Discrete Mathematics 349 (2026) 115069

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

On H-joins of complex unit gain graphs and their stability

Callum Huntington [a,*], Stefano Spessato [b], Francesco Belardo [a]

[a] Department of Mathematics and Applications R. Caccioppoli, University of Naples Federico II, Naples, Italy
[b] Department of Engineering, Università Niccolò Cusano, Rome, Italy

ARTICLE INFO

Article history:
Received 17 September 2025
Received in revised form 19 January 2026
Accepted 16 February 2026
Available online xxxx

Keywords:
Gain graph
H-join
Adjacency matrix
Balancedness
Spectrum
Switching equivalence

ABSTRACT

Let H be a graph of order k and let $\mathcal{F}=\{G_1,G_2,\dots,G_k\}$ be a family of vertex-disjoint graphs. Then the H-join of the family $\mathcal{F}$ is obtained by replacing each vertex v_i of H with the graph G_i of $\mathcal{F}$ and preserving the adjacencies existing in H. This article presents two distinct definitions of the H-join of complex unit gain graphs. The first is a direct extension of the H-join of signed graphs and, as in that setting, is switching-stable only with respect to H. The second, on the other hand, requires the fixing of a spanning forest in order to be defined but has the advantage of being stable under switchings of both H and the component G_i graphs. When applied to signed graphs this provides a new definition of the H-join which does not coincide with the usual version and has the stronger property of being switching-stable with respect to all of the graphs involved. A spectral analysis is provided for both constructions.

1. Introduction

Let $\mathbb{T}$ be the group of all complex numbers z with $|z|=1$. A complex unit gain graph, or simply a $\mathbb{T}$-gain graph, is a triple $\tilde{G}=(G,\mathbb{T},\varphi)$ consisting of an underlying graph $G=(V_G,E_G)$, the circle group $\mathbb{T}$, and a gain function $\varphi:E_G\to\mathbb{T}$ assigning a complex unit to each oriented edge such that $\varphi(v_iv_j)=\varphi(v_jv_i)^{-1}$ for any adjacent vertices v_i and v_j. We think of an ordinary graph G as the all-positive complex unit gain graph $\tilde{G}=(G,\mathbb{T},+)$ whose gain function assigns the value $+1$ to all of its edges. When two vertices v_i and v_j are adjacent, we may denote this by $v_i\sim v_j$. The graphs to be considered in this paper shall all be simple.

The adjacency matrix $A_{\tilde{G}}$ of a $\mathbb{T}$-gain graph $\tilde{G}=(G,\mathbb{T},\varphi)$ of order n is the $n\times n$ complex matrix defined by

$$(a_{ij})=\begin{cases}\varphi(v_iv_j), & \text{if } v_iv_j\in E_G,\\ 0, & \text{otherwise.}\end{cases}$$

The characteristic polynomial of $\tilde{G}$ is the characteristic polynomial of $A_{\tilde{G}}$, and it is denoted by $p_{\tilde{G}}(\lambda)=\det\big(\lambda I-A_{\tilde{G}}\big)$. The spectrum of $\tilde{G}$ is denoted by $\operatorname{spec}_{\tilde{G}}$ and is made up of the eigenvalues of $A_{\tilde{G}}$. When we say eigenvalues of $\tilde{G}$ we are referring to the eigenvalues of $A_{\tilde{G}}$.

Two $\mathbb{T}$-gain graphs $\tilde{G}=(G,\mathbb{T},\varphi)$ and $\tilde{G}'=(G,\mathbb{T},\varphi')$ on the same underlying graph are switching equivalent if there exists a function $s:V_G\to\mathbb{T}$ such that for any pair of adjacent vertices v_i and v_j we have

* Corresponding author.
E-mail addresses: callum.huntington@unina.it (C. Huntington), stefano.spessato@unicusano.it (S. Spessato), fbelardo@unina.it (F. Belardo).

https://doi.org/10.1016/j.disc.2026.115069

$$\varphi'(v_iv_j) = s(v_i)^{-1}\varphi(v_iv_j)s(v_j).$$

To represent switching equivalence between graphs or gain functions we may write $\tilde{G} \sim \tilde{G}'$ or $\varphi \sim \varphi'$.

In a graph G, a closed walk W of length k is a list of vertices $v_1v_2\dots v_{k+1}$ such that $v_{k+1} = v_1$ and $v_i \sim v_{i+1}$ for each $i = 1, \dots, k$. If G is endowed with a complex unit gain function φ, it is possible to define the gain of a closed walk as $\varphi(W) = \varphi(v_1v_2)\varphi(v_2v_3)\dots\varphi(v_kv_{k+1})$. Given in terms of the gains of closed walks, two gain functions φ and φ' are switching equivalent if and only if $\varphi(W) = \varphi'(W)$ for every closed walk W in G.

A complex unit gain graph is balanced if and only if the product of the edge gains in every closed walk is equal to the identity element of the gain group $\mathbb{T}$. Equivalently, a complex unit gain graph with gain function φ is balanced if and only if it can be switched to the all-positive underlying graph, that is, if $\varphi \sim +$ (see [29, Lemma 5.3]).

It is known that the switching equivalence relation between complex unit gain graphs can be expressed in terms of matrices. Namely, $\tilde{G}$ and $\tilde{G}'$ are switching equivalent if and only if there exists a diagonal matrix S with entries in $\mathbb{T}$ such that $A_{\tilde{G}'} = S^*A_{\tilde{G}}S$, where S^* is the conjugate transpose of S.

In graph theory there are various types of products and operations whose purpose is to produce a new graph G from given graphs $G_1, \dots, G_k$. A topic of active research interest is the extension of these operations to signed graphs [4,6,17,19,22], to complex unit gain graphs [1,3,5], and, more generally, to arbitrary gain graphs [12,14,15]. In other words, the goal is to define a desirable signature or gain function on G when the graphs $G_1, \dots, G_k$ are already equipped with such structures. Typically, a well-designed generalisation of a product or join operation on gain graphs should aim to satisfy as many of the following three properties as possible:

- If the gain functions on $G_1, \dots, G_k$ are all equal to the identity function, the gain function on G assigns the identity element to every edge of G.
- Geometric and spectral properties are preserved or suitably generalised.
- The operation is switching-stable, meaning that replacing the gain functions of $G_1, \dots, G_k$ with switching equivalent ones does not change the switching equivalence class of G.

This last property is particularly useful in spectral computations since it is well known that switching equivalent gain graphs are cospectral.

In this article, we focus on the H-join of complex unit gain graphs. The first iteration of the operation we are studying was introduced as the generalised composition of graphs by Schwenk in 1974 in [27]. It was then rediscovered in 2011 by Cardoso et al. in [10] as an application of generalising Fiedler's lemma. Since then there has been much work in the theory of both ordinary graphs and signed graphs. For more information, see [2,8,9,11,21,23,25,28,31].

In [13] the authors introduce the notion of pseudo-potential functions on signed graphs and give a solution to an open problem concerning the switching-stability of the lexicographic product of signed graphs. Following the approach of works such as [8] and [31], we begin by providing a direct extension of the H-join to the setting of complex unit gain graphs. In the second half of this paper, we extend the pseudo-potential functions of [13] to complex unit gain graphs and, together with the direct extension of the H-join from the first half, define a distinct and more robust construction that avoids the stability and balance issues present in the original setting. Since many join operations and families of graphs can be recovered as special cases or subgraphs of the H-join, H-generalised-join, and H_m-join operations, see [2], our novel approach offers a unified solution to a wide range of potential stability and balance problems.

The remainder of this paper is structured as follows. In Section 2, we collect some preliminary facts and results for later use. In Section 3, we define the $\tilde{H}$-join operation of complex unit gain graphs, determine the characteristic polynomial and spectrum, show that the operation is stable under switching of $\tilde{H}$, and give a characterisation of the balancedness of the produced graph. In Section 4, we define pseudo-potential functions on complex unit gain graphs which enable us to provide an alternative definition for the gain function of a $\tilde{H}$-join, one that is stable under switchings of both $\tilde{H}$ and the component $\tilde{G}_i$ graphs and consequently relaxes the requirements for balancedness. In Section 5, we present a chronological survey of signatures and gain functions that have been given for the lexicographic product of two graphs, an operation that can be viewed as a special case of the H-join, in which each G_i is isomorphic to the others, along with some discussion of how these relate to the constructions that we have proposed.

2. Preliminaries

We shall use the following notations throughout the rest of this paper. The identity matrix is denoted by I_n when it is of order n or simply by I if the relevant size is clear. We denote the all-one matrix by $J_{m\times n}$ if it is an $m \times n$ matrix, by J_n if it is square of order n, or just by J. Likewise we denote the all-zero matrix by $O_{m\times n}$ if it is an $m \times n$ matrix, or by O_n, or just O. In addition to this, the all-one vector $J_{n\times 1}$ may be denoted by $\mathbf{1}_n$ and the all-zero vector $O_{n\times 1}$ may be denoted by $\mathbf{0}_n$.

Throughout, we adopt the standard notations of P_n, K_n, and C_n to denote the path, complete, and cycle graphs on n vertices, respectively.

The following is the very important result known as Schur's determinant or complement formula. This was first introduced in 1917 by Issai Schur [26, pp. 215–216] and appears in many textbooks, see [16].

Lemma 2.1. *Let* $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ *be a block matrix. Let A and D be square matrices. It follows that*

(a) if A is an invertible matrix, $\det(Q) = \det(A)\det\big(D - CA^{-1}B\big)$,
(b) if D is an invertible matrix, $\det(Q) = \det(D)\det\big(A - BD^{-1}C\big)$.

The matrix $D - CA^{-1}B$ ($A - BD^{-1}C$, respectively) is usually called the Schur complement of the matrix A (the matrix D, respectively) in the matrix Q and is denoted Q/A (Q/D, respectively). Both expressions can be obtained by using block Gaussian elimination. For instance, Lemma 2.1(a) follows from the factorisation

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I & O \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B \\ O & D - CA^{-1}B \end{pmatrix}.$$

The subsequent result comes immediately from applying Schur's complement formula to particular matrices.

Lemma 2.2 *([2, Lemma 3]). Let X and Y be* $n \times m$ *matrices and let M be an* $n \times n$ *invertible matrix. Then*

(a) $\det\big(I_n + XY^T\big) = \det\big(I_m + Y^TX\big)$.
(b) $\det\big(M + XY^T\big) = \det(M)\det\big(I_m + Y^TM^{-1}X\big)$.

For further applications and historical notes pertaining to Schur's result, see [30].

The graph invariant known as the coronal was introduced in 2011 by McLeman and McNicholas in [24]. In the following we adapt their definition directly to a matrix.

Definition 2.3 *([24, Definition 1]).* Let M be an $n \times n$ matrix. The coronal $\chi_M(\lambda)$ of M is defined to be the sum of the entries of the inverse of the characteristic matrix of M. This can be calculated as

$$\chi_M(\lambda) = \mathbf{1}_n^T(\lambda I_n - M)^{-1}\mathbf{1}_n.$$

We also have the following useful result.

Lemma 2.4 *([24, Proposition 6]). Let M be an* $n \times n$ *matrix with all row sums equal to r. Then*

$$\chi_M(\lambda) = \frac{n}{\lambda - r}.$$

3. $\tilde{H}$-join operation

We shall begin with the definition of the $\tilde{H}$-join and then we detail the structuring of its adjacency matrix along with the computations of its characteristic polynomial and spectrum. To conclude this section, the stability of the $\tilde{H}$-join under switching of $\tilde{H}$ is shown as well as the conditions required on both $\tilde{H}$ and the family of vertex-disjoint $\tilde{G}_i$ graphs in order for their join to be balanced.

