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Singularity of cycle-spliced signed graphs

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Abstract

We consider the adjacency spectrum of cycle-spliced signed graphs (CSSG), i.e., signed graphs whose blocks are (independent) signed cycles. For a signed graph Σ, the nullity $\eta(\Sigma)$ is the multiplicity of the 0-eigenvalue. The adjancency spectrum of cycle-spliced (signed) graphs is studied in the literature for the relation between the nullity η and the cyclomatic number c, in particular, it is known that $0 \le \eta(\Sigma) \le c(\Sigma) + 1$. In this paper, nonsingular cycle-spliced bipartite signed graphs are characterized. For cycle-spliced signed graphs Σ having only odd cycles, we show that $\eta(\Sigma)$ is 0 or 1. Finally, we compute the nullity of CSSGs consisting of at most three cycles.

Keywords: *Nullity, Cycle-spliced bipartite signed graphs, cyclomatic number*

MSC (2020): *05C05, 05C50.*

1. Introduction

In graph theory, a *signed graph* is a graph in which each edge is labeled as positive or negative. More formally, a signed graph $\Sigma = (\Gamma, \sigma)$ consists of a simple graph $\Gamma = (V, E)$, the *underlying graph*, with set of vertices $V = \{u_1, u_2, \ldots, u_n\}$ and set of edges $E = E(\Gamma)$, and a map $\sigma : E(\Gamma) \to \{+1, -1\}$, the *signature*, assigning a value from $\{+1, -1\}$ to each edge of Γ. The study of signed graphs has attracted considerable attention, as it offers insights into a wide range of phenomena and has a large usage in classical mathematical modeling, variety of socio-psychological and physical processes. Signed graphs are very important for their connections with classical mathematical systems (see [1, 2, 21, 29]).

Signed graphs inherit most notation from unsigned graphs but possess several interesting properties that differentiate them from unsigned graphs. These differences are highlighted, for example, in the study of extremal problems with respect to a fixed spectral parameter, such as the index or the spectral radius, within a given class of signed graphs ([3, 5, 6, 8]). The adjacency matrix is a fundamental tool for analyzing the properties and behavior of (signed) graphs. The *adjacency matrix* of Σ is a $n \times n$ matrix, usually denoted by $A(\Sigma) = (a_{ij})_{n\times n}$, and defined by

$$a_{ij} = \begin{cases} \sigma(e_{ij}) & \text{if } u_i \text{ is adjacent to } u_j, \\ 0 & \text{otherwise.} \end{cases}$$

The *nullity* of Σ is the multiplicity of eigenvalue zero of $A(\Sigma)$ and is denoted by $\eta(\Sigma)$. The *rank* of a signed graph $r(\Sigma)$, is the rank of $A(\Sigma)$. Let $n(\Sigma)$ be the order of a signed graph Σ, obviously, it is $\eta(\Sigma) = n(\Sigma) - r(\Sigma)$. The *cyclomatic number* of a signed graph Σ is denoted by $c(\Sigma)$ and is defined as $c(\Sigma) = e(\Sigma) - n(\Sigma) + \Theta(\Sigma)$, where $e(\Sigma)$ and $\Theta(\Sigma)$ represent the number of edges and the number of connected components in Σ, respectively. For a connected signed graph Σ, that is, $\Theta(\Sigma) = 1$, if $c(\Sigma) = 0$, or $c(\Sigma) = 1$, or $c(\Sigma) = 2$, then Σ is, respectively, a *tree*, a *unicyclic*, a *bicyclic* signed graph. In a signed (or unsigned) graph a *block* is a maximal connected subgraph with no articulation point or cut vertex. A connected signed graph Σ is a *cycle-spliced signed graph* if every block in Σ is a cycle. In particular, for cycle-spliced signed graph, the cycles are independent, and therefore the cyclomatic number equals the number of cycles. A cycle-spliced signed graph without odd cycles is a *cycle-spliced bipartite signed graph.* A cycle C in a signed graph Σ with $c(\Sigma) \geq 2$ is a *pendant cycle* if only one of its vertices has degree greater than 2. If $c(\Sigma) = 1$, then $C = \Sigma$ is considered a

pendant cycle.
The sign of a cycle C in Σ is the product of the signatures of its edges. A cycle C in a signed graph Σ is positive if and only if C contains an even number of negative edges. We call a signed graph Σ to be *balanced* if no negative cycles in Σ exist and *unbalanced*, otherwise. A *switching* consists of switching the signs of the edges in a cut. Two signed graphs are equivalent, if one can be obtained from the other by a sequence of (sign) switchings. The following result is well-known, and it says that balanced signed graphs are equivalent to their underlying graphs; thus the eigenvalues are the same.

