Integral IDA-PBC for underactuated mechanical systems subject to matched and unmatched disturbances

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***Abstract*— This work presents a new formulation of the integral interconnection and damping-assignment passivity-based control methodology for underactuated mechanical systems subject to both matched and unmatched disturbances, either constant or position-dependent. The new controller is also applicable to systems with non-constant input matrix. Simulations results on two examples demonstrate its effectiveness.**

I. Introduction

The control of underactuated mechanical systems has been approached with various methodologies, including model predictive control [1], optimal control [2], and energy-shaping controllers [3]. Among the latter, the *Interconnection and damping assignment Passivity based control* (IDA-PBC) methodology [4] involves designing the control action such that the closed-loop dynamics preserves the port-Hamiltonian structure and is characterized by a desired total energy. The key advantages of IDA-PBC are the interpretability of the closed-loop dynamics in terms of mechanical structure, and its passivity properties. Several works have investigated the robustification of IDA-PBC through integral actions resulting in the *integral* IDA-PBC methodology (iIDA-PBC) [5]. The initial formulation of iIDA-PBC was applicable to a limited class of systems subject to constant matched disturbances, and it employed a change of coordinates [6]. Subsequent works have avoided the change of coordinates, see [7], and have extended the result to a broader class of systems [8]. More recent works have investigated the iIDA-PBC for systems with physical damping [9], [10] and with non-constant disturbances [11], [12]. In our recent works [13], [14] we have extended the iIDA-PBC to underactuated mechanical systems subject to matched disturbances, either constant or state-dependent, and constant unmatched disturbances. However, the former results are limited to mechanical systems with constant input matrix. This excludes well-known examples such as the pendulum-on-cart (POC) and the vertical takeoff and landing aircraft (VTOL) for which the existing iIDA-PBC formulations are not directly applicable.

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The main contribution of this work is a new formulation of the iIDA-PBC methodology for underactuated mechanical systems characterized by non-constant input matrix, and subject to both matched and unmatched additive disturbances, either constant or position-dependent. To the best of the authors' knowledge, this is the first iIDA-PBC design applicable to a class of systems including the POC and VTOL with unmatched disturbances. The effectiveness of the new controller is demonstrated with numerical simulations.

Notation. Function arguments are specified on first use and subsequently omitted in equations for conciseness.

II. Overview of Integral IDA-PBC

The dynamics of an underactuated mechanical system with n DOFs and the control input $u \in \mathbb{R}^m$ applied through the input matrix $G(q) \in \mathbb{R}^{n\times m}$, where $\operatorname{rank}(G) = m < n$ for all $q \in \mathbb{R}^n$, and subject to the disturbances $\delta(q) \in \mathbb{R}^n$, is described in port-Hamiltonian form as

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & -D\end{bmatrix}\begin{bmatrix}\nabla_q H\\ \nabla_p H\end{bmatrix} + \begin{bmatrix}0\\ G\end{bmatrix}(u+v) - \begin{bmatrix}0\\ \delta\end{bmatrix}, \quad (1)$$

where $v \in \mathbb{R}^m$ is an auxiliary control to add integral action, and $D = D^\top \succeq 0$ is the physical damping. The system states are the positions $q \in \mathbb{R}^n$ and the momenta $p = M(q)\dot{q} \in \mathbb{R}^n$, while $y = G^\top \nabla_p H$ is a passive output of (1). The mechanical energy of the system,

$$H(q,p) = \frac{1}{2}p^\top M(q)^{-1}p + \Omega(q), \quad (2)$$

is characterized by the inertia matrix $M(q) = M(q)^\top \succ 0$, and the potential energy $\Omega(q)$. The remaining terms in (1) are the identity matrix I, the vector of partial derivatives of H with respect to q, $\nabla_q H$, and the vector of partial derivatives of H with respect to p, $\nabla_p H$. The controller design aims at stabilizing the prescribed equilibrium $(q,p) = (q^\star, 0)$. This is achieved, in the absence of disturbances (i.e., $\delta = 0$), by using $v = 0$ and the IDA-PBC control law [4]

$$\begin{aligned} u = G^\dagger\left(\nabla_q H - M_d M^{-1}\nabla_q H_d + J_2 \nabla_p H_d\right) + u_d,\\ u_d = -K_v G^\top \nabla_p H_d, \end{aligned} \quad (3)$$

where $H_d(q,p) = \frac{1}{2}p^\top M_d(q)^{-1}p + \Omega_d(q)$, $K_v = K_v^\top \succ 0$, and $G^\dagger = \left(G^\top G\right)^{-1}G^\top$. The control law (3) exists provided that the inertia matrix $M_d(q) = M_d^\top(q) \succ 0$, the potential energy $\Omega_d(q)$, and the matrix $J_2(q,p) = -J_2^\top(q,p)$ verify for all $(q,p) \in \mathbb{R}^{2n}$ the partial differential equations (PDEs)

$$\begin{aligned} G^\perp\left(\nabla_q(p^\top M^{-1}p) - M_d M^{-1}\nabla_q(p^\top M_d^{-1}p)\right) \quad (4)\\ +G^\perp\left(2J_2 M_d^{-1}p\right) = 0, \end{aligned}$$

$$G^\perp\left(\nabla_q \Omega - M_d M^{-1}\nabla_q \Omega_d\right) = 0, \quad (5)$$

where $G^\perp$ is defined such that $G^\perp G = 0$ and $\text{rank}\left(G^\perp\right) = n - m$. To achieve the regulation goal $(q,p) = (q^\star, 0)$, the potential energy $\Omega_d(q)$ should also admit a strict minimizer in $q^\star$ hence verifying the conditions $\nabla_q \Omega_d(q^\star) = 0$ and $\nabla_q^2 \Omega_d(q^\star) \succ 0$. The desired closed-loop dynamics is thus

$$\begin{bmatrix}\dot{q}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & S_{12}\\ -S_{12}^\top & J_2 - GK_vG^\top - DS_{12}\end{bmatrix}\begin{bmatrix}\nabla_q H_d\\ \nabla_p H_d\end{bmatrix}, \quad (6)$$

where $S_{12} = M^{-1}M_d$, see [4]. Computing the time-derivative of H_d along the trajectories of (6) yields then

$$\dot{H}_d = -(\nabla_p H_d)^\top (GK_vG^\top + DM^{-1}M_d - J_2)\nabla_p H_d. \quad (7)$$