3.1. Definition

First we describe how this operation works with ordinary graphs. Let H be a graph of order k and let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of vertex-disjoint graphs. The H-join of $\mathcal{F}$ is obtained by first replacing each vertex v_i of H with the graph G_i of $\mathcal{F}$ and then, for each edge between vertices v_i and v_j in E_H, introducing new edges between every vertex in G_i and every vertex in G_j.

This procedure can be taken to the realm of complex unit gain graphs with the following definition.

Definition 3.1. Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ be a $\mathbb{T}$-gain graph with $V_H = \{v_1, \ldots, v_k\}$ and let $\mathcal{F} = \{\tilde{G}_1, \ldots, \tilde{G}_k\}$ be a family of vertex-disjoint complex unit gain graphs where $\tilde{G}_i = (G_i, \mathbb{T}, \varphi_i)$ is a $\mathbb{T}$-gain graph of order n_i for each $i = 1, \ldots, k$. Then the $\tilde{H}$-join of this family is the $\mathbb{T}$-gain graph $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \ldots, \tilde{G}_k\}$, sometimes denoted $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}}$, with vertex set $V_G = \bigcup_{i=1}^k V_{G_i}$, oriented edge set

$$E_G = \left(\bigcup_{i=1}^k E_{G_i}\right) \cup \left(\bigcup_{v_iv_j \in E_H} \{uv : u \in V_{G_i}, v \in V_{G_j}\}\right),$$

and gain function φ satisfying

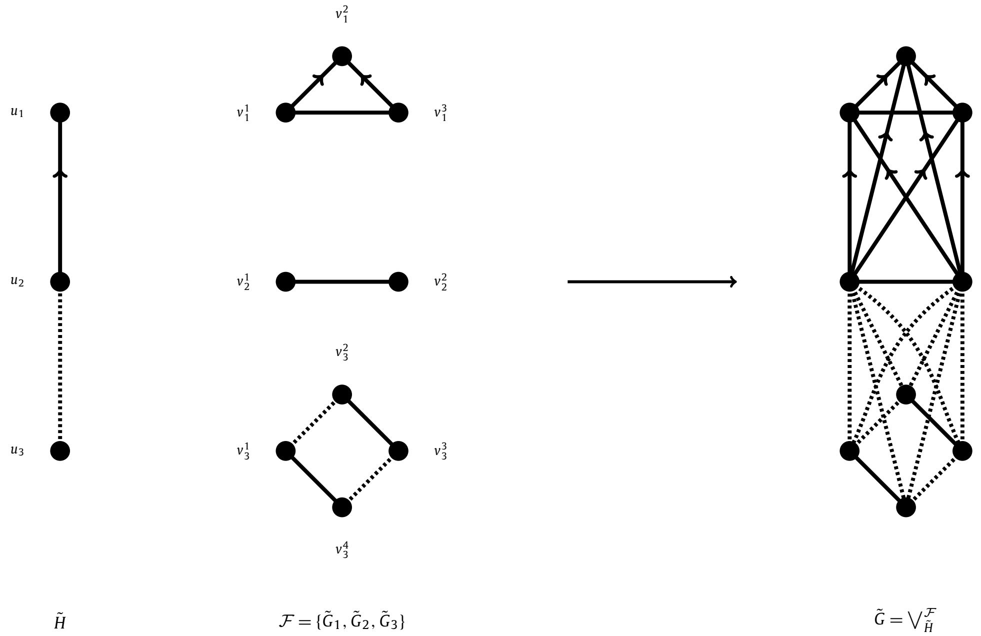

Fig. 1. Take $\tilde{H} = (P_3, \mathbb{T}, \varphi_H)$, $\tilde{G}_1 = (K_3, \mathbb{T}, \varphi_1)$, $\tilde{G}_2 = (K_2, \mathbb{T}, \varphi_2)$, and $\tilde{G}_3 = (C_4, \mathbb{T}, \varphi_3)$ with gain functions and vertex labellings as shown. Then the right hand side of the diagram depicts the $\tilde{H}$-join of $\mathcal{F} = \{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3\}$. Here and throughout this paper, solid lines represent positive edges, dashed lines represent negative edges, and lines directed from a to b represent edges with gain functions $\varphi(ab) = i$ and $\varphi(ba) = -i$.

$$\varphi(uv) = \begin{cases} \varphi_i(uv), & \text{if } u, v \in V_{G_i} \text{ and } uv \in E_{G_i}, \\ \varphi_H(v_i v_j), & \text{if } u \in V_{G_i}, v \in V_{G_j}, \text{and } v_i v_j \in E_H. \end{cases}$$

3.2. Adjacency matrix and characteristic polynomial

We can now describe the adjacency matrix of a $\mathbb{T}$-gain graph obtained via the $\tilde{H}$-join procedure and, consequently, determine its characteristic polynomial.

Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ be a $\mathbb{T}$-gain graph with $V_H = \{v_1, \ldots, v_k\}$. Consider the $\tilde{H}$-join graph $\tilde{G} = \bigvee_{\tilde{H}} \{\tilde{G}_1, \ldots, \tilde{G}_k\}$ where $\tilde{G}_i = (G_i, \mathbb{T}, \varphi_i)$ is a $\mathbb{T}$-gain graph of order n_i for each $i = 1, \ldots, k$. Then, with consistent vertex labellings, the adjacency matrix of $\tilde{G}$ can be expressed as the $k \times k$ block matrix $A_{\tilde{G}} = (A_{ij})$ with

$$A_{ij} = \begin{cases} A_{\tilde{G}_i}, & \text{if } i = j, \\ \varphi_H(v_i v_j) J_{n_i \times n_j}, & \text{if } i \neq j. \end{cases}$$

An example, with the blocks clearly depicted, is given below.

Example 3.2. Take $\tilde{H} = (P_3, \mathbb{T}, \varphi_H)$, $\tilde{G}_1 = (K_3, \mathbb{T}, \varphi_1)$, $\tilde{G}_2 = (K_2, \mathbb{T}, \varphi_2)$, and $\tilde{G}_3 = (C_4, \mathbb{T}, \varphi_3)$ with gain functions and vertex labellings as shown in Fig. 1. Then for the adjacency matrix of the $\tilde{H}$-join of $\mathcal{F} = \{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3\}$ we have

$$A_{\tilde{G}} = \left(\begin{array}{ccc|cc|cccc} 0 & i & 1 & -i & -i & 0 & 0 & 0 & 0 \\ -i & 0 & -i & -i & -i & 0 & 0 & 0 & 0 \\ 1 & i & 0 & -i & -i & 0 & 0 & 0 & 0 \\ \hline i & i & i & 0 & 1 & -1 & -1 & -1 & -1 \\ i & i & i & 1 & 0 & -1 & -1 & -1 & -1 \\ \hline 0 & 0 & 0 & -1 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -1 & 1 & 0 & -1 & 0 \end{array}\right).$$

Theorem 3.3. *Consider a complex unit gain graph $\tilde{G} = \bigvee_{\tilde{H}} \{\tilde{G}_1, \ldots, \tilde{G}_k\}$. Define the characteristic polynomial of $\tilde{G}_i$ for each $i = 1, \ldots, k$ as $p_i = \det\left(\lambda I_{n_i} - A_{\tilde{G}_i}\right)$ and denote $\chi_i(\lambda) = \mathbf{1}_{n_i}^T (\lambda I_{n_i} - A_{\tilde{G}_i})^{-1} \mathbf{1}_{n_i}$. Then, the characteristic polynomial of $\tilde{G}$ is*

$$p_{\tilde{G}}(\lambda) = \left(\prod_{i=1}^{k} p_i\right) \det(M),$$

where $M = (m_{ij})$ is the $k \times k$ matrix with

$$m_{ij} = \begin{cases} 1, & \text{if } i = j, \\ -\varphi_H(v_i v_j)\chi_i, & \text{if } i \neq j. \end{cases}$$

Proof. Within this proof we shall use η_{ij} to denote $\varphi_H(v_i v_j)$. Thus, we have

$$A_{\tilde{G}} = \begin{pmatrix} A_{\tilde{G}_1} & \eta_{12} J_{n_1 \times n_2} & \cdots & \eta_{1k} J_{n_1 \times n_k} \\ \eta_{21} J_{n_2 \times n_1} & A_{\tilde{G}_2} & \cdots & \eta_{2k} J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k1} J_{n_k \times n_1} & \eta_{k2} J_{n_k \times n_2} & \cdots & A_{\tilde{G}_k} \end{pmatrix}.$$

Since

$$p_{\tilde{G}}(\lambda) = \det \underbrace{\begin{pmatrix} \lambda I_{n_1} - A_{\tilde{G}_1} & -\eta_{12} J_{n_1 \times n_2} & \cdots & -\eta_{1,k-1} J_{n_1 \times n_{k-1}} & -\eta_{1k} J_{n_1 \times n_k} \\ -\eta_{21} J_{n_2 \times n_1} & \lambda I_{n_2} - A_{\tilde{G}_2} & \cdots & -\eta_{2,k-1} J_{n_2 \times n_{k-1}} & -\eta_{2k} J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\eta_{k-1,1} J_{n_{k-1} \times n_1} & -\eta_{k-1,2} J_{n_{k-1} \times n_2} & \cdots & \lambda I_{n_{k-1}} - A_{\tilde{G}_{k-1}} & -\eta_{k-1,k} J_{n_{k-1} \times n_k} \\ -\eta_{k1} J_{n_k \times n_1} & -\eta_{k2} J_{n_k \times n_2} & \cdots & -\eta_{k,k-1} J_{n_k \times n_{k-1}} & \lambda I_{n_k} - A_{\tilde{G}_k} \end{pmatrix}}_{Q},$$

partitioning this matrix as the block matrix $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that

$$A = \begin{pmatrix} \lambda I_{n_1} - A_{\tilde{G}_1} & -\eta_{12} J_{n_1 \times n_2} & \cdots & -\eta_{1,k-1} J_{n_1 \times n_{k-1}} \\ -\eta_{21} J_{n_2 \times n_1} & \lambda I_{n_2} - A_{\tilde{G}_2} & \cdots & -\eta_{2,k-1} J_{n_2 \times n_{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{k-1,1} J_{n_{k-1} \times n_1} & -\eta_{k-1,2} J_{n_{k-1} \times n_2} & \cdots & \lambda I_{n_{k-1}} - A_{\tilde{G}_{k-1}} \end{pmatrix},$$

$$B = \begin{pmatrix} -\eta_{1k} J_{n_1 \times n_k} \\ -\eta_{2k} J_{n_2 \times n_k} \\ \vdots \\ -\eta_{k-1,k} J_{n_{k-1} \times n_k} \end{pmatrix},$$

$$C = \left(-\eta_{k1} J_{n_k \times n_1} \quad -\eta_{k2} J_{n_k \times n_2} \quad \cdots \quad -\eta_{k,k-1} J_{n_k \times n_{k-1}}\right),$$

$$D = \left(\lambda I_{n_k} - A_{\tilde{G}_k}\right),$$

and applying Schur's determinant formula, it follows that

$$p_{\tilde{G}}(\lambda) = \det(Q) = \det(D)\det\left(A - BD^{-1}C\right) = \det(D)\det(Q/D), \tag{1}$$

where $\det(Q/D)$ is the Schur complement of the matrix D in Q.