Lemma 1.1. *Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. Then Σ is balanced if and only if $(\Gamma, \sigma) \sim (\Gamma, +)$.*

More generally, for a signed graph $\Sigma = (\Gamma, \sigma)$, the signature is determined by the signs of the independent cycles. Hence, for cycle-spliced signed graphs the signature is determined by the signs of its cycles. For all notation and results not give here, we refer the reader to [4] or to the extensive bibliography [29].

A graph is *singular* (resp. *nonsingular*) if its adjacency matrix is singular (resp. nonsingular). In [13], Collatz and Sinogowitz posed a problem to characterize all singular graphs ($\eta(\Gamma) > 0$). In [24], Ma et al., proved that $\eta(\Gamma) \leq 2c(\Gamma) + p(\Gamma) - 1$ unless Γ is a cycle of length multiple of 4, where $p(\Gamma)$ is the total number of leaves in Γ. Chang et al., [11] and Wang [25] characterized the graphs Γ with $\eta(\Gamma) = 2c(\Gamma)+p(\Gamma)-1$. In [23], Lu and Wu characterized the signed graphs and proved that there are no signed graphs with $\eta(\Gamma, \sigma) = n(\Gamma) - 2m(\Gamma) + 2c(\Gamma) - 1$, where $m(\Gamma)$ is the matching number of Γ. The nullity of unicyclic and bicyclic signed graphs were studied by Fan et al. [19, 17] respectively. In [26], Wong et al. studied the nullity and singularity of cycle-spliced bipartite graphs and gave the bounds for the nullity of cycle-spliced bipartite graphs Γ in term of $c(\Gamma)$, i.e., $0 \leq \eta(\Gamma) \leq c(\Gamma)+1$. Furthermore, they characterized the cycle-spliced bipartite graphs with $\eta(\Gamma) = c(\Gamma) + 1$ and $\eta(\Gamma) = 0$, respectively. These results are presented in Theorem 1.2.

A cycle in a cycle-spliced signed graph Σ is of $i-$type, $0 \leq i \leq 3$, if its length is equal to $i \pmod 4$.

Theorem 1.2. *([26], Theorem 1.1). Let Γ be a cycles-spliced bipartite graph with $c(\Gamma)$ cycles. Then*
(i) $0 \leq \eta(\Gamma) \leq c(\Gamma) + 1$.
(ii) $\eta(\Gamma) = c(\Gamma) + 1$ *if and only if all cycles in Σ are of $0-$type.*

(iii) Γ *is nonsingular if and only if* Γ *has a perfect matching, and* Γ *has a maximum matching* M *such that* $M \cap E(C)$ *is not a perfect matching of* C *for every 0-type cycle* C *in* Γ.

In [9], the authors prove that there is no cycle-spliced bipartite graph Σ such that $\eta(\Sigma) = c(\Sigma)$, and characterize cycle-spliced bipartite graphs Σ with $\eta(\Sigma) = c(\Sigma) - 1$.

Theorem 1.3 ([9], Theorem 1.4). *For any cycle-spliced bipartite graph* G *with* $c(G)$ *cycles,* $\eta(G) = c(G) - 1$ *if and only if* G *is a graph obtained from a cycle-spliced bipartite graph* H *with* $\eta(H) = c(H) - 1$ *in which every pendant cycle (if any) has length congruent to* $2 \pmod 4$ *by attaching* $c(G) - c(H)$ *cycles having length divisible by* 4 *on arbitrary vertex of* H.

In [10] the same authors consider the signed case: they show that for every signed cycle-spliced graph Γ, $\eta(\Gamma) \leq c(\Gamma)+1$ and the extremal graphs Γ with nullity $c(\Gamma) + 1$ are characterized, which extend the corresponding results [26] on unsigned cycle-spliced graphs.

Theorem 1.4 ([10], Theorem 1.1). *Let* Γ *be a signed cycle-spliced graph with* $c(\Gamma)$ *cycles. Then* $\eta(\Gamma) \leq c(\Gamma) + 1$ *and the equality holds if and only if all cycles in* Γ *have nullity* 2.

Moreover, in [10] Theorem 1.3, they prove that for every (not necessarily bipartite) signed cycle-spliced graph Γ, $\eta(\Gamma) \neq c(\Gamma)$. Some properties on signed cycle-spliced graphs Γ with $\eta(\Gamma) = c(\Gamma) - 1$ are explored as well, and they provide a structural characterization of signed cycle-spliced bipartite graphs Γ satisfying $\eta(\Gamma) = c(\Gamma) - 1$.