According to [15], it follows from (7) that $\dot{H}_d \le 0$ if

$$\Delta_S = GK_vG^\top + \frac{1}{2}DM^{-1}M_d + \frac{1}{2}M_dM^{-1}D \succeq 0. \quad (8)$$

If $D = 0$, $\dot{H}_d \le 0$ for all $K_v \succ 0$ and the equilibrium $(q,p) = (q^\star, 0)$ is asymptotically stable if $y_d = G^\top \nabla_p H_d$ is a detectable output of (6), that is $y_d \to 0 \implies (q,p) \to (q^\star, 0)$. In addition, $\dot{H}_d \le y_d^\top u_d$, where y_d is a passive output of (6), that is the control law (3) preserves passivity, see [4].

If the disturbances are constant and matched (i.e., $\delta = G\delta_0, \delta_0 \in \mathbb{R}^m$), the input matrix G and the matrix M_d are constant, $D = 0$, and the matrix M is independent of the unactuated coordinates, the iIDA-PBC design [5] can be used to compensate the disturbance. Then, the auxiliary control v and the time-derivative of the integral state ζ take the form

$$\begin{aligned} v &= -K_{II}K_IK_{II}^{-T}G^\top M^{-1}\nabla_q\Omega_d - K_vK_I\zeta,\\ \dot{\zeta} &= K_{II}^\top G^\top M^{-1}\nabla_q\Omega_d, \end{aligned} \quad (9)$$

with constant $K_I \succ 0$ and $K_{II} = (G^\top M^{-1}G)^{-1}$. The extended closed-loop dynamics in port-Hamiltonian form is

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}0 & S_{12} & S_{13}\\ -S_{12}^\top & -GK_vG^\top & 0\\ -S_{13}^\top & 0 & 0\end{bmatrix}\begin{bmatrix}\nabla_{z_1}W_d\\ \nabla_{z_2}W_d\\ \nabla_{z_3}W_d\end{bmatrix}, \quad (10)$$

with $S_{13} = -M^{-1}GK_{II}$, $z_1 = q$, $z_2 = p + GK_{II}K_I(\zeta - \alpha)$, $z_3 = \zeta$. Subsequent versions of iIDA-PBC have avoided the coordinate transformation [7], and have extended the results to systems with non-constant matrices M_d and G, see [8].

III. Main result

This section presents a new iIDA-PBC design for a class of mechanical systems defined by *Assumptions 1* to *4*.

Assumption 1. The PDEs (4)-(5) are solvable analytically with $M_d(q)$, $J_2(q,p)$ and $\Omega_d(q)$, where $q^\star = \text{argmin}(\Omega_d)$, and $S_{12} = M^{-1}M_d$ in (6). The output $y_d = G^\top \nabla_p H_d$ is detectable. The model parameters $D \succeq 0, G(q), M(q), \Omega(q)$ are exactly known, and the states (q,p) are measurable.
The solvability of PDEs is a fundamental step in IDA-PBC and remains a major challenge [4]. This step is beyond the scope of this paper, which focuses on the integral action design for disturbance rejection. Nevertheless, the PDEs are solvable for many examples, see e.g. [4], [16]. Differently from [13], [14], the matrix G is not required to be constant.

Assumption 2. The disturbance is parameterized as $\delta = \delta_1 GG^\top h(q) + \delta_2 G^{\perp^\top} G^\perp h(q)$, where $\delta_1, \delta_2 \in \mathbb{R}$ are unknown scalar constants, while $h(q) \in \mathbb{R}^n$ is a known globally bounded and continuously differentiable function of q. The prescribed equilibrium $q = q^\star$ is assignable for system (1), that is $G^\perp\left(\nabla_q\Omega(q^\star) + \delta_2 G^{\perp^\top} G^\perp h(q^\star)\right) = 0$.
Without loss of generality, the disturbances can be separated into matched (i.e., $\delta_1 GG^\top h(q)$) and unmatched components (i.e., $\delta_2 G^{\perp^\top} G^\perp h(q)$), where δ_1 and δ_2 are unknown constants (i.e., the disturbance bounds are unknown).

Assumption 3. There exist some $K_v \succ 0$ and some scalar constant $\Gamma_1 > 0$ such that (8) holds and $\Gamma_1 I - \Delta_S \succ 0$.

A. Controller design

The target closed-loop dynamics is defined as

$$\begin{bmatrix}\dot{q}\\ \dot{p}\\ \dot{\zeta}_1\\ \dot{\zeta}_2\end{bmatrix} = \begin{bmatrix}0 & S_{12} & S_{13} & S_{14}\\ -S_{12}^\top & -S_{22} & S_{23} & S_{24}\\ -S_{13}^\top & -S_{23}^\top & -S_{33} & S_{34}\\ -S_{14}^\top & -S_{24}^\top & -S_{43} & -S_{44}\end{bmatrix}\begin{bmatrix}\nabla_q W_d\\ \nabla_p W_d\\ \nabla_{\zeta_1} W_d\\ \nabla_{\zeta_2} W_d\end{bmatrix}, \quad (11)$$

where ζ_1 and ζ_2 are integral states introduced to reject the matched and unmatched disturbances respectively, while