Let us compute the determinant of Q/D, taking into account that $\eta_{ij}\eta_{ji} = \eta_{ij}\eta_{ij}^{-1} = 1$, $J_{n_i \times n_j} = \mathbf{1}_{n_i}\mathbf{1}_{n_j}^T$, and $\chi_i = \mathbf{1}_{n_i}^T(\lambda I_{n_i} - A_{\tilde{G}_i})^{-1}\mathbf{1}_{n_i}$. We have

$$\det(Q/D) = \det\left(A - \underbrace{\begin{pmatrix} -\eta_{1k} J_{n_1 \times n_k} \\ -\eta_{2k} J_{n_2 \times n_k} \\ \vdots \\ -\eta_{k-1,k} J_{n_{k-1} \times n_k} \end{pmatrix}}_{B} \underbrace{\left(\lambda I_{n_k} - A_{\tilde{G}_k}\right)^{-1}}_{D^{-1}} \underbrace{\begin{pmatrix} -\eta_{k1} J_{n_k \times n_1} \\ -\eta_{k2} J_{n_k \times n_2} \\ \vdots \\ -\eta_{k,k-1} J_{n_k \times n_{k-1}} \end{pmatrix}^T}_{C} \right)$$

$$= \det\left(A - \begin{pmatrix} \eta_{1k}\eta_{k1}\mathbf{1}_{n_1}\chi_k\mathbf{1}_{n_1}^T & \eta_{1k}\eta_{k2}\mathbf{1}_{n_1}\chi_k\mathbf{1}_{n_2}^T & \cdots & \eta_{1k}\eta_{k,k-1}\mathbf{1}_{n_1}\chi_k\mathbf{1}_{n_{k-1}}^T \\ \eta_{2k}\eta_{k1}\mathbf{1}_{n_2}\chi_k\mathbf{1}_{n_1}^T & \eta_{2k}\eta_{k2}\mathbf{1}_{n_2}\chi_k\mathbf{1}_{n_2}^T & \cdots & \eta_{2k}\eta_{k,k-1}\mathbf{1}_{n_2}\chi_k\mathbf{1}_{n_{k-1}}^T \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k-1,k}\eta_{k1}\mathbf{1}_{n_{k-1}}\chi_k\mathbf{1}_{n_1}^T & \eta_{k-1,k}\eta_{k2}\mathbf{1}_{n_{k-1}}\chi_k\mathbf{1}_{n_2}^T & \cdots & \eta_{k-1,k}\eta_{k,k-1}\mathbf{1}_{n_{k-1}}\chi_k\mathbf{1}_{n_{k-1}}^T \end{pmatrix} \right)$$

$$
= \begin{pmatrix} \lambda I_{n_1} - A_{\tilde{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\tilde{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\tilde{G}_{k-1}} \end{pmatrix}
$$

$$
- \begin{pmatrix} \mathbf{1}_{n_1}\chi_k\mathbf{1}_{n_1}^T & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k2}\chi_k+\eta_{12})\mathbf{1}_{n_2}^T & \cdots & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k,k-1}\chi_k+\eta_{1,k-1})\mathbf{1}_{n_{k-1}}^T \\ \mathbf{1}_{n_2}(\eta_{2k}\eta_{k1}\chi_k+\eta_{21})\mathbf{1}_{n_1}^T & \mathbf{1}_{n_2}\chi_k\mathbf{1}_{n_2}^T & \cdots & \mathbf{1}_{n_2}(\eta_{2k}\eta_{k,k-1}\chi_k+\eta_{2,k-1})\mathbf{1}_{n_{k-1}}^T \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k1}\chi_k+\eta_{k-1,1})\mathbf{1}_{n_1}^T & \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k2}\chi_k+\eta_{k-1,2})\mathbf{1}_{n_2}^T & \cdots & \mathbf{1}_{n_{k-1}}\chi_k\mathbf{1}_{n_{k-1}}^T \end{pmatrix}
$$

$$
= \begin{pmatrix} \lambda I_{n_1} - A_{\tilde{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\tilde{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\tilde{G}_{k-1}} \end{pmatrix}
$$

$$
- \left(\begin{pmatrix} \mathbf{1}_{n_1}\chi_k & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k2}\chi_k+\eta_{12}) & \cdots & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k,k-1}\chi_k+\eta_{1,k-1}) \\ \mathbf{1}_{n_2}(\eta_{2k}\eta_{k1}\chi_k+\eta_{21}) & \mathbf{1}_{n_2}\chi_k & \cdots & \mathbf{1}_{n_2}(\eta_{2k}\eta_{k,k-1}\chi_k+\eta_{2,k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k1}\chi_k+\eta_{k-1,1}) & \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k2}\chi_k+\eta_{k-1,2}) & \cdots & \mathbf{1}_{n_{k-1}}\chi_k \end{pmatrix} \cdot \begin{pmatrix} \mathbf{1}_{n_1}^T & O & \cdots & O \\ O & \mathbf{1}_{n_2}^T & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{1}_{n_{k-1}}^T \end{pmatrix} \right).
$$

At this point we can apply Lemma 2.2(b) to obtain the determinant of Q/D as

$$
\det(Q/D) = \left(\prod_{i=1}^{k-1} p_i\right) \det\left(I_{k-1} - \begin{pmatrix} \mathbf{1}_{n_1}^T & O & \cdots & O \\ O & \mathbf{1}_{n_2}^T & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \mathbf{1}_{n_{k-1}}^T \end{pmatrix} \begin{pmatrix} \lambda I_{n_1} - A_{\tilde{G}_1} & O & \cdots & O \\ O & \lambda I_{n_2} - A_{\tilde{G}_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & \lambda I_{n_{k-1}} - A_{\tilde{G}_{k-1}} \end{pmatrix}^{-1} \right.
$$

$$
\left. \cdot \begin{pmatrix} \mathbf{1}_{n_1}\chi_k & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k2}\chi_k+\eta_{12}) & \cdots & \mathbf{1}_{n_1}(\eta_{1k}\eta_{k,k-1}\chi_k+\eta_{1,k-1}) \\ \mathbf{1}_{n_2}(\eta_{2k}\eta_{k1}\chi_k+\eta_{21}) & \mathbf{1}_{n_2}\chi_k & \cdots & \mathbf{1}_{n_2}(\eta_{2k}\eta_{k,k-1}\chi_k+\eta_{2,k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k1}\chi_k+\eta_{k-1,1}) & \mathbf{1}_{n_{k-1}}(\eta_{k-1,k}\eta_{k2}\chi_k+\eta_{k-1,2}) & \cdots & \mathbf{1}_{n_{k-1}}\chi_k \end{pmatrix} \right)
$$

$$
= \left(\prod_{i=1}^{k-1} p_i\right) \det\left(I_{k-1} - \begin{pmatrix} \chi_1\chi_k & \chi_1(\eta_{1k}\eta_{k2}\chi_k+\eta_{12}) & \cdots & \chi_1(\eta_{1k}\eta_{k,k-1}\chi_k+\eta_{1,k-1}) \\ \chi_2(\eta_{2k}\eta_{k1}\chi_k+\eta_{21}) & \chi_2\chi_k & \cdots & \chi_2(\eta_{2k}\eta_{k,k-1}\chi_k+\eta_{2,k-1}) \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{k-1}(\eta_{k-1,k}\eta_{k1}\chi_k+\eta_{k-1,1}) & \chi_{k-1}(\eta_{k-1,k}\eta_{k2}\chi_k+\eta_{k-1,2}) & \cdots & \chi_{k-1}\chi_k \end{pmatrix} \right)
$$

$$
= \left(\prod_{i=1}^{k-1} p_i\right) \det\left(\begin{pmatrix} 1 & -\eta_{12}\chi_1 & \cdots & -\eta_{1,k-1}\chi_1 \\ -\eta_{21}\chi_2 & 1 & \cdots & -\eta_{2,k-1}\chi_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{k-1,1}\chi_{k-1} & -\eta_{k-1,2}\chi_{k-1} & \cdots & 1 \end{pmatrix} - \begin{pmatrix} \chi_1\chi_k & \eta_{1k}\eta_{k2}\chi_1\chi_k & \cdots & \eta_{1k}\eta_{k,k-1}\chi_1\chi_k \\ \eta_{2k}\eta_{k1}\chi_2\chi_k & \chi_2\chi_k & \cdots & \eta_{2k}\eta_{k,k-1}\chi_2\chi_k \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k-1,k}\eta_{k1}\chi_{k-1}\chi_k & \eta_{k-1,k}\eta_{k2}\chi_{k-1}\chi_k & \cdots & \chi_{k-1}\chi_k \end{pmatrix} \right)
$$

$$= \left(\prod_{i=1}^{k-1} p_i\right) \det\left(\begin{pmatrix} 1 & -\eta_{12}\chi_1 & \cdots & -\eta_{1,k-1}\chi_1 \\ -\eta_{21}\chi_2 & 1 & \cdots & -\eta_{2,k-1}\chi_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{k-1,1}\chi_{k-1} & -\eta_{k-1,2}\chi_{k-1} & \cdots & 1 \end{pmatrix} - \begin{pmatrix} \eta_{1k}\chi_1 \\ \eta_{2k}\chi_2 \\ \vdots \\ \eta_{k-1,k}\chi_{k-1} \end{pmatrix} (1) \begin{pmatrix} \eta_{k1}\chi_k \\ \eta_{k2}\chi_k \\ \vdots \\ \eta_{k,k-1}\chi_k \end{pmatrix}^T\right).$$

Employing a reversal of Schur's complement formula here gives us

$$\det(Q/D) = \left(\prod_{i=1}^{k-1} p_i\right) \det\begin{pmatrix} 1 & -\eta_{12}\chi_1 & \cdots & -\eta_{1k}\chi_1 \\ -\eta_{21}\chi_2 & 1 & \cdots & -\eta_{2k}\chi_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\eta_{k1}\chi_k & -\eta_{k2}\chi_k & \cdots & 1 \end{pmatrix}.$$

Finally, from Equation (1) and substituting $\varphi_H(v_i v_j)$ back in for η_{ij}, we have

$$p_{\tilde{G}}(\lambda) = \left(\prod_{i=1}^{k} p_i\right) \det\begin{pmatrix} 1 & -\varphi_H(v_1v_2)\chi_1 & \cdots & -\varphi_H(v_1v_k)\chi_1 \\ -\varphi_H(v_2v_1)\chi_2 & 1 & \cdots & -\varphi_H(v_2v_k)\chi_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_H(v_kv_1)\chi_k & -\varphi_H(v_kv_2)\chi_k & \cdots & 1 \end{pmatrix} = \left(\prod_{i=1}^{k} p_i\right) \det(M).$$

This concludes the proof. □

3.3. Spectrum

Here, we adapt ideas presented by the authors in [25] to our context so that we can derive a spectral description of the $\tilde{H}$-join in terms of the eigenvalues of the component $\tilde{G}_i$ graphs.

First we recall the definition of what it means for an eigenvalue to be main or non-main.

Definition 3.4 *([25, Definition 2]).* Let M be an $n \times n$ normal complex matrix and let $\mathbf{u}$ be an $n \times 1$ complex vector. An eigenvalue λ of M is called a $\mathbf{u}$-main eigenvalue if the corresponding eigenspace $\mathcal{E}_M(\lambda)$ is not orthogonal to $\mathbf{u}$. If an eigenvalue of M is not $\mathbf{u}$-main, we say it is $\mathbf{u}$-non-main. In the case of $\mathbf{u} = \mathbf{1}_n$ we do not specify the vector and simply call the eigenvalue λ a main or non-main eigenvalue of M.

Now we shall state a lemma from [25] along with its proof when $\mathbf{u} = \mathbf{1}_n$, as this will be useful later on.

