strategy p of John is given by:

$$BR_2^{GPS}(p) = \Big\{(r_1, r_2) \mid w_2^{GPS}((p, r), (p, r)) \geqslant w_2^{GPS}((p, r), (p, y))\ \forall (y_1, y_2) \in \Sigma_2\Big\}.$$

The strategy L is strictly dominated for Anne, so the first component of the best reply must be equal to 0, that is $r_1 = 0$. Therefore, we have to maximize $u_2(\tilde{q}_2, (p, (0, y_2)))$, which has the form:

$$u_2(\tilde{q}_2, (p, (0, y_2))) = py_2(1 + \tilde{q}_2) + 1 - y_2 = y_2[p(1 + \tilde{q}_2) - 1] + 1.$$

We need to distinguish three cases:

- If $p(1 + \tilde{q}_2) - 1 > 0$, then u_2 is maximized only for $y_2 = 1$. Being the corresponding correct belief $\tilde{q}_2 = 1$, then $p(1 + 1) - 1 > 0$ must be satisfied. Hence, when $p > 1/2$, it follows that
$$w_2^{GPS}((p, (0, 1)), (p, (0, 1))) \geqslant w_2^{GPS}((p, (0, 1)), (p, (y_1, y_2))) \quad \forall (y_1, y_2) \in \Sigma_2.$$
- If $p(1 + \tilde{q}_2) - 1 < 0$, then u_2 is maximized only for $y_2 = 0$. Being the corresponding correct belief $\tilde{q}_2 = 0$, then $p - 1 < 0$ must be satisfied. Hence, when $p < 1$, it follows that
$$w_2^{GPS}((p, (0, 0)), (p, (0, 0))) \geqslant w_2^{GPS}((p, (0, 0)), (p, (y_1, y_2))) \quad \forall (y_1, y_2) \in \Sigma_2.$$
- If $p(1 + \tilde{q}_2) - 1 = 0$, then every y_2 maximizes u_2 but $p(1 + \tilde{q}_2) - 1 = 0$ implies that
$$\tilde{q}_2 = \frac{1 - p}{p}.$$
Since
$$\frac{1 - p}{p} \in [0, 1] \iff p \in [1/2, 1],$$
then for every $p \in [1/2, 1]$, it follows that
$$w_2^{GPS}\left(\left(p, \left(0, \frac{1 - p}{p}\right)\right), \left(p, \left(0, \frac{1 - p}{p}\right)\right)\right) \geqslant w_2^{GPS}\left(\left(p, \left(0, \frac{1 - p}{p}\right)\right), (p, (y_1, y_2))\right)$$
for every $(y_1, y_2) \in \Sigma_2$.

Summarizing, the best reply correspondence is:

$$BR_2^{GPS}(p) = \left\{ (r_1, r_2) \in \Sigma_2 \;\middle|\; \begin{array}{ll} (r_1, r_2) = (0, 0) & \text{if } p \in [0, 1/2[\\ (r_1, r_2) \in \{(0, 0)\} \cup \{(0, 1)\} & \text{if } p = 1/2 \\ (r_1, r_2) \in \{(0, 0)\} \cup \{(0, 1)\} & \\ \quad \cup \left\{\left(0, \frac{1-p}{p}\right)\right\} & \text{if } p \in]1/2, 1] \end{array} \right\}.$$

Now, let us analyze John's utility. We know that in equilibrium $r_1 = 0$ so we look only at the best reply to Anne's mixed strategies of the form $(0, r_2)$ for every $r_2 \in [0, 1]$. John's utility function is the classical expected utility without any psychological term, so the best reply can be immediately computed:

$$BR_1^{GPS}(0, r_2) = \left\{ p \in [0, 1] \;\middle|\; \begin{array}{ll} p \in [0, 1] & \text{if } r_2 = 0 \\ p = 1 & \text{if } r_2 \in]0, 1] \end{array} \right\}. \tag{15}$$

The set of Psychological Nash Equilibria is the set of strategy profiles $(p, (r_1, r_2))$ belonging to the set

$$E^{GPS} = \{(p, (0, 0)) \mid p \in [0, 1]\} \cup \{(1, (0, 1))\}.$$

Now we introduce ambiguity and assume that John has partially specified probabilities about Anne's strategy choice. The correspondence K_1 defined in (8) takes the following form

$$K_1(r_1, r_2) = \{(y_1, y_2) \in \Sigma_2 \mid y_1 + y_2 = r_1 + r_2\}$$

and represents the situation in which John does not know correctly the strategy of Anne but knows the probability that she will play L or C. We assume that first and second-order beliefs are given by set-valued maps as defined in (11). However, only the expectations of Anne's correct second-order beliefs play a role; therefore with an abuse of notation, Anne's beliefs are given by

$$\gamma_2(p, (r_1, r_2)) = \left\{(\tilde{q}_1, \tilde{q}_2) \mid \tilde{q}_1, \tilde{q}_2 \geqslant 0,\ \tilde{q}_1 + \tilde{q}_2 = r_1 + r_2\right\},$$

meaning that Anne expects that John expects that she will play L or C with probability $r_1 + r_2$. For $p, h \in \Sigma_1$, $r = (r_1, r_2)$, $y = (y_1, y_2) \in \Sigma_2$ we get:

