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Some isoperimetric inequalities involving the boundary momentum

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Abstract

The aim of this paper is twofold. In the first part we focus on a functional involving a weighted curvature integral and the quermassintegrals. We prove upper and lower bounds for this functional in the class of convex sets, which provide a stronger form of the classical Aleksandrov-Fenchel inequality involving the $(n-1)$ and $(n-2)$-quermassintegrals, and consequently a stronger form of the classical isoperimetric inequality in the planar case. Moreover, quantitative estimates are proved. In the second part, we deal with a shape optimization problem for a functional involving the boundary momentum. It is known that in dimension two the ball is a maximizer among simply connected sets when the perimeter and centroid is fixed. We show that the result can be extended to the class of indecomposable sets. In higher dimensions, the same result does not hold and we consider a new scaling invariant functional that might be a good candidate to generalize the planar case. For this functional, we prove that the ball is a stable maximizer in the class of nearly spherical sets in any dimension.

Keywords Isoperimetric Inequalities · Boundary Momentum · Weighted Curvature Integral

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1 Introduction

In the last few years, weighted isoperimetric problems have attracted the attention of many mathematicians, due to its intrinsic mathematical interest and mainly for its wide class of applications, see [7, 8, 10, 24–26] for isoperimetric problem with densities and [8, 15] for quantitative weighted isoperimetric inequalities. In this paper, we are interested in weighted isoperimetric inequalities with density $|\cdot|^2$. For the weighted boundary integral

$$M(E) = \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1}, \tag{1}$$

Brock (see [4]) proved that $M(E)$ is minimized by the ball centered at the origin when a volume constraint is imposed. Another proof of this fact can be found in [3], where the authors prove isoperimetric inequalities for a larger class of integrals with radial weights, both on the volume and the perimeter. The quantity introduced in (1) is known as the *boundary momentum* of E. It is worth mentioning that the authors in [5] proved an isoperimetric inequality for a functional involving the boundary momentum, the perimeter and the measure, obtaining the Weinstock inequality in any dimension in the class of convex sets.

In the first part of the paper, we study a shape optimization problem involving the functional

$$\mathscr{H}(E) = \frac{1}{n} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} G_{\partial E} |x - x_0|^2 \, d\mathcal{H}^{n-1},$$

where $G_{\partial E}$ is the Gaussian curvature of ∂E and E is a convex set with not empty interior. Inequalities involving weighted curvature integrals have been studied in recent years, due to the increasing interest in isoperimetric problems in manifolds. To name some recent contributions, we mention [16], where the authors study the case where the weight is the Gaussian density, and [21, 22, 29], where weighted quermassintegrals have been studied. In fact, in [21], the authors prove that, for $E \subset \mathbb{R}^n$ convex and $k \in \{1, \ldots, n-1\}$, it holds

$$\frac{1}{2} \int_{\partial E} |x|^2 H_{\partial E,k}(x) \, d\mathcal{H}^1 + W_{k-1}(E) \geq c(n,k) W_k(E)^{\frac{n+1-k}{n-k}}, \tag{2}$$

where $c(n,k)$ is a constant depending only on n and k, $W_i(E)$ are the so called quermassintegrals and $H_{\partial E,i}$ are the the i-th elementary symmetric polynomials in the $n-1$ principal curvatures of ∂E (for the precise definition see subsection 2.2).

Our aim is to bound from below and above a slight modification of the left hand side of (2). Given $\beta \geq 0$, we define the functional

$$\mathcal{G}_\beta(E) = \mathscr{H}(E) + \beta W_{n-2}(E). \tag{3}$$

The main result of this first part is the following.

Theorem 1.1 *Let $E \subset \mathbb{R}^n$ be an open, bounded convex set. If $0 \le \beta \le n-1$, then we have*

$$\beta W_{n-2}(E) + \frac{1}{n} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 G_{\partial E} \, d\mathcal{H}^{n-1} \ge \frac{(1+\beta)}{\omega_n} W_{n-1}(E)^2. \tag{4}$$

If $\beta \ge \beta(n) := (n-1)(1 + \frac{n}{n+1})$,

$$\beta W_{n-2}(E) + \frac{1}{n} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 G_{\partial E} \, d\mathcal{H}^{n-1} \le \frac{(1+\beta)}{\omega_n} W_{n-1}(E)^2. \tag{5}$$

Moreover, equality holds if and only if $E = B_r(x_0)$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.

Therefore, for $\beta \le n-1$, balls are minimizers of $\mathcal{G}_\beta$ among convex sets of fixed $(n-1)$-quermassintegral (hence the curvature term is the dominating one), while, for $\beta \ge \beta(n)$, the ball is a maximizer in the same class (the $(n-2)$-quermassintegral term is the dominating one). We also show that in the planar case the thresholds $\beta = 1$ and $\beta = \beta(2) = 5/3$ are sharp (see Corollary 3.8). We also provide a quantitative version of Theorem 1.1. To be precise, we introduce the functional

$$\mathscr{I}_\beta(E) = \frac{1+\beta}{\omega_n} - \frac{\mathcal{G}_\beta(E)}{W_{n-1}^2(E)} \tag{6}$$

and, in the spirit of [17], we prove the following quantitative inequality.

Theorem 1.2 *Let $n \ge 2$. There exist δ and two positive constants $C_1 = C_1(\beta)$, $C_2 = C_2(\beta)$ such that, if $E \subset \mathbb{R}^n$ is an open, bounded convex set with $|\mathscr{I}_\beta(E)| < \delta$, then, for $\beta > \beta(n)$, it holds*

$$\mathscr{I}_\beta(E) \ge C_1 g(\tilde{\mathcal{A}}_{\mathcal{H}}(E))^{\frac{5}{2}}. \tag{7}$$

For $\beta < n-1$, we have

$$\mathscr{I}_\beta(E) \le -C_2 g(\tilde{\mathcal{A}}_{\mathcal{H}}(E))^{\frac{5}{2}}.$$

The function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as

$$g(s) = \begin{cases} s^2 & \text{if } n = 2 \\ f^{-1}(s^2) & \text{if } n = 3 \\ s^{\frac{n+1}{2}} & \text{if } n \ge 4, \end{cases}$$

with $f(t) = \sqrt{t \log\left(\frac{1}{t}\right)}$, for $0 < t < e^{-1}$, and $\tilde{\mathcal{A}}_{\mathcal{H}}(\cdot)$ is the asymmetry index defined in (12).

Some comments on the functional $G_\beta(\cdot)$ are in order here.

- To prove the lower bound (4), it is not necessary to be careful with respect to the position of the set in the space. On the other hand, when one is interested in an upper bound as in (5), it is necessary to understand where the set is positioned.

This explains why we need to take the infimum in the definition of the functional $\mathscr{H}(\cdot)$. In particular, the infimum is attained at a privileged point, named curvature centroid, defined in (24).

- As we point out in Remark 3.7, the inequality proven in Theorem 1.1 is stronger than the classical Aleksandrov-Fenchel inequality with indices $n-1$ and $n-2$ (see section 2) in n dimensions for convex sets. In particular, it gives a stronger form of the classical isoperimetric inequality in the plane for convex sets.
- In dimension 2, the functional $G_\beta(\cdot)$ becomes $\mathscr{H}(\cdot)+\beta\,|\cdot|$ and we observe the following. By the isoperimetric inequality, balls maximize the volume among sets of fixed perimeter, while the functional $\mathscr{H}(\cdot)$ does not admit a maximizer among convex sets of fixed perimeter, as we highlight in Proposition 3.9. Furthermore, balls are minimizers of $\mathscr{H}(\cdot)$ when the perimeter is fixed.

Looking at the proof of Proposition 3.9, it is clear that the problem comes from the fact that, to maximize $\mathscr{H}(\cdot)$, it is convenient to lose mass. For this reason, we introduce the functional $G_\beta(\cdot)$ defined in (3). In two dimensions, this corresponds to add a volume penalization to the functional $\mathscr{H}(\cdot)$. Thus, the functional $G_\beta(\cdot)$ resembles the famous Gamov model for liquid drop, since there is a competition between two functionals having opposite behavior on balls (see Corollary 3.8).

In the second part of the paper, we study the problem of bounding from above the boundary momentum. As in two dimension it is possible to bound the diameter of a sets by its perimeter, it is natural to ask if the problem

$$\sup_{E\in\mathcal{A}}\{M(E)\mid P(E)=m,\ x_E=0\} \tag{8}$$

admits a solution. In (8), $\mathcal{A}$ is a certain class of sets, $P(E)$ stands for the perimeter of E and x_E is the centroid of E (see Definition 2.1). Clearly, a maximization problem makes no sense in any class of sets that fix only the perimeter, as $M(\cdot)$ positively diverges if we translate E far away from the origin. This leads to study the shape optimization problem nailing the set in a specific point.

Our first Theorem, in this second part, is a generalization of an already known result proved in the class of simply connected sets (see [19, Pages 396-397]). We extend it to the class of indecomposable sets of finite perimeter (see Definition 2.2).

Theorem 1.3 *Let $E\subset\mathbb{R}^2$ be an indecomposable set of finite perimeter. Then,*

$$\inf_{x_0\in\mathbb{R}^2}\int_{\partial^* E}|x-x_0|^2\,d\mathcal{H}^2\le\frac{P(E)^3}{(2\pi)^2}.$$

The assumption of E being indecomposable is necessary. Indeed, if $\{E_n\}$ is a sequence given by the union of two disjoint balls with the same perimeter and symmetrical with respect to the origin that move away from each other, then the centroid remains fixed at the origin, but $M(E_n)\to\infty$. So, necessarily $\mathcal{A}$ is the class of connected and bounded open set in $\mathbb{R}^2$. We mention that a related problem has been studied in [11], where the authors prove (33) for almost every inner parallel curve of smooth bounded and simply connected sets. This result is relevant to prove an isoperimetric inequality for the first eigenvalue of the magnetic Laplacian with negative Robin boundary condition (see [20]).

In the last part of this paper, we try to generalize Theorem 1.3 to higher dimensions. Note that we can construct a sequence of bounded cylinders with fixed perimeter but second momentum as large as we wish (see the counterexample 4.3).

The problem arises because the assumption equibounded perimeter does not guarantee that the diameters of a maximizing sequence are equibounded. For this reason, we introduce the following scaling and translations invariant functional

$$\mathcal{F}(E) = \frac{|E|^{(n-2)(n+1)}}{P(E)^{n^2-1}} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^{n-1}. \tag{9}$$

In Proposition 4.5, we prove the existence of a maximizer for (9) in the class of open, bounded and convex sets, and we prove that the ball is the unique maximizer in the class of nearly spherical sets (see Subsection 2.3 for the precise definition).
We also prove a quantitative isoperimetric inequality for the quantity defined in (9).

Theorem 1.4 *Let $E \subset \mathbb{R}^n$ be an open, bounded convex set. There exists $\varepsilon > 0$ such that if $\mathcal{A}_{\mathcal{H}}(E) \leq \varepsilon$, then*

$$\frac{1}{(n\omega_n)^{n^2-2}} - \frac{|E|^{(n+1)(n-2)} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^{n-1}}{P(E)^{n^2-1}} \geq C(n) g\left(\mathcal{A}_{\mathcal{H}}(E)\right),$$

where $\mathcal{A}_{\mathcal{H}}(E)$ is an asymmetry index defined in (11) and g defined in Theorem 1.2.

As a corollary of this fact, we immediately find a quantitative version of (33) in two dimensions for convex sets.

Corollary 1.5 *There exists $\varepsilon > 0$ such that, for any open bounded convex set E, it holds*

$$\frac{1}{(2\pi)^2} - \frac{\inf_{x_0 \in \mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^1}{P(E)^3} \geq C \mathcal{A}_{\mathcal{H}}(E)^2.$$

The proof of Theorem 1.4 comes from a standard argument, introduced in [13], and successfully adapted to weighted isoperimetric inequalities in presence of a perimeter constraint (see [9, 18]) to prove the stability of some spectral inequalities.