Lemma 3.5 *([25, Lemma 2]). Let M be an $n \times n$ normal complex matrix and let $\mathbf{u}$ be an $n \times 1$ complex vector. Then the singularities of $\mathbf{u}^T(\lambda I - M)^{-1}\mathbf{u}$ are simple and correspond to the $\mathbf{u}$-main eigenvalues of M.*

Proof. We consider only the case when $\mathbf{u} = \mathbf{1}_n$. Let $\{\theta_1, \dots, \theta_k\}$ be the distinct eigenvalues of M ordered so that $\{\theta_1, \dots, \theta_m\}$, with $m \le k$, is the set of main eigenvalues of M. Suppose that the spectral decomposition of M is $M = \sum_{i=1}^k \theta_i E_{\theta_i}$, where E_{θ_i} is the orthogonal projection on the eigenspace of θ_i. Then $(\lambda I - M)^{-1} = \sum_{i=1}^k \frac{E_{\theta_i}}{\lambda - \theta_i}$ and $\chi_M(\lambda) = \mathbf{1}_n^T(\lambda I - M)^{-1}\mathbf{1}_n = \sum_{i=1}^k \frac{\mathbf{1}_n^T E_{\theta_i}\mathbf{1}_n}{\lambda - \theta_i}$. Note that $\mathbf{1}_n^T E_{\theta_i}\mathbf{1}_n \neq 0$ if and only if θ_i is a main eigenvalue of M, so $\chi_M(\lambda) = \sum_{i=1}^m \frac{\mathbf{1}_n^T E_{\theta_i}\mathbf{1}_n}{\lambda - \theta_i}$, and the result follows. □

In [25, Theorem 2], with $\rho_{i,j}$ being arbitrary complex numbers for $1 \le i, j \le k$ and $i \neq j$, they define the function $\Gamma_i(\lambda) = \Gamma_{M_i}(u_i, v_i; \lambda) = v_i^T(\lambda I - M_i)u_i$ and two matrices

$$A(\mathbf{M}, \mathbf{u}, \rho) := \begin{pmatrix} M_1 & \rho_{1,2}u_1v_2^T & \cdots & \rho_{1,k}u_1v_k^T \\ \rho_{2,1}u_2v_1^T & M_2 & \cdots & \rho_{2,k}u_2v_k^T \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{k,1}u_kv_1^T & \rho_{k,2}u_kv_2^T & \cdots & M_k \end{pmatrix}, \quad \tilde{A}(\mathbf{M}, \mathbf{u}, \rho) := \begin{pmatrix} \frac{1}{\Gamma_1(\lambda)} & -\rho_{1,2} & \cdots & -\rho_{1,k} \\ -\rho_{2,1} & \frac{1}{\Gamma_2(\lambda)} & \cdots & -\rho_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ -\rho_{k,1} & -\rho_{k,2} & \cdots & \frac{1}{\Gamma_k(\lambda)} \end{pmatrix}.$$

For full details, the reader is directed to that paper. For our purposes, we can see that when $M_i = A_{\tilde{G}_i}$, $u_i = v_i = \mathbf{1}_{n_i}$, and $\rho_{i,j} = \varphi_H(v_iv_j)$, we have $\Gamma_i(\lambda) = \chi_i(\lambda)$ and $A(\mathbf{M}, \mathbf{u}, \rho) = A_{\tilde{G}}$.

Since the adjacency matrices of the $\mathbb{T}$-gain graphs $\tilde{G}_i$ are normal, we can suppose that $\{\theta_1, \dots, \theta_{m_i}\}$ is the set of distinct main eigenvalues of $\tilde{G}_i$ for each $i = 1, \dots, k$ and then, as discussed in the proof of Lemma 3.5, we can write $\chi_i = \frac{f_i}{g_i}$ where $g_i = \prod_{j=1}^{m_i}(\lambda - \theta_j)$. Hence, by Theorem 3.3, we have

$$p_{\tilde{G}}(\lambda) = \left(\prod_{i=1}^{k} \frac{p_i}{g_i}\right) \det(\Omega(\lambda)), \tag{2}$$

where $\Omega(\lambda) = (\omega_{ij}(\lambda))$ is the $k \times k$ matrix with

$$\omega_{ij}(\lambda) = \begin{cases} g_i(\lambda), & \text{if } i = j, \\ -\varphi_H(v_i v_j) f_i(\lambda), & \text{if } i \neq j. \end{cases}$$

Consequently, we can describe the spectrum of the $\tilde{H}$-join as follows.

Theorem 3.6. *Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ be a $\mathbb{T}$-gain graph with $V_H = \{v_1, \dots, v_k\}$ and let $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$ be a family of vertex-disjoint graphs where $\tilde{G}_i = (G_i, \mathbb{T}, \varphi_i)$ is a $\mathbb{T}$-gain graph of order n_i for each $i = 1, \dots, k$. Consider the $\tilde{H}$-join of this family, $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \dots, \tilde{G}_k\}$. Then*

(a) if λ is a non-main eigenvalue of $\tilde{G}_i$ of multiplicity μ, it follows that λ is an eigenvalue of $\tilde{G}$ of multiplicity at least μ,
(b) if λ is a main eigenvalue of $\tilde{G}_i$ of multiplicity μ, it follows that λ is an eigenvalue of $\tilde{G}$ of multiplicity at least $\mu - 1$,
(c) the remaining eigenvalues of $\tilde{G}$ are precisely the roots of the polynomial $\det(\Omega(\lambda))$.

Proof. By Lemma 3.5, the singularities of χ_i are main eigenvalues and they are simple. Now the proof easily follows from Equation (2). □

In [5], Brunetti gave the following definitions.

Definition 3.7. Let u be a vertex of a $\mathbb{T}$-gain graph $\tilde{G} = (G, \mathbb{T}, \varphi)$. The numbers

$$d^{\rightarrow}_{\tilde{G}}(u) = \sum_{uv \in E_G} \varphi(uv) \qquad \text{and} \qquad d^{\leftarrow}_{\tilde{G}}(u) = \sum_{uv \in E_G} \varphi(vu)$$

are, respectively, called the $\mathbb{T}$-outgain and the $\mathbb{T}$-ingain of the vertex u.

If u is an isolated vertex, then $d^{\rightarrow}_{\tilde{G}}(u)$ and $d^{\leftarrow}_{\tilde{G}}(u)$ are assumed to be equal to zero. In all cases, $d^{\leftarrow}_{\tilde{G}}(u)$ is the complex conjugate of $d^{\rightarrow}_{\tilde{G}}(u)$.

Definition 3.8 *([5, Definition 1.2]).* An r-$\mathbb{T}$-regular graph is a $\mathbb{T}$-gain graph $\tilde{G} = (G, \mathbb{T}, \varphi)$ such that $d^{\rightarrow}_{\tilde{G}}(u) = r$ for all $u \in V_G$.

By Lemma 2.4 and Definition 3.8, we see that for an r-$\mathbb{T}$-regular graph $\tilde{G}$ on n vertices, we have $\chi_{A_{\tilde{G}}}(\lambda) = \frac{n}{\lambda - r}$.

Consider the $\tilde{H}$-join of a family of vertex-disjoint graphs $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \dots, \tilde{G}_k\}$ where for each $i = 1, \dots, k$, the graph $\tilde{G}_i$ has order n_i and is r_i-$\mathbb{T}$-regular. Since $A_{\tilde{G}_i}$ has constant row sum r_i, the all-one vector $\mathbf{1}_{n_i}$ will be an eigenvector with eigenvalue r_i, and so r_i will clearly be a main eigenvalue of $\tilde{G}_i$. Furthermore, since $A_{\tilde{G}_i}$ is Hermitian, all of the other eigenvectors will be orthogonal to $\mathbf{1}_{n_i}$, meaning all of the other eigenvalues of $\tilde{G}_i$ will be non-main.

We have the following corollary.

Corollary 3.9. *Let $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \dots, \tilde{G}_k\}$ be the $\tilde{H}$-join of a family of vertex-disjoint graphs $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$ in which $\tilde{G}_i = (G_i, \mathbb{T}, \varphi_i)$ is an r_i-$\mathbb{T}$-regular graph of order n_i for each $i = 1, \dots, k$. Then*

$$\mathrm{spec}_{\tilde{G}} = \left(\bigcup_{i=1}^{k}\left(\mathrm{spec}_{\tilde{G}_i} \setminus \{r_i\}\right)\right) \cup \mathrm{spec}_{\Psi},$$

where $\Psi = (\psi_{ij})$ is the $k \times k$ matrix with

$$\psi_{ij} = \begin{cases} r_i, & \text{if } i = j, \\ \varphi_H(v_i v_j) n_i, & \text{if } i \neq j. \end{cases}$$

3.4. A detailed example

In this section we present a fully worked example to illustrate the results we have given so far.

Example 3.10. Let $\tilde{H} = (P_3, \mathbb{T}, \varphi_H)$, $\tilde{G}_1 = (K_3, \mathbb{T}, \varphi_1)$, $\tilde{G}_2 = (K_2, \mathbb{T}, \varphi_2)$, and $\tilde{G}_3 = (C_4, \mathbb{T}, \varphi_3)$ with gain functions and vertex labellings as shown in Fig. 1. Then if we take the adjacency matrix of $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \tilde{G}_2, \tilde{G}_3\}$, which is given explicitly in Example 3.2, and compute the characteristic polynomial directly we have

$$p_{\tilde{G}}(\lambda) = \lambda^9 - 22\lambda^7 - 20\lambda^6 + 97\lambda^5 + 124\lambda^4 - 84\lambda^3 - 176\lambda^2 - 64\lambda.$$

Taking the roots of this, the spectrum is

$$\mathrm{spec}_{\tilde{G}} = \{4.548, 2, 1.456, 0, -0.741, -1, -1, -2, -3.263\}.$$

Now let us gather all of the components needed to make use of the formula in Theorem 3.3. Firstly we have

$$\varphi_H(v_1v_2) = -i,\ \varphi_H(v_2v_1) = i,\ \varphi_H(v_2v_3) = -1,\ \varphi_H(v_3v_2) = -1.$$

Then we have

$$p_1 = \lambda^3 - 3\lambda - 2,\ p_2 = \lambda^2 - 1,\ p_3 = \lambda^4 - 4\lambda^2,$$

along with

$$\chi_1 = \frac{3\lambda - 1}{(\lambda - 2)(\lambda + 1)},\ \chi_2 = \frac{2}{\lambda - 1},\ \chi_3 = \frac{4}{\lambda}.$$

Hence

$$\det(M) = \det\begin{pmatrix} 1 & \frac{(3\lambda-1)i}{(\lambda-2)(\lambda+1)} & 0 \\ -\frac{2i}{\lambda-1} & 1 & \frac{2}{\lambda-1} \\ 0 & \frac{4}{\lambda} & 1 \end{pmatrix} = \frac{\lambda^4 - 2\lambda^3 - 15\lambda^2 + 12\lambda + 16}{\lambda(\lambda-2)(\lambda-1)(\lambda+1)}.$$

Putting all of this together gives us exactly the desired characteristic polynomial

$$\begin{aligned} p_{\tilde{G}}(\lambda) &= (\lambda^3 - 3\lambda - 2)(\lambda^2 - 1)(\lambda^4 - 4\lambda^2)\left(\frac{\lambda^4 - 2\lambda^3 - 15\lambda^2 + 12\lambda + 16}{\lambda(\lambda-2)(\lambda-1)(\lambda+1)}\right) \\ &= \lambda^9 - 22\lambda^7 - 20\lambda^6 + 97\lambda^5 + 124\lambda^4 - 84\lambda^3 - 176\lambda^2 - 64\lambda. \end{aligned}$$

Now we can look to the spectral relations given in Theorem 3.6. For $\tilde{G}_1$ the spectrum is $\{2, -1, -1\}$ and we have

$$\mathcal{E}_{A_{\tilde{G}_1}}(2) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -i \\ 1 \end{pmatrix}\right\}, \quad \mathcal{E}_{A_{\tilde{G}_1}}(-1) = \mathrm{span}\left\{\begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\}.$$

Hence both 2 and -1 are main eigenvalues, of multiplicity one and two respectively. Thus -1 will be an eigenvalue of $A_{\tilde{G}}$ at least once.