$$\operatorname*{arg\,min}_{(\tilde{q}_1, \tilde{q}_2) \in \gamma_2(p, (r_1, r_2))} u_2(\tilde{q}_2, (h, (y_1, y_2))) = \operatorname*{arg\,min}_{(\tilde{q}_1, \tilde{q}_2) \in \gamma_2(p, (r_1, r_2))} \left[hy_2(1 + \tilde{q}_2) + 1 - y_1 - y_2\right]$$
$$= \begin{cases} \gamma_2(p, (r_1, r_2)) & \text{if } hy_2 = 0, \\ \tilde{q}_2 = 0 & \text{if } hy_2 > 0. \end{cases}$$

Therefore

$$w((p,r),(h,y)) = U_2(\gamma_2(p,(r_1,r_2)),(h,(y_1,y_2))) = \min_{(\tilde{q}_1,\tilde{q}_2)\in\gamma_2(p,(r_1,r_2))}\left[hy_2(1+\tilde{q}_2)+1-y_1-y_2\right] = \begin{cases} 1-y_1-y_2 & \text{if } h=0 \\ y_2[h-1]+1-y_1 & \text{if } h>0. \end{cases}$$

Let BR_2 be the best reply correspondence of Anne in the ambiguous game:

$$BR_2(p) = \{(r_1,r_2) \mid w((p,r),(p,r)) \geqslant w((p,r),(p,y)) \quad \forall (y_1,y_2) \in \Sigma_2\}.$$

It follows that

$$BR_2(p) = \left\{(r_1,r_2) \in \Sigma_2 \;\middle|\; \begin{matrix} (r_1,r_2) = (0,0) & \text{if } p \in [0,1[\\ (r_1,r_2) \in \{(0,r_2) \mid r_2 \in [0,1]\} & \text{if } p = 1 \end{matrix}\right\}.$$

The best reply of John is the same as in (15).

Therefore the set of Psychological Nash Equilibria under ambiguity is the set of strategy profiles $(p,(r_1,r_2))$ belonging to the set

$$E = \{(p,(0,0)) \mid p \in [0,1]\} \cup \{(1,(0,r_2)) \mid r_2 \in [0,1]\}.$$

It can be immediately noticed that the set of equilibria in the ambiguous game results enlarged by the interval $\{(1,(0,r_2)) \mid r_2 \in [0,1]\}$.

4. Equilibrium existence

This section is devoted to the issue of existence of psychological Nash equilibrium under ambiguity. To this purpose we need to recall some tools on set-valued maps.[18]

Preliminaries about correspondences

Consider a set-valued map $\Gamma : X \rightsquigarrow Y$ between two metric spaces X and Y, meaning that $\Gamma(x) \subseteq Y$ for every $x \in X$.

Then, the *upper limit* of Γ in $\bar{x} \in X$ is defined by

$$\operatorname{Lim\,sup}_{x\to\bar{x}} \Gamma(x) = \left\{ y \in Y \;\middle|\; \liminf_{x\to\bar{x}} d(y,\Gamma(x)) = 0 \right\},$$

where $d(y,\Gamma(x))$ denotes the distance (in the metric space Y) between y and the set $\Gamma(x)$, while the *lower limit* of Γ in $\bar{x} \in X$ is defined by

$$\operatorname{Lim\,inf}_{x\to\bar{x}} \Gamma(x) = \left\{ y \in Y \;\middle|\; \lim_{x\to\bar{x}} d(y,\Gamma(x)) = 0 \right\}.$$

Definition 4.1. The set-valued map $\Gamma : X \rightsquigarrow Y$ is said to be:

(i) *lower semicontinuous*[19] at $\bar{x} \in X$ if $\Gamma(\bar{x}) \subseteq \operatorname{Lim\,inf}_{x\to\bar{x}}\Gamma(x)$, meaning that for any $y \in \Gamma(\bar{x})$ and for any sequence $(x_\nu)_\nu \subset X$ converging to $\bar{x}$, there exists a sequence of elements $(y_\nu)_\nu \subset Y$, with $y_\nu \in \Gamma(x_\nu)$ for every $\nu \in \mathbb{N}$, that converges to y. Γ is lower semicontinuous in X if it is so in every point $x \in X$;

(ii) *closed* at $\bar{x} \in X$ if

$$\operatorname{Lim\,sup}_{x\to\bar{x}} \Gamma(x) \subseteq \Gamma(\bar{x}),$$

that is, for every sequence $(x_\nu)_\nu \subset X$ converging to $\bar{x}$ and every sequence $(y_\nu)_\nu \subset Y$, with $y_\nu \in \Gamma(x_\nu)$ for every $\nu \in \mathbb{N}$, that converges to a point $y \in Y$, it follows that $y \in \Gamma(\bar{x})$. Γ is closed in X if it is so in every point $x \in X$. Moreover Γ is closed in X if and only if $Graph(\Gamma) = \{(x,y) \mid x \in X,\ y \in \Gamma(x)\}$ is a closed subset of $X \times Y$;

(iii) *upper semicontinuous* at $\bar{x} \in X$ if for any neighborhood $\mathcal{U}$ of $\Gamma(\bar{x})$ there exists $\eta > 0$ such that $\Gamma(x) \subset \mathcal{U}$ for all $x \in B_X(\bar{x},\eta) = \{x \in X : \|x-\bar{x}\|_X < \eta\}$. Γ is upper semicontinuous in X if it is so in every point $x \in X$;

(iv) *continuous* at $\bar{x} \in X$ if it is upper and lower semicontinuous at $\bar{x}$. Γ is continuous in X if it is so in every point $x \in X$.