The paper is organized as follows. In Section 2, we fix the notation and report some classical definitions and results that are used in the manuscript. In Section 3, we deal with the weighted curvature integral, proving the aforementioned upper and lower bounds as well as the quantitative results contained in Theorem 1.2. In Section 4, we study of the isoperimetric inequality involving the boundary momentum and prove Theorems 1.3, 1.4 and Corollary 1.5. We conclude the paper with Section 5, where we provide some comments and list some open problems.

2 Preliminary results and Definitions

2.1 Basic notions and definitions

Throughout this paper, we denote by $B_R(x_0)$ and B_R the balls in $\mathbb{R}^n$ of radius $R > 0$ centered at $x_0 \in \mathbb{R}^n$ and at the origin, respectively. Moreover, the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$ will be denoted by $\mathcal{H}^{n-1}$ and the Euclidean scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$.
Let $D \subseteq \mathbb{R}^n$ be an open bounded set and let $E \subseteq \mathbb{R}^n$ be a measurable set. For the sake of completeness, we recall here the definition of the perimeter of E in D (see for instance [2, 23]), that is

$$P(E; D) = \sup \left\{ \int_E \mathrm{div}\varphi \, dx : \ \varphi \in C_c^\infty(D; \mathbb{R}^n), \ ||\varphi||_\infty \leq 1 \right\}.$$

The perimeter of E in $\mathbb{R}^n$ will be denoted by $P(E)$ and, if $P(E) < \infty$, we say that E is a set of finite perimeter. We denote by $\partial^* E$ the reduced boundary of E. Moreover, if E has Lipschitz boundary, it holds

$$P(E) = \mathcal{H}^{n-1}(\partial E).$$

The Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$ will be denoted by $|E|$. We also give the definition of centroid.

Definition 2.1 Let $E \subset \mathbb{R}^n$ be a set with finite perimeter. We define the *centroid* of E as the barycenter of its boundary, i.e.,

$$x_E = \frac{1}{P(E)} \int_{\partial^* E} x \, d\mathcal{H}^{n-1}.$$

We also recall the definition of decomposable set and indecomposable set.

Definition 2.2 A set $E \subset \mathbb{R}^n$ is said a decomposable set of finite perimeter if there exists a partition of E in two measurable sets E_1 and E_2, with strictly positive measure, such that

$$P(E) = P(E_1) + P(E_2).$$

A set is said to be indecomposable if it is not decomposable.

In two dimensions, decomposable sets of finite perimeter have a strict structure, as explained by the following theorem (see [1, Corollary 1]).

Theorem 2.3 *Let E be a indecomposable set of finite perimeter in $\mathbb{R}^2$. There exists a unique decomposition ($\mathrm{mod}(\mathcal{H}^1)$) of $\partial^* E$ into a finite or countable number of Jordan curves C_0, C_i, $i \in I$, such that $\mathrm{int}(C_i) \subset \mathrm{int}(C_0)$, the sets $\mathrm{int}(C_i)$ are pairwise disjoint and $P(E) = \mathcal{H}^1(C_0) + \sum_i \mathcal{H}^1(C_i)$.*

We define the circumradius of $E \subset \mathbb{R}^n$ as

$$R_E = \inf\{r > 0 : E \subset B_r(x), x \in \mathbb{R}^n\}.$$

We recall the following useful inequality valid for any convex set in dimension two (see [28])

$$R_E < \frac{P(E)}{4}. \tag{10}$$

The diameter of E is

$$\mathrm{diam}(E) = \sup_{x,y \in E} |x - y|.$$

In order to define the mean curvature, we now introduce some basic tools of differential geometry, such as the tangential gradient and the tangential divergence.

Let E be a set with C^2 boundary. A vector field $X \in C^1(\partial E, \mathbb{R}^n)$ can be extended in a tubular neighborhood of ∂E and such an extension can be used to define the tangential divergence of X as

$$\mathrm{div}_\tau X = \mathrm{div}\, X - \langle DX \nu_{\partial E}, \nu_{\partial E} \rangle.$$

Note that this definition does not depend on the chosen extension. In the same way, a function $u \in C^1(\partial E)$ can be extended in a tubular neighborhood of ∂E and such an extension can be used to define the tangential gradient of u at a point $x \in \partial E$ as

$$\nabla_\tau u = \nabla u - \langle \nabla u(x), \nu_{\partial E} \rangle \nu_{\partial E}.$$

For $x \in \partial E$, the normalized mean curvature of ∂E is defined as

$$H_{\partial E}(x) := \frac{1}{n-1} \mathrm{div}_\tau\, \nu_{\partial E}(x).$$

Observe that with this definition the curvature of the unit ball in $\mathbb{R}^n$ is 1. The divergence theorem on manifold reads as follows.

Theorem 2.4 *Let M be a C^2 surface in $\mathbb{R}^n$ and $V \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$. Then we get*

$$\int_M \mathrm{div}_M V \, d\mathcal{H}^{n-1} = (n-1) \int_M V \cdot \bar{H}_M \, d\mathcal{H}^{n-1},$$

where $\bar{H}_M = H_M \nu_m$ is the vectorial normalized mean curvature, which is the scalar mean curvature H_M in the direction of the outer unit normal ν_M.

2.2 Hausdorff distance and curvature measure

We also need to define the Hausdorff distance between two sets.

Definition 2.5 The Hausdorff distance between two non-empty compact sets $E, F \subset \mathbb{R}^n$ is defined by:

$$d^{\mathcal{H}}(E, F) = \inf \{ \varepsilon > 0 \ : \ E \subset F + B_\varepsilon, \ F \subset E + B_\varepsilon \}.$$

If $D \subset \mathbb{R}^n$ is a compact set and $E, F \subset D$ are two bounded open sets, we define the Hausdorff distance between the two open sets E and F by

$$d_{\mathcal{H}}(E, F) := d^{\mathcal{H}}(D \setminus E, D \setminus F).$$

This last definition is independent of the "big compact box" D. We say that a sequence of compact set (respectively, bounded open sets) $(E_j)_j$ converges to the compact set (respectively, bounded open set) E in the sense of Hausdorff if $d^{\mathcal{H}}(E_j, E) \to 0$ (respectively, $d_{\mathcal{H}}(E_j, E) \to 0$).

If E and F are open convex sets, we have

$$d_{\mathcal{H}}(E, F) = d^{\mathcal{H}}(\overline{E}, \overline{F}) = d^{\mathcal{H}}(\partial E, \partial F)$$

and the following rescaling property holds

$$d_{\mathcal{H}}(tE, tF) = t\, d_{\mathcal{H}}(E, F), \quad t > 0.$$

We define

$$\mathcal{A}_{\mathcal{H}} = \min_{x \in \mathbb{R}^n} \left\{ \frac{d_{\mathcal{H}}(E, B_r(x_0))}{r}, \ P(E) = P(B_r) \right\} \tag{11}$$

and

$$\tilde{\mathcal{A}}_{\mathcal{H}} = \min_{x \in \mathbb{R}^n} \left\{ \frac{d_{\mathcal{H}}(E, B_r(x_0))}{r}, \ W_{n-1}(E) = W_{n-1}(B_r) \right\}, \tag{12}$$

so that it is invariant under rigid motion and dilatation.

The Hausdorff convergence is a very important tool to introduce the concept of curvature measure of a generic convex set. Without aiming for completeness, we just introduce the concept and refer to [27, Section 1.8] for an exhaustive dissertation. For convex sets $E \subset \mathbb{R}^n$, the mean curvature and the Gaussian curvature can be defined also if E is not smooth and they are Radon measure. For this reason, they are also called curvature measures. We will denote the mean curvature measure and the Gaussian curvature measure, respectively, by μ_E^H and μ_E^G. Note that when E is a smooth convex set, $\mu_E^H = H_{\partial E} \mathcal{H}^{n-1} \llcorner \partial E$, with $H_{\partial E}$ defined above. We recall the following result, whose proof (actually, of a more general result regarding curvature measures) can be found in [27, Section 4.2].

Theorem 2.6 *Let $\{K_h\}_{h \in \mathbb{N}}$ be a sequence of convex bodies in $\mathbb{R}^n$ and assume that there exists a convex set $K \subset \mathbb{R}^n$, such that $K_h \to K$ in the Hausdorff sense. Then, the curvature measure $\mu_{K_h} \to \mu_K$ weakly* in the sense of measures.*

2.3 Nearly spherical sets

Here we recall the notion of a nearly spherical set and state some useful results.

Definition 2.7 Let $n \geq 2$ and ε_0 a positive small real number. We say that a set E is nearly spherical of mass m if $|E| = m$ and

$$E = \{rx(1 + u(x)),\ x \in \mathbb{S}^{n-1}\},$$

with $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon_0$.

Notice that $||u||^{n-1}_{L^\infty(\mathbb{S}^{n-1})} = d_{\mathcal{H}}(E, B_r)$, where B_r is the ball centered at the origin with the same measure as E. The Lebesgue measure, perimeter and the boundary momentum can be written in the following way (see [14])

$$\begin{aligned} |E| &= \frac{1}{n}\int_{\mathbb{S}^{n-1}} (1 + u(x))^n \, d\mathcal{H}^{n-1}, \\ P(E) &= \int_{\mathbb{S}^{n-1}} (1 + u(x))^{n-2}\sqrt{(1 + u(x))^2 + |D_\tau u(x)|^2}\, d\mathcal{H}^{n-1}, \\ \int_E |x|^2 \, d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} (1 + u(x))^n\sqrt{(1 + u(x))^2 + |D_\tau u(x)|^2}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{13}$$

In next section, it will be useful the following lemma (see [13] for a proof).

Lemma 2.8 *If* $v \in W^{1,\infty}(\mathbb{S}^{n-1})$ *and* $\int_{\mathbb{S}^{n-1}} v \, d\mathcal{H}^{n-1} = 0$, *then*

$$||v||^{n-1}_{L^\infty(\mathbb{S}^{n-1})} \leq \begin{cases} \pi \, \|\nabla_{\mathbb{S}^{n-1}} v\|_{L^2(\mathbb{S}^{n-1})} & n = 2 \\ 4||\nabla_{\mathbb{S}^{n-1}} v||^2_{L^2(\mathbb{S}^{n-1})} \log \dfrac{8e||\nabla_{\mathbb{S}^{n-1}} v||^{n-1}_{L^\infty(\mathbb{S}^{n-1})}}{||\nabla_{\mathbb{S}^{n-1}} v||^2_{L^2(\mathbb{S}^{n-1})}} & n = 3 \\ C(n)||\nabla_{\mathbb{S}^{n-1}} v||^2_{L^2(\mathbb{S}^{n-1})} ||\nabla_{\mathbb{S}^{n-1}} v||^{n-3}_{L^\infty(\mathbb{S}^{n-1})} & n \geq 4. \end{cases} \tag{14}$$

2.4 Some basic notions on convex bodies

What follows is contained in [27, Secions 4 and 8].

2.4.1 Support functions and curvatures of uniformly convex sets

Let $E \in \mathbb{R}^n$ be a convex body, which is a compact convex set with nonempty interior in $\mathbb{R}^n$. The support function of a convex body $E \subset \mathbb{R}^n$ is the 1-homogeneous function $h_E : \mathbb{R}^n \to \mathbb{R}$ defined as

$$h_E(y) = \sup_{p \in E}\{\langle p, y\rangle\}.$$

The regularity of the support function is strictly related to the regularity of the boundary of E. We say that a convex body $E \in \mathbb{R}^n$ is of class C^k, $k \in \mathbb{N}$, if its boundary is a $k-$differentiable hypersurface in the sense of differential geometry. If E is of class C^2, we can define the *Gauss map* as the C^1 function $\nu_E : \partial E \to \mathbb{S}^{n-1} \subset \mathbb{R}^n$, which maps every point $x \in \partial E$ to its unique outer unit normal $\nu(x)$. Its differential $W_x := d_x \nu_E : T_x E \to T_x E$, that maps the tangent space $T_x E$ into itself, is a linear map, known as the *Weingarten map*, and its eigenvalues are, by definition, the *principal curvatures* $\kappa_1, ..., \kappa_{n-1}$. We denote by $H_{\partial E,k}$ the k-th elementary symmetric polynomial in the $n-1$ principal curvatures $(\kappa_1, ..., \kappa_{n-1})$ of ∂E, as

$$H_{\partial E,k} = \binom{n-1}{k}^{-1} \sum_{1 \le i_1 \le i_2 \cdots\cdots i_k \le n-1} \kappa_{i_1} \cdots \kappa_{i_k}, \quad k = 1, ..., n-1, \qquad H_0 = 1. \tag{15}$$

In particular, $H_{\partial E} := H_{\partial E,1}$ and $G_{\partial E} := H_{\partial E,n-1}$ are the normalized mean curvature and the Gaussian curvature of E, respectively.
The inverse of the Gauss map plays a key role in this paper. In order to define it, we need some extra regularity assumption.