For $\tilde{G}_2$ the spectrum is $\{1, -1\}$ and we have

$$\mathcal{E}_{A_{\tilde{G}_2}}(1) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}, \quad \mathcal{E}_{A_{\tilde{G}_2}}(-1) = \mathrm{span}\left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\}.$$

Hence 1 is a main eigenvalue and -1 is a non-main eigenvalue, each of multiplicity one. Therefore -1 will be an eigenvalue of $A_{\tilde{G}}$ at least once. Combining this with what we have for $\tilde{G}_1$, we see that -1 will actually be an eigenvalue of $A_{\tilde{G}}$ at least twice.

For $\tilde{G}_3$ the spectrum is $\{2, 0, 0, -2\}$ and we have

$$\mathcal{E}_{A_{\tilde{G}_3}}(2) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}\right\}, \quad \mathcal{E}_{A_{\tilde{G}_3}}(0) = \mathrm{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}\right\}, \quad \mathcal{E}_{A_{\tilde{G}_3}}(-2) = \mathrm{span}\left\{\begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}\right\}.$$

Hence 0 is a main eigenvalue of multiplicity two, while 2 and -2 are both non-main eigenvalues of multiplicity one. Therefore each of 2, 0, and -2 will appear as an eigenvalue of $A_{\tilde{G}}$ at least once.

So far we have collected the eigenvalues $\{2, 0, -1, -1, -2\}$. For those that remain, we need to turn our attention to the polynomial $\det(\Omega(\lambda))$. We have

$$\det(\Omega(\lambda)) = \det\begin{pmatrix} (\lambda-2)(\lambda+1) & (3\lambda-1)i & 0 \\ -2i & \lambda - 1 & 2 \\ 0 & 4 & \lambda \end{pmatrix} = \lambda^4 - 2\lambda^3 - 15\lambda^2 + 12\lambda + 16,$$

which has roots $\lambda = 4.548, 1.456, -0.741, -3.263$.

Again, putting all of this together gives us precisely the desired result

$$\mathrm{spec}_{\tilde{G}} = \{4.548, 2, 1.456, 0, -0.741, -1, -1, -2, -3.263\}.$$

3.5. Stability under switching of $\tilde{H}$

It is shown in [21] that the $\dot{H}_m$-join of signed graphs remains stable under switching of $\dot{H}$. This means that if we perform the operation on any two switching equivalent signed graphs $\dot{H}$ and $\dot{H}'$, the two signed graphs that we produce shall also be switching equivalent.

Now we can show that there is an analogous result for the $\tilde{H}$-join operation.

Proposition 3.11. *Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ and $\tilde{H}' = (H, \mathbb{T}, \varphi'_H)$ be switching equivalent $\mathbb{T}$-gain graphs of order k and let $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$. Then $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}}$ and $\tilde{G}' = \bigvee_{\tilde{H}'}^{\mathcal{F}}$ are also switching equivalent.*

Proof. Switching equivalence between $\tilde{H}$ and $\tilde{H}'$ implies that for some diagonal matrix $S = \mathrm{diag}(s_1, \dots, s_k)$ with entries in $\mathbb{T}$, we have $S^* A_{\tilde{H}} S = A_{\tilde{H}'}$. Using this matrix to give the switching operation in terms of the gain functions φ_H and φ'_H we have $s_i^* \varphi_H(v_i v_j) s_j = \varphi'_H(v_i v_j)$ for all $1 \le i, j \le k$.

Let $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}}$ and $\tilde{G}' = \bigvee_{\tilde{H}'}^{\mathcal{F}}$. As in a previous proof, from here we shall use η_{ij} to denote $\varphi_H(v_i v_j)$ and η'_{ij} to denote $\varphi'_H(v_i v_j)$. By considering the adjacency matrix of $\tilde{G}'$ we can see that

$$\begin{aligned}
A_{\tilde{G}'} &= \begin{pmatrix} A_{\tilde{G}_1} & \eta'_{12} J_{n_1 \times n_2} & \cdots & \eta'_{1k} J_{n_1 \times n_k} \\ \eta'_{21} J_{n_2 \times n_1} & A_{\tilde{G}_2} & \cdots & \eta'_{2k} J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \eta'_{k1} J_{n_k \times n_1} & \eta'_{k2} J_{n_k \times n_2} & \cdots & A_{\tilde{G}_k} \end{pmatrix} \\
&= \begin{pmatrix} A_{\tilde{G}_1} & s_1^* \eta_{12} s_2 J_{n_1 \times n_2} & \cdots & s_1^* \eta_{1k} s_k J_{n_1 \times n_k} \\ s_2^* \eta_{21} s_1 J_{n_2 \times n_1} & A_{\tilde{G}_2} & \cdots & s_2^* \eta_{2k} s_k J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ s_k^* \eta_{k1} s_1 J_{n_k \times n_1} & s_k^* \eta_{k2} s_2 J_{n_k \times n_2} & \cdots & A_{\tilde{G}_k} \end{pmatrix} \\
&= \begin{pmatrix} s_1 I_{n_1} & O & \cdots & O \\ O & s_2 I_{n_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & s_k I_{n_k} \end{pmatrix}^* \begin{pmatrix} A_{\tilde{G}_1} & \eta_{12} J_{n_1 \times n_2} & \cdots & \eta_{1k} J_{n_1 \times n_k} \\ \eta_{21} J_{n_2 \times n_1} & A_{\tilde{G}_2} & \cdots & \eta_{2k} J_{n_2 \times n_k} \\ \vdots & \vdots & \ddots & \vdots \\ \eta_{k1} J_{n_k \times n_1} & \eta_{k2} J_{n_k \times n_2} & \cdots & A_{\tilde{G}_k} \end{pmatrix} \begin{pmatrix} s_1 I_{n_1} & O & \cdots & O \\ O & s_2 I_{n_2} & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & s_k I_{n_k} \end{pmatrix}.
\end{aligned}$$

Hence $A_{\tilde{G}'} = S^* A_{\tilde{G}} S$ for some diagonal matrix S and so by definition $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}}$ is switching equivalent to $\tilde{G}' = \bigvee_{\tilde{H}'}^{\mathcal{F}}$ and we are done. □

One way to think of the switching here is as follows: whenever we perform switching on a vertex $v_i \in V_H$, it yields the same outcome as performing switchings with the same element on every vertex of V_{G_i} in the compound graph produced by the $\tilde{H}$-join operation.

3.6. Balancedness of $\tilde{H}$-join

In [31, Theorem 1], the authors give conditions under which a graph produced by the signed $\dot{H}$-join operation is balanced. We can state a similar theorem for the complex unit $\tilde{H}$-join.

Theorem 3.12. *Let $\tilde{H}$ be a $\mathbb{T}$-gain graph of order k with no isolated vertex and let $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$ be a family of vertex-disjoint $\mathbb{T}$-gain graphs such that the union $E_{G_1} \cup \cdots \cup E_{G_k}$ is nonempty. If $\tilde{G} = \bigvee_{\tilde{H}} \{\tilde{G}_1, \dots, \tilde{G}_k\}$, then $\tilde{G}$ is balanced if and only if $\tilde{H}$ is balanced and $\tilde{G}_i$ is all-positive for each $i = 1, \dots, k$.*

Proof. If $\tilde{H}$ is balanced then it can be switched to the all-positive graph. From Proposition 3.11 we know that for switching equivalent $\tilde{H}$ and $\tilde{H}'$, the join graphs $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}}$ and $\tilde{G}' = \bigvee_{\tilde{H}'}^{\mathcal{F}}$ are also switching equivalent. If each $\tilde{G}_i \in \mathcal{F}$ is all-positive, it is easy to see that for a balanced $\tilde{H}$, the join graph $\tilde{G}$ is switching equivalent to the all-positive graph, and so it is balanced.

Conversely, suppose that $\tilde{G}$ is a balanced $\mathbb{T}$-gain graph. For $i = 1, \dots, k$, we can take a single vertex from each $\tilde{G}_i$ and see that the induced subgraph is balanced. Since this subgraph is isomorphic to $\tilde{H}$, it is the case that $\tilde{H}$ must be balanced.

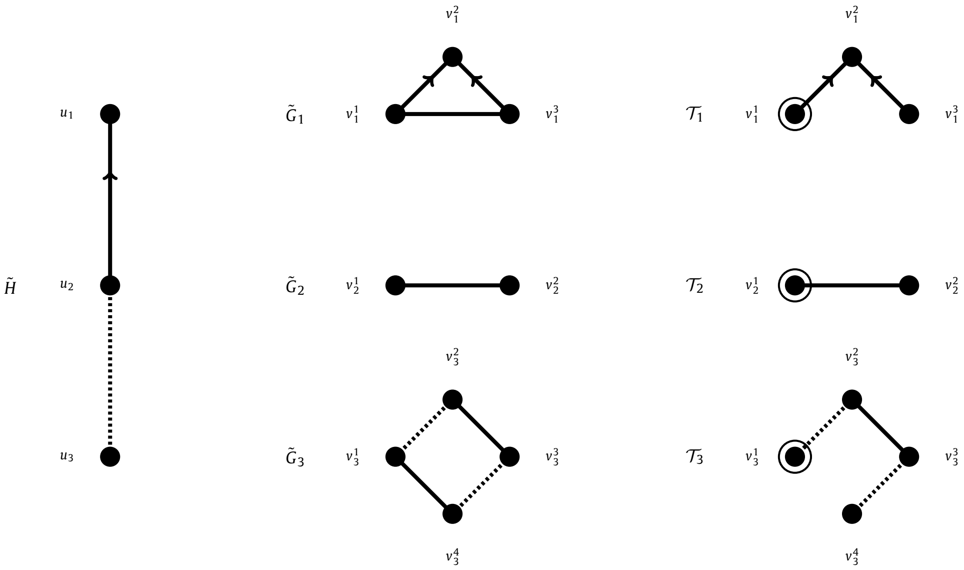

Fig. 2. The $\mathbb{T}$-gain graphs $\tilde{H}$, $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$ from Fig. 1 and the spanning trees $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ with fixed vertices v_1^1, v_2^1, and v_3^1.

Additionally, since $\tilde{G}$ is balanced, the product of the edge gains in every closed walk is equal to 1. Suppose we have $u, v \in V_{G_i}$ with $\varphi_i(uv) = \alpha$. Let $w \in V_{G_j}$ and let $v_iv_j \in E_H$. Then in $\tilde{G}$ we will have the triangle uvw and this will have gain $\varphi_i(uv)\varphi_H(v_iv_j)\varphi_H(v_jv_i) = \varphi_i(uv) = \alpha$. Clearly, for this triangle to be balanced we need $\alpha = 1$, hence $\varphi_i(uv) = 1$ for all edges in each $\tilde{G}_i$, making each $\tilde{G}_i$ all-positive. □

4. $\tilde{H}_{\mathcal{T},\overline{v}}$-join operation

As demonstrated, the graph produced by the $\tilde{H}$-join operation is stable under switching of $\tilde{H}$. However, the gain function in Definition 3.1 does not maintain stability under switchings of the individual component $\tilde{G}_i$ graphs. For instance, consider $\tilde{H} = \tilde{G}_1 = \tilde{G}_2 = (K_2, \mathbb{T}, +)$. Then the graph obtained will be an all-positive, hence balanced, complete graph on four vertices. However, taking $\tilde{G}_2' = (K_2, \mathbb{T}, -)$ (the complete graph on two vertices joined by an edge having gain function equal to -1, which is clearly switching equivalent to $(K_2, \mathbb{T}, +)$) means that the operation now produces an unbalanced complete graph on four vertices, and so is evidently not stable.