Recall that if X is closed, Y is compact and Γ has closed values then Γ is upper semicontinuous if and only if it is closed (see Proposition 1.4.8 in Aubin and Frankowska (1990)). We will see that every set-valued map introduced in this paper satisfies these properties, therefore, in our setting, upper semicontinuity and closedness are equivalent notions.

We conclude this section recalling some useful and well-known results. The first result is a version of the Berge's maximum theorem as presented in Aubin and Frankowska (1990, Theorem 1.4.16).

Theorem 4.2. *Let X, Y be two metric spaces, $\Gamma : X \rightsquigarrow Y$ a set-valued map and $f : Graph(\Gamma) \to \mathbb{R}$ a function. Let $g : X \to \overline{\mathbb{R}}$ be the marginal function defined by*

$$g(x) = \sup_{y\in\Gamma(x)} f(x,y) \quad \forall x \in X.$$

Then,

(i) If f is a lower semicontinuous function and Γ a lower semicontinuous set-valued map then g is a lower semicontinuous function;

(ii) If f is an upper semicontinuous function and Γ an upper semicontinuous set-valued map with compact images then g is an upper semicontinuous function.

The second result is the well known Kakutani fixed point theorem.

Theorem 4.3 (*Kakutani Fixed Point Theorem*)**.** *Let X be an Euclidean finite dimensional space. Let K be a non-empty compact convex subset of X. If $\Gamma : K \rightsquigarrow K$ is an upper semicontinuous mapping such that, for all $x \in K$, the set $\Gamma(x)$ is convex, closed and non-empty, then there exists a fixed point for Γ, that is a point $\bar{x} \in K$ such that $\bar{x} \in \Gamma(\bar{x})$.*

The existence theorem

Theorem 4.4. *Consider a psychological game under ambiguity $G = (A_1,\dots,A_n,U_1,\dots,U_n)$ as presented in Definition 2.6. Assume that, for every player $i \in I$,*

(i) $u_i : \overline{B}_i \times \Sigma \to \mathbb{R}$ is a continuous function in $\overline{B}_i \times \Sigma$;

(ii) $u_i(b_i,(\cdot,\tau_{-i})) : \Sigma_i \to \mathbb{R}$ is a quasi-concave function[20] *in Σ_i, for every $b_i \in \overline{B}_i$ and every $\tau_{-i} \in \Sigma_{-i}$;*

(iii) $\gamma_i : \Sigma \rightsquigarrow \overline{B}_i$ is a continuous set-valued map in Σ with not empty, convex and compact images $\gamma_i(\sigma)$, for every $\sigma \in \Sigma$.

Then there exists $\sigma^* \in \Sigma$ such that $(\gamma(\sigma^*),\sigma^*)$ is a psychological Nash equilibrium under ambiguity for the game G.

Proof. Consider the summary form $\hat{G} := (A_1,\dots,A_n,w_1,\dots,w_n)$ of the game G. Let $BR_i : \Sigma \rightsquigarrow \Sigma_i$ be the set-valued map defined

[18] We refer mainly to Aubin and Frankowska (1990) and references therein.

[19] The terms *lower (upper) semicontinuous set-valued map* and *lower hemi-continuous correspondence* are synonyms in the game theory literature and, in particular, in this paper.

[20] Here we refer to the classical definition of quasi-concavity: a function $g : X \to \mathbb{R}$ (where X is convex) is quasi-concave in X if and only if the upper level sets are convex subsets of X.

by:

$$BR_i(\sigma) := \{\tau_i \in \Sigma_i \mid w_i(\sigma, (\tau_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \quad \forall y_i \in \Sigma_i\} \quad \forall \sigma \in \Sigma,$$

and $BR : \Sigma \rightsquigarrow \Sigma$ the set-valued map defined by:

$$BR(\sigma) = \Pi_{i=1}^n BR_i(\sigma) \quad \forall \sigma \in \Sigma.$$

Lemma 2.8 guarantees that σ^* is a fixed point for BR, i.e. $\sigma^* \in BR(\sigma^*)$, if and only if $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity for G. Therefore, our proof reduces to verify the existence of such a fixed point. To this aim we apply Theorem 4.3 to the correspondence $BR : \Sigma \rightsquigarrow \Sigma$.

From the assumptions it follows that the summary utility function w_i defined in (4) is well posed; moreover, $w_i(\sigma, \tau) = \min_{b_i \in \gamma_i(\sigma)} u_i(b_i, \tau)$, for all $(\sigma, \tau) \in \Sigma \times \Sigma$. Theorem 4.2 ensures that w_i is continuous in $\Sigma \times \Sigma$, hence the best reply correspondence BR_i is upper semicontinuous[21] with not empty and compact images $BR_i(\sigma)$ for every $\sigma \in \Sigma$. It follows that BR is upper semicontinuous[22] with not empty and compact images $BR(\sigma)$ for every $\sigma \in \Sigma$.

