Quantum Science and Technology

PAPER • **OPEN ACCESS**

Continuous variable structured collision models

Quantum Science and Technology

OPEN ACCESS

RECEIVED
27 March 2025

REVISED
10 September 2025

ACCEPTED FOR PUBLICATION
23 September 2025

PUBLISHED
8 October 2025

PAPER

Continuous variable structured collision models

Anton Corr[1,*], **Stefano Cusumano**[2,3] **and Gabriele De Chiara**[1,4]

[1] Centre for Quantum Materials and Technology, Queen's University Belfast, BT7 1NN Belfast, United Kingdom
[2] Dipartimento di Fisica 'Ettore Pancini', Universitá degli Studi di Napoli Federico II, Via Cintia, 80126 Napoli, Italy
[3] INFN, Sezione di Napoli, Napoli, Italy
[4] Física Teórica: Informació i Fenómens Quántics, Departament de Física, Universitat Autónoma de Barcelona, 08193 Bellaterra, Spain
[*] Author to whom any correspondence should be addressed.

E-mail: acorr09@qub.ac.uk

Keywords: quantum thermodynamics, open quantum systems, collision model, continuous variable

Abstract

Quantum collision models allow for the dynamics of open quantum systems to be described by breaking the environment into small segments, typically consisting of non-interacting harmonic oscillators or two-level systems. This work introduces structure within these environmental units via spring-like interactions between N coupled oscillators in a ring structure, initially prepared in a thermal state. Two models of interest are examined. The first highlights a case in which a continuous time evolution can be obtained, wherein the system interacts with the environment via a beam-splitter-like, energy-preserving, interaction. The resulting dynamics are analogous to those due to interactions with unstructured units prepared as squeezed thermal states. The second model highlights a case in which the continuous time limit for the evolution cannot be taken generally, requiring instead discrete-time propagation. Special cases in which the continuous time limit can be taken are also investigated, alongside the addition of a secondary environment to induce a steady state. The first and second laws of thermodynamics are verified for both examples.

1. Introduction

The study of open quantum systems [1, 2], i.e. systems that exchange energy, information and particles with their surrounding environments, plays a crucial role in the description of realistic quantum systems, as it is often impractical, or impossible, to neglect the effects of the environment. These environments typically contain a large number of degrees of freedom, making an exact analysis of such dynamics at best computationally difficult, and at worst impossible.

Quantum collision models [3–6] offer a method of simulating open dynamics by segmenting the large environment into smaller environmental units. These units then interact with the system sequentially for short timesteps, before being discarded post-interaction. These models allow for the dynamics of a large compilation of physical systems and environments to be examined while maintaining a small number of degrees of freedom.

Due to their versatility, collision models have become a popular method of investigating a wide variety of phenomena in both Markovian [3, 5, 7, 8] and non-Markovian regimes [9–12], including quantum thermal machines [13–21] and the behavior of time crystals [22, 23]. Applications of such models can be found in the field of quantum optics [24–27], quantum resources and information [28–31], and many others [32–36].

A key advantage of these models is the ability to study the thermodynamic properties of open systems [37–44], allowing for the investigation of the flow of the system's internal energy, alongside the corresponding thermodynamic heat and work flows. Furthermore, the entropy and entropy production are readily available [45, 46], allowing for the verification of the laws of thermodynamics.

In a typical collision model, the system and environmental units are considered to be composed of independent objects, such as harmonic oscillators or two-level systems. Recently, however, focus has been drawn to the idea of structured collision models, wherein interactions within the system or environment are included [47]. In this work, we present the case of a structured continuous-variable environment, studying

the dynamics of a system of harmonic oscillators interacting with environmental units composed of an arbitrary number of interacting harmonic oscillators.

We examine two examples of interest in this work. In both, the environmental units consist of N_E harmonic oscillators coupled via spring-like interactions in a ring structure (see figure 1). The first example investigates the effects of coupling the system and environment through beamsplitter-like interactions. In this example, the continuous time limit for the evolution of the reduced state of the system can be taken, allowing for the corresponding Lyapunov equation for the covariance matrix to be derived. We find that preparing the structured environmental units in a thermal state leads to dynamics analogous to those produced through interactions with an unstructured environment initially prepared in a squeezed thermal state, with a modified effective temperature.

The second example considers spring-like interactions between the system and environment, which may result in divergences when taking the continuous time limit. We find that discrete time propagation must be used in the general case, with the continuous time limit only possible in special cases. We emphasise, however, that these cases are not achievable without the use of environmental units composed of a minimum of two oscillators. While this example does not result in a steady state, a method of inducing a steady state through the inclusion of a secondary unstructured environment interacting with the system via a beamsplitter-like interaction is also presented.

This work is organized such that the general model is derived in section 2, including the structure of the collision model, the corresponding dynamics and the methods of deriving the relevant thermodynamic quantities. Section 3 introduces the physical examples, with section 3.2 presenting the beamsplitter-like system-environment interactions. Section 3.3 then presents the spring-like system-environment interactions, including the general discrete time propagation and continuous time special cases. Outlooks on future work, alongside the conclusions drawn from the models, will be presented in section 4. Some details on the analysis and notation used can be found in the appendices.

2. General model

Collision models, in general, allow for the study of the dynamics of a system $\mathcal{S}$ interacting with an environment $\mathcal{E}$. As the environment is typically large and complicated, tracking the interactions with the total environment may be conceptually difficult and computationally expensive. To combat this, collision models treat the environment as an infinitely large collection of, generally identically prepared, environmental units E_i. In the Markovian regime, the system interacts with each unit sequentially for a time δt, and the unit is discarded after the interaction to ensure the system's evolution is Markovian, see figure 1. Often, it then becomes possible to derive a continuous Lyapunov equation for the system by taking the continuous limit of $\delta t \to 0$ after imposing some conditions on the interaction between the system and the environment.

2.1. System and environment structure

As a general case, we will consider a system of N_S oscillators and an environment made of an infinite number of units, each composed of N_E coupled oscillators. The system then interacts with each of these units sequentially for a time δt. The total Hamiltonian H_{tot} of such a model is composed of the free Hamiltonian of the system oscillators, H_S, the Hamiltonian of the environmental unit, H_E, which includes the coupling terms present within the environment, and the system-environment interaction Hamiltonian, H_{SE}, such that

$$H_{\text{tot}} = H_S + H_E + H_{SE}. \tag{1}$$

The Hamiltonian of the system can be described in terms of the position and momentum operators of each oscillator i, namely x_{S_i} and p_{S_i} respectively, such that

$$H_S = \frac{1}{2}\sum_{i=1}^{N_S}\left(\omega_{S_i}^2 x_{S_i}^2 + p_{S_i}^2\right). \tag{2}$$

For simplicity, in this work we consider uncoupled system oscillators of unit mass m, with $\hbar = m = 1$. Furthermore, the position and momentum operators obey the standard commutation relation $[x_{S_i}, p_{S_j}] = \mathrm{i}\delta_{ij}$. Equation (2) can be expressed in matrix notation as

$$H_S = \frac{1}{2} R_S^{\intercal} M_S R_S, \tag{3}$$

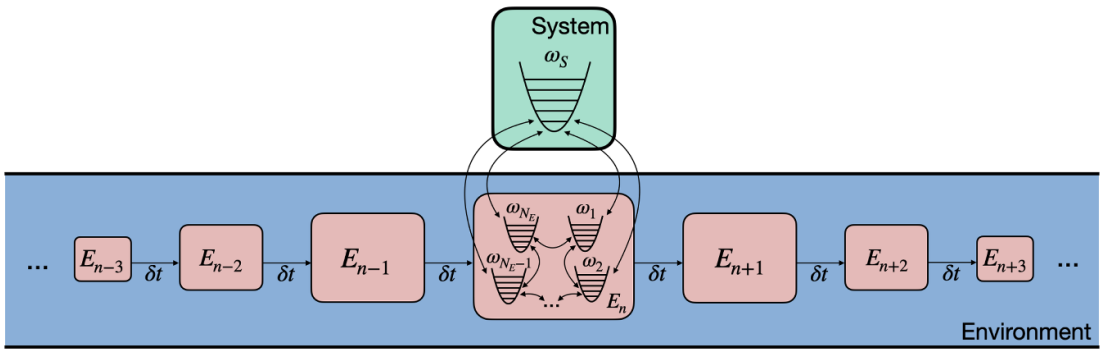

Figure 1. A system composed of a harmonic oscillator interacting with an environment composed of structured units, each containing N_E interacting oscillators arranged in a ring, via a collision model. Each environmental unit interacts with the system for a timestep δt before being replaced by an identical unit.

with R_S denoting a column vector containing the position and momentum operators such that $R_S^\intercal = \left(x_{S_1}, p_{S_1}, x_{S_2}, p_{S_2}, \ldots, x_{S_{N_S}}, p_{S_{N_S}}\right)$, and

$$M_S = \bigoplus_{i=1}^{N_S} \begin{pmatrix} \omega_{s_i}^2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4}$$

The Hamiltonian for each of the environmental units can be separated into two terms, namely the free Hamiltonian H_E^F and the interaction Hamiltonian describing the internal interactions within the unit H_E^I, where the total H_E is

$$H_E = H_E^F + H_E^I. \tag{5}$$

The free Hamiltonian of each environmental unit can be written in a form analogous to equation (2):

$$H_E^F = \frac{1}{2}\sum_{i=1}^{N_E} \left(\omega_{E_i}^2 x_{E_i}^2 + p_{E_i}^2\right). \tag{6}$$

with x_{E_i} denoting the position operator of each environmental oscillator i, and p_{E_i} the corresponding momentum operator obeying the analogous commutation relations, $[x_{E_i}, p_{E_j}] = i\delta_{ij}$. The Hamiltonians H_E^I and H_{SE} are then dependent on the choice of internal environmental interactions and system-environment interactions, respectively. The environmental Hamiltonian can also be cast in matrix form, with

$$H_E = \frac{1}{2} R_E^\intercal M_E R_E, \tag{7}$$

where $R_E^\intercal = \left(x_{E_1}, p_{E_1}, x_{E_2}, p_{E_2}, \ldots, x_{E_{N_E}}, p_{E_{N_E}}\right)$, and M_E as the corresponding matrix of coefficients.

In this work, we assume each environmental unit to be prepared initially in a thermal state of the form $\rho_E = \exp(-\beta_E H_E)/Z_E$ where $Z_E = \mathrm{Tr}[\exp(-\beta_E H_E)]$ and $\beta_E = 1/T_E$. For simplicity, units have been chosen such that $k_\mathrm{B} = 1$, where k_B is the Boltzmann constant. As only Gaussian states will be considered, it is convenient to introduce the covariance matrix, σ, defined through

$$\sigma_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i\rangle - \langle R_i\rangle\langle R_j\rangle. \tag{8}$$

Here, R_i refers to the system or environment's operators, while the notation $\langle R_i\rangle = \mathrm{Tr}[\rho R_i]$ denotes the average value of an operator, where ρ is the relevant density matrix of the system, ρ_S, environmental unit ρ_E, or the combination of the system and environmental unit ρ_{tot}.