In this section we will show that by imposing certain conditions, it is possible define gain functions on $\tilde{H}$-joins such that they are also stable under switchings of their constituents. Furthermore, we can show that this process eases the constraints on which graphs will return a balanced product and note that the spectral analysis we have just done can also be applied in this case.

4.1. Pseudo-potentials

In [13, Definition 2.1], the authors define pseudo-potentials on signed graphs. Here we extend the notion to complex unit gain graphs in the natural way.

Definition 4.1. Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a connected $\mathbb{T}$-gain graph. Fix a spanning tree $\mathcal{T}$ of G and fix a vertex v in V_G. The pseudo-potential $U_{\mathcal{T},v,\varphi} : V_G \to \mathbb{T}$ is the function defined, for each w in V_G, as

$$U_{\mathcal{T},v,\varphi}(w) = \varphi(\gamma_{vw}),$$

where γ_{vw} is any walk contained in $\mathcal{T}$ from v to w.

The definition of a pseudo-potential on a gain graph $\tilde{G}$ which is not connected requires a maximal spanning forest instead of a single spanning tree and a list of fixed vertices, say $\overline{v} = \{v_1, \dots, v_r\}$, in V_G (one for each connected component) instead of a single vertex.

In the following, with a slight abuse of notation, we will use the same symbol $U_{\mathcal{T},\overline{v},\varphi}$ to denote the diagonal matrix with entries satisfying $\left(U_{\mathcal{T},\overline{v},\varphi}\right)_{ww} = U_{\mathcal{T},\overline{v},\varphi}(w)$.

Lemma 4.2. *Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph and let $\mathcal{T}$ be a spanning forest of G. Then* *(a) if G is connected, there is an element $z \in \mathbb{T}$ such that*

$$U_{\mathcal{T},v,\varphi} = zU_{\mathcal{T},v',\varphi} \qquad \forall\, v, v' \in V_G,$$

(b) if $\tilde{G}$ is balanced, for any spanning tree $\mathcal{T}'$ of G we have

$$U_{\mathcal{T},\overline{v},\varphi} = U_{\mathcal{T}',\overline{v},\varphi},$$

(c) if φ and φ' are two switching equivalent $\mathbb{T}$-gain functions on G, there exists a diagonal matrix S with diagonal entries in $\mathbb{T}$ such that

$$A_{\varphi'} = S^* A_\varphi S \quad \text{and} \quad U_{\mathcal{T},\overline{v},\varphi'} = S^* U_{\mathcal{T},\overline{v},\varphi} = U_{\mathcal{T},\overline{v},\varphi} S^*$$

where A_φ and $A_{\varphi'}$ are the adjacency matrices related to φ and φ', respectively.

Proof. The element z in the first point is given by $\varphi(\gamma_{vv'})$. The remainder of the proof is essentially the same as is given in [13, Lemma 2.3]. □

Definition 4.3. Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph and let $\mathcal{T}$ be a spanning forest of G. Fix a vertex v_i in each connected component of G and denote by $\overline{v} = \{v_1, \dots, v_r\}$. Then the complete graph induced by $\tilde{G}$ and $\mathcal{T}$ is the graph $K_{\tilde{G}\mathcal{T}} = (K_G, \mathbb{T}, \overline{\varphi}_{\mathcal{T},\overline{v}})$ whose vertex set is V_G and

$$\overline{\varphi}_{\mathcal{T},\overline{v}}(xy) = \begin{cases} \varphi(xy), & \text{if } xy \in E_G, \\ U_{\mathcal{T},\overline{v},\varphi}(x)^{-1} U_{\mathcal{T},\overline{v},\varphi}(y), & \text{otherwise.} \end{cases}$$

Remark 4.4. Notice that the adjacency matrix of $K_{\tilde{G}\mathcal{T}}$ is given by

$$A_{K_{\tilde{G}\mathcal{T}}} = A_\varphi + U_{\mathcal{T},\overline{v},\varphi} A_{G^c} U^{-1}_{\mathcal{T},\overline{v},\varphi},$$

where A_{G^c} is the adjacency matrix of the graph G^c, the complement of G.

Proposition 4.5. *Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph and let $\mathcal{T}$ be a spanning forest of G. Then the switching equivalence class of $K_{\tilde{G}\mathcal{T}}$ depends on neither the choice of fixed vertices $\overline{v}$ nor on the choice of the representative for the switching equivalence class of $\tilde{G}$. Finally, $\tilde{G}$ is balanced if and only if $K_{\tilde{G}\mathcal{T}}$ is balanced.*

Proof. Suppose that $\tilde{G}$ has exactly r connected components and let $\overline{v} = \{v_1, \dots, v_r\}$ and $\overline{v}' = \{v'_1, \dots, v'_r\}$ be two lists of vertices, one for each connected component. Let φ_1 and φ_2 be switching equivalent gain functions on $\tilde{G}$. Let us denote by $\overline{\varphi_1}_{\mathcal{T}}$ (respectively $\overline{\varphi_1}'_{\mathcal{T}}$) the gain function on the complete graph described in Definition 4.3 constructed from φ_1 and $\overline{v}$ (respectively $\overline{v}'$). Let us denote by $\overline{\varphi_2}_{\mathcal{T}}$ the gain function constructed from φ_2 and $\overline{v}$.

Following the same reasoning as in the proof of [13, Theorem 2.9], we aim to show that these three gain functions belong to the same switching equivalence class. The difference between the signed case and the complex unit gain case lies in the definition of the matrix S, which is no longer a diagonal matrix with entries in $\{\pm 1\}$ but, as established by Lemma 4.2, is a diagonal matrix with entries in $\mathbb{T}$. It holds that for any two vertices v and w belonging to the same connected component of G we have $S_{vv} = S_{ww}$, and thanks to this property S commutes with A_φ. With this, we can proceed in combining Lemma 4.2 and Definition 4.3 in precisely the same manner as in the case of signed graphs in order to conclude the proof. An interested reader may check [13] for the full details. □

Definition 4.6. Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph and let $\mathcal{T}$ be a spanning tree (or forest) of G. Then $\tilde{G}$ is $\mathcal{T}$-positive if $\varphi(v_i v_j) = 1$ for each edge $v_i v_j \in \mathcal{T}$.

Notice that for each switching equivalence class $[\varphi]$ and each spanning tree $\mathcal{T}$, there is exactly one representative of the class which is $\mathcal{T}$-positive.

Definition 4.7. Let $\tilde{G} = (G, \mathbb{T}, \varphi)$ be a $\mathbb{T}$-gain graph and let $\mathcal{T}$ be a spanning tree (or forest) of G. The $\mathcal{T}$-representative of $[\varphi]$ is the $\mathbb{T}$-gain graph $\tilde{G} = (G, \mathbb{T}, \varphi_{\mathcal{T}})$ that is $\mathcal{T}$-positive and whose gain function $\varphi_{\mathcal{T}}$ belongs to the class $[\varphi]$.

4.2. Switching-stable definition

Before providing the definition for the $\tilde{H}_{\mathcal{T},\overline{v}}$-join, also referred to as the stable $\tilde{H}$-join, we illustrate a scheme for labelling the vertices. Let $\tilde{H}$ be a $\mathbb{T}$-gain graph of order k with vertex set $V_H = \{u_1, \dots, u_k\}$ and let $\tilde{G}_i$ be a $\mathbb{T}$-gain graph of order n_i with vertex set $V_{G_i} = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$ for each $i = 1, \dots, k$. Then we can take the vertex set of $\tilde{G} = \bigvee_{\tilde{H}}\{\tilde{G}_1, \dots, \tilde{G}_k\}$ as

$$V_G = \bigcup_{i=1}^{k} \left(\{u_i\} \times V_{G_i}\right). \tag{3}$$

Example 4.8. Let $H = P_3$ and $V_H = \{u_1, u_2, u_3\}$. Let $G_1 = K_3$ and $V_{G_1} = \{v_1^1, v_1^2, v_1^3\}$, $G_2 = K_2$ and $V_{G_2} = \{v_2^1, v_2^2\}$, $G_3 = C_4$ and $V_{G_3} = \{v_3^1, v_3^2, v_3^3, v_3^4\}$. Then, according to Equation (3), we have

$$V_G = \{(u_1, v_1^1), (u_1, v_1^2), (u_1, v_1^3), (u_2, v_2^1), (u_2, v_2^2), (u_3, v_3^1), (u_3, v_3^2), (u_3, v_3^3), (u_3, v_3^4)\}.$$

With this labelling we can proceed to defining the stable $\tilde{H}$-join.

Definition 4.9. Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ be a $\mathbb{T}$-gain graph with $V_H = \{u_1, \dots, u_k\}$ and let $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$ be a family of vertex-disjoint graphs where $\tilde{G}_i = (G_i, \mathbb{T}, \varphi_i)$ is a $\mathbb{T}$-gain graph of order n_i with vertex set $V_{G_i} = \{v_i^1, v_i^2, \dots, v_i^{n_i}\}$ for each $i = 1, \dots, k$.

Fix a spanning tree $\mathcal{T}_i$ of each G_i, fix a vertex v_i^s in each V_{G_i}, and take the pseudo-potential functions $U_i = U_{\mathcal{T}_i, v_i^s, \varphi_i} : V_{G_i} \to \mathbb{T}$ as $U_i(w_i^t) = \varphi_i(\gamma_{v_i^s w_i^t})$ for each w_i^t in V_{G_i}.

Then the $\tilde{H}_{\mathcal{T},\overline{v}}$-join of this family is the $\mathbb{T}$-gain graph $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{T}}\{\tilde{G}_1, \dots, \tilde{G}_k\}$, sometimes denoted $\tilde{G} = \bigvee_{\tilde{H}}^{\mathcal{F}\mathcal{T}}$, with vertex set $V_G = \bigcup_{i=1}^{k}\left(\{u_i\} \times V_{G_i}\right)$, oriented edge set

$$E_G = \left(\bigcup_{i=1}^{k} E_{G_i}\right) \cup \left(\bigcup_{u_i u_j \in E_H} \{v_i^s v_j^t : v_i^s \in V_{G_i}, v_j^t \in V_{G_j}\}\right),$$

and gain function φ satisfying

$$\varphi((u_i, v_i^s), (u_j, v_j^t)) = \begin{cases} \varphi_i(v_i^s v_i^t), & \text{if } i = j \text{ and } v_i^s v_i^t \in E_{G_i}, \\ \varphi_H(u_i u_j) U_i(v_i^s)^{-1} U_j(v_j^t), & \text{if } u_i u_j \in E_H. \end{cases}$$

We can see that by this definition, the $\tilde{H}_{\mathcal{T},\overline{v}}$-join of a family which is $\mathcal{T}$-positive coincides with the basic definition from the previous section.

4.3. Stability under switchings of $\tilde{G}_i$

With this new definition, the operation remains stable under switching of $\tilde{H}$ as before and is also stable under switchings of each component graph $\tilde{G}_i$, as we shall prove in this section.