$$\begin{aligned} S_{13} &= S_{12}\nabla_p\Psi_1,\ S_{14} = S_{12}\nabla_p\Psi_2,\\ S_{22} &= GK_vG^\top - J_2 + DM^{-1}M_d,\\ S_{23} &= \nabla_p\Psi_1\Gamma_1 - S_{22}\nabla_p\Psi_1 - S_{12}^\top\nabla_q\Psi_1,\\ S_{24} &= \nabla_p\Psi_2\Gamma_1 - S_{22}\nabla_p\Psi_2 - S_{12}^\top\nabla_q\Psi_2,\\ S_{33} &= S_{13}^\top\nabla_q\Psi_1 + S_{23}^\top\nabla_p\Psi_1,\\ S_{34} &= -S_{13}^\top\nabla_q\Psi_2 - S_{23}^\top\nabla_p\Psi_2,\\ S_{43} &= S_{14}^\top\nabla_q\Psi_1 + S_{24}^\top\nabla_p\Psi_1,\\ S_{44} &= S_{14}^\top\nabla_q\Psi_2 + S_{24}^\top\nabla_p\Psi_2. \end{aligned} \quad (12)$$

The storage function $W_d(q,p,\zeta_1,\zeta_2)$ is defined as

$$\begin{aligned} W_d &= H_d^\star + \frac{k_1}{2}(\zeta_1 - \Psi_1 - \alpha)^2 + \frac{k_2}{2}(\zeta_2 - \Psi_2 - \beta)^2,\\ H_d^\star(q,p,\zeta_2) &= \Omega_d(q) + \Phi(q,p,\zeta_2) + \frac{1}{2}p^\top M_d^{-1}p + k_0,\\ \Psi_1(q,p) &= h(q)^\top GG^\top p,\ \Psi_2(q,p) = h(q)^\top G^{\perp^\top} G^\perp p, \end{aligned} \quad (13)$$

where $\alpha = \delta_1/(k_1\Gamma_1) \in \mathbb{R}$ and $\beta = \delta_2/(k_2\Gamma_1) \in \mathbb{R}$, with k_1, k_2, k_0, Γ_1 positive scalar constants. In particular, $H_d^\star(q,p,\zeta_2)$ is an extended Hamiltonian, with $M_d(q)$ and $\Omega_d(q)$ that solve the PDEs (4) and (5), see *Assumption 1*.
The scalar function $\Phi(q,p,\zeta_2)$ represents the mechanical work of the closed-loop non-conservative forces resulting from the unmatched disturbance, see [13], and it is defined by the following assumption.

Assumption 4. Given the assignable equilibrium $q^\star$ of (1), there exists a scalar function $\Phi(q,p,\zeta_2)$ that verifies

$$\begin{aligned} &G^\perp\left(\nabla_p\Psi_2\Gamma_1k_2(\zeta_2 - \Psi_2) - M_dM^{-1}\nabla_q\Phi\right)\\ &+G^\perp\left(\nabla_p\Psi_2\Gamma_1 - M_dM^{-1}\nabla_q\Psi_2\right)\nabla_{\zeta_2}\Phi = 0, \quad (14a)\\ &\nabla_p\Phi + \nabla_p\Psi_2\nabla_{\zeta_2}\Phi = 0, \quad (14b)\\ &\nabla_q\Omega_d + \nabla_q\Phi = 0 \quad q = q^\star, \quad (14c)\\ &\nabla_q^2\Omega_d + \nabla_q^2\Phi \succ 0 \quad q = q^\star. \quad (14d) \end{aligned}$$

This assumption is a bottleneck of the proposed approach, since solving the PDEs (14a) to (14d) can be challenging. The new control input is given by

$$\begin{aligned} u = G^{\dagger}\left(\nabla_q H - S_{12}^{\top}\left(\nabla_q H_d^{\star} + \nabla_q \Psi_2 \nabla_{\zeta_2}\Phi\right)\right) \\ + G^{\dagger}\left(-S_{22}M_d^{-1}p + \nabla_p \Psi_1 \Gamma_1 k_1\left(\zeta_1 - \Psi_1\right)\right). \end{aligned} \tag{15}$$

The time-derivatives of the new integral states are

$$\dot{\zeta}_1 = -(\nabla_p \Psi_1)^{\top} S_{12}^{\top}\left(\nabla_q H_d^{\star} + \nabla_q \Psi_2 \nabla_{\zeta_2}\Phi\right) \\ -((\nabla_p \Psi_1)^{\top}(\Gamma_1 I - S_{22}^{\top}) - (\nabla_q \Psi_1)^{\top} S_{12}) M_d^{-1} p, \tag{16a}$$

$$\dot{\zeta}_2 = -(\nabla_p \Psi_2)^{\top} S_{12}^{\top}\left(\nabla_q H_d^{\star} + \nabla_q \Psi_2 \nabla_{\zeta_2}\Phi\right) \\ -((\nabla_p \Psi_2)^{\top}(\Gamma_1 I - S_{22}^{\top}) - (\nabla_q \Psi_2)^{\top} S_{12}) M_d^{-1} p. \tag{16b}$$

Proposition 1. The system (1) with *Assumptions 1* to *4* in closed-loop with the new control law (15) and the time-derivatives of the integral states (16a) and (16b) yields (11) with the parameters (12). The proof is given in Appendix A.