Definition 2.9 We say that the convex body E is of class C^2_+ if it is of class C^2 and all the principal curvatures are different from zero at any point (or, equivalently, $G_{\partial E} \neq 0$).

If $E \in C^2_+$, the inverse of the Gauss map $x_{\partial E} := \nu_E^{-1} : \mathbb{S}^{n-1} \to \partial E$ is well defined and of class C^1. It is known as *reverse Gauss map* and we have

$$h_E(y) = \langle y, x_{\partial E}(y) \rangle, \qquad y \in \mathbb{S}^{n-1}.$$

This shows that h_E is differentiable on $\mathbb{S}^{n-1}$, hence on $\mathbb{R}^n \setminus \{0\}$. In particular, $x \in \partial E$ can be characterized as the only one point of ∂E such that

$$\nu_E(x_{\partial E}(y)) = \frac{y}{|y|}.$$

Since the map $y : \mathbb{R}^n \setminus \{0\} \mapsto x_{\partial E}(y) \in \partial E$ is $0-$homogeneous, we have

$$\nabla h_E(y) = x_{\partial E}(y),$$

proving that actually h_E is of class C^2. Moreover, by the homogeneity of h_E, we have that

$$x_{\partial E}(y) = \nabla_{\mathbb{S}^{n-1}} h_E(y) + h_E(y) y. \tag{16}$$

The differential of the inverse of the Gauss map $\overline{W_y} := d_y x_{\partial E} : T_y\mathbb{S}^{n-1} \to T_y\mathbb{S}^{n-1}$ is a linear map, known as the *reverse Weingarten map*, and its eigenvalues $r_1, ..., r_{n-1}$ are the *principal radii of curvature* of E at y. Analogously, we denote by $s_{\partial E,k}$ the k-th elementary symmetric polynomial in the $n-1$ principal radii of curvatures $(r_1, ..., r_{n-1})$ of ∂E, i.e.,

$$s_{\partial E,k} = \binom{n-1}{k}^{-1} \sum_{1 \le i_1 < i_2 \cdots\cdots < i_k \le n-1} r_{i_1} \cdots r_{i_k}, \quad k = 1, ..., n-1.$$

$H_{\partial E,k}$ and $s_{\partial E,k}$ are connected by the following relationships

$$H_{\partial E,k}(x_{\partial E}(y)) = \frac{s_{\partial E,n-1-k}}{s_{\partial E,n-1}}(y), \qquad k = 1, ..., n-1, \tag{17}$$

and

$$s_{\partial E,k}(y) = \frac{H_{\partial E,n-1-k}}{H_{\partial E,n-1}}(x_E(y)), \qquad k = 1, ..., n-1. \tag{18}$$

We also recall the following fact (see [27, Corollary 2.5.3]).

Corollary 2.10 *Let E be a convex body of class C^2_+, so that h_E is of class C^2. For $k = 1, ..., n-1$, we have that $\binom{n-1}{k} s_{\partial E,k}(y)$ is equal to the sum of the principal minors of order k of the Hessian matrix (with respect to an orthonormal basis in $\mathbb{R}^n$) of h_E at y.*

In particular, we get

$$s_{\partial E,1}(y) = \frac{\Delta h_E}{n-1},$$

where Δ denotes the Laplace operator in $\mathbb{R}^n$. While if we choose an orthonormal basis $\{e_1, ..., e_n\}$ in $\mathbb{R}^n$, such that $e_n = y \in \mathbb{S}^{n-1}$ (i.e., $\{e_1, ..., e_{n-1}\}$ is an orthonormal basis in $T_y\mathbb{S}^{n-1}$), then

$$s_{\partial E,n-1}(y) = \det(\nabla^2 h_E(y)).$$

Equivalently, if we consider the restriction of the support function h_E of E to $\mathbb{S}^{n-1}$ (always denoted by h_E), we have

$$s_{\partial E,1}(y) = \frac{\Delta_{\mathbb{S}^{n-1}} h_E(y)}{n-1} + h_E(y)$$

and, by (16), we also get

$$s_{\partial E,n-1}(y) = \det(\nabla_{\mathbb{S}^{n-1}} h_E(y) + h_E(y)I),$$

where $\nabla_{\mathbb{S}^{n-1}} h_E$ and I are the Jacobian of h_E and the identity matrix.
In this case, for $k = n-2$, equation (17) has the nice form

$$H_{\partial E,n-2}(x(y)) = \frac{\Delta_{\mathbb{S}^{n-1}} h_E(y) + (n-1)h_E(y)}{(n-1)\det(\nabla_{\mathbb{S}^{n-1}} h_E(y) + h_E(y)I)}.$$

$H_{\partial E,k}$ and $s_{\partial E,k}$ are useful tools for treating surface integrals. In particular (see [27, equations 2.61-2.62]), for any integrable function f on ∂E we have

$$\int_{\partial E} f(x)\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{S}^{n-1}} f(x_E(y)) s_{\partial E,n-1}\, d\mathcal{H}^{n-1}(y),$$

and for every integrable function g on $\mathbb{S}^{n-1}$,

$$\int_{\mathbb{S}^{n-1}} g(y)\, d\mathcal{H}^{n-1}(y) = \int_{\partial E} g(\nu_E(x)) H_{\partial E,n-1}\, d\mathcal{H}^{n-1}(x).$$

2.4.2 Quermassintegrals: definition and properties

Let $\emptyset \neq E \subset \mathbb{R}^n$ be an open, bounded convex set and let $\rho > 0$ be a positive real number. We consider the outer parallel set E_ρ, defined as

$$E_\rho := E + B_\rho = \{x + \rho y \ : \ x \in E, y \in B_1\},$$

where "$+''$ is the Minkowski sum of two sets and B_1 is the unit ball centered at the origin. In particular, by Steiner formula, we can write the measure of E_ρ as a polynomial in ρ as follows

$$\left|E_\rho\right| = \sum_{i=0}^{n} \binom{n}{i} W_i(E)\rho^i,$$

where the coefficients $W_i(E)$ are known as *quermassintegrals*. It is well known that these coefficients have an immediate geometrical interpretation when E is C^2_+. Indeed,

$$W_0(E) = |E|, \qquad nW_i(E) = \int_{\partial E} H_{\partial E, i-1}\, d\mathcal{H}^{n-1},\ i = 1, ..., n,$$

where $H_{\partial E,i-1}$ is defined in (15). Hence, in this case

$$nW_1(E) = P(E),\ \ n(n-1)W_2(E) = \int_{\partial E} H_{\partial E}\, d\mathcal{H}^{n-1},\ \ldots\ , nW_n(E) = \int_{\partial E} G_{\partial E}\, d\mathcal{H}^{n-1},$$

where $H_{\partial E}$ and $G_{\partial E}$ are, respectively, the normalized mean curvature and the Gaussian curvature of ∂E. Another useful integral representation is given by (see [27, Section 5.3])

$$W_i(E) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_E(y) s_{\partial E, n-1-i}(y)\, d\mathcal{H}^{n-1}(y),$$

or, using (18), by

$$W_i(E) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_E(y) \frac{H_{\partial E,i}(x_E(y))}{H_{\partial E,n-1}(x_E(y))}\, d\mathcal{H}^{n-1}(y). \tag{19}$$

In the particular case $i = n-1$, (19) gives, up to a multiplicative constant, the mean width of Ω and when $i = n-2$ takes the nice form

$$W_{n-2}(\Omega) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_E(y)(h_E(y) + \frac{1}{n-1}\Delta_{\mathbb{S}^{n-1}} h_E(y)) d\mathcal{H}^{n-1}(y). \tag{20}$$

Lastly, we recall the well known Aleksandrov-Fenchel inequalities: for any $0 \le i < j \le n-1$ it holds

$$\left(\frac{W_j(E)}{\omega_n}\right)^{\frac{1}{n-j}} \ge \left(\frac{W_i(E)}{\omega_n}\right)^{\frac{1}{n-i}} \tag{21}$$

and equality holds if and only if E is a ball. We stress that for $j = 1$ and $i = 0$, (21) is the classical isoperimetric inequality.

3 Isoperimetric inequalities involving curvature and boundary momentum

In this section, we prove an upper and a lower bound for a weighted curvature integral for convex sets. We remark that Proposition 3.1 still holds in higher dimension, provided that we restrain ourselves to the class of convex sets (see Proposition 3.3). We start with the lower bound, which is easier as it comes just from an integration by parts, after writing the functional $\mathscr{H}(\cdot)$ in polar coordinates.

3.1 Lower bounds on the weigthed curvature

To prove the lower bound, no assumptions on the position of the set are needed, as it will be clear in Corollary 3.2.

Proposition 3.1 *Let $r > 0$ and $E \subset \mathbb{R}^2$ be an open, starshaped $C^{1,1}$ set with respect to the origin, such that $|E| = |B_r|$. Then*

$$\int_{\partial E} H_{\partial E} |x|^2 \, d\mathcal{H}^1 \geq \int_{\partial B_r} H_{\partial B_r} |x|^2 \, d\mathcal{H}^1 = 2|E|. \tag{22}$$

Moreover, equality holds if and only if $E = B_r$.

Proof We parameterize the boundary of E by means of the radial function ρ, i.e., let $\rho : [0, 2\pi] \mapsto \mathbb{R}_+$ be such that $\partial E = \{\rho(\theta)(\cos\theta, \sin\theta),\ \theta \in [0, 2\pi]\}$. The formulas expressing the area the perimeter and the mean curvature are

$$|E| = \frac{1}{2} \int_0^{2\pi} \rho^2(\theta) \, d\theta,$$

$$P(E) = \int_0^{2\pi} \sqrt{\rho^2(\theta) + \dot{\rho}(\theta)^2} \, d\theta$$

and

$$H_{\partial E} = \frac{\rho^2 + 2\dot{\rho}^2 - \rho\ddot{\rho}}{(\rho^2 + \dot{\rho}^2)^{\frac{3}{2}}}.$$

Hence, a simple integration by parts allows to arrive to the conclusion

$$\begin{aligned}
&\int_{\partial E} H_{\partial E}|x|^2\, d\mathcal{H}^{n-1} \\
&= \int_0^{2\pi} \rho^2 \frac{\rho^2+2\dot\rho^2-\rho\ddot\rho}{\rho^2+\dot\rho^2}\, d\theta = \int_0^{2\pi} \rho^2\, d\theta + \int_0^{2\pi} \rho^2 \frac{\dot\rho^2-\rho\ddot\rho}{\rho^2+\dot\rho^2}\, d\theta \\
&= 2|E| - \int_0^{2\pi} \rho^2 \frac{d}{d\theta} \arctan\left(\frac{\dot\rho}{\rho}\right) d\theta \\
&= 2|E| + 2\int_0^{2\pi} \rho\dot\rho \arctan\left(\frac{\dot\rho}{\rho}\right) \geq 2|E| = \int_{\partial B_r} H_{\partial B_r}|x|^2\, d\mathcal{H}^1,
\end{aligned}$$

where we used that $t \arctan t \geq 0$ for all $t \in \mathbb{R}$. If the equality sign holds, then it must be

$$\int_0^{2\pi} \rho\dot\rho \arctan\left(\frac{\dot\rho}{\rho}\right) = 0,$$

which in turn implies $\dot\rho = 0$ a.e. and consequently $\rho = const$. □

As we said before, in higher dimensions Proposition 3.1 remains valid for generic convex sets.