As M_E is necessarily non-diagonal in the basis of R_E, due to the internal environmental interactions, the covariance matrix of the environment σ_E in the thermal state is not diagonal. It is possible, however, to define a normal-mode coordinate system $Q_E^\intercal = \left(\tilde{x}_{E_1}, \tilde{p}_{E_1}, \tilde{x}_{E_2}, \tilde{p}_{E_2}, \ldots, \tilde{x}_{E_{N_E}}, \tilde{p}_{E_{N_E}}\right)$, through a transformation G such that $Q_E = GR_E$, for which the covariance matrix is diagonal. The corresponding Hamiltonian can be expressed as

$$H_E = \frac{1}{2} Q_E^\intercal \tilde{M}_E Q_E, \tag{9}$$

where $\tilde{M}_E$ is a diagonal matrix defined by

$$\tilde{M}_E = G M_E G^{\intercal}. \tag{10}$$

This transformation matrix G must be symplectic to ensure the commutation relations are preserved under the change of basis, i.e. $G^{\intercal}\Omega_{N_E}G = \Omega_{N_E}$, where

$$\Omega_{N_E} = \bigoplus_{i=1}^{N_E} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{11}$$

In general, $\tilde{M}_E$ takes the form

$$\tilde{M}_E = \bigoplus_{m=1}^{N_E} \begin{pmatrix} \tilde{\omega}_{E_m} & 0 \\ 0 & 1 \end{pmatrix}, \tag{12}$$

where $\tilde{\omega}_{E_m}$ denotes the normal mode frequencies of the environmental unit. For the interactions considered in this work, we find $G^{\intercal}G = \mathbb{1}_{2N_E}$, with $\mathbb{1}_{2N_E}$ as the identity matrix of dimension $2N_E$, where the columns of G correspond to the orthogonal eigenvectors of M_E. One can then recover the basis of R_E through

$$\sigma_E = G^{\intercal}\tilde{\sigma}_E G. \tag{13}$$

The total state of the system at time t and an environmental unit can then be described through the covariance matrix $\sigma_{\text{tot}}(t)$,

$$\sigma_{\text{tot}}(t) = \sigma_S(t) \oplus \sigma_E, \tag{14}$$

where $\sigma_S(t)$ is the covariance matrix corresponding to the system at a time t. In equation (14), we employed the direct sum as a consequence of assuming no initial correlations between the system and the unit.

For completeness, one can also recast the system-environment interaction Hamiltonian, H_{SE} into its matrix form, producing

$$H_{SE} = \frac{1}{2} R_{\text{tot}}^{\intercal} M_{SE} R_{\text{tot}}, \tag{15}$$

where $R_{\text{tot}} = R_S \oplus R_E$.

2.2. System dynamics

The Markovian collision model framework describes the system's interaction with an environment through a series of short interactions of duration δt with the units of the environment. Each unit is discarded after the collision to ensure Markovianity. As in this work we consider the dynamics of Gaussian states, we are interested in the propagation of the system's first and second moments. The evolution of the second moments can be described through a symplectic operator U such that

$$\sigma_{\text{tot}}(t + \delta t) = U \sigma_{\text{tot}}(t) U^{\dagger} \tag{16}$$

where $\sigma_{\text{tot}}(t)$ is defined in equation (14). The time evolution operator U is described as

$$U = \mathrm{e}^{\delta t D_{\text{tot}}}, \tag{17}$$

where the propagation matrix D_{tot} satisfies

$$\langle \dot{R} \rangle_{\text{tot}} = D_{\text{tot}} \langle R \rangle_{\text{tot}} \tag{18}$$

with $\langle \dot{R} \rangle_{\text{tot}} = \frac{\mathrm{d}}{\mathrm{d}t} \langle R \rangle_{\text{tot}}$. The elements of this propagation matrix D_{tot} can be found directly using the Heisenberg equation for the system and environment position and momentum operators, or equivalently through $D_{\text{tot}} = \Omega_{N_{\text{tot}}} M_{\text{tot}}$, where

$$H_{\text{tot}} = \frac{1}{2} R_{\text{tot}}^{\intercal} M_{\text{tot}} R_{\text{tot}} \tag{19}$$

and $N_{\text{tot}} = N_S + N_E$.

Expanding equation (16) through the Taylor expansion to second order in δt produces three superoperators:

$$U\sigma_{\text{tot}}(t)\,U^{\dagger} \simeq \mathcal{U}_0[\sigma_{\text{tot}}(t)] + \mathcal{U}_1[\sigma_{\text{tot}}(t)] + \mathcal{U}_2[\sigma_{\text{tot}}(t)]. \tag{20}$$

The superoperator $\mathcal{U}_i[\sigma_{\text{tot}}(t)]$ describes the ith order contributions from the Taylor expansion, with

$$\mathcal{U}_0[\sigma_{\text{tot}}(t)] = \sigma_{\text{tot}}(t), \tag{21}$$
$$\mathcal{U}_1[\sigma_{\text{tot}}(t)] = \delta t[D_{\text{tot}}\sigma_{\text{tot}}(t) + \sigma_{\text{tot}}(t)\,D_{\text{tot}}^{\intercal}], \tag{22}$$
$$\mathcal{U}_2[\sigma_{\text{tot}}(t)] = \frac{\delta t^2}{2}\left[2D_{\text{tot}}\sigma_{\text{tot}}(t)\,D_{\text{tot}}^{\intercal} + D_{\text{tot}}^2\sigma_{\text{tot}}(t) + \sigma_{\text{tot}}(t)\,D_{\text{tot}}^{\intercal 2}\right]. \tag{23}$$

As the environmental units are replaced between collisions, it is only necessary to retain the effect these superoperators have on the block of the covariance matrix that describes the system, i.e. $\sigma_S(t)$. This is mathematically equivalent to tracing out the degrees of freedom of the relevant environmental unit after each collision. By discarding the components outside of this block, these superoperators can be simplified to retain only the required matrices of dimensions $2N_S$. The dynamics of the system can then be described through $\sigma_S(t+\delta t) = \mathcal{L}[\sigma_{\text{tot}}(t)]$ where $\mathcal{L} \simeq \mathcal{L}_0 + \mathcal{L}_1 + \mathcal{L}_2$ to second order in δt, where

$$\mathcal{L}_0[\sigma_S(t)] = \sigma_S(t), \tag{24}$$
$$\mathcal{L}_1[\sigma_S(t)] = \delta t\left\{D_S\sigma_S(t) + \sigma_S(t)\,D_S^{\intercal} + [D_{SE}\sigma_{\text{tot}}(t)]_S + [\sigma_{\text{tot}}(t)\,D_{SE}^{\intercal}]_S\right\}, \tag{25}$$
$$\mathcal{L}_2[\sigma_S(t)] = \frac{\delta t^2}{2}\left\{\mathcal{L}_2^{S}[\sigma_S(t)] + \mathcal{L}_2^{SE}[\sigma_S(t)] + \mathcal{L}_2^{S/SE}[\sigma_S(t)]\right\}. \tag{26}$$

The detailed derivation of these superoperators, along with the explicit expressions of $\mathcal{L}_2^{S}, \mathcal{L}_2^{SE}$ and $\mathcal{L}_2^{S/SE}$, can be found in appendix A. We have introduced the notation $[A]_S$ to denote the partial trace of the environment through the discarding of elements of the covariance matrix outside of those corresponding to the system, retaining only matrix elements A_{ij} for which $1 \leqslant i \leqslant 2N_S$ and $1 \leqslant j \leqslant 2N_S$. Here, D_S is a matrix of dimension $2N_S$ describing the dynamics of the free Hamiltonian of the system such that $D_S = \Omega_{N_S}M_S$. D_{SE} is then a matrix of dimension $2N_{\text{tot}}$, such that $D_{SE} = \Omega_{N_{\text{tot}}}M_{SE}$.

As we will see throughout the examples in section 3, D_{SE} is dependent on the choice of system-environment interaction Hamiltonian. The terms relating to the free evolution of the environment do not survive the tracing out of the environmental unit; instead the internal environmental couplings only affect the system through the initial state $\sigma_{\text{tot}}(t)$ and the D_{SE} terms.

The discrete propagation of the system can then be described through

$$\frac{\sigma_S(t+\delta t) - \sigma_S(t)}{\delta t} = \frac{\mathcal{L}_1(\sigma_{\text{tot}}(t)) + \mathcal{L}_2(\sigma_{\text{tot}}(t))}{\delta t}. \tag{27}$$

The continuous time limit of this propagation is obtained by rescaling the coupling constants g_{E_i} within D_{SE} before taking the limit of $\delta t \to 0$, as shown in section 3.2, to form a Lyapunov equation. This is not always possible without divergences, however the following conditions are sufficient to ensure a finite Lyapunov equation:

(i) The second order term $\mathcal{L}_2$ must be a homogeneous function of the coupling constants g_{E_i}, where g_{E_i} represents the coupling strength between the system and each oscillator i within the environmental unit.
(ii) The first order corrections to the system's free evolution appearing in $\mathcal{L}_1$ (see equation (25)) must be zero: $[D_{SE}\sigma_{\text{tot}}(t)]_S = [\sigma_{\text{tot}}(t)D_{SE}^{\intercal}]_S = 0$.

In this work, we consider quadratic coupling Hamiltonians for which condition (i) is always satisfied. In cases for which condition (ii) is satisfied, we find that the terms corresponding to the first order expansion in δt, $\mathcal{L}_1(\sigma_{\text{tot}}(t))$, describe the free evolution of the system, while the second order terms, $\mathcal{L}_2(\sigma_{\text{tot}}(t))$, result from the interaction of the system with the environment.

2.3. Thermodynamic quantities

The change in the system's internal energy $\Delta U(t)$, the heat $\Delta Q(t)$ exchanged with the environment and the associated thermodynamic work $\Delta W(t)$ are natural quantities of interest [16, 37, 46–49] that can be extracted from the dynamics described in section 2.2. These can be expressed through

$$\Delta U(t) = \frac{1}{2}\text{Tr}\left[\Delta\sigma_{\text{tot}}(t)\,(M_S \oplus O_{2N_E})\right], \tag{28}$$

$$\Delta Q(t) = -\frac{1}{2}\mathrm{Tr}\left[\Delta\sigma_{\mathrm{tot}}(t)\left(O_{2N_S} \oplus M_E\right)\right], \tag{29}$$

$$\Delta W(t) = \frac{1}{2}\mathrm{Tr}\left[\Delta\sigma_{\mathrm{tot}}(t)\left(M_S \oplus M_E\right)\right], \tag{30}$$

where $\Delta\sigma_{\mathrm{tot}}(t) = \sigma_{\mathrm{tot}}(t+\delta t) - \sigma_{\mathrm{tot}}(t)$ and O_N denotes the zero matrix of dimension N. These quantities satisfy the first law of thermodynamics by construction, i.e.

$$\Delta U(t) = \Delta Q(t) + \Delta W(t). \tag{31}$$

One can also consider the von Neumann entropy of the system

$$S_{\mathrm{VN}}(\rho_S) = -\mathrm{Tr}\left[\rho_S \ln \rho_S\right], \tag{32}$$

where ρ_S is the reduced density operator of the system. To remain consistent with the covariance matrix formalism described in section 2.1, it is instructive to cast equation (32) in terms of the symplectic eigenvalues ν_k of $\sigma_S(t)$, where ν_k are defined as the absolute value of the eigenvalues of $\mathrm{i}\Omega_{N_S}\sigma_S(t)$. As the eigenvalues of σ_S are paired such that each ν_k is 2-fold degenerate, we consider only the $N_S/2$ unique elements. This allows for the entropy to be calculated as [50]

$$S_{\mathrm{VN}}(\rho_S(t)) = \sum_{k=1}^{N_S/2} S_e(\nu_k). \tag{33}$$

$S_e(\nu_k)$ takes the form

$$S_e(\nu_k) = \left(\nu_k + \frac{1}{2}\right)\ln\left(\nu_k + \frac{1}{2}\right) - \left(\nu_k - \frac{1}{2}\right)\ln\left(\nu_k - \frac{1}{2}\right). \tag{34}$$

As information is lost between collisions due to the discarding and replacing of the environmental units, it is important to focus the analysis on the entropy production per collision, defined as

$$\Sigma(t) = \Delta S_{\mathrm{VN}}(t) - \frac{\Delta Q(t)}{T_E}. \tag{35}$$

The entropy production then allows for the verification of the second law of thermodynamics,

$$\Sigma(t) \geqslant 0, \tag{36}$$

namely that the entropy production must be non-negative during time evolution.