Theorem 1.5 ([10], Theorem 1.4). *Let* Γ *be a signed cycle-spliced bipartite graph with* $c(\Gamma) \geq 2$ *and all pendant cycles have nullity 0. Then* $\eta(\Gamma) = c(\Gamma) - 1$ *if and only if the distance between any two cut vertices of* Γ *is even.*

Theorem 1.6 ([10], Theorem 1.5). *For any signed cycle-spliced bipartite graph* Γ *with* $c(\Gamma)$ *cycles,* $\eta(\Gamma) = c(\Gamma) - 1$ *if and only if* Γ *is a signed graph obtained from a signed cycle-spliced bipartite graph* (H, σ) *with* $\eta(H, \sigma) = c(H, \sigma) - 1$ *in which every pendant cycle (if any) has nullity 0 by attaching* $c(\Gamma) - c(H, \sigma)$ *cycles with nullity 2 on arbitrary vertex of* (H, σ).

In this article we characterize nonsingular cycle-spliced bipartite signed graphs.

A *matching* of Σ is a collection of non-adjacent edges of Σ. A *maximum matching* is a matching with the maximum possible number of edges. The size of a maximum matching of Σ is denoted by $m(\Sigma)$ and is called the *matching number* of Σ. A matching covering all vertices of Σ is called a *perfect matching.*

Theorem 1.7. *Let $\Sigma = (\Gamma, \sigma)$ be a cycle-spliced bipartite graph with $c(\Gamma)$ cycles. Then Σ is nonsingular if and only if it has a perfect matching, and a maximum matching M such that $M \cap E(C)$ is not a perfect matching of C for every balanced 0-type or unbalanced 2-type cycle C in Σ.*

Examples of nonsingular cycle-spliced signed graphs are also among those cycle-spliced graphs with only odd cycles. We get the following result about this kind of cycle-spliced signed graphs.

Theorem 1.8. *Let Σ be a cycle-spliced signed graph in which all cycles are odd.*

- *(i) If $c(\Sigma)$ is odd, then Σ is nonsingular.*
- *(ii) If $c(\Sigma)$ is even, then $\eta(\Sigma)$ is 0 or 1.*
- *(iii) If every cycle of Σ has at most two cut-vertices of Σ, then Σ is singular if and only if $p^+ + q^- = p^- + q^+$, where p^+ and p^- are the number of positive and the number of negative 1-type cycles respectively, while q^+ and q^- are the number of positive and the number of negative 3-type cycles respectively.*

The cases when Σ has both even cycles and odd cycles still eludes us. We just analyze the case of a general wedge of balanced or unbalanced cycles, of $i-$type for $i = 0, 1, 2, 3$.

The rest of the article is organized as follows. In Section 1 we provide some notations and useful results about signed cycles.

Section 2 is devoted to characterizing nonsingular cycle-spliced bipartite signed graphs. We also consider the nullity of cycle-spliced graphs having only odd cycles. In Section 3 we discuss the particular case of a wedge of both even and odd, balanced and unbalanced cycles and pose some open questions. Further, we give a complete description of all cycle-spliced signed graphs Σ such that $c(\Sigma) \leq 3$.

2. Preliminaries on signed cycles

Let Σ be a signed cycle. For $v \in V(\Sigma)$, let $N_\Sigma(v)$ be the set of neighbors of v and let $d_\Sigma(v) = |N_\Sigma(v)|$ be the *degree* of v.

If we consider an induced subgraph M of Σ, then the neighbors of v in M is denoted by $N_M(v)$. If $D \subseteq V(\Sigma)$, then the deletion of D together with all incidence edges is the induced subgraph of Σ denoted by $\Sigma - D$. If $D = \{v_1\}$ or $\{v_1, v_2\}$, then $\Sigma - D$ is denoted by $\Sigma - v_1$ or $\Sigma - v_1 - v_2$. If v is a vertex in M, then we denote by $\Sigma - M + v$ the subgraph of Σ induced by $V(\Sigma - M) \cup \{v\}$.

A pendant vertex of a signed graph is a vertex of degree 1 of the underlying graph. If u is a pendant vertex of a graph Σ and v is its unique neighbor in Σ, then the operation that gives $\Sigma - \{u, v\}$ from Σ is called a *pendant K_2 deletion*.