Corollary 3.2 *Let $E \subset \mathbb{R}^2$ be an open, bounded convex set. Then*

$$\int_{\partial E} |x|^2 d\mu_E^H \geq \int_{\partial B_r} H_{\partial B_r}|x|^2\, d\mathcal{H}^1.$$

Proof We start observing that (22) holds for $C^{1,1}$ convex sets (hence, we start removing the constraint on the *position* of the set). If the origin belongs to the interior of E, then Proposition 3.1 holds and, by continuity, it is clear that it also holds when the origin belongs to the boundary of E. In the case $0 \in E^c$, we let x_0 be the nearest point of E from the origin, hence $|x_0| = \min_{x\in\overline{E}} |x|$. By convexity, we have that E is contained in the half plane tangent to E in x_0, which means that $\langle x - x_0, x_0\rangle \geq 0$, or equivalently, $|x_0|^2 \leq \langle x, x_0\rangle$, for all $x \in \overline{E}$. Thus, we find

$$|x - x_0|^2 = |x|^2 + |x_0|^2 - 2\langle x, x_0\rangle \leq |x|^2 - |x_0|^2 \leq |x|^2.$$

Therefore,

$$\int_{\partial E} |x|^2 H_{\partial E}(x)\, d\mathcal{H}^1 \geq \int_{\partial E} |x-x_0|^2 H_{\partial E}(x)\, d\mathcal{H}^1 = \int_{\partial(x_0+E)} |x|^2 H_{\partial(x_0+E)}(x)\, d\mathcal{H}^1.$$

Observing that $0 \in \overline{x_0 + E}$, then inequality (22) holds, and we have the result for C^1 convex sets. To prove the result in the general case, we recall that for a generic open, bounded convex set E, there exists a curvature measure μ_E with supp $\mu_E \subset \partial E$, such that for any sequence of smooth convex sets $\{E_h\}_{h\in\mathbb{N}}$ converging in the Hausdorff sense to E, we have $H_{\partial E_h} \llcorner \partial E \rightharpoonup \mu_E$. □

We note that in higher dimension the same lower bound continues to hold.

Proposition 3.3 *Let $E \subset \mathbb{R}^n$ be a convex body. Then*

$$\int_{\mathbb{R}^n} |x|^2 d\mu_E^H \geq n(n-1)|E|.$$

Proof The proof of this inequality is just a consequence of the divergence theorem. First, we observe that if E is a smooth open convex set and $x \in \partial E$, we get

$$\nabla_{\partial E}|x| = (I - \nu_{\partial E} \otimes \nu_{\partial E}) \frac{x}{|x|} = \frac{x}{|x|} - \left\langle \frac{x}{|x|}, \nu_{\partial E} \right\rangle \nu_{\partial E},$$

which in turn implies

$$\langle x, \nabla_{\partial E}|x|\rangle = \frac{1}{|x|}(|x|^2 - \langle x, \nu_{\partial E}\rangle^2) \geq 0.$$

The convexity of E implies $H_{\partial E} \geq 0$ and this leads to

$$\begin{aligned} \int_{\partial E} H_{\partial E}|x|^2 \, d\mathcal{H}^{n-1} &\geq \int_{\partial E} H_{\partial E}|x|\langle x, \nu\rangle \, d\mathcal{H}^{n-1} = \int_{\partial E} \mathrm{div}_{\partial E}(x|x|) d\mathcal{H}^{n-1} \\ &= (n-1)\int_{\partial E} |x| \, d\mathcal{H}^{n-1} + \int_{\partial E} \langle x, \nabla_{\partial E}|x|\rangle \, d\mathcal{H}^{n-1} \geq (n-1) \\ &\quad \int_{\partial E} |x| \, d\mathcal{H}^{n-1}. \end{aligned} \tag{23}$$

By applying the divergence theorem, we have

$$\int_{\partial E} |x| \, d\mathcal{H}^{n-1} \geq \int_{\partial E} \langle x, \nu_{\partial E}\rangle \, d\mathcal{H}^{n-1} = n|E|.$$

The result thus follows from the last inequality and (23). For a generic convex set, the proof follows verbatim the arguments of Corollary 3.2. □

3.2 Upper bound for the weighted curvature

The proof of the second inequality of this section is a bit more delicate. In fact, we are going to prove a sharp upper bound for the functional $\mathscr{H}(\cdot)$.

Before proving it, we first define the curvature centroid of a set and prove some elementary properties. For a convex set E, we denote by μ_E^G the Gaussian curvature measure associated with E.

Definition 3.4 Let $E \subset \mathbb{R}^n$ be an open, bounded set. We define the Gaussian curvature centroid of E the following point

$$x_E^G = \frac{1}{n\omega_n} \int_{\partial E} x \, d\mu_E^G. \tag{24}$$

Of course, when E is smooth enough, say $C^{1,1}$, we have

$$x_E^G = \frac{1}{n\omega_n}\int_{\partial E} x G_{\partial E}\, d\mathcal{H}^{n-1}.$$

Using Theorem 2.6, it is immediate to check that the curvature centroid is continuous with respect to the Hausdorff convergence. With this observation in mind, we now prove the next Lemmata.

Lemma 3.5 *Let $E \subset \mathbb{R}^n$ be an open, bounded convex set. Then*

$$\inf_{y\in\mathbb{R}^n}\int_{\partial E} |x-y|^2 d\mu_E^G = \int_{\partial E} |x-x_E^G|^2\, d\mu_E^G.$$

Proof Define the functional

$$L(y) = \int_{\partial E} |x-y|^2\; d\mu_E^G.$$

For $y \neq x_E^G$, we get

$$L(y) = \int_{\partial E} (|x-x_E^G|^2 + |y-x_E^G|^2 - 2\langle y-x_E^G, x-x_E^G\rangle\, d\mu_E^G$$

$$= \int_{\partial E} (|x-x_E^G|^2 + |y-x_E^G|^2)\, d\mu_E^G,$$

where in the last equality we used (24). The result easily follows as $L(y) \geq \int_{\partial\Omega} |x - x_E^G|^2\, d\mu_E^G$, with the equality sign if and only if $y = x_E^G$. □

When E is convex x_E^G lies inside E, as shown in the next Lemma.

Lemma 3.6 *Let $E \subset \mathbb{R}^n$ be an open, bounded convex set. Then $x_E^G \in E$.*

Proof By contradiction, assume that $x_E^G \notin \overline{E}$. Then, we can find a positive real number $\delta > 0$, such that $\overline{B_\delta(x_E^G)} \cap \overline{E} = \emptyset$. Thus, by Hahn-Banach separation theorem, $\overline{E}$ and $\overline{B_\delta(x_E^G)}$ are strictly separated. This means that there exist a linear map $f : \mathbb{R}^n \to \mathbb{R}$ and a real number $t \in \mathbb{R}$, such that

$$f(x) < t < f(y), \qquad \forall x \in \overline{E},\; y \in \overline{B_\delta(x_E^G)}.$$

Therefore, using the Gauss-Bonnet Theorem together with the linearity of f, we get

$$f(x_E^G) = f\left(\frac{1}{n\omega_n}\int_{\partial E} x\, d\mu_E^G\right) = \frac{1}{n\omega_n}\int_{\partial E} f(x)\, d\mu_E^G < t,$$

which is clearly a contradiction. To prove that x_E^G is in fact an interior point, we assume that $x_E^G \in \partial E$ and let l be a supporting hyperplane for E at x_E^G, i.e., a hyperplane such that E lies on one side of it. Since $\mathscr{H}(\cdot)$ is invariant under rotation, without loss of generality, we can assume that $x_E^G = (x_0, 0, \dots, 0)$, $l = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 = x_0\}$ for some $x_0 \in \mathbb{R}$ and $E \subset \{x_1 > x_0\}$. Since E is convex and bounded, we have that $\mu_E^G(\{x_1 > x_0\}) > 0$. Therefore, $\int_{\partial E}(x_1 - x_0)\, d\mu_E^G > 0$, which implies that $(x_0, 0, \dots, 0)$ does not satisfy (24). □

3.3 Proof of Theorem 1.1 and consequences.

Proof of Theorem 1.1 First, we prove the result for a smooth uniformly convex set and then we obtain the general case by approximation. Without loss of generality, we assume that

$$\inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 G_{\partial E}\, d\mathcal{H}^{n-1} = \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1}.$$

Therefore, we have that the origin belongs to E by Lemma 3.6. Let $h : \mathbb{S}^{n-1} \to \mathbb{R}_+$ be the support function of E. Recalling (16), the map

$$x : \mathbb{S}^{n-1} \longmapsto \partial E, \qquad x(\omega) = \nabla_{\mathbb{S}^{n-1}} h(\omega) + \omega h(\omega)$$

is the inverse of the Gauss map $x \in \partial E \mapsto \nu(x) \in \mathbb{S}^{n-1}$. Note that the Gaussian curvature is the Jacobian of the Gauss map. Therefore, the area formula implies

$$\int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} h(\omega)|^2 + h(\omega)^2\, d\mathcal{H}^{n-1}, \tag{25}$$

where we used that $\langle \omega, \nabla_{\mathbb{S}^{n-1}} h(\omega)\rangle = 0$. Thus, by (20) and (25), we have

$$\begin{aligned}
&n\Big(\beta W_{n-2}(E) + \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1} - \frac{1+\beta}{\omega_n} W_{n-1}(E)^2\Big)\\
=&\frac{\beta}{n-1}\int_{\mathbb{S}^{n-1}} h((n-1)h + \Delta_{\mathbb{S}^{n-1}} h)\, d\mathcal{H}^{n-1}\\
&\quad + \int_{\mathbb{S}^{n-1}} (h^2 + |\nabla_{\mathbb{S}^{n-1}} h|^2)\, d\mathcal{H}^{n-1} - \frac{1+\beta}{n\omega_n}\left(\int_{\mathbb{S}^{n-1}} h\right)^2\\
=&\frac{n-1-\beta}{n-1}\int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} h|^2\, d\mathcal{H}^{n-1} + (1+\beta)\left(\int_{\mathbb{S}^{n-1}} h^2\, d\mathcal{H}^{n-1} - \frac{1}{n\omega_n}\left(\int_{\mathbb{S}^{n-1}} h\right)^2\right),
\end{aligned}$$

where in the last line we integrated by parts and rearranged terms.
For $0 \le \beta \le n-1$, the desired inequality (4) follows immediately by applying Jensen's inequality in the last line above:

$$\begin{aligned}
\beta W_{n-2}(E) + \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1} - \frac{1+\beta}{n\omega_n} W_{n-1}(E)^2 \ge \frac{n-1-\beta}{n-1}\\
\int_{\mathbb{S}^{n-1}} |\nabla_{\mathbb{S}^{n-1}} h|^2\, d\mathcal{H}^{n-1} \ge 0.
\end{aligned}$$

Recall that the space $L^2(\mathbb{S}^{n-1})$ admits the set of spherical harmonics $\{Y_{k,i},\, 1 \le i \le N_k,\, k \in \mathbb{N}\}$, i.e., the restriction to $\mathbb{S}^{n-1}$ of homogeneous harmonic polynomials in $\mathbb{R}^n$, as an orthonormal basis. For $k \in \mathbb{N}$ and $i \le N_k$, we have

$$-\Delta_{\mathbb{S}^{n-1}} Y_{k,i} = k(n+k-2) Y_{k,i}.$$

Hence, we can write h as

$$h = \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} a_{k,i} Y_{k,i},$$

where $a_{k,i} = \int_{\mathbb{S}^{n-1}} h Y_{k,i}\, d\mathcal{H}^{n-1}$. Since $\{Y_{k,i},\, 1 \le i \le N_k,\, k \in \mathbb{N}\}$ is an orthonormal basis, we have

$$\|h\|^2_{L^2(\mathbb{S}^{n-1})} = a_0^2 + \sum_{k=0}^{\infty} \sum_{i=1}^{N_k} a_{k,i}^2,$$

and, using the properties of $Y_{k,i}$, it holds

$$\|\nabla h\|^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} \sum_{i=1}^{N_k} k(n+k-2) a_{k,i}^2.$$