3. Examples

In order to examine the effects of the interactions present within the environment, two key examples will be discussed in detail here. For each example, the results presented will consider a system consisting of a single harmonic oscillator of frequency ω_S (thus $N_S = 1$), initially prepared in a thermal state of temperature T_S. The environmental units shall consist of N_E oscillators, each of frequency ω_E, coupled in a ring structure through spring-like interactions. Each unit as a whole is considered to be prepared in a thermal state at temperature T_E. The system-environment interactions will then be described as beamsplitter-like in section 3.2, and spring-like in section 3.3.

3.1. Initial states

In both examples, we consider the same initial covariance matrix describing the system and environment $\sigma_{\mathrm{tot}}(0)$. The system, described by the Hamiltonian defined in equation (2), is initially prepared in a thermal state of the form $\rho_{S,\mathrm{th}} = \exp(-\beta_S H_S)/Z_S$, where $Z_S = \mathrm{Tr}[\exp(-\beta_S H_S)]$ and $\beta_S = 1/T_S$. The corresponding initial covariance matrix for the single oscillator system can then be written as

$$\sigma_S(0) = \begin{pmatrix} \frac{1}{2\omega_S}\coth\left(\frac{\omega_S}{2T_S}\right) & 0 \\ 0 & \frac{\omega_S}{2}\coth\left(\frac{\omega_S}{2T_S}\right) \end{pmatrix}, \tag{37}$$

where the terms proportional to $\coth(\frac{\omega_S}{2T_S})$ relate to the thermal population of the system.

We assume the environmental units to be arranged along a linear chain with nearest-neighbor interactions and periodic boundary conditions, i.e. a ring-like structure. The corresponding Hamiltonian H_E can be described by equations (5) and (6), where

$$H_E^I = \lambda_I \sum_{i=1}^{N_E} \left(x_{E_i} - x_{E_{i+1}}\right)^2. \tag{38}$$

Here, λ_I is the coupling constant controlling the strength of the interactions within the environment. We also take $x_{N_E+1} \equiv x_1$ to satisfy the periodicity of the boundary conditions. Following the steps outlined in section 2.1, this can be recast in matrix form akin to equation (10).

The covariance matrix σ_E of the environmental unit before each collision can then be written as

$$\sigma_E = G^{\intercal} \left(\bigoplus_{m=1}^{N_E} \tilde{\sigma}_{E_m} \right) G, \tag{39}$$

where

$$\tilde{\sigma}_{E_m} = \begin{pmatrix} \frac{1}{2\tilde{\omega}_{E_m}} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) & 0 \\ 0 & \frac{\tilde{\omega}_{E_m}}{2} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) \end{pmatrix}, \tag{40}$$

As each unit is replaced between collisions, we can neglect the time dependence of σ_E. The frequencies $\tilde{\omega}_{E_m}$ correspond to the normal mode frequencies of the environmental unit, taking values of

$$\tilde{\omega}_{E_m} = \sqrt{\omega_E^2 + 8\lambda_I \sin^2\left[\frac{\pi}{N_E}(m-1)\right]}, \tag{41}$$

with $m = 1, 2, \ldots, N_E$. The full derivation of this can be found in appendix B. These normal mode frequencies can be understood as the frequencies $\tilde{\omega}_{E_m}$ that satisfy

$$H_E = \frac{1}{2} \sum_m \left(\tilde{\omega}_{E_m}^2 \tilde{x}_{E_m}^2 + \tilde{p}_{E_m}^2\right). \tag{42}$$

Due to the thermal nature of the environmental units and the system's initial state, each element of $\langle R \rangle_{\text{tot}}$ is identically zero and remains zero throughout the propagation as $\langle \dot{R} \rangle_{\text{tot}}$ is described through equation (18), i.e. as a linear combination of the elements of $\langle R \rangle_{\text{tot}}$. The dynamics of the system can therefore be fully described through the dynamics of its corresponding covariance matrix.

3.2. Example I: beamsplitter interactions

As a first example, we consider the interactions between the system and environment to be beamsplitter-like, such that the interaction Hamiltonian H_{SE} takes the form

$$H_{SE} = \sum_{i=1}^{N_E} g_{E_i} \left(\sqrt{\omega_S \omega_E}\, x_S x_{E_i} + \frac{p_S p_{E_i}}{\sqrt{\omega_S \omega_E}} \right), \tag{43}$$

where g_{E_i} are the coupling constants controlling the strength of the beamsplitter interaction between the system and each environmental oscillator E_i. The propagation matrix D_{tot} can then be described through $D_{\text{tot}} = \Omega_{N_{\text{tot}}} M_{\text{tot}}$ [51].

Equation (43) can be recast in terms of the normal mode coordinate system Q_E, defined in section 2.1, such that

$$H_{SE} = \sum_{m=1}^{N_E} \tilde{g}_{E_m} \left(\sqrt{\omega_S \omega_E}\, x_S \tilde{x}_{E_m} + \frac{p_S \tilde{p}_{E_m}}{\sqrt{\omega_S \omega_E}} \right). \tag{44}$$

Here, we have defined new coupling constants $\tilde{g}_{E_m}$ through

$$\tilde{g}_{E_m} = \sum_{i=1}^{N_E} G_{mi} g_{E_i}. \tag{45}$$

The corresponding Lyapunov equation can then be derived via equation (27), leading to a differential equation in the continuous time limit, i.e. $\delta t \to 0$,

$$\dot{\sigma}_S(t) = D_{BS}\sigma_S + \sigma_S D_{BS}^{\intercal} + V_{BS}, \tag{46}$$

with

$$D_{BS} = D_S - \frac{\gamma_{\hat{E}}}{2}\mathbb{1}_2 \tag{47}$$

and

$$V_{BS} = \sum_{m=1}^{N_E} \frac{\tilde{\gamma}_{E_m}}{2} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) \begin{pmatrix} \frac{\tilde{\omega}_{E_m}}{\omega_S\,\omega_E} & 0 \\ 0 & \frac{\omega_S\,\omega_E}{\tilde{\omega}_{E_m}} \end{pmatrix}. \tag{48}$$

Here, $\tilde{\gamma}_{E_m}$ denotes a rescaled coupling constant such that $\tilde{g}_{E_m}^2 \delta t \to \tilde{\gamma}_{E_m}$ as $\delta t \to 0$, corresponding to the strength of the interactions with an individual, uncoupled normal mode m in the continuous time limit. The effective dissipation rate $\gamma_{\hat{E}}$ is defined through

$$\gamma_{\hat{E}} = \sum_{m=1}^{N} \tilde{\gamma}_{E_m}. \tag{49}$$

Following the derivations outlined in section 2.3, we obtain expressions for the transient thermodynamic quantities of interest, namely the flow of internal energy $\dot{U}(t)$, heat $\dot{Q}(t)$ and work $\dot{W}(t)$, described through

$$\dot{U}(t) = \sum_{m=1}^{N_E} \frac{\tilde{\gamma}_{E_m}}{4} \left(\frac{\left(\omega_E^2 + \tilde{\omega}_{E_m}^2\right)\omega_S}{\omega_E\,\tilde{\omega}_{E_m}} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) - 2\omega_S^2\sigma_{xx}(t) - 2\sigma_{pp}(t) \right), \tag{50}$$

$$\dot{Q}(t) = \sum_{m=1}^{N_E} \frac{\tilde{\gamma}_{E_m}}{2} \left(\tilde{\omega}_{E_m} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) - \omega_S\,\omega_E\sigma_{xx}(t) - \frac{\tilde{\omega}_{E_m}^2}{\omega_S\,\omega_E}\sigma_{pp}(t) \right), \tag{51}$$

and

$$\dot{W}(t) = \dot{U}(t) - \dot{Q}(t). \tag{52}$$

The dynamics of the system are found to be analogous to an effective setup in which the system interacts through a beamsplitter-like interaction via a collision model, with environmental units composed of a single oscillator prepared in a squeezed thermal state with squeezing rate r, such that

$$\langle x_{\hat{E}}^2 \rangle_{\mathrm{sq}} = \frac{1}{2\mathrm{e}^{2r}\omega_E} \coth\left(\frac{\omega_E}{2T_{\hat{E}}}\right) \tag{53}$$

and

$$\langle p_{\hat{E}}^2 \rangle_{\mathrm{sq}} = \frac{\mathrm{e}^{2r}\omega_E}{2} \coth\left(\frac{\omega_E}{2T_{\hat{E}}}\right), \tag{54}$$

where the subscript $\hat{E}$ denotes the single oscillator effective environment and $\langle A \rangle_{\mathrm{sq}}$ denotes the expectation value of an operator A prepared in the squeezed state. We find a squeezing parameter of

$$r = \frac{1}{4}\ln\left(\frac{\alpha_p}{\alpha_x}\right), \tag{55}$$

where

$$\alpha_p = \sum_{m=1}^{N_E} \frac{\tilde{\gamma}_{E_m}\tilde{\omega}_{E_m}}{\omega_E} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) \tag{56}$$

and

$$\alpha_x = \sum_{m=1}^{N_E} \frac{\tilde{\gamma}_{E_m}\omega_E}{\tilde{\omega}_{E_m}} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right). \tag{57}$$

As $r > 0$ for $\omega_E > \tilde{\omega}_{E_m}$ and $r < 0$ for $\omega_E < \tilde{\omega}_{E_m}$ when $m \neq 1$, it is clear that this squeezing decreases the variance of the x quadrature for $\lambda_I < 0$, while $\lambda_I > 0$ decreases the variance of the p quadrature.

The effective temperature $T_{\hat{E}}$ can be shown to be

$$T_{\hat{E}} = \frac{\omega_E}{2\coth^{-1}\left(\frac{\sqrt{\alpha_x\alpha_p}}{\gamma_{\hat{E}}}\right)}. \tag{58}$$

This effective temperature $T_{\hat{E}}$ is then controlled by λ_I, allowing it to be greater than, less than or equal to T_E.