Lemma 2.1 ([19]). *Let Σ be a signed graph. If u is a pendant vertex of Σ and v is its unique neighbor, then $\eta(\Sigma) = \eta(\Sigma - \{u, v\})$.*

Lemma 2.2 ([27, 16]). *Let (P_n, σ) be a (signed) path. Then $\eta(P_n, \sigma) = 1$ if n is odd, and $\eta(P_n, \sigma) = 0$ if n is even.*

Lemma 2.3 ([26]). *Let Γ be a graph and C be a pendant cycle of Γ with a cut vertex y. Let $M = \Gamma - C + y$. If C is a cycle of $0-$type, then $\eta(\Gamma) = \eta(M) + 1$, and if C is a cycle of $2-$type, then $\eta(\Gamma) \leq \eta(M) + 1$.*

In [15], D. Cvetkoviċ et.al., obtained the following result for a balanced signed cycle.

Lemma 2.4. *Let C_n be a balanced signed cycle. Then $\eta(C_n) = 2$ if C_n is a cycle of $0-$type, and 0 otherwise.*

Y. Fan proved the following result when C_n is unbalanced.

Lemma 2.5 ([20]). *Let C_n be a unbalanced signed cycle. Then it has eigenvalues $2cos\frac{(2k-1)\pi}{n}$, $k = 1, 2, \ldots, n$.*

More precisely, we can write $\eta(C_n) = 2$ if C_n is of $2-$type (i.e., $n \equiv 2 (\mathrm{mod}\ 4)$), and 0 otherwise. In particular, we can say that for all unbalanced cycles of $0-$type it is $\eta(C_n) = 0$.

These results can be summarized in the following Lemma (see also [16, 27]).

Lemma 2.6. *Let (C_n, σ) be a signed cycle. Then $\eta(C_n, \sigma) = 2$ if and only if (C_n, σ) is balanced of 0-type or (C_n, σ) is unbalanced of 2-type. $\eta(C_n, \sigma) = 0$ otherwise.*

Recall that M is a pendant subgraph of a signed graph Σ with root y if the removal of the vertex y disconnects Σ; thus y is a cut vertex of Σ.

Lemma 2.7. *([28]) Let Σ_1 be a pendant subgraph of Σ with root y.*
(i) If $\eta(\Sigma_1 - y) = \eta(\Sigma_1) + 1$, then $\eta(\Sigma) = \eta(\Sigma_1) + \eta(\Sigma - \Sigma_1)$.
(ii) If $\eta(\Sigma_1 - y) = \eta(\Sigma_1) - 1$, then $\eta(\Sigma) = \eta(\Sigma_1) + \eta(\Sigma - \Sigma_1 + y) - 1$.

Corollary 2.8. *Let C be a pendant cycle of a signed graph Σ with root y.*
(i) If C is a balanced (resp. unbalanced) cycle of $0-$type (resp. $2-$type), then $\eta(\Sigma) = \eta(\Sigma - C + y) + 1$.
(ii) If C is a balanced (resp. unbalanced) cycle of $2-$type (resp. $0-$type), then $\eta(\Sigma) = \eta(\Sigma - C)$.

For any natural number $t \geq 2$, let $\Sigma_1, \ldots, \Sigma_t$ be signed rooted graphs with root v_i, respectively. We denote by $\bigvee_{i=1}^{t} \Sigma_i$ or $\bigvee_v \Sigma_i$ the *wedge* of the Σ_i's, that is, the graph obtained by identifying their roots at a unique vertex v. If the rooted graphs are signed cycles, then the wedge of signed cycles is equivalent to a cycle-spliced signed graph with exactly one cut-vertex v.

The following proposition is about the nullity of a wedge of signed cycles of even length.

Proposition 2.9. *Let Σ be a signed graph obtained from the wedge of n cycles C_i of even length, $i = 1, \ldots, n$. Hence,*
(i) $\eta(\Sigma) = c(\Sigma) + 1 = n + 1$ if and only if all cycles C_i are balanced of $0-$type or unbalanced of $2-$type.
(ii) $\eta(\Sigma) = c(\Sigma) - 1 = n - 1$ if and only if at least one of the cycles C_i is balanced of 2-type or unbalanced of 0-type.

Proof. (i) This comes from Theorem 1.4.
(ii) Assume that, say, C_1 is either balanced of 2-type, or unbalanced of 0-type, and let v be the unique cut vertex of Σ. By Corollary 2.8 part(ii), $\eta(\Sigma) = \eta(\Sigma - C_1) = \cup_{i=2}^{n}(C_i - v) = n - 1$. If $\eta(\Sigma) = c(\Sigma) - 1 = n - 1$, then at least one cycle of the C_i's is balanced and $2-$type or unbalanced and $0-$type by Theorem 1.4. □

Example 2.10. *In Fig. 2.1 a signed cycle-spliced graph is depicted. According to the above given notation, the unbalanced C_3 and the balanced*

C_5 are signed cycles of 0-type; the balanced C_1 is a cycle of 1-type; the balanced C_4 is cycle of 2-type; the unbalanced C_2 is a cycle of 3-type. The pendant cycles are C_1, C_2, and C_5.