Note that

$$a_0^2 = \frac{1}{n\omega_n} \left(\int_{\mathbb{S}^{n-1}} h\, d\mathcal{H}^{n-1} \right)^2.$$

Moreover, since

$$\inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 G_{\partial E}\, d\mathcal{H}^{n-1} = \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1},$$

we get

$$0 = \int_{\partial E} x G_{\partial E}\, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} \omega h(\omega) + \nabla_{\mathbb{S}^{n-1}} h(\omega)\, d\mathcal{H}^{n-1}. \tag{26}$$

For $i = 1, \ldots, n$, we let e_i the standard orthonormal basis of $\mathbb{R}^n$ and denote by $(\nabla_{\mathbb{S}^{n-1}})_j u(\omega) = \langle \nabla_{\mathbb{S}^{n-1}} u, e_i \rangle$. It is quickly checked that $(\nabla_{\mathbb{S}^{n-1}})_j u(\omega) = \mathrm{div}_{\mathbb{S}^{n-1}} U_i$, where U_i is the vector field such that $\langle U_i, e_j \rangle = \delta_{ij} u$. Therefore, the divergence theorem on $\mathbb{S}^{n-1}$ yields

$$\int_{\mathbb{S}^{n-1}} \nabla_{\mathbb{S}^{n-1}} h\, d\mathcal{H}^{n-1} = (n-1) \int_{\mathbb{S}^{n-1}} h(\omega) \omega\, d\mathcal{H}^{n-1}.$$

Hence, substituting the above equality in (26), we find

$$0 = n \int_{\mathbb{S}^{n-1}} \omega h(\omega)\, d\mathcal{H}^{n-1}.$$

Recalling that $Y_{1,i}(\omega) = c_n\omega_i$, we have $a_{1,i} = 0$ for $i \in \{1, \dots, n\}$. Therefore,

$$\beta W_{n-2}(E) + \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1} - \frac{(1+\beta)}{n\omega_n} W_{n-1}(E)^2$$
$$= \sum_{k\geq 2}\sum_{i=1}^{N_k}((1+\beta) + \frac{n-1-\beta}{n-1}k(n+k-2))a_{k,i}^2.$$

If $\beta \geq \beta(n)$, using $k \geq 2$, we have

$$\begin{aligned}\beta W_{n-2}(E) &+ \int_{\partial E} |x|^2 G_{\partial E}\, d\mathcal{H}^{n-1} - \frac{(1+\beta)}{n\omega_n} W_{n-1}(E)^2 \\ &\leq \sum_{k\geq 2}\sum_{i=1}^{N_k}\left(\frac{1+\beta}{2n} + \frac{n-1-\beta}{n-1}\right)k(n+k-2)a_{k,i}^2 \\ &= (n+1)(\beta(n)-\beta)\|\nabla h\|^2_{L^2(\mathbb{S}^{n-1})}.\end{aligned} \tag{27}$$

For the general case, we can proceed by approximation: given an open convex set E, there exists a sequence of smooth strictly convex sets $E_n \subset E$ such that $E_n \to E$ in the Hausdorff sense. This fact can be proved by using the result in [6], which states that for any bounded convex set $E \subset \mathbb{R}^n$ (actually the result works in every dimension), there exists a smooth strictly convex exhaustion, i.e., a strictly convex function $v : E \to \mathbb{R}$, such that $v \in C^\infty(E) \cap C(\overline{E})$ and $v(x) = 0$ for $x \in \partial\Omega$. Therefore, defining $E_k = \{x \in E : v(x) < -\frac{1}{k}\}$, we have that E_k is a sequence of smooth strictly convex sets. The continuity of all the quantities involved and the weak convergence of the curvature measures with respect to the Hausdorff convergence imply the result for a generic convex set. □

Remark 3.7 We remark that Theorem 1.1 implies the classical Aleksandrov-Fenchel inequality (21), when $j = n-1$ and $i = n-2$. Indeed, by taking $\beta = 0$, Theorem 1.1 gives

$$\inf_{x_0} \int_{\partial E} |x-x_0|^2 G_{\partial E}\, d\mathcal{H}^{n-1} \geq \frac{W^2_{n-1}(E)}{\omega_n}. \tag{28}$$

Moreover, for $\beta \geq \beta(n)$, we have

$$\inf_{x_0} \int_{\partial E} |x-x_0|^2 G_{\partial E}\, d\mathcal{H}^{n-1} \leq \frac{1+\beta}{\omega_n} W_{n-1}(E)^2 - \beta W_{n-2}(E),$$

which, together with (28), gives

$$\frac{W^2_{n-1}(E)}{\omega_n} \geq W_{n-2}(E).$$

In particular, when $n = 2$, the above inequality is precisely the isoperimetric inequality.

3.4 Special cases: two and three dimension

In low dimensions,

The functional $G_\beta(\cdot)$ has an immediate interpretation. We start with the planar case.

Corollary 3.8 *Let $\beta > 0$ and $E \subset \mathbb{R}^2$ an open, bounded convex set. For $\beta \leq 1$ it holds*

$$\int_{\mathbb{R}^2} |x|^2 \, d\mu_E^G + 2\beta|E| \geq (1+\beta)\, \frac{P^2(E)}{2\pi}, \tag{29}$$

while, for $\beta \geq \frac{5}{3}$, we have

$$\inf_{x_0 \in \mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mu_E^G + 2\beta|E| \leq (1+\beta)\, \frac{P^2(E)}{2\pi}. \tag{30}$$

The equality sign in (29) and (30) holds if and only if $E = B_r(\overline{x})$ for some $r > 0$ and $\overline{x} \in \mathbb{R}^2$. Moreover, the thresholds $\beta = \frac{5}{3}$ and $\beta = 1$ are sharp.

Proof We just need to prove the sharpness of the thresholds.
Sharpness of the maximality threshold: For $\varepsilon > 0$ small enough, we denote by $\mathcal{E}$ the ellipse of semiaxes $1+\varepsilon$ and $1-\varepsilon$, whose boundary is parameterized, for $\theta \in [0, 2\pi]$, by

$$\begin{cases} x(\theta) = (1+\varepsilon)\cos\theta \\ y(\theta) = (1-\varepsilon)\sin\theta. \end{cases}$$

The measure and the curvature of $\mathcal{E}$ are given respectively by

$$|\mathcal{E}| = \pi(1-\varepsilon^2),$$
$$H_{\partial\mathcal{E}} = \frac{1-\varepsilon^2}{[(1+\varepsilon)^2\sin^2\theta + (1-\varepsilon)^2\cos^2\theta]^{\frac{3}{2}}}.$$

Regarding the perimeter of $\mathcal{E}$, we use Taylor expansion, with respect to the parameter ε, to find

$$\begin{aligned} P(\mathcal{E}) &= \int_0^{2\pi} \sqrt{(1+\varepsilon)^2\sin^2\theta + (1-\varepsilon)^2\cos^2\theta}\, d\theta = \int_0^{2\pi} \sqrt{1 - 2\cos 2\theta\varepsilon + \varepsilon^2}\, d\theta \\ &= \int_0^{2\pi} \left(1 + \frac{1}{2}(-2\cos 2\theta\varepsilon + \varepsilon^2) - \frac{1}{8}(-2\cos 2\theta\varepsilon + \varepsilon^2)^2 + o(\varepsilon^2)\right) d\theta \\ &= 2\pi + \frac{\pi}{2}\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

Therefore, we get

$$\frac{P^2(\mathcal{E})}{2\pi} = 2\pi + \pi\varepsilon^2 + o(\varepsilon^2).$$

Moreover, we compute

$$\int_{\partial\mathcal{E}} H_{\partial E}\,|x|^2\,d\mathcal{H}^1 = (1-\varepsilon^2)\int_0^{2\pi} \frac{(1+\varepsilon)^2\cos^2\theta + (1-\varepsilon)^2\sin^2\theta}{(1+\varepsilon)^2\sin^2\theta + (1-\varepsilon)^2\cos^2\theta}\,d\theta$$
$$= (1-\varepsilon^2)\int_0^{2\pi} \frac{1+2\cos 2\theta\varepsilon+\varepsilon^2}{1-2\cos 2\theta\varepsilon+\varepsilon^2}\,d\theta$$
$$= (1-\varepsilon^2)\int_0^{2\pi} (1+2\cos 2\theta\varepsilon+\varepsilon^2)(1+2\cos 2\theta\varepsilon+(4\cos^2 2\theta-1)\varepsilon^2+o(\varepsilon^2))\,d\theta$$
$$= (1-\varepsilon^2)\int_0^{2\pi} (1+8\cos^2 2\theta\varepsilon^2+o(\varepsilon^2))\,d\theta = 2\pi+6\pi\varepsilon^2+o(\varepsilon^2).$$

If B is the ball centered at the origin such that $P(B) = P(\mathcal{E})$, then

$$\int_{\partial B} H_{\partial B}|x|^2\,d\mathcal{H}^1 + 2\beta|B| - (1+\beta)\,\frac{P^2(B)}{2\pi} = 0,$$

while, for $\varepsilon > 0$ small enough and $\beta < 5/3$, it holds

$$\int_{\partial\mathcal{E}} H_{\partial\mathcal{E}}|x|^2\,d\mathcal{H}^1 + 2\beta|\mathcal{E}| - (1+\beta)\,\frac{P^2(\mathcal{E})}{2\pi}$$
$$= 2\pi+6\pi\varepsilon^2+2\pi\beta(1-\varepsilon^2)$$
$$-2\pi(1+\beta)(\varepsilon^2/2+1)+o(\varepsilon^2) = \pi(5-3\beta)\varepsilon^2+o(\varepsilon^2) > 0,$$

proving the sharpness of the threshold $\beta = 5/3$.

Sharpness of the minimality threshold. The sharpness of $\beta = 1$ comes directly from the proof. In fact, for $\beta > 1$, let $k_0^2 = \lfloor(1+\beta)/(\beta-1)\rfloor + 1$, where $\lfloor\cdot\rfloor$ denotes the integer part, and set $p_1(\theta) = 1 + \frac{\varepsilon}{k_0^2}\sin(k_0\theta) > 0$. It is immediate to check that, for ε small enough, $p_1 + \ddot{p}_1 > 0$. Thus, let E_1 the convex set having p_1 as support function. We have

$$\int_{\partial E_1} H_{\partial E_1}|x|^2\,d\mathcal{H}^1 + 2\beta|E_1| - (1+\beta)\,\frac{P^2(E_1)}{2\pi} = ((1+\beta)+(1-\beta)k_0^2)\pi\varepsilon^2 < 0.$$

□

The next proposition shows that the functional $\mathscr{H}(\cdot)$ in two dimensions does not attain a maximizer among open convex sets.

Proposition 3.9 *Let $l > 0$ and $E \subset \mathbb{R}^2$ be a convex set with $P(E) = l$. It holds*

$$\mathscr{H}(E) < \frac{\pi}{8}l^2 \tag{31}$$

and the inequality is sharp.