These dynamics then follow an effective Lyapunov equation for the system's covariance matrix of

$$\dot{\sigma}_S = D_{\text{eff}}\sigma_S + \sigma_S D_{\text{eff}}^{\mathsf{T}} + V_{\text{eff}}, \tag{59}$$

where

$$D_{\text{eff}} = D_{BS}, \tag{60}$$

and

$$V_{\text{eff}} = \frac{\gamma_{\hat{E}}}{2}\coth\left(\frac{\omega_E}{2T_{\hat{E}}}\right)\begin{pmatrix}\frac{e^{2r}}{\omega_S} & 0\\ 0 & \frac{\omega_S}{e^{2r}}\end{pmatrix}. \tag{61}$$

We consider the steady state of the system that satisfies the equation $\dot{\sigma}_S = 0$. The corresponding steady state energy of the system $\langle H_S\rangle_{\text{Steady}}$ reads

$$\langle H_S\rangle_{\text{Steady}} = \frac{\omega_S}{2}\coth\left(\frac{\omega_E}{2T_{\hat{E}}}\right)\cosh(2r). \tag{62}$$

As with the effective temperature, the steady state energy is controlled by the choice of λ_I, and is independent of the initial state of the system.

Although we have assumed a generic model so far, in which the system is potentially coupled to all normal modes of the environmental unit, it is also possible to choose the couplings g_{E_i} in equation (43) such that the system interacts solely with one specific mode with frequency $\tilde{\omega}_{E_{m'}}$. This is achieved by choosing $g_{E_i} = gG_{m'i}$, where g is the overall coupling strength. With this choice, the system-environment interaction of equation (43) becomes:

$$H_{SE} = g\left(\sqrt{\omega_S\omega_E}x_S\tilde{x}_{E_{m'}} + \frac{p_S\tilde{p}_{E_{m'}}}{\sqrt{\omega_S\omega_E}}\right). \tag{63}$$

In this case, the expressions for the effective squeezing rate and temperature simplify to

$$r = \frac{1}{2}\ln\left(\frac{\tilde{\omega}_{E_{m'}}}{\omega_E}\right) \tag{64}$$

and

$$T_{\hat{E}} = \frac{\omega_E}{\tilde{\omega}_{m'}}T_E. \tag{65}$$

We shall consider two examples in the following sections: one in which the system interacts exclusively with the center of mass mode frequency $\tilde{\omega}_{\text{CoM}} = \omega_E$, and one in which it interacts with the normal mode frequency corresponding to the largest wavevector within the Brillouin zone and the shortest wavelength supported by the environmental unit.

3.2.1. Center of mass

The first case of interest is when the system interacts exclusively with the center of mass of each environmental unit, with frequency $\tilde{\omega}_{\text{CoM}} = \omega_E$, obtained from equation (41) with $m = 1$. In this scheme, the system's covariance matrix follows equation (59), with a squeezing parameter of $r = 0$ (since $\alpha_p = \alpha_x$). The resulting dynamics are equivalent to that of a system interacting with units consisting of a single oscillator of frequency ω_E prepared in a thermal state at temperature T_E via a collision model through a beamsplitter-like interaction, see for example reference [14].

The flow of heat, work and the internal energy are found to be

$$\dot{U}(t) = \omega_S f(t), \tag{66}$$

$$\dot{Q}(t) = -\omega_E f(t)\,, \tag{67}$$

and

$$\dot{W}(t) = (\omega_S - \omega_E) f(t)\,, \tag{68}$$

where

$$f(t) = \tilde{\gamma}_{E_1}\left[\omega_S \sigma_{xx}(t) + \frac{1}{\omega_S}\sigma_{pp}(t) - \coth\left(\frac{\omega_E}{2T_E}\right)\right]. \tag{69}$$

Immediately, one can see agreement with the first law of thermodynamics (see equation (31)). For the case of $\omega_S = \omega_E$, one can recognize an inability to produce or absorb work in the center of mass case, explained through the commutator $[H_S + H_E, H_{SE}]$. Following reference [37], the condition $[H_S + H_E, H_{SE}] = 0$ is sufficient to ensure $\dot{W}(t) = 0$. The fact that $\dot{U}(t), \dot{Q}(t)$ and $\dot{W}(t)$ can be expressed as the product of a common function $f(t)$ and the frequencies ω_S and ω_E is reminiscent of similar results for a continuously operated Otto cycle (see for instance reference [14]). This is a consequence of the linearity of the coupling between the harmonic oscillators in the environment and the system. For the $\omega_S = \omega_E$ case, one finds in the steady state

$$\dot{U} = 0, \tag{70}$$

$$\dot{Q} = 0, \tag{71}$$

$$\dot{W} = 0. \tag{72}$$

Since, in this case, the steady state of the system is a thermal state at temperature T_E, it follows that for long times $f(t) \to 0$ because the detailed balance condition is satisfied. Under these conditions $\dot{U}(t), \dot{Q}(t)$ and $\dot{W}(t)$ vanish as expected for an equilibrium state.

Due to the lack of squeezing or change in the effective temperature, for $\omega_S = \omega_E$ the dynamics of the system follows the intuitive path: $T_E > T_S$ leads to the system heating until a steady state at temperature T_E is reached; $T_E < T_S$ leads to the system cooling until a steady state at temperature T_E and the $T_E = T_S$ case produces no changes within the system. As there is no contribution from any other normal mode frequencies, and $\tilde{\omega}_{E_{\mathrm{CoM}}}$ does not depend on λ_I, it is clear in this case that the structure within the environment does not affect the dynamics or steady state of the system.

Following section 2.3, one also finds $\Sigma(t) \geqslant 0$, showing agreement with the second law of thermodynamics.

3.2.2. Largest wavevector within the Brillouin zone

The second case of interest is when the system interacts solely with the normal mode frequency $\tilde{\omega}_{\mathrm{Max}}$ corresponding to the largest wavevector within the Brillouin zone. This corresponds to the shortest wavelength mode with the frequency $\tilde{\omega}_{E_{m'}}$, where m' is chosen to be the value of m for which the $\sin^2\left[\frac{\pi}{N_E}(m-1)\right]$ term in equation (41) is maximized. This frequency is found to be

$$\tilde{\omega}_{\mathrm{Max}} = \begin{cases} \sqrt{\omega_E^2 + 8\lambda_I}, & \text{for even } N_E \\ \sqrt{\omega_E^2 + 8\lambda_I \cos^2\left(\frac{\pi}{2N_E}\right)}, & \text{for odd } N_E. \end{cases} \tag{73}$$

One must take care to choose appropriate values of λ_I such that $\tilde{\omega}_{\mathrm{Max}}$ remains real-valued.

In this regime, the system's dynamics can be mapped to that of a quantum harmonic oscillator interacting with environmental units prepared in a squeezed thermal state with effective temperature $T_{\hat{E}}$ (see equation (58)) and squeezing parameter r (see equation (55)). This is chosen as a case of particular interest as it allows for control over the effective temperature of the environment. We plot the dependence of the effective temperature $T_{\hat{E}}$ on the environment's internal coupling strength λ_I for $N_E = 3, 4, 5$ in figure 2. The temperature $T_{\hat{E}}$ is always monotonically decreasing with λ_I, such that $T_{\hat{E}} > T_E$ for $\lambda_I < 0$ and $T_{\hat{E}} \leq T_E$ for $\lambda_I \leq 0$. The dependence of the effective temperature on N_E is very small as $\tilde{\omega}_{\mathrm{Max}}$ depends only weakly on the number of environment oscillators. In the absence of internal interactions within the environment, i.e. $\lambda_I \to 0$, one recovers the center of mass dynamics with $T_{\hat{E}} = T_E$, analogous to those for beamsplitter-like interactions via a collision model between the system and single oscillator environmental units prepared in a thermal state.

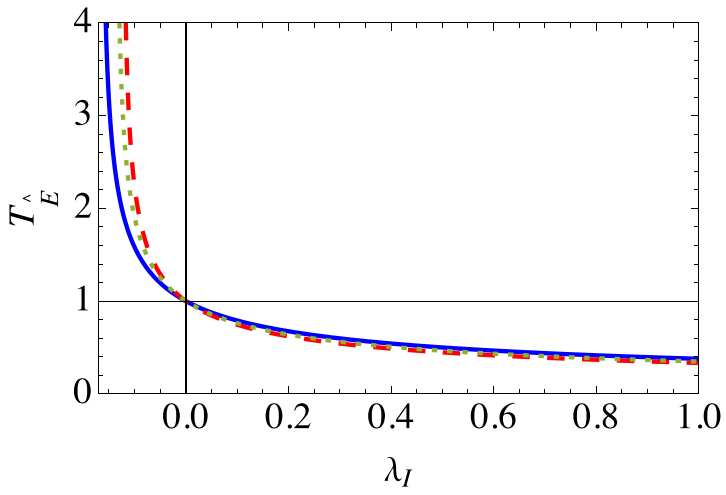

Figure 2. Effective temperature of the environmental units $T_{\hat{E}}$ interacting with the system as a function of the internal coupling strength λ_I within the environment. The blue (solid), red (dashed) and green(dotted) lines represent the effective temperature of an environmental unit composed of 3, 4 and 5 oscillators respectively, where the system interacts solely with the normal mode frequency of the environmental units $\tilde{\omega}_{E_{\text{Max}}}$ corresponding to the largest wavevector within the Brillouin zone. Parameters: $\omega_E = 1$, $T_E = 1$, $\tilde{\gamma}_{E_m,} = 0.5$.

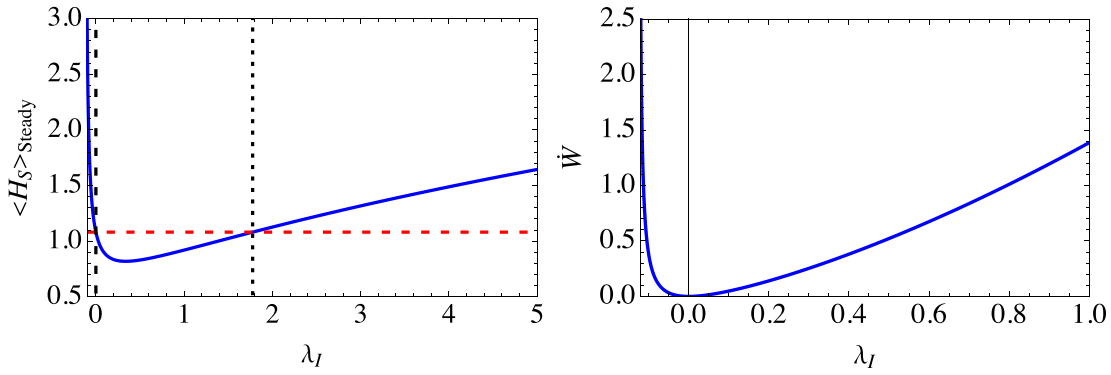

Figure 3. The steady-state energy H_{Steady} (left) and work $\dot{W}$ (right) applied to a system interacting with the normal mode frequency of the environmental units $\tilde{\omega}_{E_{\text{Max}}}$ corresponding to the largest wavevector within the Brillouin zone. Each unit is composed of 4 oscillators prepared in a thermal state at a temperature T_E as a function of the internal coupling strength λ_I (solid line). The horizontal dashed line represents the energy of a thermal state E_{eq}. The vertical lines act as a guide to the eye, with the dashed line at $\lambda_I = 0$ and the dotted line at $\lambda_I = \lambda_{I_C}$. The steady state internal energy is independent of internal coupling, such that $\dot{U} = 0$ for all λ_I, while the heat flow can be found through $\dot{Q} = -\dot{W}$. Parameters: $\omega_S = \omega_E = 1$, $T_E = 1$, $\tilde{\gamma}_{E_m,} = 0.5$.