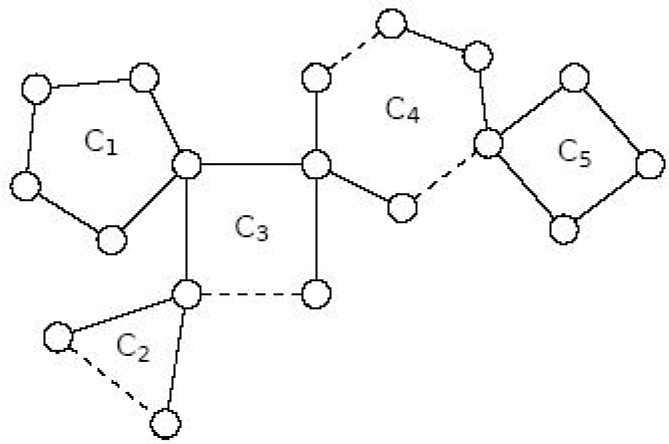

Figure 2.1: A signed cycle-spliced graph.

3. Extremal graphs Σ with $\eta(\Sigma) = 0$

The purpose of this section is to identify cycle-spliced signed graphs Σ that are nonsingular, that is $\eta(\Sigma) = 0$, the minimum possible value for nullity. In the bipartite case, we have a characterization result. Other nonsingular graphs can be found among those having only odd cycles.

3.1. Characterizing nonsingular cycle-spliced bipartite graphs

Let $\Sigma = (\Gamma, \sigma)$ be a signed graph. The characteristic polynomial of Σ is

$$\Phi_\Sigma(x) = |xI - A(\Sigma)| = x^n + a_{n-1}(\Sigma)x^{n-1} + \cdots + a_1(\Sigma)x + a_0(\Sigma)$$

In analogy to Sachś formula ([14], p. 32), one can easily derive (using the Coates formula [12]) the following characterization of the coefficients:

$$a_{n-i}(\Sigma) = \sum_{H \in \mathcal{H}_i} \sigma(H_c)((-1)^{p(H)} 2^{q(H)}), \quad i = 1, \ldots, n$$

where H is a signed subgraph of Σ, spanned over i vertices, whose components are edges or cycles (of length at least 3), $p(H)$ is the number of components of H, $q(H)$ is the number of cycles in H, H_c is a signed subgraph of H containing only its cycles, $\sigma(H_c)$ is the product of the signs of all cycles in H_c and $\mathcal{H}_i$ is the set of all subgraphs of H of order i.

If n is even, then a_0 in $|xI - A(\Sigma)|$ is equal to $|A(\Sigma)|$; if n is odd, then a_0 in $|xI - A(\Sigma)|$ is equal to $-|A(\Sigma)|$.

The following result gives a sufficient condition for a signed graph Σ, not necessarily cycle-spliced, to be nonsingular.

Lemma 3.1. *Every signed graph Σ with an odd number of perfect matchings is nonsingular.*

Proof. Since Σ has a perfect matching, then $n = 2m(\Sigma)$. We need to prove that the constant term $a_0(\Sigma)$ of the characteristic polynomial $\Phi_\Sigma(x)$ is non zero. Recall that

$$a_0(\Sigma) = \sum_{H \in \mathcal{H}_n} \sigma(H_c)((-1)^{p(H)} 2^{q(H)}),$$

where H is a signed subgraph of Σ, spanned over all the n vertices of Σ. Since Σ has an odd number, say $2h+1$, of basic subgraphs of order n with $\frac{n}{2}$ copies of K_2, there are $2h+1$ copies of $(-1)^{m(\Sigma)}$ in the expression of $a_0(\Sigma)$. All the other nonzero terms in the above expression are even, since they are of the form $\sigma(H_c)((-1)^{p(H)}2^{q(H)})$ where $q(H) \geq 1$. Hence $a_0(\Sigma) \neq 0$ because it is an odd number. $\square$

For Σ a signed cycle-spliced bipartite graph, let Y be the set of all 0-type balanced and all 2-type unbalanced cycles of Σ. We say that a maximum matching M of Σ intersects Y if $M \cap E(C)$ is not a perfect matching of C for each $C \in Y$.