Lastly, it remains to prove that $BR(\sigma)$ is a convex subset of Σ. It is sufficient to verify that each $BR_i(\sigma)$ is a convex subset of Σ_i, as the finite product of convex sets is obviously convex. Take $\lambda \in [0, 1]$ and $\overline{\tau}_i, \hat{\tau}_i \in BR_i(\sigma)$. We will prove that $\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i \in BR_i(\sigma)$. Since $\overline{\tau}_i, \hat{\tau}_i \in BR_i(\sigma)$, then

$$w_i(\sigma, (\overline{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i,$$

$$w_i(\sigma, (\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i,$$

which implies that

$$u_i(b_i, (\overline{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i, \forall b_i \in \gamma_i(\sigma)$$

$$u_i(b_i, (\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})), \qquad \forall y_i \in \Sigma_i, \forall b_i \in \gamma_i(\sigma).$$

Therefore, for every $b_i \in \gamma_i(\sigma)$, it follows that

$$\alpha_{b_i} := \min\{u_i(b_i, (\overline{\tau}_i, \sigma_{-i})), u_i(b_i, (\hat{\tau}_i, \tau_{-i}))\} \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \quad \forall y_i \in \Sigma_i.$$

Now, since the function $u_i(b_i, (\cdot, \tau_{-i}))$ is quasi-concave, it follows that

$$u_i(b_i, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) \geqslant \alpha_{b_i} \geqslant w_i(\sigma, (y_i, \sigma_{-i})) \qquad \forall y_i \in \Sigma_i.$$

Since the previous inequality holds for every $b_i \in \gamma_i(\sigma)$, we finally get

$$w_i(\sigma, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) = \inf_{b_i \in \gamma_i(\sigma)} u_i(b_i, (\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i, \sigma_{-i})) \geqslant w_i(\sigma, (y_i, \sigma_{-i})).$$

Hence, $\lambda\overline{\tau}_i + (1-\lambda)\hat{\tau}_i \in BR_i(\sigma)$. □

5. Ambiguous trembles and stability

As already mentioned in the Introduction, the classical theory of refinements of Nash equilibria deals with the problem of equilibrium selection based on properties of stability of the equilibria (Van Damme, 1989). It is well known that, in case of games with multiple equilibria, some of them may not be robust with respect to perturbations on the strategies or on the payoffs of the players, so that it is possible to restrict significatively the set of equilibria on the basis of some stability property. This approach arises with the concept of *trembling hand perfect equilibrium* for Nash equilibria introduced in the seminal paper by Selten (1975). The main idea underlying this concept is that players believe that their opponents can make mistakes playing their equilibrium strategies, therefore each equilibrium strategy should be *close* to the best reply against perturbed expectations about opponents' behavior, if the perturbation is small enough. As already mentioned in the Introduction, the concept of trembling hand perfect equilibrium is extended to psychological games in other papers: in Geanakoplos et al. (1989), the idea is that strategies are perturbed as in Selten (1975) and hierarchies of beliefs are consistent with the (unperturbed) equilibrium strategies, along the perturbations; in Battigalli and Dufwenberg (2009), strategies are perturbed in the same way and hierarchies of beliefs are perturbed accordingly, by means of the *consistency condition* between strategies and hierarchies of beliefs. In this paper, we take into account a different perspective as, on the one hand, we look at the stability with respect to perturbations on the entire hierarchies of beliefs and, on the other hand, we allow for ambiguous perturbations, that (can) take the form of sets of hierarchies of beliefs.

To better understand the problem, we look at Example 3.4 in Section 3. It turns out that, when the correct belief function β_A is perturbed by ambiguous trembles so that beliefs are represented by the set-valued map γ_A^ε, the set of equilibria reduces to just one out of the two equilibria that we find in the non ambiguous case. Namely, the presence of ambiguity destroys the psychological equilibrium $(p, r) = (1, 0)$ and selects only the equilibrium $(p, r) = (0, 0)$. Now, when ε converges to 0, the set-valued map γ_A^ε converges (in a suitable way) to β_A. Taking the sequence of the corresponding psychological Nash equilibria under ambiguity (the constant sequence obtained for $p = 0$ and $r = 0$), we get that, as $\varepsilon \to 0$, the limit process obviously selects $(p, r) = (0, 0)$ and not $(p, r) = (1, 0)$. So we have constructed a selection mechanism for psychological equilibria based on ambiguous trembles.

The arguments above have an underlying problem that concerns the way ambiguous belief should converge to correct beliefs in such a way that sequences of psychological equilibria under perturbations converge to psychological equilibria of the unperturbed game. Below we look at this problem that we embody in a larger one in which the unperturbed game can be itself ambiguous[23] and utilities can be perturbed as well.[24] We give a general limit theorem that gives conditions on the convergence of psychological games under ambiguity to an unperturbed one in such a way that corresponding sequences of equilibria under perturbation converge to unperturbed equilibria. Then we apply the theorem to construct selection criteria for classical psychological equilibria.