Remark 1. The PDEs (4) and (5) are preserved by design, thus the controller (15) is modular with respect to the IDA-PBC (3). In addition, the proposed design contains our previous implementation [13] as a special case: if the input matrix G and the disturbance are constant (i.e., $h(q) = \kappa \in \mathbb{R}^n$) we have $\Psi_2 = G^{\perp}p$ and therefore $\nabla_q \Psi_2 = 0, \nabla_p \Psi_2 = G^{\perp\top}$ for $\kappa = 1$, recovering the PDE (13a) in [13], that is

$$G^{\perp}\left(G^{\perp\top}\Gamma_1\left(k_2(\zeta_2 - G^{\perp}p) + \nabla_{\zeta_2}\Phi\right) - S_{12}^{\top}\nabla_q \Phi\right) = 0.$$

If in addition $G^{\perp}S_{12}^{\top} = G^{\perp}M_d M^{-1}$ is constant, then the former PDE has constant coefficients, and $\Phi(p, \zeta_2)$ can be expressed as $\Phi = \Lambda^{\top}(q - q^{\star})$, where $\Lambda(p, \zeta_2)$ is a vector of closed-loop non-conservative forces, see [17], [18].

B. Stability analysis

Proposition 2. Consider the system (1) with *Assumptions 1* to *4* in closed-loop with the new control law (15) and the time-derivatives of the integral states (16a) and (16b). Then the equilibrium point $(q, p, \zeta_1, \zeta_2) = (q^{\star}, 0, \alpha, \beta)$ of the closed-loop system (11) is locally asymptotically stable.

Proof. It follows from (13) that $W_d \geq 0$ for some $k_0 > 0$ in proximity of $q^{\star}$. Computing the time-derivative of W_d along the trajectories of the closed-loop system (11) yields

$$\begin{aligned} \dot{W}_d = -\nabla_p W_d^{\top} S_{22} \nabla_p W_d - \nabla_{\zeta_1} W_d^{\top} S_{33} \nabla_{\zeta_1} W_d \\ -\nabla_{\zeta_2} W_d^{\top} S_{44} \nabla_{\zeta_2} W_d + \nabla_{\zeta_1} W_d^{\top} S_{34} \nabla_{\zeta_2} W_d \\ -\nabla_{\zeta_2} W_d^{\top} S_{43} \nabla_{\zeta_1} W_d. \end{aligned} \tag{17}$$

Computing the symmetric part of S_{22} as $\Delta_S = \frac{1}{2}S_{22} + \frac{1}{2}S_{22}^{\top}$ yields (8), which is verified by *Assumption 3*. If follows from (12) that $S_{34} - S_{43}^{\top} = -2(\nabla_p \Psi_1)^{\top}(\Gamma_1 I - \Delta_S)(\nabla_p \Psi_2)$. Substituting S_{33} and S_{44} in (17) while omitting skew-symmetric terms yields

$$\begin{aligned} \dot{W}_d = -\nabla_p W_d^{\top} \Delta_S \nabla_p W_d \\ -\nabla_{\zeta_1} W_d^{\top} (\nabla_p \Psi_1)^{\top} (\Gamma_1 I - \Delta_S)(\nabla_p \Psi_1) \nabla_{\zeta_1} W_d \\ -\nabla_{\zeta_2} W_d^{\top} (\nabla_p \Psi_2)^{\top} (\Gamma_1 I - \Delta_S)(\nabla_p \Psi_2) \nabla_{\zeta_2} W_d \\ -2\nabla_{\zeta_2} W_d^{\top} (\nabla_p \Psi_2)^{\top} (\Gamma_1 I - \Delta_S)(\nabla_p \Psi_1) \nabla_{\zeta_1} W_d. \end{aligned} \tag{18}$$

Refactoring terms in (18) yields finally

$$\begin{aligned} \dot{W}_d = -\nabla_p W_d^{\top} \Delta_S \nabla_p W_d - \eta^{\top}(\Gamma_1 I - \Delta_S)\eta, \\ \eta = \nabla_p \Psi_1 \nabla_{\zeta_1} W_d + \nabla_p \Psi_2 \nabla_{\zeta_2} W_d. \end{aligned} \tag{19}$$

It follows from *Assumption 3* and (19) that $\dot{W}_d \leq 0$, hence the equilibrium is stable and all states are bounded.
Case 1. If $\Delta_S \succ 0$ (i.e., $D \succ 0$), it follows from LaSalle's theorem (see Theorem 3.4 in [19]) that the trajectories of the closed-loop system (11) converge asymptotically to the set $\nabla_p W_d = 0 \cap \eta = 0$. Combining the former expressions while substituting (14b), which is verified by *Assumption 4*, and the partial derivatives of W_d (see (20) in Appendix A) yields $p = 0$. Computing $\dot{p}$ from (11) yields then (14c), which is verified by *Assumption 4*, thus the equilibrium point $(q, p) = (q^{\star}, 0)$ is locally asymptotically stable.
Case 2. If $D = 0$ then $\Delta_S = GK_vG^{\top} \succeq 0$ and $\dot{W}_d = 0 \implies G^{\top}\nabla_p W_d = 0 \cap \eta = 0$. Combining the former expressions and substituting (14b), which is verified by *Assumption 4*, yields $y_d = G^{\top}M_d^{-1}p = 0$. Since the output y_d is detectable by *Assumption 1*, the equilibrium point $(q, p) = (q^{\star}, 0)$ is locally asymptotically stable, see [4] □

Remark 2. While the proposed controller (15) has a wider applicability compared to our prior work, [13], [14], the stability conditions are more stringent. This is apparent if $D = 0$, for which *Assumption 3* yields $\Gamma_1 I - GK_vG^{\top} \succ 0$, while the corresponding condition in [13], [14] is simply $\Gamma_1 > 0$. Physical damping $D \succ 0$ further restricts the stability conditions, in accordance with [10], [15].