Lemma 3.2. *Let Σ be an order n cycle-spliced bipartite graph with a perfect matching, and Y the set of all 0-type balanced and all 2-type unbalanced cycles of Σ. Then*

$$\sum_H (-1)^{s(H)} 2^{t(H)} \sigma(H) = 0,$$

where H goes over all basic subgraphs of order n whose perfect matchings fail to intersect Y.

Proof. Let Δ be a basic subgraph of order n whose perfect matchings fail to intersect Y. The matching numbers $m(\Delta)$ and $m(\Sigma)$ coincide and a perfect matching of Δ is also a perfect matching of Σ. Let M be a perfect matching of Δ. As M fails to intersect Y, there is $C \in Y$ such that $M \cap E(C)$ is a perfect matching of C. Thus we get three related basic subgraphs of order n: One is Δ, which contains C as a component, and the other two, Δ' and Δ'' are obtained by replacing C with two distinct perfect matchings of C keeping all other components unchanged. Δ' and Δ'' have $s(\Delta) - 1 + \frac{|V(C)|}{2}$ components and have one cycle less than those of Δ. The contributions to the sum of the three basic subgraphs are the following three terms:

$$(-1)^{s(\Delta)}\sigma(\Delta)2^{t(\Delta)}, \quad (-1)^{s(\Delta)-1+\frac{|V(C)|}{2}}\sigma(\Delta)\sigma(C)2^{t(\Delta)-1},$$

$$(-1)^{s(\Delta)-1+\frac{|V(C)|}{2}}\sigma(\Delta)\sigma(C)2^{t(\Delta)-1}.$$

Next, we distinguish two cases:

(i) C is balanced of 0-type. $\sigma(C) = 1$ and $\frac{|V(C)|}{2}$ is even, so the sum of the above three terms is zero.

(ii) C is unbalanced of 2-type. $\sigma(C) = -1$ and $\frac{|V(C)|}{2}$ is odd, so the sum of the above three terms is zero.

Thus the sum $\sum_H (-1)^{s(H)}\sigma(H)2^{t(H)} = 0$ is zero when H goes over all basic subgraphs of order n whose perfect matchings fail to intersect Y. $\square$

Σ is nonsingular if and only if it has a perfect matching, and a maximum matching M such that $M \cap E(C)$ is not a perfect matching of C for every balanced 0-type or unbalanced 2-type cycle C in Σ.

We are ready to prove Theorem 3.4.

Proof. [Proof of Theorem 3.4] Suppose that Σ is nonsingular, then $\eta(\Sigma) = 0$ and $\mathrm{rk}(\Sigma) = n$, where n is the order of Σ. Thus Σ has a basic subgraph of order n. In particular, Σ has a basic subgraph whose edge set forms a perfect matching of Σ.

The signed graph Σ has a perfect matching which intersects Y, otherwise a_0 would be null by Lemma 3.2.

Vice versa, let $a_0 = \sum_H (-1)^{s(H)}2^{t(H)}\sigma(H) = 0$, where $n = 2m(\Sigma)$ and H goes over all basic subgraphs of order n. Let $\mathcal{H}$ be the set of all such basic subgraphs. Then $\mathcal{H} = \mathcal{D} \cup \mathcal{W}$, where $\mathcal{D}$ is the set of all basic subgraphs of order n whose perfect matchings fail to intersect Y and $\mathcal{W}$ is the set of all basic subgraphs of order n whose perfect matchings intersect Y. Then

$$a_0 = \sum_{\Delta \in \mathcal{D}} (-1)^{s(\Delta)} \sigma(\Delta) 2^{t(\Delta)} + \sum_{W \in \mathcal{W}} (-1)^{s(W)} \sigma(W) 2^{t(W)}.$$

By Lemma 3.2 the sum over $\mathcal{D}$ is zero, thus

$$a_0 = \sum_{W \in \mathcal{W}} (-1)^{s(W)} 2^{t(W)} \sigma(W),$$

and this sum contains at least one term by hypothesis. Let $W \in \mathcal{W}$. It contains balanced of 2-type, unbalanced 0-type cycles and copies of K_2. We will not find balanced of 0-type and unbalanced of 2-type cycles because W intersects Y. Let t_0^- and t_2^+ be the number of unbalanced 0-type cycles and of balanced 2-type cycles in W respectively. Then the contribution given by W to a_0 is

$$(-1)^{s(W)} 2^{t_2^+} (-1)^{t_0^-} 2^{t_0^-} = (-1)^{s(W)+t_0^-} 2^{t_2^+ + t_0^-}.$$

If $t_0^- = 0$, W doesn't contain any unbalanced 0-type cycle and $s(W) \equiv m(\Sigma) \mod 2$ since the order of its components is $4k_i + 2$ and $n = 2m(\Sigma) = \sum_{i=1}^{s(W)} (4k_i + 2)$, that is $m(\Sigma) = \sum_{i=1}^{s(W)} (2k_i) + s(W)$. All such subgraphs as W contribute to a_0 with the same sign. Let

$$a_0' = \sum_{W' \in \mathcal{W}'} (-1)^{s(W')} 2^{t(W')},$$

where $\mathcal{W}'$ is the set of all $W' \in \mathcal{W}$ containing only balanced 2-type cycles ($t(W') = t_2^+(W')$ and $t_0^-(W') = 0$). We get $a_0' \neq 0$ if $t_2^+(W') \geq 1$.