5.1. The limit theorem

In this subsection we show what conditions must be imposed in order that sequences of psychological equilibria under ambiguity of perturbed games converge to psychological equilibria under ambiguity of the unperturbed game, as the perturbation vanishes. In order to state and prove this limit result, we need firstly to recall definitions on variational convergence of sequences of functions and set-valued maps.

21 The continuity of w_i guarantees immediately that BR_i is a closed set-valued map, hence it is upper semicontinuous.

22 Berge (1997, Theorem 4' page 114) shows that the cartesian product of a finite number of upper semicontinuous set-valued map is an upper semicontinuous map.

23 This means that we allow for perturbations of ambiguous beliefs.

24 The theory of refinements of Nash equilibria studies stability with respect to perturbations on payoffs as well (see, for instance the property of *essentiality* in Van Damme (1989)).

Technical tools

We referred mainly to the paper (Lignola and Morgan, 1992) for the following definitions and results.

Definition 5.1. Let X be a topological space. Consider a sequence of functions[25] $\{g_\nu\}_{\nu\in\mathbb{N}}$ with $g_\nu : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$ for every $\nu \in \mathbb{N}$ and a function $g : X \subset \mathbb{R}^k \to \overline{\mathbb{R}}$.

(i) The sequence of functions $\{g_\nu\}_{\nu\in\mathbb{N}}$ *epiconverges* to the function g if:

(1) for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have

$$g(x) \leqslant \liminf_{\nu\to\infty} g_\nu(x_\nu);$$

(2) for every $x \in X$ there exists a sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X such that

$$\limsup_{\nu\to\infty} g_\nu(x_\nu) \leqslant g(x).$$

(ii) The sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ *hypoconverges* to the function g if the sequence of functions $\{-g_\nu\}_{\nu\in\mathbb{N}}$ epiconverges to the function $-g$.

(iii) The sequence $\{g_\nu\}_{\nu\in\mathbb{N}}$ *sequentially converges* (or continuously converges) to the function g if it epiconverges and hypoconverges to the function g, i.e. if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$g(x) = \lim_{\nu\to\infty} g_\nu(x_\nu) = \limsup_{\nu\to\infty} g_\nu(x_\nu) = \liminf_{\nu\to\infty} g_\nu(x_\nu). \quad (16)$$

The next definition is devoted to set-valued maps.

Definition 5.2. Let X and Y be metric spaces. Let $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ be a sequence of set-valued maps, with $\Gamma_\nu : X \rightsquigarrow Y$ for every $\nu \in \mathbb{N}$ and let $\Gamma : X \rightsquigarrow Y$ be a set-valued map. Let $S(y,\varepsilon)$ be the ball in Y with center in y and radius ε and

$$\operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \exists\overline{\nu} \text{ s.t. for all } \nu \geq \overline{\nu},\ S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\},$$

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) = \{y \in Y \mid \forall\varepsilon > 0,\ \forall\overline{\nu},\ \exists\nu \geq \overline{\nu} \text{ s.t. } S(y,\varepsilon) \cap \Gamma_\nu(x_\nu) \neq \emptyset\}.$$

Then

(i) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially lower convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\Gamma(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu);$$

(ii) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially upper convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) \subseteq \Gamma(x);$$

(iii) $\{\Gamma_\nu\}_{\nu\in\mathbb{N}}$ is *sequentially convergent* to Γ if for every $x \in X$ and for every sequence $\{x_\nu\}_{\nu\in\mathbb{N}} \subset X$ converging to x in X we have:

$$\operatorname*{Lim\,sup}_{\nu\to\infty} \Gamma_\nu(x_\nu) \subseteq \Gamma(x) \subseteq \operatorname*{Lim\,inf}_{\nu\to\infty} \Gamma_\nu(x_\nu).$$

The result

Now we can state the limit theorem.

Theorem 5.3. *Let $G = \{A_1, \ldots, A_n, U_1, \ldots, U_n\}$ be a psychological game under ambiguity. For every player i, let*

(a) $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ be a sequence of functions with $u_{i,\nu} : \overline{B}_i \times \Sigma \to \mathbb{R}$ for every $\nu \in \mathbb{N}$;
(b) $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ be a sequence of set-valued maps $\gamma_{i,\nu} : \Sigma \rightsquigarrow \overline{B}_i$, for every $\nu \in \mathbb{N}$;
(c) $\{U_{i,\nu}\}_{\nu\in\mathbb{N}}$ be the sequence of functions $U_{i,\nu} : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined by

$$U_{i,\nu}(K_i,\sigma) = \inf_{b_i\in K_i} u_{i,\nu}(b_i,\sigma) \quad \forall(K_i,\sigma) \in \mathscr{K}_i \times \Sigma$$

for every $\nu \in \mathbb{N}$;
(d) $\{G_\nu\}_{\nu\in\mathbb{N}}$ be the sequence of games where $G_\nu = \{A_1, \ldots, A_n, U_{1,\nu}, \ldots, U_{n,\nu}\}$ for every $\nu \in \mathbb{N}$.