The steady-state energy, $\langle H_S \rangle_{\text{Steady}}$, can be obtained through equation (62) and compared to the thermal energy $E_{\text{eq}} = \omega_S \coth[\omega_E/2T_E]/2$ of a quantum harmonic oscillator in equilibrium with an unstructured environment ($\lambda_I = 0$). The plot of figure 3 shows three different regions of interest. The first of these, in which $\lambda_I < 0$, produces a steady-state energy larger than E_{eq}, with small changes in λ_I leading to large changes in the steady-state energy. The second region, where $0 < \lambda_I < \lambda_{I_C}$, produces a steady-state energy lower than the unstructured environment or the center of mass cases. The quantity λ_{I_C} is the value $\lambda_I \neq 0$ for which $\langle H_S \rangle_{\text{Steady}}$ is equal to E_{eq}. In this region the system is effectively cooled to a temperature lower than that of the environment thanks to the internal structure of the environmental units and their thermal coherences. The third region, with $\lambda_I > \lambda_{I_C}$, again produces a steady-state energy larger than the unstructured case, however the magnitude of the change of the steady state energy with respect to changes in λ_I is much smaller than that of first region.

Following the derivations outlined in section 2.3, one can obtain expressions for particular thermodynamic quantities of interest, namely the internal energy $\dot{U}(t)$, heat $\dot{Q}(t)$ and work $\dot{W}(t)$ flows involved in this process. Focusing on the steady state, these quantities can be examined as a function of

internal coupling within the environment, λ_I, with $\dot{U}=0$, $\dot{Q}=-\dot{W}$ and

$$\dot{W}=\frac{\gamma_{\hat{E}}\,\omega_S^2\left(\omega_E^2-\tilde{\omega}_{E_{\mathrm{Max}}}^2\right)^2}{2\,\omega_E^2\,\tilde{\omega}_{E_{\mathrm{Max}}}\left(\gamma_E^2+4\omega_S^2\right)}\coth\left(\frac{\tilde{\omega}_{E_{\mathrm{Max}}}}{2T_E}\right). \tag{74}$$

One finds consistently that $\dot{W}\geqslant 0$, with the equality holding for $\lambda_I=0$, wherein the dynamics reduce to those of the center of mass regime. Work is therefore always injected into the system for $\lambda_I\neq 0$ (see figure 3). Power always increases monotonically with $|\lambda_I|$ due to the resulting larger frequency detuning $\omega_E-\tilde{\omega}_{E_{\mathrm{Max}}}$. Similarly, one finds $\dot{Q}\leqslant 0$, with the same conditions on the equality, such that for $\lambda_I\neq 0$, heat is always extracted from the system and damped into the environment and the first law of thermodynamics is always satisfied. One can contextualize this into the framework of quantum thermodynamic machine cycles by adding a second environment, interacting with the system through beamsplitter-like interactions via a collision model, with each unit consisting of a single harmonic oscillator. With the inclusion of this secondary environment, one can construct a regime with $\omega_S\neq\omega_E$ in which this model acts as a refrigerator, consuming work for heat to flow from a cold to a hot reservoir.

Following the procedure outlined in section 2.3, one finds that while the entropy may increase or decrease to a steady state depending on the choice of λ_I, the entropy production is always positive, ensuring the second law of thermodynamics is consistently satisfied.

This scheme lends itself well to physical implementation, such as through trapped ion platforms using Coulomb crystals [52]. One can see from equation (73) that this model is identical for all even N_E, with the exception of the special case $N_E=2$. Therefore, one only requires $N_E=4$ to achieve the maximum deviation from ω_E for a given λ_I. Experimentally, this can be achieved by 'resetting' an environmental unit of 4 oscillators to the initial state between interactions to avoid preparing a large number of units, provided the reset time τ_{Reset} is faster than the relaxation time of the system τ_{Relax}, i.e. $\tau_{\mathrm{Reset}}\ll\tau_{\mathrm{Relax}}$. We emphasize, however, that precise engineering of the environmental units is not crucial to mimic a squeezed bath and that a high-frequency mode coupled to the system's degrees of freedom would be sufficient.

3.3. Example II: spring-like interactions

The second example presented in this work considers spring-like system-environment interactions, such that H_{SE} takes the form

$$H_{SE}=\sum_{i=1}^{N_E}\lambda_{E_i}\left(x_S-x_{E_i}\right)^2, \tag{75}$$

where λ_{E_i} are the coupling constants controlling the strength of the spring-like interaction between the system and each environmental oscillator E_i.

This can be recast in terms of the normal mode coordinate system Q_E defined in section 2.1, as with the beamsplitter example, to produce

$$H_{SE}=H_S^{(SE)}+H_E^{(SE)}+H_{SE}^{(SE)}, \tag{76}$$

where $H_S^{(SE)}=\sum_{i=1}^{N_E}\lambda_{E_i}x_S^2$ and gives rise to a renormalized system's frequency $\bar{\omega}_S^2=\omega_S^2+2\sum_{i=1}^{N_E}\lambda_{E_i}$. The term $H_E^{(SE)}$ is then given by

$$H_E^{(SE)}=\sum_{m,m'=1}^{N_E}\tilde{\lambda}_{E_{mm'}}\tilde{x}_{E_m}\tilde{x}_{E_{m'}}, \tag{77}$$

where $\tilde{\lambda}_{E_{mm'}}=\sum_{i=1}^{N_E}\lambda_{E_i}G_{mi}G_{m'i}$ is an effective coupling strength between normal modes, while

$$H_{SE}^{(SE)}=-2\sum_{m=1}^{N_E}\tilde{\lambda}_{E_m}x_S\tilde{x}_{E_m} \tag{78}$$

with

$$\tilde{\lambda}_{E_m}=\sum_{i=1}^{N_E}\lambda_{E_i}G_{mi}. \tag{79}$$

The dynamics of the system can then be described in the form of equation (27). We consider two approaches to study these dynamics. The first is by propagation through discrete time steps wherein δt is kept finite. One can begin with

$$\frac{\sigma_S(t+\delta t)-\sigma_S(t)}{\delta t} = D_{\mathrm{Sp}}\sigma_S(t)+\sigma_S(t)D_{\mathrm{Sp}}^{\intercal}+\delta t D_S^{\mathrm{RN}}\sigma_S(t)D_S^{\intercal\mathrm{RN}}+V_{\mathrm{Sp}}+O\left(\delta t^2\right), \tag{80}$$

with

$$D_{\mathrm{Sp}} = D_S^{\mathrm{RN}} - \frac{\bar{\omega}_S^2\delta t}{2}\mathbb{1}_2, \tag{81}$$

where D_S^{RN} represents the free evolution of the system's oscillator with the renormalized frequency $\bar{\omega}_S$, which depends on the coupling constants λ_{E_i}. The matrix V_{Sp} reads

$$V_{\mathrm{Sp}} = 2\delta t\sum_{m=1}^{N_E}\tilde{\lambda}_{E_m}^2\begin{pmatrix}0 & 0\\ 0 & \frac{1}{\tilde{\omega}_{E_m}}\coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right)\end{pmatrix}. \tag{82}$$

Some of the terms in equation (80) may lead to difficulties in taking the continuous limit, as the method of rescaling used in section 3.2 may lead to divergences. In this model, the rescaling method would require taking $\tilde{\lambda}_{E_m}^2\delta t\to$ const. as $\delta t\to 0$. However when doing so, the term D_S^{RN} would diverge through its dependence on $\bar{\omega}_S$, due to the divergence of $\tilde{\lambda}_{E_m}$. We remark, however, that using a finite δt, as in the discrete evolution, would not lead to any mathematical issue. As the continuous time limit is not possible in the general case, we instead propagate the system through equation (16) to avoid the approximation due to the Taylor expansion in δt.

While the discrete propagation model has been presented as generic so far, we shall focus on interactions with the center of mass frequency, as this case requires discrete propagation. Such interactions, i.e. those with an individual chosen normal mode frequency $\tilde{\omega}_{E_{m'}}$, can be achieved by choosing couplings $\lambda_{E_i}=\lambda_E G_{m'i}$, where λ_E is the overall coupling strength.

For the second approach, we will consider the continuous limit $\delta t\to 0$ in the specific case $\sum_i^{N_E}\lambda_{E_i}=0$ which leaves the system's frequency unaltered, i.e. $\bar{\omega}_S=\omega_S$, avoiding any divergence.

3.3.1. Discrete time

As discussed above, for generic λ_{E_i} in the spring-like interaction model, it is not possible to derive a continuous Lyapunov equation for the covariance matrix akin to equation (59). However, one can still propagate the system through discrete time steps δt according to equation (16). Here, we present this discrete-time evolution resulting from interactions between a single oscillator system, prepared initially in a thermal state at temperature T_S and frequency ω_S, and the center of mass mode of the environmental units, each composed of $N_E=4$ oscillators, i.e. we choose homogeneous interaction couplings $\lambda_{E_i}=\lambda_E G_{m'i}$ with $m'=1$. Furthermore we assume each environmental unit to be initially prepared in a thermal state at a temperature T_E.

We propagate the system's covariance matrix numerically using equation (16), with the corresponding results for the system's energy shown in figure 4. We observe that, unlike the beamsplitter case, the system does not tend to a steady state. Instead, the energy oscillates around a linearly increasing average. This originates from a persistent injection of energy from the environment to the system due to the spring-like form of interaction.

This is confirmed by looking at the heat, $\Delta Q(t)$, and work, $\Delta W(t)$, per collision computed using the methods outlined in section 2.3. As shown in figure 5, work is periodically injected and extracted from the system, with a net positive injection over long time frames, accounting for the increase in the systems energy between each period. One also finds dissipation through the heat ΔQ, which remains negative throughout the propagation.

The entropy of the system $S(t)$ also increases as one would expect, as the system's energy increases throughout the process. Once again, the second law of thermodynamics can be verified by examining the entropy production $\Sigma(t)$, where one requires $\Sigma(t)\geqslant 0$.

3.3.2. Continuous time

It is possible to choose interaction constants λ_{E_i} such that divergences do not appear, allowing for the continuous-time limit to be taken [53]. Applying the rescaling $\lambda_{E_i}\lambda_{E_j}\delta t\to\Lambda_{E_{ij}}$ as $\delta t\to 0$, can in general produce $\mathcal{L}_1(\sigma_{\mathrm{tot}})\to\infty$ however this can be avoided provided $\sum_{i=1}^{N_E}\lambda_{E_i}=0$.

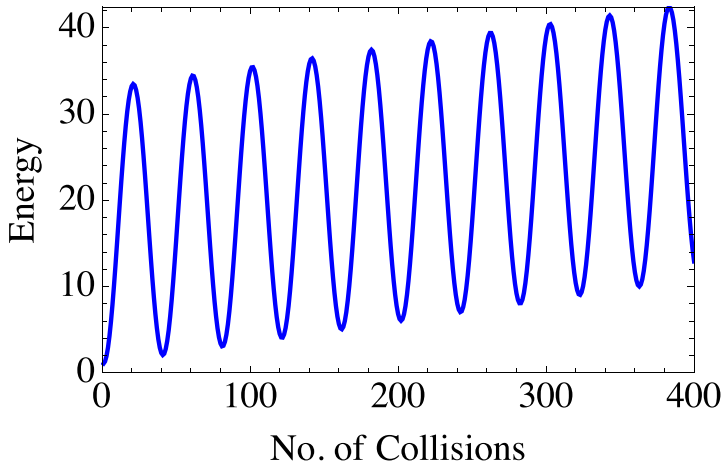

Figure 4. Dynamics of the energy of a single oscillator system prepared in the thermal state as a function of time, with discrete time-steps $\delta t = 0.01$, interacting with the center of mass frequency of environmental units composed of $N_E = 4$ oscillators. Parameters: $\omega_S = \omega_E = 1$, $T_S = T_E = 1$, $\lambda_I = 0.67$, $\tilde{\lambda}_{E_i} = 15$.