IV. Simulation results

A. Pendulum-on-cart system

The POC consists of a pendulum of length l_0 with a point mass m_0 at the tip, mounted on an actuated cart, thus $n = 2$ and $m = 1$, see Fig. 1. The equations of motion after partial feedback-linearization, see [16], are given by (1) with

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \; M = I, \; D = 0, \; G = \begin{bmatrix} -b\cos(q_1) \\ 1 \end{bmatrix}, \\ \Omega = a(1 - \cos(q_1)),$$

where a, b are positive constants depending on m_0 and l_0. The angle q_1 of the unactuated pendulum is measured from the vertical, while the position q_2 of the actuated cart is measured from an arbitrary origin. The matrices M_d, J_2 and the potential energy Ω_d that solve the PDEs (4) and (5) are

$$M_d = \begin{bmatrix} \frac{1}{3}b^2 k\cos(q_1)^3 & -\frac{1}{2}bk\cos(q_1)^2 \\ -\frac{1}{2}bk\cos(q_1)^2 & m_{20} + k\cos(q_1) \end{bmatrix},$$

$$J_2 = j_0 \frac{bk\sin(q_1)\cos(q_1)}{24m_{20} + 6k\cos(q_1)} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$j_0 = p_2 bk\cos(q_1)^2 + 3p_1 k\cos(q_1) + 6m_{20}p_1,$$

$$\Omega_d = \frac{3a}{b^2 k\cos(q_1)^2} + \frac{k_p}{2}(\gamma_1)^2,$$

$$\gamma_1 = q_2 + \frac{6m_{20}\tan(q_1)}{bk} + \frac{3}{b}\log\left(\frac{\cos(q_1) + \sin(q_1)}{\cos(q_1)}\right),$$

where k_p, k, m_{20} are tuning parameters, and $M_d \succ 0$ for all $-\pi/2 < q_1 < \pi/2$. For illustrative purposes, the system is subjected to a position-dependent matched disturbance and a constant unmatched disturbance, that is $\delta = \delta_1 G G^\top q + \delta_2 G^{\perp^\top}$. The assignable equilibrium is $(q_1, q_2) = (q_1^\star, q_2^\star)$, where $a \sin(q_1^\star) = \delta_2(1 + b^2 \cos(q_1^\star)^2)$, and it exists provided that $|\delta_2| \leq a/(1+b^2)$. To implement the controller (15), $q_1^\star$ is computed from the equation

$$\sin(q_1^\star) = \frac{(1 + b^2 \cos(q_1^\star)^2)}{a} \Gamma_1 k_2 (\zeta_2 - \Psi_2).$$

In addition, $\Psi_1 = (p_2 - p_1 b \cos(q_1))(q_2 - b q_1 \cos(q_1))$ and $\Psi_2 = p_1 + p_2 b \cos(q_1)$. The scalar function $\Phi(q, p, \zeta_2)$ that verifies (14a) to (14d) locally at the assignable equilibrium $(q, p) = (q^\star, 0)$ with $q_2^\star = 0$ is given in Appendix B.

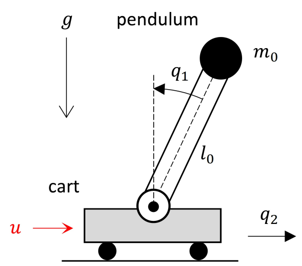

Fig. 1. Schematic of the POC system.

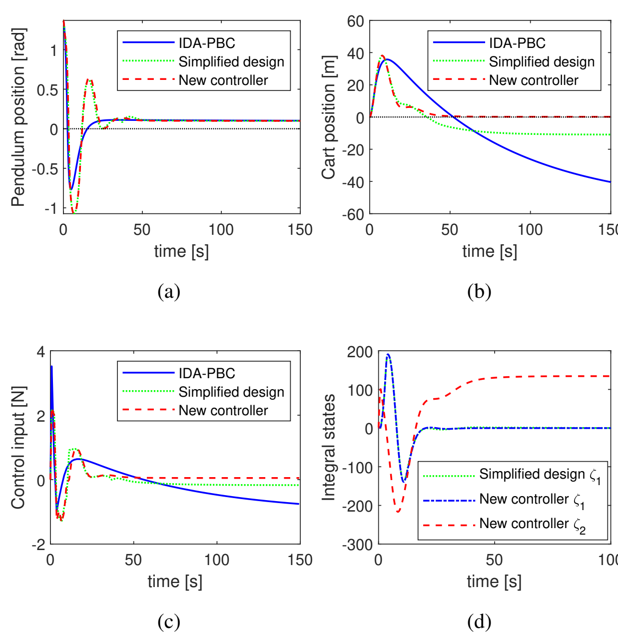

Fig. 2. Simulation results for POC with position-dependent matched disturbance and constant unmatched disturbance: (a) pendulum position; (b) cart position; (c) control input; (d) integral states (16a) and (16b). "Simplified design" imposes $\zeta_2 = 0$, similarly to [14].

The simulations were performed in MATLAB with an ODE23 solver using the parameters $a = 1, b = 1$ and the initial conditions $(q_1, q_2, p_1, p_2, \zeta_1, \zeta_2) = (1.37, -0.1, 0, 0, 0, 0)$. The tuning parameters, $k = m_{20} = 0.01, k_p = 1, K_v = 0.01$ and $k_1 = 10, k_2 = 0.0075, \Gamma_1 = 0.05$, verify *Assumption 3*. Figure 2 shows the system response with $\delta_1 = 0.02$ and $\delta_2 = 0.05$. Using the controller (15) with the integral states (16a) and (16b), the position reaches the assignable equilibrium $(q_1^\star, q_2^\star) = (0.1, 0)$. Instead, either ignoring the unmatched disturbance (i.e., see "Simplified design" in Fig. 2) or employing the baseline IDA-PBC (3) yields large errors on the cart position.