Remark 4.10. While considering only the underlying graphs, without applying any gain functions, notice that each closed walk W on the H-join of a family $\mathcal{F} = \{G_1, \dots, G_k\}$ has the form

$$(u_{i_1}, v_{i_1}^{1_1}) \dots (u_{i_1}, v_{i_1}^{k_1})(u_{i_2}, v_{i_2}^{1_2}) \dots (u_{i_2}, v_{i_2}^{k_2})(u_{i_3}, v_{i_3}^{1_3}) \dots (u_{i_j}, v_{i_j}^{k_j})(u_{i_1}, v_{i_1}^{1_{(j+1)}}) \dots (u_{i_1}, v_{i_1}^{1_1})$$

where $u_{i_s} \sim u_{i_{s+1}}$ for each $s = 1, \dots, j-1$, $u_{i_j} \sim u_{i_1}$, and $v_{i_s}^{r_s} \sim v_{i_s}^{r+1_s}$ for each $s = 1, \dots, j$. In the following proofs we use W_H to denote the closed walk on H induced by W, which is defined as

$$W_H = u_{i_1} u_{i_2} \dots u_{i_j} u_{i_1}.$$

Moreover, W can be viewed as a closed walk in the complete graph K_G, obtained by taking the union of $G = \sqcup_{i=1}^{k} G_i$ with its complement.

Lemma 4.11. *Let $\tilde{H} = (H, \mathbb{T}, \varphi_H)$ be a $\mathbb{T}$-gain graph of order k and let $\mathcal{F} = \{\tilde{G}_1, \dots, \tilde{G}_k\}$ be a family of vertex-disjoint complex unit gain graphs. Fix a spanning forest $\mathcal{T}$ of $G = \sqcup_{i=1}^{k} G_i$, and choose a pair of vertices v_i and v_i' in each connected component of G. Let $\overline{v} = \{v_1, \dots, v_r\}$ and denote by U_i the pseudo-potential on $\tilde{G}_i$ induced by $\mathcal{T}$ and $\overline{v}$. Let W be a closed walk in the $\tilde{H}_{\mathcal{T},\overline{v}}$-join of the family $\mathcal{F}$. Then*

$$\varphi(W)=\varphi_H(W_H)\overline{\varphi}_{\mathcal{T},\overline{v}}(W),$$

where φ *is the gain function in the* $\tilde{H}_{\mathcal{T},\overline{v}}$*-join and* $\overline{\varphi}_{\mathcal{T},\overline{v}}$ *is the gain function induced by the* $\tilde{G}_i$ *graphs,* $\mathcal{T}$*, and* $\overline{v}$ *in the complete graph* K_G*.*

Proof. It follows by an easy computation. We have

$$\begin{aligned}
\varphi(W)&=\varphi((u_{i_1},v_{i_1}^{1_1})\cdots(u_{i_1},v_{i_1}^{k_1})(u_{i_2},v_{i_2}^{1_2})\cdots(u_{i_j},v_{i_j}^{k_j})(u_{i_1},v_{i_1}^{1_{(j+1)}})\cdots(u_{i_1},v_{i_1}^{1_1}))\\
&=\varphi_{i_1}(v_{i_1}^{1_1},v_{i_1}^{2_1})\cdots\varphi_{i_1}(v_{i_1}^{(k-1)_1},v_{i_1}^{k_1})\cdot\varphi_H(u_{i_1},u_{i_2})U_{i_1}(v_{i_1}^{k_1})^{-1}U_{i_2}(v_{i_2}^{1_2})\\
&\quad\cdots\varphi_{i_1}(v_{i_1}^{1_{(j+1)}},v_{i_1}^{2_{(j+1)}})\cdots\varphi_{i_1}(v_{i_1}^{(k-1)_{(j+1)}},v_{i_1}^{1_1})\\
&=[\varphi_H(u_{i_1},u_{i_2})\cdots\varphi_H(u_{i_j},u_{i_1})]\\
&\quad\cdot[\varphi_{i_1}(v_{i_1}^{1_1},v_{i_1}^{2_1})\cdots\varphi_{i_1}(v_{i_1}^{(k-1)_1},v_{i_1}^{k_1})\cdot U_{i_1}(v_{i_1}^{k_1})^{-1}U_{i_2}(v_{i_2}^{1_2})\cdots\varphi_{i_1}(v_{i_1}^{(k-1)_{(j+1)}},v_{i_1}^{1_1})]\\
&=\varphi_H(W_H)\overline{\varphi}_{\mathcal{T},\overline{v}}(W).\quad\square
\end{aligned}$$

The following theorem presents the main result of this section, establishing the stability of the $\tilde{H}_{\mathcal{T},\overline{v}}$-join.

Theorem 4.12. *Let* $\tilde{H}=(H,\mathbb{T},\varphi)$ *be a* $\mathbb{T}$*-gain graph of order* k *and let* $\mathcal{F}=\{\tilde{G}_1,\dots,\tilde{G}_k\}$ *be a family of vertex-disjoint* $\mathbb{T}$*-gain graphs. Fix a spanning forest* $\mathcal{T}$ *of* $G=\sqcup_{i=1}^k G_i$ *and choose a vertex* v_i *in each connected component of* G*. Let* $\overline{v}=\{v_1,\dots,v_r\}$*. Then*

- *the switching equivalence class of the* $\tilde{H}_{\mathcal{T},\overline{v}}$*-join of* $\mathcal{F}$ *depends only on* $\mathcal{T}$ *and is independent of the choice of vertices,*
- *the* $\tilde{H}_{\mathcal{T},\overline{v}}$*-join operation is switching-stable on both* $\tilde{H}$ *and on each member* $\tilde{G}_i$ *of the family* $\mathcal{F}$*, that is, replacing any* $\tilde{G}_i$ *with some switching equivalent* $\tilde{G}_i'$ *leaves the switching equivalence class of the* $\tilde{H}_{\mathcal{T},\overline{v}}$*-join unchanged.*

Proof. This is an immediate consequence of Proposition 4.5 and Lemma 4.11. $\square$

With this we can directly give a characterisation of the balancedness of a stable $\tilde{H}$-join.

Corollary 4.13. *The* $\tilde{H}_{\mathcal{T},\overline{v}}$*-join is a balanced* $\mathbb{T}$*-gain graph, whose gain function is independent of the choice of* $\mathcal{T}$*, if and only if* $\tilde{H}$ *and all component graphs* $\tilde{G}_i$ *are balanced.*

Remark 4.14. It is clear from Theorem 4.12 and Corollary 4.13 that, when applied to the setting of signed graphs, Definition 4.9 will provide a switching-stable signature for the signed $\dot{H}$-join (see [31]), $\dot{H}$-generalised-join (see [23]), and $\dot{H}_m$-join (see [21]) operations.

Since the operation is switching-stable, to compute the spectrum of the $\tilde{H}_{\mathcal{T},\overline{v}}$-join, one can, without loss of generality, consider the $\tilde{H}_{\mathcal{T},\overline{v}}$-join obtained by replacing each $\tilde{G}_i$ with the $\mathcal{T}$-positive representatives of each switching equivalence class involved. In this specific case the $\tilde{H}_{\mathcal{T},\overline{v}}$-join and the $\tilde{H}$-join coincide. As a result, the spectral analysis done in the previous section also applies in this case.

Now we present an example to illustrate explicitly how to compute this new gain function and to show that from balanced components we obtain a balanced product.

Example 4.15. Take $\tilde{H}$, $\tilde{G}_1$, $\tilde{G}_2$, and $\tilde{G}_3$ from Example 3.10 and note that each of these graphs is balanced. Let $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}_3$ be the spanning trees depicted in Fig. 2. Fix the vertex v_i^1 in each $\mathcal{T}_i$ and denote $U_i=U_{\mathcal{T}_i,v_i^1,\varphi_i}$. Then for the pseudo-potentials we have

$$\begin{aligned}
&U_1(v_1^1)=1,\quad U_1(v_1^2)=i,\quad U_1(v_1^3)=1,\\
&U_2(v_2^1)=1,\quad U_2(v_2^2)=1,\\
&U_3(v_3^1)=1,\quad U_3(v_3^2)=-1,\quad U_3(v_3^3)=-1,\quad U_3(v_3^4)=1.
\end{aligned}$$

Using these we can compute the gain function according to Definition 4.9. We know that the edges already present in the $\tilde{G}_i$ graphs will remain the same so let us look at (u_2,v_2^1) and see how we create the remaining adjacencies with respect to the gains and pseudo-potentials. We have

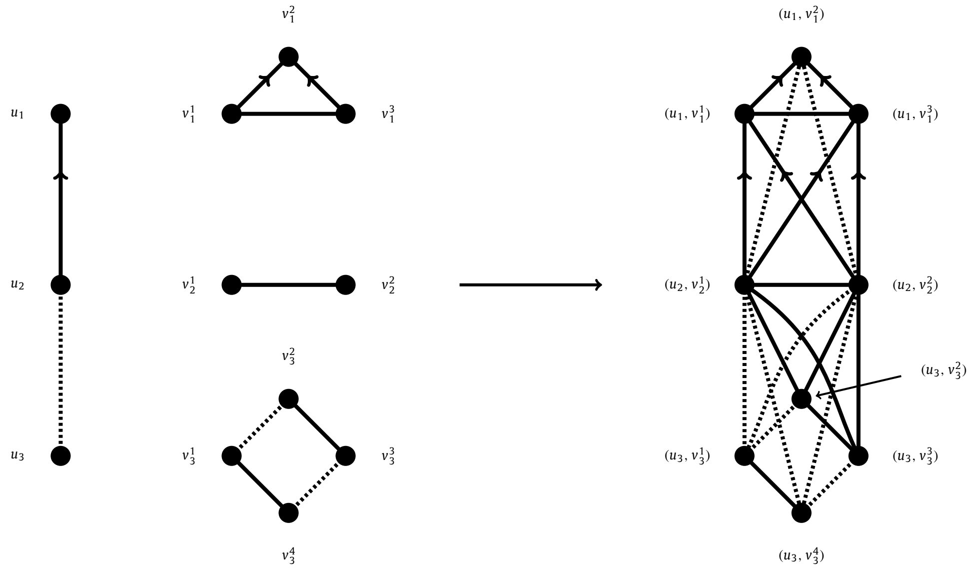

Fig. 3. The stable $\tilde{H}$-join of the graphs, spanning trees, and fixed vertices presented in Fig. 2. The vertices are labelled according to Equation (3) and the graph produced is balanced.

$$\varphi((u_2, v_2^1), (u_1, v_1^1)) = \varphi_H(u_2u_1)U_2(v_2^1)^{-1}U_1(v_1^1) = i \cdot 1 \cdot 1 = i,$$
$$\varphi((u_2, v_2^1), (u_1, v_1^2)) = \varphi_H(u_2u_1)U_2(v_2^1)^{-1}U_1(v_1^2) = i \cdot 1 \cdot i = -1,$$
$$\varphi((u_2, v_2^1), (u_1, v_1^3)) = \varphi_H(u_2u_1)U_2(v_2^1)^{-1}U_1(v_1^3) = i \cdot 1 \cdot 1 = i,$$
$$\varphi((u_2, v_2^1), (u_3, v_3^1)) = \varphi_H(u_2u_3)U_2(v_2^1)^{-1}U_3(v_3^1) = -1 \cdot 1 \cdot 1 = -1,$$
$$\varphi((u_2, v_2^1), (u_3, v_3^2)) = \varphi_H(u_2u_3)U_2(v_2^1)^{-1}U_3(v_3^2) = -1 \cdot 1 \cdot -1 = 1,$$
$$\varphi((u_2, v_2^1), (u_3, v_3^3)) = \varphi_H(u_2u_3)U_2(v_2^1)^{-1}U_3(v_3^3) = -1 \cdot 1 \cdot -1 = 1,$$
$$\varphi((u_2, v_2^1), (u_3, v_3^4)) = \varphi_H(u_2u_3)U_2(v_2^1)^{-1}U_3(v_3^4) = -1 \cdot 1 \cdot 1 = -1.$$

Doing this again for (u_2, v_2^2) completes the gain function and gives us the complex unit gain graph shown in Fig. 3, which is balanced.