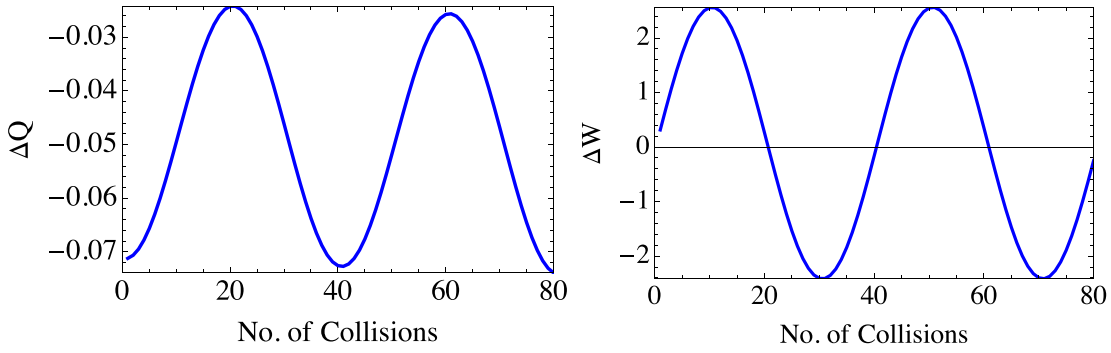

Figure 5. The heat current ΔQ (left) and work ΔW (right) applied to the system the system with same parameters as figure 4. The change in the internal energy of the system ΔU can be found as the sum ΔQ and ΔW.

The resulting Lyapunov equation can then be stated as

$$\dot{\sigma}_S = D_S\sigma_S + \sigma_S D_S^{\intercal} + V_{\mathrm{Sp}}, \tag{83}$$

where V_{Sp} is given by equation (82) after replacing $\tilde{\lambda}_{E_m}^2 \delta t \to \tilde{\Lambda}_{E_m}$ and taking the continuous limit $\delta t \to 0$. Notice that the trick we employed to obtain an effective continuous time dynamics would not be possible if the environmental units consisted of only a single oscillator through solely spring-like interactions outside of the trivial $\lambda_{E_i} = 0$ case, as a minimum of two oscillators are necessary to fulfill the condition of $\sum_{i=1}^{N_E} \lambda_{E_i} = 0$.

To maximize the effects of the environment, we present the dynamics due to interactions of the system with the normal mode frequency $\tilde{\omega}_{E_{\mathrm{Max}}}$ described in equation (73). In figure 6, we plot the second moments, $\langle x_S^2 \rangle$ and $\langle p_S^2 \rangle$, which oscillate around a linearly increasing average value. The almost linear increase in these quantities is reminiscent of a diffusion model, in which the variances increase linearly with respect to time. Similarly to the discrete case, this is due to a persistent energy injection from the environment.

One can again calculate the thermodynamic quantities described in section 2.3 to find that work must always be injected into the system, as $\dot{W}(t) > 0$, while heat consistently flows out of the system, as $\dot{Q}(t) < 0$, also shown in figure 6. The internal energy increases linearly over time, with $\dot{U}(t)$ equal to a positive constant (whose expression can be found but it does not give additional information). Because of this linear increase in the internal energy, it is clear that a model using solely spring-like interactions does not admit a steady state.

One may investigate the entropy and entropy production of the system in this case, resulting again in the verification of the second law of thermodynamics, wherein the change in both the entropy $S(t)$ and entropy production $\Sigma(t)$ between collisions is positive.

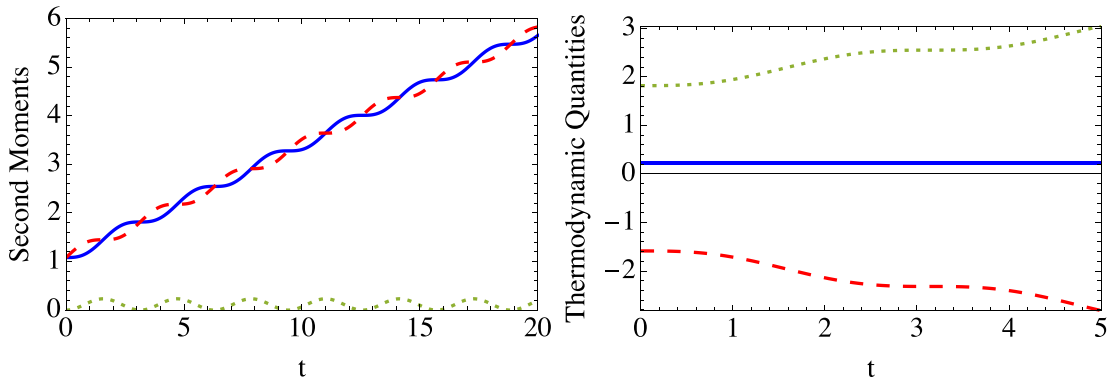

Figure 6. Dynamics of a system interacting with the normal mode frequency $\tilde{\omega}_{E_{\text{Max}}}$ corresponding to the largest wavevector within the Brillouin zone in the continuous-time limit. The second moments (left) and thermodynamic quantities (right) are plotted as a function of time t. The second moments are described by the elements of the covariance matrix, particularly $\langle x_S^2 \rangle$ (blue, solid), $\langle p_S^2 \rangle$ (red, dashed)) and $\langle x_S p_S + p_S x_S \rangle$ (green, dotted). The thermodynamic quantities presented are the flow of internal energy $\dot{U}(t)$ (blue, solid), heat $\dot{Q}(t)$ (red, dashed) and work $\dot{W}(t)$ (green, dotted). Parameters: $\omega_S = \omega_E = 1$, $T_S = T_E = 1$, $\lambda_I = 0.67$, $\tilde{\Lambda}_{E_m'} = 0.5$.

It is possible to induce a steady state through the use of a secondary environment. As a simple example, we assume this additional environment to consist of units composed of a single oscillator in a thermal state, with frequency ω_{B} and temperature T_B interacting with the system through beamsplitter-like interactions via a collision model, as in section 3.2 ($N_E = 1$ and $\lambda_I = 0$), see equations (46)–(48).

The total propagation takes the form of

$$\dot{\sigma}_S(t) = D_L \sigma_S(t) + \sigma_S(t) D_L^{\intercal} + V_{\text{Sp}} + V_{\text{B}}, \tag{84}$$

with

$$D_L = D_S - \frac{\gamma_B}{2} \mathbb{1}_2, \tag{85}$$

and

$$V_{\text{B}} = \frac{1}{2} \begin{pmatrix} \frac{\gamma_B}{\omega_S} \coth\left(\frac{\omega_{\text{B}}}{2T_B}\right) & 0 \\ 0 & \gamma_B \omega_S \coth\left(\frac{\omega_{\text{B}}}{2T_B}\right) \end{pmatrix}, \tag{86}$$

where γ_B is related to the coupling of this secondary environment to the system.

In this case, the system tends to a steady state described through

$$\langle H_S \rangle_{\text{Steady}} = \sum_{m=2}^{N_E} \frac{\tilde{\Lambda}_{E_m}}{\gamma_B \tilde{\omega}_{E_m}} \coth\left(\frac{\tilde{\omega}_{E_m}}{2T_E}\right) + \frac{\omega_S}{2} \coth\left(\frac{\omega_{\text{B}}}{2T_B}\right). \tag{87}$$

From figure 7, one can see that the primary environment interacting with the system via spring-like interactions injects a constant amount of energy into the system with each collision. For the secondary environment interacting via beamsplitter-like interactions, the amount of energy injected or extracted by each unit is dependent on the energy of the system at the time of collision. The steady state then occurs when the rate at which energy is drained by the secondary environment is equal to the constant amount of energy injected by the primary environment. When both the primary and secondary environment are initially at the same temperature with oscillators of equal frequency, taking $\lambda_I = 0$ still results in an increase to the system's steady-state energy. The coupling λ_I therefore does not initiate the change, but rather allows for control of its magnitude, where $\lambda_I < 0$ allows for an increase in the steady-state energy that is large in magnitude, while $\lambda_I > 0$ allows for an increase that is small and controlled.

In the steady state, one finds a dissipator-like behavior, where work is constantly injected into the system, as $\dot{W} > 0$, while heat flows out of the system, as $\dot{Q} < 0$. In such a steady state, one finds the internal energy is constant, i.e. $\dot{U} = 0$.

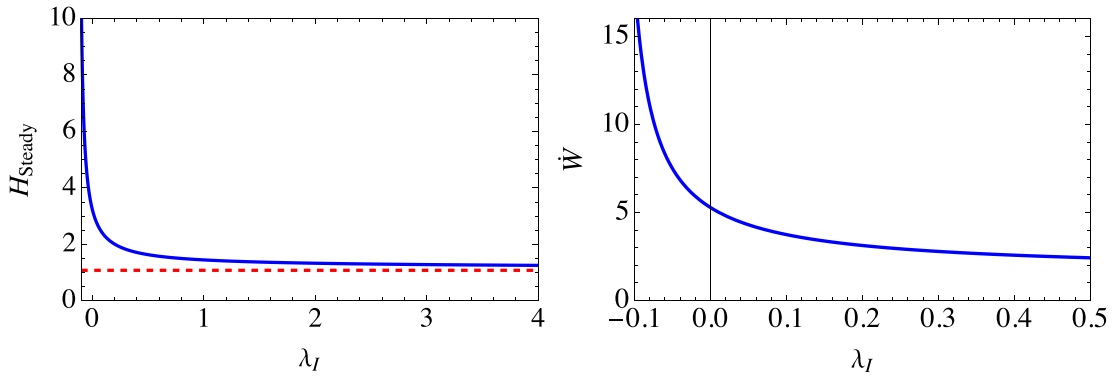

Figure 7. The steady state energy H_{Steady} (left) and work $\dot{W}$ (right) as a function of the internal environmental coupling strength λ_I of a structured primary environment. The system also interacts with a secondary 'drain' environment. Each unit of the primary environment is composed of $N_E = 4$ oscillators interacting with the system via spring-like interactions. The units of the secondary environment are composed of $N_E = 1$ oscillator interacting with the system via beamsplitter-like interactions. In the steady-state, $\dot{U} = 0$ for all λ_I, and the heat flow can be found through $\dot{Q} = -\dot{W}$. The initial energy (dashed, red) of the system with these parameters is also plotted. Parameters: $\omega_S = \omega_E = \omega_B = 1$, $T_S = T_E = T_B = 1$, $\tilde{\Lambda}_{E_m} = 0.5$, $\gamma_B = 0.5$.