B. Vertical-take-off and landing aircraft

The VTOL is characterized by $m = 2$ actuators, that is (u_1, u_2), and $n = 3$ DOF, that is the horizontal and vertical coordinates of the center of mass (x, y), and the roll angle θ, see Fig. 3. For conciseness, the equations of motion and the details of the IDA-PBC implementation (3) are omitted, and the reader is referred to [16]. For illustrative purposes, the system is subjected to a constant matched disturbance and a constant unmatched disturbance, that is $\delta = \delta_1 G + \delta_2 G^{\perp^\top}$. This results in $\Psi_1 = p_1 + p_2 + p_3 \frac{1}{\epsilon}(\cos(q_3) + \sin(q_3))$ and $\Psi_2 = p_1 \cos(q_3) - \epsilon p_3 + p_2 \sin(q_3)$, where $0 \leq \epsilon \leq 1$ is a parameter that captures the effect of the "slopped" wings, inducing a coupling between the vertical and the roll dynamics. The assignable equilibrium is $(q_1, q_2, q_3) = (q_1^\star, q_2^\star, q_3^\star)$, where $-g \sin(q_3^\star) = (\epsilon^2 + 1)\delta_2$, and it exists provided that $|\delta_2| \leq g/(1+\epsilon^2)$. To implement the controller (15), $q_3^\star$ is computed from the equation

$$\sin(q_3^\star) = -\frac{1}{g}(\epsilon^2 + 1)\Gamma_1 k_2 (\zeta_2 - \Psi_2).$$

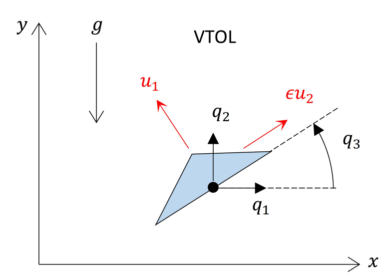

Fig. 3. Schematic of the VTOL system with $(q_1, q_2, q_3) = (x, y, \theta)$.

The simulations were performed in MATLAB with an ODE23 solver and the parameters $\epsilon = 1$, $K_p = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.2 \end{bmatrix}$, $K_v = K_0 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$, $k_1 = 1$, $k_2 = 1, \Gamma_1 = 3.1$, which verify *Assumption 3*. The initial conditions are $(q_1, q_2, q_3, p_1, p_2, p_3, \zeta_1, \zeta_2) = (-5, 0, 0.1, -0.1, -0.1, 0.1, 0, 0)$ (i.e., not corresponding to steady state). Figure 4 shows the system response with $\delta_1 = 0.5$ and $\delta_2 = -0.2$. Employing the new controller (15) with the integral states (16a) and (16b), the position reaches the assignable equilibrium $(q_1^\star, q_2^\star, q_3^\star) = (0, 0, 0.04)$. Either ignoring the unmatched disturbance (i.e., see "Simplified design" in Fig.4) or employing the baseline IDA-PBC (3), while setting $K_v = 10K_0$ to reduce oscillations, yields large steady-state errors on the position (x, y).

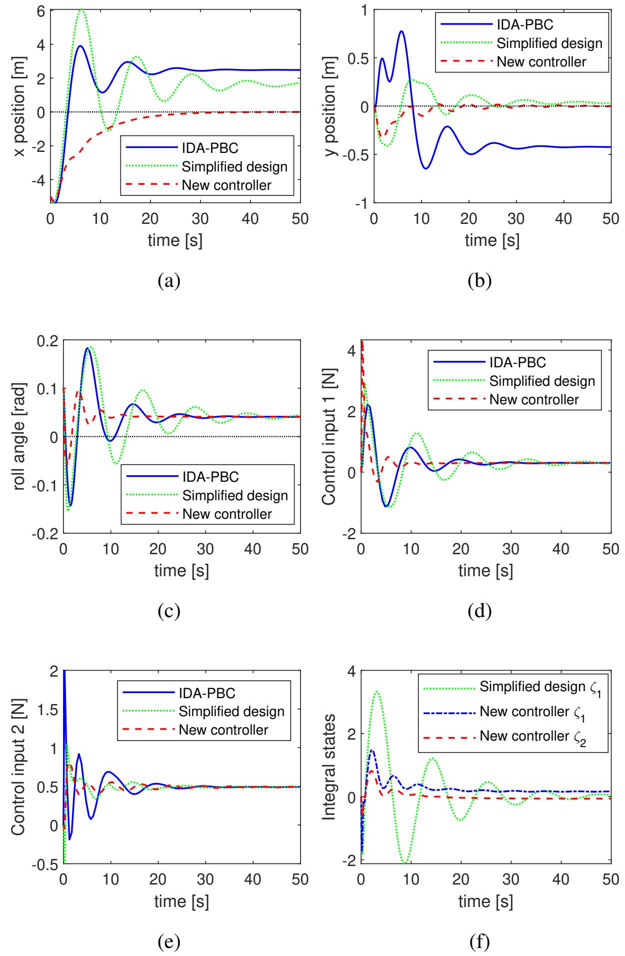

Fig. 4. Simulation results for VTOL with constant matched and unmatched disturbances: (a) x position; (b) y position; (c) roll angle θ; (d) control input u_1 and (e) u_2; (f) integral states (16a) and (16b). "Simplified design" imposes $\zeta_2 = 0$.

The MATLAB code of both examples, including the analytical expression of $M, \Omega, M_d, \Omega_d, J_2$, and Φ that solves (14a) to (14d) locally at $q^\star$ for the VTOL, are available on IEEE Code Ocean.

V. Conclusion

This work introduces a novel iIDA-PBC design for underactuated mechanical systems with a non-constant input matrix and subject to both matched and unmatched disturbances, either constant or position-dependent. The proposed controller design is more general than existing implementations, but it imposes stricter stability conditions. In addition, rejecting unmatched disturbances requires solving additional PDEs, which poses practical challenges. Simulation results on two examples with various types of disturbances demonstrate the effectiveness of the new controller.

Future work will explore methodologies for solving the PDEs to provide constructive solutions for a broad class of systems and will investigate different classes of disturbances.