Proof Let E be an open bounded convex set and let R_E denote its circumradius. Since $\mathscr{H}$ is invariant under translations, we may assume without loss of generality that the smallest ball containing E is centered at the origin. We have

$$\mathscr{H}(E) \leq \int_{\mathbb{R}^2} |x|^2 \, d\mu_E^G \leq 2\pi R_E^2 < \frac{\pi}{8} P^2(E) = \frac{\pi}{8} l^2, \tag{32}$$

where we used inequality (10): $R_E < P(E)/4$. To prove the sharpness of (31), we now construct a sequence of convex sets E_n with $P(E_n) = l$ and $\mathscr{H}(E_n) \to \pi l^2/8$. Let $\alpha \in (0, \pi)$ and consider the rhombus centered at the origin $R_{l,\alpha}$ of perimeter l, with vertices P_1, P_2, $P_3 = -P_1$ and $P_4 = -P_2$, and angles $\alpha = \alpha_1$ at the vertices P_1 (and $P_3 = -P_1$) and $\alpha_2 = \pi - \alpha_1$ at vertices P_2 (and $P_4 = -P_2$). The curvature measure $\mu_{R_{l,\alpha}}$ associated with $R_{l,\alpha}$ is

$$\mu_{R_{l,\alpha}} = \sum_{i=1}^{4} (\pi - \alpha_i) \delta_{P_i},$$

where δ_{P_i} is the Dirac delta centered at P_i. Therefore, by symmetry

$$\mathscr{H}(R_{l,\alpha}) = \sum_{i=1}^{4} |P_i|^2 (\pi - \alpha_i)$$

$$= 2(|P_1|^2(\pi - \alpha) + (\pi - \alpha_2)|P_2|^2 = 2(|P_1|^2(\pi - \alpha) + \alpha|P_2|^2).$$

Since the rhombus has perimeter equal to l, we get

$$|P_1| = \frac{l}{4} \cos \frac{\alpha}{2} \quad \text{and} \quad |P_2| = \frac{l}{4} \sin \frac{\alpha}{2},$$

which gives

$$\mathscr{H}(R_{l,\alpha}) = \frac{l^2}{8} \left((\pi - \alpha) \cos^2 \frac{\alpha}{2} + \alpha \sin^2 \frac{\alpha}{2} \right) := \frac{l^2}{8} f(\alpha).$$

It is straightforward to show that $\alpha = \frac{\pi}{2}$ is minimum of f when $\alpha \in [0, \pi]$ and $f(\alpha) \leq f(0) = f(\pi) = \pi$. Therefore, we have constructed a sequence of sets $R_{l,\alpha}$ such that $\mathscr{H}(R_l) \to \frac{\pi}{8} l^2$, as $\alpha \to 0$, which, together with (32), proves that the upper bound is sharp and is not attained. ◻

To conclude this section, we state the result in dimension three, because of the natural interpretation of the geometric quantities involved in (4) and (5).

Corollary 3.10 *Let $\beta > 0$ and $E \subset \mathbb{R}^2$ an open, bounded convex set. For $\beta \leq 2$ it holds*

$$\int_{\mathbb{R}^3} |x|^2 \, d\mu_E^G + \beta P(E) \geq (1 + \beta) \frac{(\int_{\mathbb{R}^3} d\mu_E^H)^2}{4\pi^2},$$

while, for $\beta \geq \frac{7}{2}$, we have

$$\int_{\mathbb{R}^3} |x|^2 \, d\mu_E^G + \beta P(E) \leq (1+\beta) \frac{(\int_{\mathbb{R}^3} d\mu_E^H)^2}{4\pi^2}.$$

Moreover, equality holds if and only if $E = B_r(\overline{x})$ for some $r > 0$ and $\overline{x} \in \mathbb{R}^3$.

3.5 Quantitative version

In this subsection, we prove a quantitative version of the inequality contained in Theorem 1.1. To this aim, we recall the scaling invariant functional introduced in (6). The proof of Theorem 1.1 leads to a quantitative inequality for $\mathscr{I}(\cdot)$.

Proof of Theorem 1.2 We prove only (7), since the proof of the second statement follows without modifications. We use (27) and Lemma 2.8, applied to the function $\tilde{h} = h - \frac{1}{n\omega_n} \int_{\mathbb{S}^{n-1}} h \, d\mathcal{H}^{n-1}$, to find

$$\begin{aligned}\mathscr{I}_\beta(E) &\leq (\beta(n) - \beta) \|\nabla_{\mathbb{S}^{n-1}} h\|_{L^2(\mathbb{S}^{n-1})} \\ &\leq C(n)(\beta(n) - \beta)(n-1) g(\|\tilde{h}\|_{L^\infty}) = C(n)(\beta(n) - \beta)(n-1) g(\tilde{\mathcal{A}}_{\mathcal{H}}(E)).\end{aligned}$$

□

4 Boundary momentum inequalities

The aim of this section is to provide upper bounds for the boundary momentum. The next proposition is essentially Theorem 1.3 in a very particular case. In fact, we now prove that the thesis Theorem 1.3 is true when the set E is assumed to be a kind of combination of polygons touching in a finite number of points. We refer to Figure 1 for a better visualization of the assumptions.

Proposition 4.1 *Let $\ell, m \geq 0$ a natural number, $E_1, \dots E_m$ and $F_1 \dots F_\ell$ open polygons such that*

- *$F_j \subset \bigcup_{i=1}^m E_i$ for all $j \in \{1, \dots \ell\}$ and $F_i \cap F_j = \emptyset$ for $1 \leq i < j \leq \ell$;*
- *$E_i \cap E_j = \emptyset$ for all $1 \leq i < j \leq m$;*
- *$\min_{k \neq i}\{\text{dist}(\partial E_i, \partial E_k)\} = 0$ for all $i \in \{1, \dots, m\}$, and $\mathcal{H}^1(\partial E_i \cap \partial E_k) = 0$ for all $1 \leq i < k \leq m$;*
- *$\min_i\{\text{dist}(\partial E_i, \partial F_j)\} = 0$ for all $j \in \{1, \dots, \ell\}$, and $\mathcal{H}^1(\partial E_i \cap \partial F_j) = 0$ for all $i \in \{1, \dots m\}$ and $j \in \{1, \dots, \ell\}$;*
- *$\mathcal{H}^1(\partial F_i \cap \partial F_j) = 0$ for all $i \neq j$, $i, j \in \{1, \dots, \ell\}$.*

Let $E = \bigcup_{i=1}^m E_i \setminus \overline{\bigcup_{j=1}^\ell F_j}$. Then

$$\inf_{x_0 \in \mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^1 \leq \frac{P(E)^3}{(2\pi)^2}. \tag{33}$$

Inequality (33) *also holds if E is a simply connected open set with Lipschitz boundary. Moreover, the equality sign holds if and only $E = B_r(x_1)$, for some $r > 0$ and $x_1 \in \mathbb{R}^2$.*

Proof We follow the proof in [19]. Without loss of generality, we assume $x_E = 0$. Therefore,

$$\inf_{x_0 \in \mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^1 = \int_{\partial E} |x|^2 \, d\mathcal{H}^1.$$

We parameterize ∂E by the arc length. Hence, for $s \in [0, P(E)]$, we get

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^1 = \int_0^{P(E)} (x(s)^2 + y(s)^2) ds.$$

We now use the Fourier decomposition of periodic functions $x(s)$, $y(s)$ in the interval $[0, P(E)]$ to write

$$x(s) = a_0 + \sum_{k \geq 1} a_k \cos\left(\frac{2\pi}{P(E)} ks\right) + b_k \sin\left(\frac{2\pi}{P(E)} ks\right)$$

and

$$y(s) = c_0 + \sum_{k \geq 1} c_k \cos\left(\frac{2\pi}{P(E)} ks\right) + d_k \sin\left(\frac{2\pi}{P(E)} ks\right),$$

with

$$a_k = \sqrt{\frac{2}{P(E)}} \int_0^{P(E)} x(s) \cos\left(\frac{2\pi}{P(E)} ks\right) ds,$$

$$b_k = \sqrt{\frac{2}{P(E)}} \int_0^{P(E)} x(s) \sin\left(\frac{2\pi}{P(E)} ks\right) ds.$$

The same expression with $y(s)$ in place of $x(s)$ gives the formula of c_k and d_k. Note that the assumption that the centroid of E is at the origin implies $a_0 = c_0 = 0$. Next, we note that

$$\int_0^{P(E)} x(s)^2 + y(s)^2 ds = \sum_{k \geq 1} a_k^2 + b_k^2 + c_k^2 + d_k^2 \leq \sum_{k \geq 1} k^2 (a_k^2 + b_k^2 + c_k^2 + d_k^2)$$
$$= \frac{P(E)^2}{(2\pi)^2} \int_0^{P(E)} \dot{x}(s)^2 + \dot{y}(s)^2 ds = \frac{P(E)^3}{(2\pi)^2}.$$

Moreover, equality holds if and only if the coefficients a_k, b_k, c_k, d_k are equal to zero for $k > 1$, hence if and only if

$$x(s) = a_1 \cos\left(\frac{2\pi}{P(E)} s\right) + b_1 \sin\left(\frac{2\pi}{P(E)} s\right) = \sqrt{a_1^2 + b_1^2} \cos\left(\frac{2\pi}{P(E)} s - \theta_1\right)$$

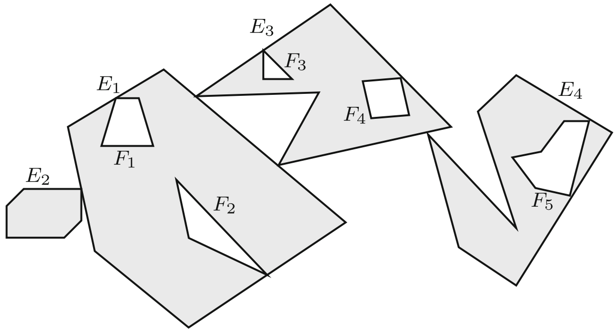

Fig. 1 The set E (in grey) is the set difference between the union of a finite number of open disjoint polygons E_i, for which each of them touches at least another one, and the union of a finite number of open disjoint polygons F_j, which are contained in $\bigcup E_i$ and must touch its boundary in a finite number of points

and

$$y(s) = c_1 \cos\left(\frac{2\pi}{P(E)}s\right) + d_1 \sin\left(\frac{2\pi}{P(E)}s\right) = \sqrt{c_1^2 + d_1^2}\sin\left(\frac{2\pi}{P(E)}s + \theta_2\right),$$

with $\theta_1 = \arcsin \frac{a_1}{\sqrt{a_1^2+b_1^2}}$ and $\theta_2 = \arccos \frac{c_1}{\sqrt{c_1^2+d_1^2}}$. Eventually, since s is the arc length, we must have $\sqrt{a_1^2 + b_1^2} = \sqrt{c_1^2 + d_1^2}$, while the angles $\frac{2\pi}{P(E)}s + \theta_2$ and $\frac{2\pi}{P(E)}s - \theta_1$ must be the same for any s. This immediately implies that E is a ball. $\square$

Before proving Theorem 1.3, we give the following definition.

Definition 4.2 Let $E \subset \mathbb{R}^2$ be an indecomposable set of finite perimeter and let $\partial^* E = C_0 \cup \bigcup_{i \in I} C_i$, where the parameter set I and the Jordan curves C_i are the ones provided by Theorem 2.3. For $k \in \mathbb{N}$, we say that E has k interior separate holes if $E = \tilde{G}_0 \setminus \bigcup_{i=1}^k \tilde{G}_i$, $|\tilde{G}_i| > 0$, $\mathrm{dist}(G_i, G_j) > 0$ for any $0 < i < j \le k$, $\mathrm{dist}(G_0^c, G_i) > 0$ and the boundary of G_i can be represented by a continuous curve.

Proof of Theorem 1.3 Assume that x_E is at the origin. By Theorem 2.3, the reduced boundary of E can be written as the union of at most countable many Jordan curves, i.e.,

$$\partial^* E = C_0 \cup \bigcup_{i \in I} C_i.$$

For $i = 0$ or $i \in I$, we set $G_i = \mathrm{int}(C_i)$ and we denote by x_i the centroid of G_i.
Step one: G_i is a polygon for all $i \ge 0$. We argue by induction on the number of interior holes. We start with the basis of the induction, that is, the set E contains one separate interior hole, which means $E = \tilde{G}_0 \setminus \tilde{G}_1$. With an abuse of notation, we also use the name C_i for the curves obtained as the boundary of the interior hole $\tilde{G}_i$.
Case one: $x_0 = 0$. In this case, we have $x_1 = 0$, since $x_E = 0$. In fact, it holds

$$0 = \int_{\partial E} x \, d\mathcal{H}^1 = \int_{C_0} x \, d\mathcal{H}^1 + \int_{C_1} x \, d\mathcal{H}^1.$$

Since C_0, C_1 are Lipschitz curves, we are in position to apply Proposition 4.1 to the sets G_0 and G_i, to get

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^1 = \int_{C_0} |x|^2 \, d\mathcal{H}^1 + \int_{C_1} |x|^2 \, d\mathcal{H}^1 \le \frac{\mathcal{H}^1(C_0)^3 + \mathcal{H}^1(C_1)^3}{(2\pi)^2} \le \frac{P(E)^3}{(2\pi)^2},$$

where we used the inequality $a^3 + b^3 \le (a+b)^3$ and the fact $P(E) = \mathcal{H}^1(C_0) + \mathcal{H}^1(C_1)$.