5. Lexicographic product

Let H and G be two ordinary graphs. The lexicographic product $H[G]$ was introduced by Harary in 1959 [20] and is the graph whose set of vertices is $V_H \times V_G$, with (u, v) adjacent to (u', v') whenever u is adjacent to u' (edges of type I) or $u = u'$ and v is adjacent to v' (edges of type II). It can be interpreted as the H-join of a homogeneous family, that is, a family of graphs consisting of copies of the same graph G.

In this section, we provide a survey of all of the definitions of the lexicographic product that have been introduced so far for signed graphs and complex unit gain graphs, along with some characterisations of when they are balanced and how these relate to the gain functions we have defined for the $\tilde{H}$-join.

In 2012, Hameed and Germina [19] defined an initial lexicographic product (let us call it HG) for signed graphs and provided a characterisation of its balancedness. In particular, given two signatures σ_H on H and σ_G on G, the HG-lexicographic product assigns the value $\sigma_H(uu')$ to edges of type I and $\sigma_G(vv')$ to edges of type II.

In 2019, Brunetti, Cavaleri, and Donno [6] defined another lexicographic product (we will refer to it as BCD) for signed graphs. They claimed that, differently from HG, their operation was switching-stable. However, they later disproved this claim by giving a counterexample in an erratum [7]. The BCD-lexicographic product divides the edges of type I into two families: edges of type Ia, where $u \sim u'$ and $v \sim v'$, and the remainder, where $u \sim u'$ and $v \nsim v'$, which we shall call edges of type Ib. The signature of an edge of type Ia is given by $\sigma_H(uu')\sigma_G(vv')$, while for those of type Ib the signature coincides with HG.

In 2021, Zhang, Wu, and He [31] defined the $\dot{H}$-join on signed graphs (let us denote it ZWH) and provided a characterisation of its balancedness. This definition coincides with the HG-lexicographic product when all the graphs $\dot{G}_i$ in $\mathcal{F} = \{\dot{G}_1, \ldots, \dot{G}_k\}$ are identical, say, when $\mathcal{F}$ is homogeneous.

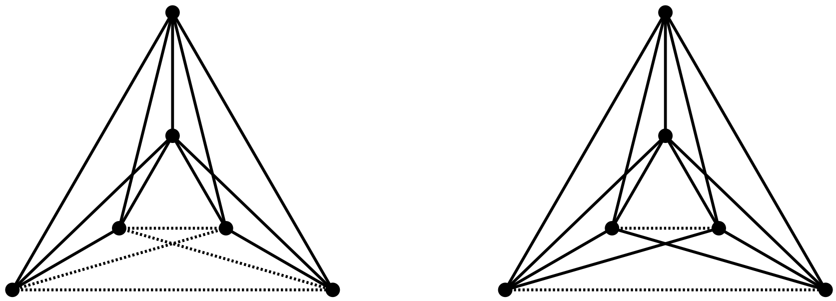

Fig. 4. On the left, the CDS-lexicographic product of a positive K_2 and $\tilde{C}_3$, a signed 3-cycle with exactly two positive edges. In this specific case the CDS product does not depend on the choice of the spanning tree $\mathcal{T}$. On the right, the stable K_2-join of $\mathcal{F} = \{\tilde{C}_3, \tilde{C}_3\}$ where the spanning trees are, in each copy of $\tilde{C}_3$, the positive subgraph of $\tilde{C}_3$.

In 2024, Brunetti [5], in a paper on complex unit gain graphs, introduced the complex versions of both HG and BCD. The complex gain functions here are defined by simply replacing the signatures σ_H and σ_G with the corresponding complex gain functions φ_H and φ_G.

In 2025, Cavaleri, Donno, and Spessato [13] defined a switching-stable version of the lexicographic product (which we shall call CDS) for signed graphs. This definition requires fixing a spanning tree and divides the edges of type Ib into a further two families, which we denote with Ib_1 and Ib_2. The edges of type Ib_1 are the edges where $u \sim u'$, $v \nsim v'$, and $v \neq v'$. For these edges, the signature is $\sigma_H(uu')\sigma^c_{G\mathcal{T}}(vv')$. Those edges left over from Ib, where $u \sim u'$ and $v = v'$, are the edges of type Ib_2, and their signature is equal to that of BCD.

In the present paper, we have introduced two complex functions on the H-join of a family of vertex-disjoint graphs, in Definitions 3.1 and 4.9. The first is the complex version of ZWH, which, in the case of a homogeneous $\mathcal{F}$, becomes the complex version of HG defined by Brunetti in 2024. The second, which we call the stable $\tilde{H}$-join, is a switching-stable definition that depends on the choice of a spanning forest.

Although they may appear similar, the stable $\tilde{H}$-join and CDS do not generally coincide, even when restricted to the subcase of signed graphs with a homogeneous family $\mathcal{F}$. In fact, they are not even switching equivalent. An example of this dissimilarity is depicted in Fig. 4.

On the other hand, these two operations do coincide when all of the graphs involved are balanced and one selects the same vertex/vertices in each copy of the relevant components.

Regardless, we have discovered a new switching-stable method for defining the lexicographic product of two signed (and complex unit gain) graphs, thus giving another answer to the question posed in the erratum of Brunetti, Cavaleri, and Donno [7].

This confirms, as already shown in [13], that there are several nontrivial switching-stable ways to define such stable products. Furthermore, we have discovered that these methods do not necessarily have to derive from the same underlying technique described in [13].

Declaration of competing interest

The authors report there are no competing interests to declare.

Acknowledgements

The authors would like to thank the anonymous referees for their insightful comments and suggestions, which contributed to improvements in the final version of this article, especially the proof of Theorem 3.3.

The research that led to this article was partially supported by a grant from the GNSAGA group of INdAM (Italy).

This work was also supported by the ISARP Grant of the National Research Foundation (NRF) of South Africa (Ref. No. ITAL22051410615) in collaboration with the Italian Ministry of Foreign Affairs and International Cooperation (MAECI) (Ref. No. ZA23MO01).

Callum Huntington and Francesco Belardo acknowledge the support of Project ASpecT3G from the University of Naples Federico II.

Data availability

No data was used for the research described in the article.

References

[1] A. Alazemi, M. Anđelić, F. Belardo, M. Brunetti, C.M. da Fonseca, Line and subdivision graphs determined by $\mathbb{T}_4$-gain graphs, Mathematics 7 (2019) 926.

[2] G. Arunkumar, R. Ganeshbabu, On the spectrum of generalized H-join operation constrained by indexing maps – I, arXiv:2402.10557 [math.CO], https://arxiv.org/abs/2402.10557, 2024.
[3] F. Belardo, M. Brunetti, S. Suliman, NEPS of complex unit gain graphs, Electron. J. Linear Algebra 39 (2023) 621–643.
[4] F. Belardo, Z. Stanić, T. Zaslavsky, Total graph of a signed graph, Ars Math. Contemp. 23 (2023), #P1.02.
[5] M. Brunetti, Eigenvalues of complex unit gain graphs and gain regularity, Spec. Matrices 12 (2024) 20240005.
[6] M. Brunetti, M. Cavaleri, A. Donno, A lexicographic product for signed graphs, Australas. J. Comb. 74 (2019) 332–343.
[7] M. Brunetti, M. Cavaleri, A. Donno, Erratum to the article "A lexicographic product for signed graphs", Australas. J. Comb. 75 (2019) 256–258.
[8] D.M. Cardoso, M.A. de Freitas, E.A. Martins, M. Robbiano, Spectra of graphs obtained by a generalization of the join graph operation, Discrete Math. 313 (2013) 733–741.
[9] D.M. Cardoso, H. Gomes, S.J. Pinheiro, The H-join of arbitrary families of graphs – the universal adjacency spectrum, Linear Algebra Appl. 648 (2022) 160–180.
[10] D.M. Cardoso, I. Gutman, E.A. Martins, M. Robbiano, A generalization of Fiedler's lemma and some applications, Linear Multilinear Algebra 59 (2011) 929–942.
[11] D.M. Cardoso, E.A. Martins, M. Robbiano, O. Rojo, Eigenvalues of a H-generalized join graph operation constrained by vertex subsets, Linear Algebra Appl. 438 (2013) 3278–3290.
[12] M. Cavaleri, A. Donno, S. Spessato, The NEPS of gain graphs over arbitrary groups and its spectra, Adv. Group Theory Appl. 19 (2024) 105–153.
[13] M. Cavaleri, A. Donno, S. Spessato, On the complement of a signed graph, Discrete Math. 348 (2025) 114433.
[14] M. Cavaleri, A. Donno, S. Spessato, On the wreath product of signed and gain graphs and its spectrum, Ars Math. Contemp. 25 (2025), #P3.10.
[15] M. Cavaleri, A. Donno, S. Spessato, Gain total graphs and their spectra via G-phases and group representations, Linear Algebra Appl. 721 (2025) 484–512.
[16] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Applications, 3rd edition, J.A. Barth Verlag, Leipzig, 1995.
[17] K.A. Germina, K.S. Hameed, T. Zaslavsky, On products and line graphs of signed graphs, their eigenvalues and energy, Linear Algebra Appl. 435 (10) (2011) 2432–2450.
[18] I. Gohberg (Ed.), I. Schur Methods in Operator Theory and Signal Processing, Operator Theory: Advances and Applications, vol. 0T18, Birkhäuser Verlag, Basel, 1986.
[19] K.S. Hameed, K.A. Germina, On composition of signed graphs, Discuss. Math., Graph Theory 32 (2012) 507–516.
[20] F. Harary, On the group of the composition of two graphs, Duke Math. J. 26 (1959) 29–34.
[21] C. Huntington, The $\dot{H}$-join operation of signed graphs constrained by indexing maps, RAIRO-Oper. Res. 59 (2025) 2501–2515.
[22] D. Lajou, On Cartesian products of signed graphs, Discrete Appl. Math. 319 (2022) 533–555.
[23] Y. Li, P. Zhang, K. Xu, The spectra of signed graphs obtained by $\dot{H}$-(generalized) join operation, Comput. Appl. Math. 42 (2023) 77.
[24] C. McLeman, E. McNicholas, Spectra of coronae, Linear Algebra Appl. 435 (2011) 998–1007.
[25] M. Saravanan, S.P. Murugan, G. Arunkumar, A generalization of Fiedler's lemma and the spectra of H-join graphs, Linear Algebra Appl. 625 (2021) 20–43.
[26] I. Schur, Über Potenzreihen, die im Innern des Einheitskreises beschrankt sind, J. Reine Angew. Math. 147 (1917) 205–232 (Translated into English in [18, pp. 31–59]).
[27] A.J. Schwenk, Computing the characteristic polynomial of a graph, in: R.A. Bari, F. Harary (Eds.), Graphs Combin., vol. 406, Springer, Berlin, 1974, pp. 153–172.
[28] B. Wu, Y. Lou, C. He, Signless Laplacian and normalized Laplacian on the H-join operation of graphs, Discrete Math. Algorithms Appl. 6 (2014) 1450046.
[29] T. Zaslavsky, Biased graphs. I. Bias, balance, and gains, J. Comb. Theory, Ser. B 47 (1989) 32–52.
[30] F. Zhang (Ed.), The Schur Complement and Its Applications, Numer. Methods Algorithms, vol. 4, Springer, New York, 2005.
[31] P. Zhang, B. Wu, C. He, Balancedness and spectra of signed graphs obtained by $\dot{H}$-join operation, Comput. Appl. Math. 40 (2021) 90–100.