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APPENDIX A

Proof of Proposition 1.
Computing the partial derivatives of W_d from (13) yields

$$\begin{aligned}
\nabla_q W_d &= \nabla_q \left(\Omega_d + \frac{1}{2}p^\top M_d^{-1}p + \Phi\right) \\
&\quad -k_1\nabla_q\Psi_1(\zeta_1 - \Psi_1 - \alpha) - k_2\nabla_q\Psi_2(\zeta_2 - \Psi_2 - \beta), \\
\nabla_p W_d &= M_d^{-1}p - k_1\nabla_p\Psi_1(\zeta_1 - \Psi_1 - \alpha) \\
&\quad +\nabla_p\Phi - k_2\nabla_p\Psi_2(\zeta_2 - \Psi_2 - \beta), \\
\nabla_{\zeta_1} W_d &= k_1\left(\zeta_1 - \Psi_1 - \alpha\right), \\
\nabla_{\zeta_2} W_d &= k_2\left(\zeta_2 - \Psi_2 - \beta\right) + \nabla_{\zeta_2}\Phi.
\end{aligned} \tag{20}$$

Equating the corresponding rows of (1) and of (11) yields

$$M^{-1}p = S_{12}\nabla_p W_d + S_{13}\nabla_{\zeta_1}W_d + S_{14}\nabla_{\zeta_2}W_d, \tag{21a}$$

$$\begin{aligned}
&-\nabla_q H - D\nabla_p H + Gu - \delta_1 GG^\top h(q) - \delta_2 {G^\perp}^\top G^\perp h(q) = \\
&-S_{12}^\top\nabla_q W_d - S_{22}\nabla_p W_d + S_{23}\nabla_{\zeta_1}W_d + S_{24}\nabla_{\zeta_2}W_d,
\end{aligned} \tag{21b}$$

$$\dot{\zeta}_1 = -S_{13}^\top\nabla_q W_d - S_{23}^\top\nabla_p W_d - S_{33}\nabla_{\zeta_1}W_d + S_{34}\nabla_{\zeta_2}W_d, \tag{21c}$$

$$\dot{\zeta}_2 = -S_{14}^\top\nabla_q W_d - S_{24}^\top\nabla_p W_d - S_{43}\nabla_{\zeta_1}W_d - S_{44}\nabla_{\zeta_2}W_d. \tag{21d}$$

Step 1. Substituting S_{13}, S_{14} from (12) and the partial derivatives of W_d from (20) into (21a) yields

$$\begin{aligned}
M^{-1}p &= S_{12}\left(M_d^{-1}p + \nabla_p\Phi\right) + S_{12}\nabla_p\Psi_1 k_1\left(\zeta_1 - \Psi_1 - \alpha\right) \\
&\quad -S_{12}\left(k_1\nabla_p\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_p\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&\quad +S_{12}\nabla_p\Psi_2\left(k_2\left(\zeta_2 - \Psi_2 - \beta\right) + \nabla_{\zeta_2}\Phi\right).
\end{aligned}$$

Refactoring the former expression and subtracting the PDE (14b), which is verified by *Assumption 4*, yields the equation $M^{-1}p = M^{-1}M_d M_d^{-1}p$, which is verified for all $M_d \succ 0$.
Step 2. Substituting S_{23} and S_{24} from (12) and the partial derivatives of W_d from (20) into (21b) yields

$$\begin{aligned}
&-\nabla_q H - D\nabla_p H + Gu - \delta_1 GG^\top h(q) - \delta_2 {G^\perp}^\top G^\perp h(q) = \\
&-S_{12}^\top\nabla_q\left(\Omega_d + \frac{1}{2}p^\top M_d^{-1}p + \Phi\right) - S_{22}\left(M_d^{-1}p + \nabla_p\Phi\right) \\
&+S_{12}^\top\left(k_1\nabla_q\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_q\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&+S_{22}\left(k_1\nabla_p\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_p\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&+\left(\nabla_p\Psi_1\Gamma_1 - S_{22}\nabla_p\Psi_1 - S_{12}^\top\nabla_q\Psi_1\right)k_1\left(\zeta_1 - \Psi_1 - \alpha\right) \\
&+\left(\nabla_p\Psi_2\Gamma_1 - S_{22}\nabla_p\Psi_2 - S_{12}^\top\nabla_q\Psi_2\right)k_2\left(\zeta_2 - \Psi_2 - \beta\right) \\
&+\left(\nabla_p\Psi_2\Gamma_1 - S_{22}\nabla_p\Psi_2 - S_{12}^\top\nabla_q\Psi_2\right)\nabla_{\zeta_2}\Phi.
\end{aligned}$$

Substituting α and β in the previous expression and noting that $\nabla_p\Psi_1 = GG^\top h(q), \nabla_p\Psi_2 = {G^\perp}^\top G^\perp h(q)$ cancels the disturbance δ_1 and δ_2 yielding

$$\begin{aligned}
&-\nabla_q H - D\nabla_p H + S_{12}^\top\nabla_q\left(\Omega_d + \frac{1}{2}p^\top M_d^{-1}p + \Phi\right) \\
&+Gu + S_{22}M_d^{-1}p - \nabla_p\Psi_1\Gamma_1 k_1\left(\zeta_1 - \Psi_1\right) = \\
&-S_{22}\left(\nabla_p\Phi + \nabla_p\Psi_2\nabla_{\zeta_2}\Phi\right) + \nabla_p\Psi_2\Gamma_1 k_2\left(\zeta_2 - \Psi_2\right) \\
&+\left(\nabla_p\Psi_2\Gamma_1 - S_{12}^\top\nabla_q\Psi_2\right)\nabla_{\zeta_2}\Phi.
\end{aligned}$$