Assume that, for every player i,

(i) the sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to the function u_i;[26]
(ii) each function $u_{i,\nu}$ and the function u_i are continuous in $\overline{B}_i \times \Sigma$;
(ii) the sequence $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ sequentially converges to the set-valued map γ_i. Suppose additionally that each $\gamma_{i,\nu}$ and γ_i have compact and not-empty values for every $\sigma \in \Sigma$.

If the sequence $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}} \subset \Sigma$ converges to $\sigma^* \in \Sigma$ and, for every $\nu \in \mathbb{N}$, $(\gamma_\nu(\sigma^*_\nu), \sigma^*_\nu)$ is a psychological Nash equilibrium of G_ν, then it follows that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity of G.

Proof. For every player i and every $\nu \in \mathbb{N}$ let $w_{i,\nu}$ be the summary utility function of the game G_ν, that is

$$w_{i,\nu}(\sigma,\tau) := \inf_{b_i\in\gamma_{i,\nu}(\sigma)} u_{i,\nu}(b_i,\tau) \quad \forall(\sigma,\tau) \in \Sigma \times \Sigma,$$

and w_i be the summary utility function of the game G, that is

$$w_i(\sigma,\tau) := \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) \quad \forall(\sigma,\tau) \in \Sigma \times \Sigma.$$

The continuous convergence of the sequence of functions $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ to the function w_i for every $i \in I$ guarantees the result. In fact, if $\{\sigma^*_\nu\}_{\nu\in\mathbb{N}} \subset \Sigma$ is a sequence converging to $\sigma^* \in \Sigma$ such that, for every $\nu \in \mathbb{N}$, $(\gamma_\nu(\sigma^*_\nu), \sigma^*_\nu)$ is a psychological Nash equilibrium of G_ν, then from Lemma 2.8 it follows that, for every player i,

$$w_{i,\nu}(\sigma^*_\nu, \sigma^*_\nu) \geqslant w_{i,\nu}(\sigma^*_\nu, (y_i, \sigma^*_{-i,\nu})) \quad \forall y_i \in \Sigma_i.$$

Applying the continuous convergence of $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ to w_i we get

$$\begin{aligned} w_i(\sigma^*, \sigma^*) &= \lim_{\nu\to\infty} w_{i,\nu}(\sigma^*_\nu, \sigma^*_\nu) \geqslant \lim_{\nu\to\infty} w_{i,\nu}(\sigma^*_\nu, (y_i, \sigma^*_{-i,\nu})) \\ &= w_i(\sigma^*, (y_i, \sigma^*_{-i})) \quad \forall y_i \in \Sigma_i. \end{aligned}$$

This latter inequality implies that $(\gamma(\sigma^*), \sigma^*)$ is a psychological Nash equilibrium under ambiguity of G. Therefore, the proof reduces in verifying the continuous convergence of $\{w_{i,\nu}\}_{\nu\in\mathbb{N}}$ to w_i. To this aim, as defined in (16), we need to check that for every $(\sigma,\tau) \in \Sigma \times \Sigma$ and for every sequence $\{(\sigma_\nu,\tau_\nu)\}_{\nu\in\mathbb{N}}$ converging to (σ,τ) we get the inequalities

$$\limsup_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu) \leqslant w_i(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu).$$

Consider $(\sigma,\tau) \in \Sigma \times \Sigma$ and take a sequence $\{(\sigma_\nu,\tau_\nu)\}_{\nu\in\mathbb{N}}$ converging to (σ,τ). The proof is organized in two steps.

[25] For technical reasons, we consider the case where functions take values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

[26] The function u_i is the one appearing in the construction of U_i (see Eq. (3)).

Step 1: $w_i(\sigma,\tau) \leqslant \lim\inf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu)$.
Suppose by contradiction that

$$w_i(\sigma,\tau) > \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu). \tag{17}$$

This means that there exists a converging subsequence $\{(\sigma_{\nu_k},\tau_{\nu_k})\}_{k\in\mathbb{N}}$ such that

$$\lim_{k\to\infty} w_{i,\nu_k}(\sigma_{\nu_k},\tau_{\nu_k}) < w_i(\sigma,\tau). \tag{18}$$

Additionally, continuity of u_i and $u_{i,\nu}$ for every ν and compactness of the images of γ_i and $\gamma_{i,\nu}$, for every ν, guarantee that there exist $b_i^* \in \gamma_i(\sigma)$ and a $b_{i,\nu}^* \in \gamma_{i,\nu}(\sigma_\nu)$ for every ν, such that

$$u_i(b_i^*,\tau) = \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) = w_i(\sigma,\tau), \tag{19}$$

$$u_{i,\nu}(b_{i,\nu}^*,\tau_\nu) = \inf_{b_{i,\nu}\in\gamma_{i,\nu}(\sigma_\nu)} u_{i,\nu}(b_{i,\nu},\tau_\nu) = w_{i,\nu}(\sigma_\nu,\tau_\nu). \tag{20}$$