Now suppose that $t_0^-(W) = 1$ so that W has just one unbalanced 0-type cycle C. Taking into account the sign $\sigma(C) = -1$, W contributes to a_0 with the term

$$a_W = (-1)^{s(W)} (-1) 2^{t_2^+(W)+1} = (-1)^{s(W)+1} 2^{t_2^+(W)+1},$$

but when we replace in W the cycle C with its two possible matchings (made by an even number $2l$ of copies of K_2), we get two subgraphs W_1' and W_2' in $\mathcal{W}'$ whose contributions $a_{W_1'}$ and $a_{W_2'}$ to a_0 are contained in a_0' and have total value

$$a_{W_1'} + a_{W_2'} = 2 \cdot [(-1)^{s(W)-1+2l} 2^{t_2^+(W)}]$$

$$= (-1)^{s(W)-1} 2^{t_2^+(W)+1} = (-1)^{s(W)+1} 2^{t_2^+(W)+1} = a_W.$$

In particular a_W has the same sign as the summands in a_0'. Let t_0 be the number of unbalanced 0-type cycles of G and suppose that for any order n basic subgraph W with $1 \leq t_0^- < t_0$ unbalanced 0-type cycles, the sign $\operatorname{sgn}(a_W)$ is the same as $\operatorname{sgn}(a_{W'})$ for any order n basic subgraph W' containing only balanced 2-type cycles (eventually $t_2^+ = 0$)(induction hypothesis). Let W be a basic subgraph containing all the t_0 unbalanced 0-type cycles that are in G and let C be on of these cycles. Then

$$a_W = (-1)^{s(W)}(-1)^{t_0} 2^{t_2^+(W)+t_0} = (-1)^{s(W)+t_0} 2^{t_2^+(W)+t_0}$$

is the contribution to $a_0(\Sigma)$ given by W. Now replace in W the cycle C with its two possible matchings (made by an even number $2l$ of copies of K_2) and we get two subgraphs W_1 and W_2 in $\mathcal{W}$ whose contributions $a_{W_1} = a_{W_2} = a$ to $a_0(G)$ have total value

$$a_{W_1'} + a_{W_2'} = 2a = 2 \cdot [(-1)^{s(W)-1+2l+t_0-1} 2^{t_2^+(W)+t_0-1}]$$

$$= (-1)^{s(W)+t_0} 2^{t_2^+(W)+t_0} = a_W.$$

Since W_1 and W_2 have $t_0 - 1$ unbalanced cycles of 0-type, by the induction hypothesis, the sign $\operatorname{sgn}(a_{W_i}) = \operatorname{sgn}(a)$ $(i = 1, 2)$ is the same as $\operatorname{sgn}(a_{W'})$ for any order n basic subgraph W' containing only balanced 2-type cycles. This proves that all the summands in a_0 have the same sign and $a_0 \neq 0$. $\square$

3.2. Cycle-spliced signed graphs with only odd cycles

In this section we consider cycle-spliced signed graphs with only odd cycles. First we present some preliminary results to apply in the proof of the main theorem. Let $\Sigma = C_n$ be an odd cycle (n odd), then

$$|A(C_n)| = -a_0 = \begin{cases} 2 & \text{if} \quad \sigma(C_n) = +1 \\ -2 & \text{if} \quad \sigma(C_n) = -1 \end{cases}$$

Hence, an odd cycle is always nonsingular. If x is a vertex of an odd cycle C_n, then $C_n - x$ has an even number of vertices, so $a_0(C_n - x) = |A(C_n - x)| = 1$

$$|A(C_n - x)| = a_0(C_n - x) = \begin{cases} 1 & \text{if} \quad C_n \text{ is a 1-type cycle} \\ -1 & \text{if} \quad C_n \text{ is a 3-type cycle} \end{cases}$$

Lemma 3.3. *If Σ is a coalescence of H and K at a vertex v, then*

$$det(\Gamma) = \; det(H)\; det(K - v) + \; det(H - v)\; det(K).$$

The following two results are contained in [26]. They also hold for signed graphs since they are independent of the sign of edges.