Multiplying the above by $G^\dagger$ and substituting the control law (15) yields the PDE (14b) (i.e., pre-multiplied by $G^\dagger S_{22}$), which is verified by *Assumption 4*. Multiplying it instead by $G^\perp$ yields the sum of the PDEs (4), (5), (14a), and (14b) (i.e., pre-multiplied by $G^\perp S_{22}$), which are all verified by *Assumption 1* and *Assumption 4*.
Step 3. Substituting S_{33}, S_{34} from (12) and the partial derivatives of W_d from (20) into (21c) yields

$$\begin{aligned}
\dot{\zeta}_1 &= -S_{13}^\top\nabla_q\left(\Omega_d + \frac{1}{2}p^\top M_d^{-1}p + \Phi\right) \\
&+S_{13}^\top\left(k_1\nabla_q\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_q\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&+S_{23}^\top\left(k_1\nabla_p\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_p\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&-\left(S_{13}^\top\nabla_q\Psi_1 + S_{23}^\top\nabla_p\Psi_1\right)k_1\left(\zeta_1 - \Psi_1 - \alpha\right) \\
&+\left(-S_{13}^\top\nabla_q\Psi_2 - S_{23}^\top\nabla_p\Psi_2\right)k_2\left(\zeta_2 - \Psi_2 - \beta\right) \\
&-S_{23}^\top\left(M_d^{-1}p + \nabla_p\Phi\right) - \left(S_{13}^\top\nabla_q\Psi_2 + S_{23}^\top\nabla_p\Psi_2\right)\nabla_{\zeta_2}\Phi.
\end{aligned}$$

Refactoring terms in the former expression cancels α and β. Substituting (16a) yields (14b) (i.e., pre-multiplied by $S_{23}^\top$), which is verified by *Assumption 4*.
Step 4. Substituting S_{43}, S_{44} from (12) and the partial derivatives of W_d from (20) into (21d) yields

$$\begin{aligned}
\dot{\zeta}_2 &= -S_{14}^\top\nabla_q\left(\Omega_d + \frac{1}{2}p^\top M_d^{-1}p + \Phi\right) \\
&+S_{14}^\top\left(k_1\nabla_q\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_q\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&+S_{24}^\top\left(k_1\nabla_p\Psi_1(\zeta_1 - \Psi_1 - \alpha) + k_2\nabla_p\Psi_2(\zeta_2 - \Psi_2 - \beta)\right) \\
&-\left(S_{14}^\top\nabla_q\Psi_1 + S_{24}^\top\nabla_p\Psi_1\right)k_1\left(\zeta_1 - \Psi_1 - \alpha\right) \\
&-\left(S_{14}^\top\nabla_q\Psi_2 + S_{24}^\top\nabla_p\Psi_2\right)k_2\left(\zeta_2 - \Psi_2 - \beta\right) \\
&-S_{24}^\top\left(M_d^{-1}p + \nabla_p\Phi\right) - \left(S_{14}^\top\nabla_q\Psi_2 + S_{24}^\top\nabla_p\Psi_2\right)\nabla_{\zeta_2}\Phi.
\end{aligned}$$

Refactoring terms in the former expression cancels α and β. Substituting (16b) yields (14b) (i.e., pre-multiplied by $S_{24}^\top$), which is verified by *Assumption 4*, concluding the proof □

APPENDIX B

Scalar function $\Phi(q, p, \zeta_2)$ for the POC system.

$$\begin{aligned}
\Phi &= \frac{3(q_1^\star - q_1)}{b^2 k}\left(\frac{2a\tan\left(q_1^\star\right) + 3k_p\gamma_3(2m_{20} + k\cos\left(q_1^\star\right))}{k\cos\left(q_1^\star\right)^2}\right. \\
&\quad \left. - \frac{3k_p q_2\gamma_3}{bk} - \gamma_2\frac{p_1 - \zeta_2 + bp_2\cos\left(q_1\right)}{2\Gamma_1 k\cos\left(q_1^\star\right)^3\left(1 + b^2\cos\left(q_1\right)^2\right)}\right. \\
&\quad \left. - \frac{a\sin\left(q_1^\star\right)\gamma_2}{2\Gamma_1^2 k_2 k\cos\left(q_1^\star\right)^3\left(1 + b^2\cos\left(q_1^\star\right)^2\right)\left(1 + b^2\cos\left(q_1\right)^2\right)}\right),
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= \left(2ak + 12k_p m_{20}^2\right)\cos\left(q_1\right)^3\sin\left(q_1^\star\right) \\
&+2\Gamma_1 k_2 k\cos\left(q_1^\star\right)^3(p_1 - \zeta_2) - 3\gamma_4 k^2 k_p\cos\left(q_1\right)^2\cos\left(q_1^\star\right)^3 \\
&+2\Gamma_1 k_2 bkp_2\cos\left(q_1\right)\cos\left(q_1^\star\right)^3 + 3\gamma_4 k^2 k_p\cos\left(q_1\right)^3\cos\left(q_1^\star\right)^2 \\
&-6k_p m_{20}\cos\left(q_1\right)\cos\left(q_1^\star\right)^2\sin\left(q_1\right)(2m_{20} + k\cos\left(q_1\right)) \\
&+6kk_p m_{20}\cos\left(q_1\right)^3\cos\left(q_1^\star\right)\sin\left(q_1\right) \\
&+6k_p km_{20}\gamma_4\cos\left(q_1\right)\cos\left(q_1^\star\right)(\cos\left(q_1\right)^2 - \cos\left(q_1^\star\right)^2) \\
&+2\Gamma_1 k_2 bk\cos\left(q_1\right)^2\cos\left(q_1^\star\right)^3(p_1 + p_2 b\cos\left(q_1\right) - \zeta_2),
\end{aligned}$$

$$\gamma_3 = k\gamma_4 + 2m_{20}\tan\left(q_1^\star\right),\ \ \gamma_4 = \log\left(\frac{1 + \sin\left(q_1^\star\right)}{\cos\left(q_1^\star\right)}\right).$$