Case two: $x_0 \neq 0$. In this case we observe that, for $i = 0, 1$, it holds

$$\int_{C_i} |x|^2 \, d\mathcal{H}^1 = \int_{C_i} |x - x_i|^2 \, d\mathcal{H}^1 + |x_i|^2 \mathcal{H}^1(C_i).$$

Therefore,

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^1 = \sum_{i=0,1} \int_{C_i} |x - x_i|^2 \, d\mathcal{H}^1 + |x_i|^2 \mathcal{H}^1(C_i).$$

Now we show that we can slightly rotate the interior hole in such a way that C_0 and C_1 do not have any parallel side. Indeed, since C_1 is a polygon, the set $\{N \in \mathbb{S}^1 : \mathcal{H}^1(\{x \in C_1 : \nu_{C_1}(x) = N\}) > 0\}$ is at most countable. Observe that, if we rotate C_1 about its centroid, the value of the functional does not change. In fact, let D_1 the polygon such that $\partial D_1 = x_1 + O(C_1 - x_1)$, where O is a rotation matrix chosen such that C_0 and ∂D_1 do not have parallels sides and $\mathrm{dist}(G_0^c, D_1) > 0$. We have $x_{D_1} = x_1$ and

$$\begin{aligned} \int_{\partial D_1} |x|^2 \, d\mathcal{H}^1 &= \int_{\partial D_1} |x - x_{D_1}|^2 \, d\mathcal{H}^1 + |x_{D_1}|^2 \mathcal{H}^1(\partial D_1) \\ &= \int_{C_1} |x - x_1|^2 \, d\mathcal{H}^1 + |x_1|^2 \mathcal{H}^1(C_1) = \int_{C_1} |x|^2 \, d\mathcal{H}^1. \end{aligned}$$

Hence, setting $E_1 = \tilde{G}_0 \setminus D_1$, we have $x_{E_1} = 0$ and

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^1 = \int_{\partial E_1} |x|^2 \, d\mathcal{H}^1.$$

Since $x_{E_1} = x_E = 0$, we have $\mathcal{H}^1(C_0)x_0 = -\mathcal{H}^1(\partial D_1)x_1$. Moreover, by using that $\mathrm{dist}(\tilde{G}_0, \tilde{G}_1) > 0$, for $t > 0$ small we consider the curves

$$C_{0,t} = C_0 + t x_0 \quad \text{and} \quad C_{1,t} = \partial D_1 + t x_1$$

and set $\tilde{G}_{i,t} = \mathrm{int}(C_{i,t})$ and $E_t = \tilde{G}_{0,t} \setminus \tilde{G}_{1,t}$. Let $\bar{t} = \sup\{t : \mathrm{dist}(C_{0,t}, C_{1,t}) > 0\}$. Consider the set

$$F = \mathrm{int}(C_{0,\bar{t}}) \setminus \mathrm{int}(C_{1,\bar{t}}).$$

Notice that F is such that its boundary is a Lipschitz curve satisfying $x_F = 0$ and $P(F) = P(E)$. Using $|x_{C_{0,\bar{i}}}| \geq |x_0|$ and $|x_{C_{1,\bar{i}}}| \geq |x_1|$, we infer

$$\int_{\partial F} |x|^2 \, d\mathcal{H}^1 > \int_{\partial E} |x|^2 \, d\mathcal{H}^1$$

and finally

$$\int_{\partial E} |x|^2 \, d\mathcal{H}^1 = \int_{\partial E_1} |x|^2 \, d\mathcal{H}^1 \leq \int_{\partial F} |x|^2 \, d\mathcal{H}^1 \leq \frac{P(F)^3}{(2\pi)^2} = \frac{P(E)^3}{(2\pi)^2}.$$

This proves the basis of the induction. The induction argument then follows by arguing as above and we briefly explain it: if the centroid of all the curves is at the origin, we apply (33) to each set $\tilde{G}_i$ and by using the inequality $\sum_i P(\tilde{G}_i)^3 \leq (\sum_i P(\tilde{G}_i))^3$ we are done. If there is $\bar{i} \in \{0, 1, \ldots, k\}$ such that the centroid of $\tilde{G}_{\bar{i}}$ is not at the origin, then we eventually rotate the curve C_i so that it does not have any side parallel with the other curves and then use the translation argument provided before, so that the value of the functional is increased, until they touch. With this procedure, we have decreased by (at least) one the number of interior holes and we use the inductive assumption to get the result.

We are left to show the validity of (33) when the number of holes is countable. To this aim, we observe that $P(E) = l$ implies $\operatorname{diam}(E) \leq \frac{l}{2}$, as we are in the plane. Since $x_E = 0$ we infer that

$$|x_i| = \frac{1}{\mathcal{H}^1(C_i)} \left| \int_{C_i} x \, d\mathcal{H}^1 \right| \leq \frac{l}{2}.$$

Moreover, given $\varepsilon > 0$, there exists $\nu \in \mathbb{N}$ such that $\sum_{i>\nu} \mathcal{H}^1(C_i) < \varepsilon$. Set $F_1 = \operatorname{int}(C_0) \setminus \overline{\bigcup_{i=1}^{\nu} \operatorname{int}(C_i)}$ and $T = \bigcup_{i>\nu} \operatorname{int}(C_i)$. We have $E = F_1 \setminus \overline{T}$ and

$$\int_{\partial T} |x|^2 \, d\mathcal{H}^1 \leq \frac{l^2}{4}\varepsilon.$$

Finally, we get

$$\begin{aligned} \inf_{x_0 \in \mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^1 &\leq \int_{\partial E} |x - x_{F_1}|^2 \, d\mathcal{H}^1 \\ &= \int_{\partial F_1} |x - x_{F_1}|^2 \, d\mathcal{H}^1 + \int_{\partial T} |x - x_{F_1}|^2 \, d\mathcal{H}^1 \\ &\leq \frac{P(F_1)^3}{2\pi} + l^2 \varepsilon \leq \frac{P(E)^3}{2\pi} + l^2 \varepsilon. \end{aligned}$$

As the left hand side does not depend on ε, sending ε to zero gives (33).

Step two: E is a indecomposable set of finite perimeter.

This follows by approximation, as the class of polygonal sets is dense in the class of sets of finite perimeter (see [23, Theorem 13.8 and Remark 13.13]) and by semi-continuity for the boundary momentum (see Lemma 3.5 in [18]). □

Inequality (33) does not hold in higher dimensions, even when the analysis is restricted to convex set. This is illustrated by the following counterexample.

Counterexample 4.3 For any $L > 0$, we consider the cylinder $C = B^2_\varepsilon \times [-L/2, L/2] \subset \mathbb{R}^3$, where B^2_ε is the two-dimensional ball centered at the origin of radius ε. Given $\varepsilon > 0$, we choose $L = L(\varepsilon)$ so that $P(C) = 2\pi L\varepsilon + 2\pi\varepsilon^2 = 2\pi$, hence $L = \varepsilon^{-1} - \varepsilon$. The boundary of the cylinder C can be decomposed as $\partial C = \partial C_l \cup \partial C_+ \cup \partial C_-$, where ∂C_l is the lateral surface of ∂C and $\partial C_\pm = B_\varepsilon(0, 0, \pm L/2)$ the basis of C. Using cylindrical coordinates, we parametrize the lateral surface of ∂C as

$$\begin{cases} x_1 = \varepsilon \cos\theta \\ x_2 = \varepsilon \sin\theta \\ x_3 = z, \end{cases}$$

with $(\theta, z) \in [0, 2\pi] \times [-L/2, L/2]$. Recalling that $L = \varepsilon^{-1} - \varepsilon$, as $\varepsilon \to 0^+$, we get

$$\int_{\partial C_l} |x|^2 \, d\mathcal{H}^{n-1} = \int_0^{2\pi} \int_{-L/2}^{L/2} \varepsilon^3 + \varepsilon z^2 \, dz \, d\theta = 2\pi \left[\varepsilon^3 L + \varepsilon L^3/3 \right] \to +\infty.$$

The basis can be parameterized as

$$\begin{cases} x_1 = r \cos\theta \\ x_2 = r \sin\theta \\ x_3 = \pm L/2, \end{cases}$$

with $(r, \theta) \in [0, \varepsilon] \times [0, 2\pi]$. We have

$$\int_{\partial C_\pm} |x|^2 d\mathcal{H}^{n-1} = \int_0^{2\pi} \int_0^{\varepsilon} r^3 + rL^2/4 \, dr \, d\theta = \frac{\pi}{2} \left[\varepsilon^4 + L^2\varepsilon^2 \right] \to \frac{\pi}{2},$$

as $\varepsilon \to 0$. In conclusion, as $\varepsilon \to 0$, we find

$$\int_{\partial C} |x|^2 \, d\mathcal{H}^{n-1} \to \infty.$$

This counterexample hints that some modifications of the functional are in order. Otherwise, there is no hope even of proving that the functional is bounded from above. Thus, we study the functional $\mathcal{F}(\cdot)$, defined in (9). We recall that $\mathcal{F}(\cdot)$ is defined as

$$\mathcal{F}(E) = \frac{|E|^{(n-2)(n+1)}}{P(E)^{(n-1)(n+1)}} \inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^{n-1}.$$

As an immediate consequence of its definition, the functional $\mathcal{F}(\cdot)$ is invariant under translation and dilation. The choice of exponents comes from the following lemma (see [12, Lemma 4.1]).

Lemma 4.4 *Let $E \subset \mathbb{R}^n$ be any open convex set. Then, we have*

$$\operatorname{diam}(E) \leq c(n)\frac{P(E)^{n-1}}{|E|^{n-2}}, \tag{34}$$

where $c = c(n)$ is a positive dimensional constant.

As an application of Lemma 4.4, we prove that $\mathcal{F} : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ admits a maximizer.

Proposition 4.5 *Let $n \geq 3$. There exists a convex set K such that*

$$\sup_{E \text{ convex}} \mathcal{F}(E) = \mathcal{F}(K).$$

Proof We first show that the functional $\mathcal{F}$ is bounded among convex sets. Since the functional is scaling and translation invariant, we may assume that $|E| = 1$ and

$$\inf_{x_0 \in \mathbb{R}^n} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^{n-1} = \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1},$$

i.e., fixing the centroid of the set E at the origin. Now we use Lemma 4.4 to infer

$$\mathcal{F}(E) \leq \frac{\operatorname{diam}(E)^2 P(E)}{4P(E)^{(n-1)(n+1)}} \leq \frac{C(n)}{P(E)^{n^2-2n}}. \tag{35}$$

The isoperimetric inequality $P(E) \geq (n\omega_n)^{\frac{1}{n}} |E|^{\frac{n-1}{n}}$ implies that there exists a constant, depending only on the dimension, such that

$$\mathcal{F}(E) \leq C,$$

for all convex sets E. Let $\{E_k\}_{k \in \mathbb{N}}$ be a a maximizing sequence, i.e., a sequence of convex sets such that

$$\mathcal{F}(E_k) \geq \sup_{E \text{ convex}} \mathcal{F}(E) - \frac{1}{k}.$$

Again, we assume that the sets E_k satisfy $|E_k| = 1$ and have centroids at the origin. We apply the inequality (35) to E_k to infer

$$\sup_{E \text{ convex}} \mathcal{F}(E) - \frac{1}{k} \leq \frac{C(n)}{P(E_k)^{n^2-2n}},$$

which immediately implies that E_k is a sequence of convex sets with equibounded perimeters and, by (34), the sets E_k are equibounded. This implies the existence of a set E such that, up to a subsequence, $|E \triangle E_k| \to 0$ and $P(E_k) \to P(E)$. Moreover, this also implies the convergence of the Green measure relative to E_k to the Green measure relative to E (see [2, Prop. 1.80]). Hence, we get

$$\int_{\partial E_k} |x|^2 \, d\mathcal{H}^{n-1} \to \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1},$$

which is the continuity of the second momentum with respect to the L^1-convergence of convex sets. As a consequence, we obtain

$$\sup_{G \text{ convex}} \mathcal{F}(G) \leq \lim_{k \to +\infty} \mathcal{F}(E_k) = \mathcal{F}(E),$$

which proves the existence of a maximizer among convex sets. □

We now provide a Fuglede-type argument, which guarantees that in any dimension balls centered at the origin are stable local maximizers of $\mathcal{F}(\cdot)$ among open convex sets. Of course, this is the main tool to prove the stability result in dimension two.