4. Conclusions and outlook

This work presents a general method of designing a collision model that includes structure within the environmental units using continuous-variable quantum systems, namely harmonic oscillators. In some cases, it is possible to construct a Lyapunov equation for the continuous-time evolution of the system's covariance matrix. We demonstrated the effect of an internal structure within the environment may have over the system's dynamics and thermodynamics, particularly with respect to the effective temperature of the environmental units as a function of their internal coupling strength λ_I. The dynamics of a system interacting through beamsplitter-like interactions via a collision model with units composed of spring-like coupled oscillators, initially prepared in a thermal state, were found to be analogous to the dynamics due to interactions with single oscillator environmental units prepared in a squeezed thermal state. The magnitude of this squeezing also corresponded to a function of the internal coupling strength. This effective squeezing arises naturally from environmental units prepared in thermal equilibrium, thus considered free within the resource theory of quantum thermodynamics. This is in stark contrast to previous works which considered environmental units which are artificially prepared in squeezed states [32, 53, 54], thus necessarily employing additional thermodynamic resources.

In other cases, in which the system interacts with the environment through spring-like interactions, we found that discrete time propagation is necessary, as taking the continuous-time limit leads to divergences. In this example, the system does not inherently reach a steady state due to constant injections of energy from the environment with each collision. Nevertheless, we showed that one can include a secondary thermal environment (corresponding to a collision model whose units are made of a single oscillator and interacting with the system via beamsplitter-like interactions) to lead the system to a steady state.

In all cases, we proved that the first and second laws of thermodynamics are fulfilled. This shows once more that structured collision models can play an important role in the context of thermodynamic investigations, as they allow for a thermodynamically consistent description even in presence of coherence, in contrast with other models where thermodynamic consistence must be enforced.

Verification of these results may be currently achievable through various experimental methods, such as through trapped ions [55–57] or photonic systems [58, 59]. The effect of the topology of the structure within the environmental units would be of keen interest for further study, alongside the effects of varying the initial states, interactions between the system and environment, and of course the internal interactions within the environmental units. Finally, it would be of practical interest to examine the effects of a structured environment to the performances of heat engines and refrigerators, as well as in the charging process of quantum batteries.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Acknowledgments

A C acknowledges support from the UK EPSRC through Grant No. EP/W52444X/1. S C acknowledges support from the PNRR MUR project PE0000023NQSTI. G D C acknowledges support from the UK EPSRC through Grant No. EP/S02994X/1.

Appendix A. Superoperator derivation

The derivation of the superoperators relating the total dynamics of the system and environment to the reduced dynamics of solely the system, $\mathcal{L}_0(\sigma_{\text{tot}}(t))$, $\mathcal{L}_1(\sigma_{\text{tot}}(t))$, $\mathcal{L}_2(\sigma_{\text{tot}}(t))$ can be found by considering the set of elements i, j of the matrices produced by the total dynamic operators $\mathcal{U}_0(\sigma_{\text{tot}}(t))$, $\mathcal{U}_1(\sigma_{\text{tot}}(t))$, $\mathcal{U}_2(\sigma_{\text{tot}}(t))$, for which $i = 1, 2\ldots, 2N_S$ and $j = 1, 2\ldots, 2N_S$. More intuitively, one can consider this as selecting only the 'top-left' $2N_S \times 2N_S$ block of the total system-environment covariance matrix after applying the total evolution operators. The action of extracting the properties and dynamics of solely the system will be denoted here as $[\cdot]_S$, such that $\mathcal{L}_i(\sigma_{\text{tot}}(t)) = [\mathcal{U}_i(\sigma_{\text{tot}}(t))]_S$.

To find $\mathcal{L}_0(\sigma_{\text{tot}}(t))$, one can recognize that the corresponding operator $\mathcal{U}_0(\sigma_{\text{tot}}(t))$ can be described simply as the application of the identity $\mathbb{1}_{2N_{\text{tot}}}$ to $\sigma_{\text{tot}}(t)$, hence,

$$\mathcal{L}_0\left(\sigma_S\left(t\right)\right) = \sigma_S\left(t\right). \tag{A.1}$$

The behavior due to $\mathcal{L}_1(\sigma_S(t))$ can be found by separating the propagation matrix D_{tot} into interactions due the the system, $D'_S = D_S \oplus O_{2N_E}$, the environment, $D'_E = O_{2N_S} \oplus D_E$ and the system-environment interactions, D_{SE}, each of dimension $2N_{\text{tot}}$, such that

$$D_{\text{tot}} = D'_S + D'_E + D_{SE}, \tag{A.2}$$

with O_N denoting the zero matrix of dimension N.

In order to describe the system's dynamics due to $\mathcal{U}_1(\sigma_{\text{tot}}(t))$, we consider the contributions from each of these terms separately. First, we see the D_E terms do not contribute to the system's dynamics by recognizing $D'_{E_{ij}} = 0$ for all $i = 1, 2, \ldots, 2N_S$, and for $j = 1, 2, \ldots, N_S$, and therefore

$$\sum_k D'_{E_{ik}} \sigma_{\text{tot}_{kj}}\left(t\right) = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.3}$$

$$\sum_k \sigma_{\text{tot}_{ik}}\left(t\right) D'^{\intercal}_{E_{kj}} = 0; \qquad 1 \leqslant j \leqslant 2N_S. \tag{A.4}$$

One can treat the D_{SE} terms using the notation of $[\cdot]_S$, retaining only the $[D_{SE}\sigma_{\text{tot}}(t)]_S$ and $[\sigma_{\text{tot}}(t)D^{\intercal}_{SE}]_S$ contributions. Similarly, applying this operation on the D'_S terms produces $[D'_S\sigma_S(t)]_S = D_S\sigma_S(t)$ and $[\sigma_{\text{tot}}(t)D'^{\intercal}_S]_S = \sigma_S(t)D^{\intercal}_S$. The dynamics of the system due to the first order δt terms can then be written as

$$\mathcal{L}_1\left(\sigma_S\left(t\right)\right) = \delta t\left\{D_S\sigma_S\left(t\right) + \sigma_S\left(t\right)D^{\intercal}_S + \left[D_{SE}\sigma_{\text{tot}}\left(t\right)\right]_S + \left[\sigma_{\text{tot}}\left(t\right)D^{\intercal}_{SE}\right]_S\right\}. \tag{A.5}$$

The second-order terms can be treated through the expansion of D_{tot} described in equation (A.2). The terms involving solely the environmental dynamics D_E and the system once again provide no contribution, as

$$\sum_{kl} D'_{E_{ik}} D'_{E_{kl}} \sigma_{\text{tot}_{lj}}\left(t\right) = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.6}$$

$$\sum_{kl} \sigma_{\text{tot}_{ik}}\left(t\right) D'^{\intercal}_{E_{kl}} D'^{\intercal}_{E_{lj}} = 0; \qquad 1 \leqslant j \leqslant 2N_S, \tag{A.7}$$

$$\sum_{kl} D'_{E_{ik}} \sigma_{\text{tot}_{kl}}\left(t\right) D'^{\intercal}_{E_{lj}} = 0; \qquad 1 \leqslant i \leqslant 2N_S. \tag{A.8}$$

One also finds no contribution from terms consisting solely of D_S and D_E terms, as $D'_S D'_E = D'_E D'_S = O_{2N_{\text{tot}}}$, and

$$\sum_{kl} D'_{S_{ik}} \sigma_{\text{tot}_{kl}}\left(t\right) D'^{\intercal}_{E_{lj}} = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.9}$$

$$\sum_{kl} D'_{E_{ik}} \sigma_{\text{tot}_{kl}}\left(t\right) D'^{\intercal}_{S_{lj}} = 0; \qquad 1 \leqslant j \leqslant 2N_S. \tag{A.10}$$

Finally, there are no contributions from the terms containing solely D_{SE} and D'_E, as

$$\sum_{kl} D'_{E_{ik}} D_{SE_{kl}} \sigma_{\mathrm{tot}_{lj}}(t) = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.11}$$

$$\sum_{kl} \sigma_{\mathrm{tot}_{ik}}(t) D'^{\intercal}_{E_{kl}} D^{\intercal}_{SE_{lj}} = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.12}$$

$$\sum_{kl} D'_{E_{ik}} \sigma_{\mathrm{tot}_{kl}}(t) D^{\intercal}_{SE_{lj}} = 0; \qquad 1 \leqslant i \leqslant 2N_S, \tag{A.13}$$

$$\sum_{kl} D_{SE_{ik}} D'_{E_{kl}} \sigma_{\mathrm{tot}_{lj}}(t) = 0; \qquad 1 \leqslant j \leqslant 2N_S, \tag{A.14}$$

$$\sum_{kl} \sigma_{\mathrm{tot}_{ik}}(t) D^{\intercal}_{SE_{kl}} D'^{\intercal}_{E_{lj}} = 0; \qquad 1 \leqslant j \leqslant 2N_S, \tag{A.15}$$

$$\sum_{kl} D_{SE_{ik}} \sigma_{\mathrm{tot}_{kl}}(t) D'^{\intercal}_{E_{lj}} = 0; \qquad 1 \leqslant j \leqslant 2N_S. \tag{A.16}$$

The remaining nonzero terms are those related to the unitary dynamics of the system, terms involving the system dynamics and the system-environment interaction, and terms solely involving the system-environment interaction. The second order terms of the system's dynamics with respect to δt are then given by

$$\mathcal{L}_2(\sigma_S(t)) = \frac{\delta t^2}{2}\left(\mathcal{L}_2^S(\sigma_S(t)) + \mathcal{L}_2^{SE}(\sigma_S(t)) + \mathcal{L}_2^{S/SE}(\sigma_S(t))\right), \tag{A.17}$$

with

$$\mathcal{L}_2^S(\sigma_S(t)) = 2D_S\sigma_S(t)D_S^{\intercal} + D_S^2\sigma_S(t) + \sigma_S(t)D_S^{\intercal 2}, \tag{A.18}$$

$$\mathcal{L}_2^{SE}(\sigma_S(t)) = 2\left[D_{SE}\sigma_{\mathrm{tot}}(t)D_{SE}^{\intercal}\right]_S + \left[D_{SE}^2\sigma_{\mathrm{tot}}(t)\right]_S + \left[\sigma_{\mathrm{tot}}(t)D^{\intercal 2}_{SE}\right]_S, \tag{A.19}$$

and

$$\begin{aligned}\mathcal{L}_2^{S/SE}(\sigma_S(t)) = {} & 2\left[D'_S\sigma_{\mathrm{tot}}(t)D^{\intercal}{}_{SE}\right]_S + 2\left[D_{SE}\sigma_{\mathrm{tot}}(t)D'^{\intercal}_S\right]_S + \left[D'_S D_{SE}\sigma_{\mathrm{tot}}(t)\right]_S \\ & + \left[D_{SE}D'_S\sigma_{\mathrm{tot}}(t)\right]_S + \left[\sigma_{\mathrm{tot}}(t)D'^{\intercal}_S D^{\intercal}_{SE}\right]_S + \left[\sigma_{\mathrm{tot}}(t)D^{\intercal}_{SE}D'^{\intercal}_S\right]_S\end{aligned} \tag{A.20}$$

Appendix B. Environmental unit normal-mode decomposition

Decomposing the structured environmental units into their corresponding normal modes greatly simplifies the derivation of the system's dynamics. Here, we consider an environmental unit of N_E oscillators arranged in a ring, coupled via spring-like interactions, with a total Hamiltonian H_E described by equation (5), (6) and (38), such that

$$H_E = \sum_{n=1}^{N_E}\left[\frac{1}{2}\left(\omega_E^2 x_{E_n}^2 + p_{E_n}^2\right) + \lambda_I\left(x_{E_n} - x_{E_{n+1}}\right)^2\right], \tag{B.1}$$

with periodic boundary conditions ($N_E + 1 \equiv 1$). Furthermore, since the kinetic-energy terms are already non-interacting (and the environmental oscillators have all the same mass), we can look for a transformation to a basis in which the position operators are non-interacting, i.e. focusing only on the terms of the Hamiltonian corresponding to the potential energy,

$$H_{E_x} = \sum_{n=1}^{N_E}\left[\frac{1}{2}\omega_E^2 x_{E_n}^2 + \lambda_I\left(x_{E_n} - x_{E_{n+1}}\right)^2\right].$$

This simplifies the search for a description of the normal modes, leaving only the constraints that the transformation must be symplectic, without inducing interactions between the momentum operators.