Consider the sequence of beliefs $\{b_{i,\nu_k}^*\}_{k\in\mathbb{N}}$ obtained from $w_{i,\nu_k}(\sigma_{\nu_k},\tau_{\nu_k})$ as in (20): since $\overline{B}_i$ is compact, it has a subsequence $\{b_{i,\nu_h}^*\}_{h\in\mathbb{N}}$ which converges to a point $\hat{b}_i \in \overline{B}_i$. This point $\hat{b}_i$ actually belongs to $\gamma_i(\sigma)$ since, by definition, the upper limit $\mathrm{Lim\,sup}_{\nu\to\infty}\gamma_{i,\nu}(\sigma_\nu)$ contains the limit of every converging subsequence of $\{b_{i,\nu}^*\}_{\nu\in\mathbb{N}}$; that is

$$\hat{b}_i \in \mathop{\mathrm{Lim\,sup}}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu).$$

On the other hand, the sequence of set-valued maps $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is supposed to be sequentially upper convergent to γ_i, that is,

$$\mathop{\mathrm{Lim\,sup}}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu) \subseteq \gamma_i(\sigma).$$

Therefore, $\hat{b}_i \in \gamma_i(\sigma)$. Eq. (19) implies that $u_i(b_i^*,\tau) \leqslant u_i(\hat{b}_i,\tau)$.
Now, the sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ epiconverges to u_i, therefore since $(b_{i,\nu_h}^*,\tau_{\nu_h})$ converges to $(\hat{b}_i,\tau)$, we have:

$$u_i(\hat{b}_i,\tau) \leqslant \liminf_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^*,\tau_{\nu_h}).$$

We finally get

$$w_i(\sigma,\tau) = u_i(b_i^*,\tau) \leqslant u_i(\hat{b}_i,\tau) \leqslant \liminf_{h\to\infty} u_{i,\nu_h}(b_{i,\nu_h}^*,\tau_{\nu_h}) = \liminf_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}) = \lim_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}).$$

Inequality (18) implies that

$$w_i(\sigma,\tau) \leqslant \lim_{h\to\infty} w_{i,\nu_h}(\sigma_{\nu_h},\tau_{\nu_h}) < w_i(\sigma,\tau),$$

which results in a contradiction. So

$$w_i(\sigma,\tau) \leqslant \liminf_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu).$$

Step 2: $w_i(\sigma,\tau) \geqslant \lim\sup_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau)$.
Let $b_i^* \in \gamma_i(\sigma)$ be such that

$$u_i(b_i^*,\tau) = \inf_{b_i\in\gamma_i(\sigma)} u_i(b_i,\tau) = w_i(\sigma,\tau).$$

Such a b_i^* exists because of the continuity of u_i and the compactness of $\gamma_i(\sigma)$ for every $\sigma \in \Sigma$. Since the sequence $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$ is sequentially lower convergent to γ_i, that is,

$$\gamma_i(\sigma) \subseteq \mathop{\mathrm{Lim\,inf}}_{\nu\to\infty} \gamma_{i,\nu}(\sigma_\nu),$$

then there exists a sequence $\{b_{i,\nu}\}_{\nu\in\mathbb{N}}$ converging to b_i^* such that, for every ν, $b_{i,\nu} \in \gamma_{i,\nu}(\sigma_\nu)$.
The sequence $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ hypoconverges to u_i; it follows that

$$\limsup_{\nu\to\infty} u_{i,\nu}(b_{i,\nu},\tau_\nu) \leqslant u_i(b_i^*,\tau).$$

Moreover, by construction $w_{i,\nu}(\sigma_\nu,\tau_\nu) \leqslant u_{i,\nu}(b_{i,\nu},\tau_\nu)$ for every $\nu \in \mathbb{N}$. This finally implies that

$$\limsup_{\nu\to\infty} w_{i,\nu}(\sigma_\nu,\tau_\nu) \leqslant \limsup_{\nu\to\infty} u_{i,\nu}(b_{i,\nu},\tau_\nu) \leqslant u_i(b_i^*,\tau) = w_i(\sigma,\tau). \quad \square$$

Remark 5.4. The proof of the previous theorem is self contained. An alternative proof could be obtained by applying the stability results for marginal functions under constraints as considered in Lignola and Morgan (1992).

5.2. Equilibrium selection

Building upon the previous result, in this subsection we show how to construct selection mechanism for psychological Nash equilibria based on ambiguous trembles. Let $G^{GPS} = \{A_1,\dots,A_n,u_1,\dots,u_n\}$ be a non-ambiguous psychological game having GPS psychological Nash equilibria. The selection mechanism works as follows:

- for every player i, choose a sequence of beliefs correspondences $\{\gamma_{i,\nu}\}_{\nu\in\mathbb{N}}$, with $\gamma_{i,\nu} : \Sigma \rightsquigarrow \overline{B}_i$ that sequentially converges to the function β_i (which represents the hierarchies of correct beliefs of the game G^{GPS});
- for every player i, choose a sequence of utility functions $\{u_{i,\nu}\}_{\nu\in\mathbb{N}}$ with $u_{i,\nu} : \overline{B}_i \times \Sigma \to \mathbb{R}$ that sequentially converges to the function u_i;
- let $\{U_{i,\nu}\}_{\nu\in\mathbb{N}}$ be the sequence of functions $U_{i,\nu} : \mathscr{K}_i \times \Sigma \to \mathbb{R}$ defined, for every player i, by

  $$U_{i,\nu}(K_i,\sigma) = \inf_{b_i\in K_i} u_{i,\nu}(b_i,\sigma) \quad \forall (K_i,\sigma) \in \mathscr{K}_i \times \Sigma,$$

  and consider the corresponding sequence of ambiguous games $\{G_\nu\}_{\nu\in\mathbb{N}}$ where
  $G_\nu = \left\{A_1,\dots,A_n,U_{1,\nu},\dots,U_{n,\nu}\right\}$ for every $\nu \in \mathbb{N}$;
- let $\{(\gamma_\nu(\sigma_\nu),\sigma_\nu)\}_{\nu\in\mathbb{N}}$ be a sequence where each $(\gamma_\nu(\sigma_\nu),\sigma_\nu)$ is a psychological Nash equilibrium under ambiguity of G_ν. Since Σ is compact, then $\{\sigma_\nu\}_\nu$ has a converging subsequence $\{\sigma_{\nu_k}\}_{k\in\mathbb{N}}$ whose limit is σ^*. Consequently, the subsequence $\{(\gamma_{\nu_k}(\sigma_{\nu_k}),\sigma_{\nu_k})\}_{k\in\mathbb{N}}$ converges to $(\beta(\sigma^*),\sigma^*)$, which is a psychological Nash equilibrium of G^{GPS} in light of Theorem 5.3. Hence, the psychological Nash equilibrium $(\beta(\sigma^*),\sigma^*)$ is *stable* with respect to the perturbation given by the sequence of games $\{G_{\nu_k}\}_{k\in\mathbb{N}}$;
- if the set of limit points of all the sequences of equilibria corresponding to the sequence of games $\{G_{\nu_k}\}_{k\in\mathbb{N}}$ is a proper subset of the set of equilibria of G^{GPS} then the selection method is *effective*.

Remark 5.5. An underlying assumption is required so that the selection mechanism previously presented makes sense: it consists in the existence of psychological Nash equilibria at least for a subsequence of the sequence games $\{G_\nu\}_{\nu\in\mathbb{N}}$. Nevertheless, the examples in Section 3 show that it is reasonably simple to construct sequences of psychological games under ambiguity with nonempty sets of equilibria.

Remark 5.6. At first sight, it might seem surprising that an equilibrium is selected if it is a limit point for just one sequence of perturbed equilibria. However, this is precisely what happens for trembling hand perfect equilibria. In fact, even in the classical game theory (with no psychological effects) it turns out that there exist entire classes of games in which no equilibrium is stable with respect to every possible perturbation. Therefore, the weaker assumption that we use is much more likely to be applicable; moreover, Example 3.4 shows that it can provide an effective selection mechanism in simple games.

6. Conclusion

The present paper aims to jointly take into account two issues that arise from different strands of literature. On the one hand, the studies on psychological games point out that players' preferences might depend on the hierarchies of beliefs. On the other hand, the literature on *strategic ambiguity* in classical games suggests that beliefs might be ambiguous (or imprecise) in equilibrium. In this paper we deal with simultaneous-move psychological games characterized by ambiguous beliefs that are represented as multiple hierarchies of beliefs. In the new concept of psychological Nash equilibrium under ambiguity, the correct belief function of each player is replaced by a set-valued map that specifies the set of hierarchies of beliefs that the corresponding player perceives to be consistent with the equilibrium played; moreover ambiguity is solved by considering the classical *maxmin* preferences. It follows that this concept generalizes the standard psychological Nash equilibrium defined in Geanakoplos et al. (1989) in a natural way and it embodies different models of strategic ambiguity as the *partially specified probability model*. The theory shows that continuity of the beliefs correspondences is the key for equilibrium existence. In addition, examples highlight that the presence of ambiguity may alter significantly the equilibria of the game: either they can be totally different from the unambiguous case or we can run into equilibrium selection.

The role of ambiguity as equilibrium selector puts our paper in relation with the theory of Nash equilibria refinements: we look at the problem of stability of psychological equilibria when perturbations affect the entire hierarchy of correct beliefs. Firstly we show that, under suitable assumptions, we can obtain the convergence of equilibria of perturbed game to those of the unperturbed one. As a consequence, it is possible to refine psychological Nash equilibria by constructing selection mechanisms based on properties of stability with respect to ambiguous trembles on the hierarchies of beliefs.

This paper is just a first step in the study of ambiguity in psychological games. As such, there are some limitations in our analysis; hence, many relevant and intricate questions remain open. On the one hand, the extension to different types of preferences under ambiguity and the effects on the theoretical results should be explored. On the other hand, our analysis focus on the specific class of psychological games involving simultaneous moves so that only initial beliefs are allowed to affect utility. In the dynamic psychological game model studied in Battigalli and Dufwenberg (2009), both sequential play and updated beliefs are allowed; so a relevant issue would be to study the problem of sequential rationality and updated beliefs in case of ambiguous *conditional probability systems*. The issues previously mentioned are certainly theoretically and technically challenging and will be the focus of future research.

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