Proposition 4.6 *Let $n \geq 2$. There exists $\varepsilon_0 > 0$ such that if E is a nearly spherical set as in Definition 2.7 with centroid at the origin and if $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon_0$, then*

$$\frac{1}{(n\omega_n)^{n^2-2}} - \frac{|E|^{(n+1)(n-2)} \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1}}{P(E)^{n^2-1}} \geq C(n) \|\nabla h\|^2_{L^2(\mathbb{S}^{n-1})}.$$

Proof Let E a convex set with $|E| = 1$ and having centroid at the origin. Let u the height function with respect to the unit ball, i.e., $\partial E = \{y = x(1+u(x)), \ x \in \mathbb{S}^{n-1}\}$. Looking at (13), the three quantities involved in $\mathcal{F}(E)$ can be expressed in terms of the height function as

$$\begin{aligned}
\int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1} &= \int_{\mathbb{S}^{n-1}} (1+h)^n \sqrt{(1+u)^2 + |\nabla u|^2} \, d\mathcal{H}^{n-1} \\
&= n\omega_n + (n+1) \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + \frac{n(n+1)}{2} \int_{\mathbb{S}^{n-1}} h^2 \, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}), \\
P(E) &= \int_{\mathbb{S}^{n-1}} (1+u)^{n-2} \sqrt{(1+u)^2 + |\nabla u|^2} \, d\mathcal{H}^{n-1} \\
&= n\omega_n + (n-1) \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + \frac{(n-1)(n-2)}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \\
&\quad + \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}),
\end{aligned} \tag{36}$$

and

$$
\begin{aligned}
n|E| &= \int_{\mathbb{S}^{n-1}} (1+u)^n \, d\mathcal{H}^{n-1} \\
&= n\omega_n + n \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + \frac{n(n-1)}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}).
\end{aligned} \tag{37}
$$

Since the functional is scaling invariant, we assume without loss of generality that $|E| = \omega_n$. The assumption $|E| = \omega_n$, together with (37), gives

$$
\int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^n = -\frac{n-1}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^n + o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}). \tag{38}
$$

Now we use (36) and (38) to infer

$$
\begin{aligned}
\frac{P(E)^{n^2-1}}{(n\omega_n)^{n^2-1}} &= 1 + (n^2-1)\Big(\frac{n-1}{n\omega_n} \int_{\mathbb{S}^{n-1}} u \, d\mathcal{H}^{n-1} + \frac{(n-1)(n-2)}{2n\omega_n} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} \\
&+ \frac{1}{2n\omega_n} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1}\Big) + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}) \\
=& 1 + (n^2-1)\left(-\frac{n-1}{2n\omega_n} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} + \frac{1}{2n\omega_n} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1}\right) + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}).
\end{aligned} \tag{39}
$$

Hence, using again (37), the boundary momentum becomes

$$
\int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1} = n\omega_n + \frac{(n+1)}{2} \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} + \frac{1}{2} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}). \tag{40}
$$

Since $|E| = \omega_n$, we just need to estimate

$$
\begin{aligned}
\frac{\mathcal{F}(E) - \mathcal{F}(B)}{\omega_n^{(n-2)(n+1)}} &= \frac{\int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1}}{P(E)^{n^2-1}} - \frac{1}{(n\omega_n)^{n^2-2}} \\
&= \frac{(n\omega_n)^{n^2-2} \int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1} - P(E)^{n^2-1}}{(n\omega_n)^{n^2-2} P(E)^{n^2-1}}.
\end{aligned}
$$

We use (39) and (40) to get

$$
\begin{aligned}
\int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1} - \frac{P(E)^{n^2-1}}{(n\omega_n)^{n^2-2}} &= \left(\frac{n+1}{2} + \frac{(n-1)^2(n+1)}{2}\right) \int_{\mathbb{S}^{n-1}} u^2 \, d\mathcal{H}^{n-1} \\
&- \frac{(n^2-1)-1}{2} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}).
\end{aligned} \tag{41}
$$

Now, we exploit the assumption that the centroid of E coincides with the origin. We decompose h in terms of the spherical harmonics, already defined in the previous section. Note that harmonic polynomials corresponding to $k = 0$ are constants, while

$Y_{1,i} = x_i$ for $i \le n$. Thus, (38) gives

$$|a_0|^2 = o(\|u\|^2_{L^2(\mathbb{S}^{n-1})}),$$

while, exploiting

$$0 = \int_{\partial E} x_i \, d\mathcal{H}^{n-1} = \int_{\mathbb{S}^{n-1}} x_i (1+h)^{n-1} \sqrt{(1+u)^2 + |\nabla u|^2} \, d\mathcal{H}^{n-1},$$

we get a smallness condition on the coefficients corresponding to $k = 1$, which reads as

$$|a_{1,i}|^2 = o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}).$$

Therefore, we arrive at

$$\begin{aligned} &2n\|u\|^2_{L^2(\mathbb{S}^{n-1})} \\ &= o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}) + 2n \sum_{k=2}^{\infty} \sum_{i=1}^{N_k} a^2_{k,i} \le \sum_{k=2}^{\infty} \sum_{i=1}^{N_k} k(n+k-2) a^2_{k,i} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}) \\ &= \|\nabla u\|^2_{L^2(\mathbb{S}^{n-1})} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}). \end{aligned} \tag{42}$$

A combination of (41) and (42) gives

$$\begin{aligned} &\int_{\partial E} |x|^2 \, d\mathcal{H}^{n-1} - \frac{P(E)^{n^2-1}}{(n\omega_n)^{n^2-2}} \\ &\le -\frac{n^3+n^2-4n-2}{4n} \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|_{W^{1,2}(\mathbb{S}^{n-1})}) \\ &= -C(n) \int_{\mathbb{S}^{n-1}} |\nabla u|^2 \, d\mathcal{H}^{n-1} + o(\|u\|^2_{W^{1,2}(\mathbb{S}^{n-1})}), \end{aligned}$$

which yields the result. □

Note that in dimension two, we have $C_2 = \frac{1}{4}$. We now prove the quantitative result stated in Theorem 1.4.

Proof of Theorem 1.4 Since the functional is invariant under translations and dilations, we assume that the centroid of E is at the origin and $P(E) = P(B)$. Hence, we have

$$\partial E = \{x(1+u(x)), \ \ x \in \mathbb{S}^{n-1}\}.$$

Let ρ be such that $|B_\rho| = |E|$. Since E and B_ρ have the same measure, we have

$$\frac{1}{n} \int_{\mathbb{S}^{n-1}} (1+u)^n d\mathcal{H}^{n-1} = \omega_n \rho^n. \tag{43}$$

Formula (43) leads to

$$1 - \rho^n = -\frac{1}{\omega_n} \int_{\mathbb{S}^{n-1}} \sum_{k=1}^{n} \binom{n}{k} u^k \, d\mathcal{H}^{n-1},$$

which implies $|1 - \rho| < C(n)\|u\|_\infty$. Let $v = \rho(1+u)^n - 1$. Then, it holds

$$C_3\|v\|_\infty \leq \|u\|_\infty \leq C_4\|v\|_\infty,$$

where C_3 and C_4 are constant depending only on the dimension. Moreover, the Leibniz rule yields $D_\tau v = n\rho^n(1+u)^{n-1} D_\tau u$ and thus

$$C_5\|D_\tau u\|_2 \leq \|D_\tau v\|_2 \leq C_6\|D_\tau u\|_2,$$

where C_5, C_6 depend on the dimension and on ρ. From (43), v has zero integral. Thus, we can apply Lemma 2.8 to v and use the above inequality to infer

$$||v||_\infty \leq \begin{cases} \pi\|D_\tau v\|_2 & n = 2 \\ 4\|D_\tau v\|_2^2 \log \frac{8e\|D_\tau v\|_\infty^{n-1}}{||D_\tau v||_2^2} & n = 3 \\ C(n)||D_\tau v||_2^2||D_\tau v||_\infty^{n-3} & n \geq 4. \end{cases}$$

Recalling that $\|u\|_\infty = \mathcal{A}_{\mathcal{H}}(E)$, we get the result. □

Proof of Corollary 1.5 Since we are dealing with an isoperimetric problem involving the boundary momentum with a perimeter constraint, the result can be proven by following the argument of the main result in [18] (in higher dimension) and we just highlight the main steps.

Step one: Continuity of the functional. The first thing we need to show is that given a sequence of convex sets $\{E_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^2$, with $P(E_n) = L$, there exists a convex set E such that $E_n \to E$, $P(E) = L$ and

$$\inf_{x_0\in\mathbb{R}^2} \int_{\partial E} |x - x_0|^2 \, d\mathcal{H}^1 = \lim_{n\to+\infty} \inf_{x_0\in\mathbb{R}^2} \int_{E_n} |x - x_0|^2 \, d\mathcal{H}^1.$$

This fact follows by reproducing the argument provided to conclude the proof Proposition 4.5.

Step two: Qualitative result. As a byproduct of the continuity shown in step one and Proposition 4.1 (the second part of the statement), it is easy to check that for every $\varepsilon > 0$ there exists $\delta > 0$, such that if $P(E) = L$ and

$$\frac{1}{(2\pi)^2} - \frac{\inf_{x_0\in\mathbb{R}^2} \int_{\partial E} |x - x_0|^2}{P(E)^3} \leq \delta, \tag{44}$$

then $A_{\mathcal{H}}(E) \leq \varepsilon$.

Step three: Conclusion. From step one and step two, we know that we can choose $\delta > 0$ small enough such that any convex set satisfying (44) also satisfies $\mathcal{A}_{\mathcal{H}}(E) \leq \varepsilon$. The conclusion follows directly from Theorem 1.4. □

5 Final comments and open problems

In this section, we offer some remarks and highlight open questions left by the present manuscript.

(1) In Theorem 1.4, we prove that, in any dimension, the ball is a local maximizer of the functional $\mathcal{F}(\cdot)$ defined in (9). Is it true that balls are global maximizers of $\mathcal{F}(\cdot)$ among convex sets?

(2) Let $n = 2$, $l > 0$ and $\beta \geq 0$. Consider the functional $\mathcal{G}_\beta$ defined in (3). Denote by $\mathcal{C}_l$ the class of convex sets of perimeter l, i.e.,

$$\mathcal{C}_l = \{F \subset \mathbb{R}^2 : F \text{ is convex and } P(F) = l\}.$$

In corollary 3.8, we proved that balls are extremal sets in $\mathcal{C}_l$, when $0 < \beta \leq 1$ and $\beta \geq \frac{5}{3}$, while in Proposition 3.9 we showed that there are no maximizers when $\beta = 0$.

Does there exist $\beta_1 \in (0, 5/3)$ such that the functional G_β does not attain a maximizer in $\mathcal{C}_l$ for $\beta \leq \beta_1$? Does there exist $\beta_2 \in (1, \infty)$ such that $\mathcal{G}_\beta$ does not attain a maximizer in $\mathcal{C}_l$ for $\beta \geq \beta_2$?

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Declarations

Conflicts of Interest The authors declare that there is no conflict of interest.

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