We can obtain the normal modes via a Fourier decomposition described by

$$x_j = \frac{1}{\sqrt{N_E}}\sum_{m=0}^{N_E-1} \mathrm{e}^{-\mathrm{i}jk_m}\tilde{x}_m, \tag{B.2}$$

where $k_m = \frac{2\pi}{N_E} m$ for $m = 0, 1, 2, \ldots, N_E - 1$ and $\tilde{x}_m$ denotes the normal mode coordinates. As all operators in this appendix refer to the environment, the subscripts E have been dropped for brevity.

The potential-energy Hamiltonian H_{E_x} can then be rewritten in terms of the normal mode coordinates through equation (B.2), producing

$$H_{E_x} = \frac{1}{N_E} \sum_{j,m,m'} \mathrm{e}^{-\mathrm{i}j(k_m + k_{m'})} \left[\frac{\omega_E^2}{2} \tilde{x}_m \tilde{x}_{m'} + \lambda_I \left(1 + \mathrm{e}^{-\mathrm{i}(k_m + k_{m'})} - \mathrm{e}^{-\mathrm{i}k_m} - \mathrm{e}^{-\mathrm{i}k_{m'}} \right) \tilde{x}_m \tilde{x}_{m'} \right]. \tag{B.3}$$

This can be simplified further through the standard result

$$\sum_{j=1}^{N_E} \mathrm{e}^{-\mathrm{i}j(k_m + k_{m'})} = N_E \delta_{k_m, -k_{m'}}, \tag{B.4}$$

to obtain

$$H_{E_x} = \sum_{m=0}^{N_E - 1} \left(\frac{\omega_E^2}{2} + 2\lambda_I - \lambda_I \left(\mathrm{e}^{\mathrm{i}k_m} + \mathrm{e}^{-\mathrm{i}k_m} \right) \right) \tilde{x}_m \tilde{x}_{-m}. \tag{B.5}$$

Finally, using trigonometric relations and noting both $\tilde{x}_{-m} = \tilde{x}_m^\dagger$ and $\tilde{x}_m^\dagger = \tilde{x}_m$, H_{E_x} can be rewritten as

$$H_{E_x} = \sum_{m=1}^{N_E} \left[\frac{\omega_E^2}{2} + 4\lambda_I \sin^2 \left(\frac{\pi}{N_E} (m-1) \right) \right] \tilde{x}_m^2. \tag{B.6}$$

One can compare this to the potential-energy Hamiltonian of a set of uncoupled oscillators of frequency $\tilde{\omega}_{E_m}$ to find the frequency of the normal modes, i.e.

$$\frac{\tilde{\omega}_{E_m}^2}{2} = \frac{\omega_E^2}{2} + 4\lambda_I \sin^2 \left(\frac{\pi}{N_E} (m-1) \right), \tag{B.7}$$

such that equation (41) is satisfied.

Equation (B.7) is valid for all $N_E > 2$, however it is important to note the special case of $N_E = 2$. Typically, one does not impose ring-like periodic boundary conditions in this case, but rather the oscillators are arranged in a line structure. There are therefore $N_E - 1$ interactions in this case (just 1), rather than N_E, resulting in a center of mass frequency $\tilde{\omega}_{\mathrm{CoM}} = \omega_E$ and a relative motion frequency of $\tilde{\omega}_{\mathrm{RM}} = \sqrt{4\lambda_I + \omega_E^2}$.

ORCID iDs

Anton Corr 0009-0003-3629-7870
Stefano Cusumano 0000-0002-4379-3739
Gabriele De Chiara 0000-0003-3265-9021

References

[1] Breuer H P and Petruccione F 2007 *The Theory of Open Quantum Systems* (Oxford University Press)
[2] Bertolotti M 2021 *Contemp. Phys.* **62** 57–58
[3] Ciccarello F, Lorenzo S, Giovannetti V and Palma G M 2022 *Phys. Rep.* **954** 1–70
[4] Cusumano S 2022 *Entropy* **24** 1258
[5] Cattaneo M, Giorgi G L, Zambrini R and Maniscalco S 2022 *Open Syst. Inf. Dyn.* **29** 2250015
[6] Campbell S and Vacchini B 2021 *Europhys. Lett.* **133** 60001
[7] Cattaneo M, De Chiara G, Maniscalco S, Zambrini R and Giorgi G L 2021 *Phys. Rev. Lett.* **126** 130403
[8] Seah S, Nimmrichter S and Scarani V 2019 *Phys. Rev. E* **99** 042103
[9] Camasca R R and Landi G T 2021 *Phys. Rev. A* **103** 022202
[10] Magalhães W F, Neto C O R and de Lima Bernardo B 2023 *Phys. Open* **15** 100144
[11] Şenyaşa H T, Kesgin Ş, Karpat G and Çakmak B 2022 *Entropy* **24** 824
[12] Ciccarello F and Giovannetti V 2013 *Phys. Scr.* **T153** 014010
[13] De Chiara G and Antezza M 2020 *Phys. Rev. Res.* **2** 033315
[14] Leitch H, Piccione N, Bellomo B and De Chiara G 2022 *AVS Quantum Sci.* **4** 012001
[15] Santiago-García M, Pusuluk O, Müstecaplioğlu O E, Çakmak B and Román-Ancheyta R 2025 *Quantum Sci. Technol.* **10** 025018
[16] Scarani V, Ziman M, Gisin N and Bužek V 2002 *Phys. Rev. Lett.* **88** 097905
[17] Seah S, Perarnau-Llobet M, Haack G, Brunner N and Nimmrichter S 2021 *Phys. Rev. Lett.* **127** 100601
[18] Molitor O A D and Landi G T 2020 *Phys. Rev. A* **102** 042217
[19] Piccione N, De Chiara G and Bellomo B 2021 *Phys. Rev. A* **103** 032211
[20] Hammam K, Leitch H, Hassouni Y and De Chiara G 2022 *New J. Phys.* **24** 113053
[21] Hammam K, Manzano G and De Chiara G 2024 *Phys. Rev. Res.* **6** 013310

[22] Campbell S, Eisert J and Guarnieri G 2025 *New J. Phys.* **27** 023001
[23] Carollo F, Lesanovsky I, Antezza M and De Chiara G 2024 *Quantum Sci. Technol.* **9** 035024
[24] Ciccarello F 2017 *Quantum Meas. Quantum Metrol.* **4** 53–63
[25] Cilluffo D, Carollo A, Lorenzo S, Gross J A, Palma G M and Ciccarello F 2020 *Phys. Rev. Res.* **2** 043070
[26] Cusumano S, Mari A and Giovannetti V 2017 *Phys. Rev. A* **95** 053838
[27] Cusumano S, Mari A and Giovannetti V 2018 *Phys. Rev. A* **97** 053811
[28] Lorenzo S, McCloskey R, Ciccarello F, Paternostro M and Palma G M 2015 *Phys. Rev. Lett.* **115** 120403
[29] O'Connor E, Campbell S and Landi G T 2024 *New J. Phys.* **26** 033048
[30] Bozkurt A M, López R and Ryu S 2025 *Phys. Rev. Res.* **7** 013145
[31] Di Bartolomeo G, Vischi M, Feri T, Bassi A and Donadi S 2024 *Phys. Rev. Res.* **6** 043321
[32] Alves G O, Santos M A F and Landi G T 2024 *Phys. Rev. A* **110** 052421
[33] Cattaneo M 2025 *Phys. Rev. A* **111** 022209
[34] Myers N M, Sable H and Scarola V W 2025 *Phys. Rev. E* **111** 014115
[35] Campbell S, Mehboudi M, Chiara G D and Paternostro M 2017 *New J. Phys.* **19** 103003
[36] Fiusa G and Landi G T 2025 *Phys. Rev. A* **111** 032220
[37] De Chiara G, Landi G, Hewgill A, Reid B, Ferraro A, Roncaglia A J and Antezza M 2018 *New J. Phys.* **20** 113024
[38] Lorenzo S, Farace A, Ciccarello F, Palma G M and Giovannetti V 2015 *Phys. Rev. A* **91** 022121
[39] Macchiavello C, Riccardi A and Sacchi M F 2020 *Phys. Rev. A* **101** 062326
[40] Cusumano S and Rudnicki Ł 2021 *Entropy* **23** 1198
[41] Bäumer E, Perarnau-Llobet M, Kammerlander P, Wilming H and Renner R 2019 *Quantum* **3** 153
[42] Shu A, Cai Y, Seah S, Nimmrichter S and Scarani V 2019 *Phys. Rev. A* **100** 042107
[43] Rodrigues F L, De Chiara G, Paternostro M and Landi G T 2019 *Phys. Rev. Lett.* **123** 140601
[44] Shaghaghi V, Palma G M and Benenti G 2022 *Phys. Rev. E* **105** 034101
[45] Cusumano S, Cavina V, Keck M, De Pasquale A and Giovannetti V 2018 *Phys. Rev. A* **98** 032119
[46] Strasberg P, Schaller G, Brandes T and Esposito M 2017 *Phys. Rev. X* **7** 021003
[47] Cusumano S and De Chiara G 2024 *New J. Phys.* **26** 023001
[48] Alicki R 1979 *J. Phys. A: Math. Gen.* **12** L103
[49] Manatuly A, Niedenzu W, Román-Ancheyta R, Çakmak B, Müstecaplğlu O E and Kurizki G 2019 *Phys. Rev. E* **99** 042145
[50] Serafini A 2017 *Quantum Continuous Variables : a Primer of Theoretical Methods / Alessio Serafini* (CRC Press)
[51] Reid B, Pigeon S, Antezza M and De Chiara G 2017 *Europhys. Lett.* **120** 60006
[52] Monroe C *et al* 2021 *Rev. Mod. Phys.* **93** 025001
[53] Grimmer D, Brown E, Kempf A, Mann R B and Martín-Martínez E 2018 *Phys. Rev. A* **97** 052120
[54] Correa L A, Palao J P, Alonso D and Adesso G 2014 *Sci. Rep.* **4** 3949
[55] Hou P Y *et al* 2024 *Nat. Phys.* **20** 1636–41
[56] Home J P, Hanneke D, Jost J D, Leibfried D and Wineland D J 2011 *New J. Phys.* **13** 073026
[57] Brown K R, Ospelkaus C, Colombe Y, Wilson A C, Leibfried D and Wineland D J 2011 *Nature* **471** 196–9
[58] Cuevas A, Geraldi A, Liorni C, Bonavena L D, De Pasquale A, Sciarrino F, Giovannetti V and Mataloni P 2019 *Sci. Rep.* **9** 3205
[59] Jin J, Giovannetti V, Fazio R, Sciarrino F, Mataloni P, Crespi A and Osellame R 2015 *Phys. Rev. A* **91** 012122