Lemma 3.4 ([26]). *Let Γ be a cycle-spliced graph with only odd cycles, v an arbitrary vertex in Γ. Then $\Gamma - v$ has a unique perfect matching.*

Lemma 3.5 ([26]). *Let Γ be an order n cycle-spliced graph with only odd cycles, v a vertex of a pendant cycle of Γ. If every cycle of Γ has at most two cut-vertices of Γ, then*

(i) $\Gamma - v$ *has a unique basic subgraph of order* $n - 1$*;*

(ii) $det(\Gamma - v) = (-1)^q$, *where q is the number of 3-type cycles in Γ.*

Lemma 3.6. *Let Σ be a cycle-spliced graph with only odd cycles. If every cycle of Σ has at most two cut-vertices of Σ, then $det(\Sigma) = (-1)^q(2p^+ - 2p^- - 2q^+ + 2q^-)$, where p^+ (p^-) is the number of positive (negative) 1-type cycles, q^+ (q^-) is the number of positive (negative) 3-type cycles in Σ and $q = q^+ + q^-$.*

Proof. Let n be the order of Σ. We use induction on the number of cycles $c(\Sigma)$ in Σ. If $\Sigma = C_n$, then $\det(\Sigma) = -a_0 = \sigma(C_n) \cdot 2$ which proves the assertion. Let $c(\Sigma) > 1$ and let C be a pendant cycle having a vertex x in common with other cycles of Σ. By Lemma **??**,

$$x\; \det(\Sigma) = \; \det(C - x)\; \det(\Sigma - C + x) + \; \det(C)\; \det(\Sigma - C).$$

If C is a 1-type cycle, then $\det(C - x) = 1$, $\det(\Sigma - C + x) = (-1)^q(2p^+ - 2p^- - \sigma(C \cdot 2) - 2q^+ + 2q^-)$ (induction hypothesis), $\det(\Sigma - C) = (-1)^q$ by Lemma 3.5 (ii) and $\det(C) = \sigma((C)) \cdot 2$, thus we have

$$\det(\Sigma) = 1 \cdot (-1)^q(2p^+ - 2p^- - \sigma(C) \cdot 2 - 2q^+ + 2q^-) + \sigma(C) \cdot 2(-1)^q =$$
$$(-1)^q(2p^+ - 2p^- - 2q^+ + 2q^-).$$

If C is a 3-type cycle, then $\det(C - x) = -1$, $\det(\Sigma - C + x) = (-1)^{q-1}(2p^+ - 2p^- + \sigma(C) \cdot 2 - 2q^+ + 2q^-)$ (induction hypothesis), $\det(\Sigma - C) = (-1)^{q-1}$ by Lemma 3.5 (ii) and $\det(C) = \sigma(C) \cdot 2$, thus we have

$$\det(\Sigma) = (-1) \cdot (-1)^{q-1}(2p^+ - 2p^- + \sigma(C) \cdot 2 - 2q^+ + 2q^-) + \sigma(C) \cdot 2 \cdot (-1)^{q-1} =$$

$$(-1)^q(2p^+ - 2p^- - 2q^+ + 2q^-).$$

□

These results are exploited to prove the second of our main results.

Proof. [Proof of Theorem 1.8] Suppose the order of Σ is n, and suppose Σ has $p = p^+ + p^-$ 1-type cycles and $q = q^+ + q^-$ 3-type cycles. Since $p+q$ is odd by hypothesis, n is odd, thus Σ has no perfect matching and a basic subgraph of Σ of order n must have an odd number of cycles. Consider a basic subgraph H of Σ of order n with exactly one cycle as a component, say C. By Lemma 3.4, precisely one such basic subgraph of order n contains C as a component (noting that $\Sigma - C$ has a unique perfect matching). This subgraph has $(n - V(C))/2$ copies of K_2. If C is of 1-type, then the contribution of H to a_0 is

$$(-1)^{(n-V(C))/2+1}\sigma(C)\cdot 2 = (-1)^{(n-4h-1)/2+1}\sigma(C)\cdot 2 = (-1)^{(n-1)/2+1}\sigma(C)\cdot 2.$$

The p basic subgraphs with a 1-type cycle totally contribute

$$(2p^+ - 2p^-)(-1)^{(n-1)/2+1}$$

to a_0. If C is of 3-type, then the contribution of H to a_0 is

$$(-1)^{(n-V(C))/2+1}\sigma(C)\cdot 2 = (-1)^{(n-4h-3)/2+1}\sigma(C)\cdot 2 = (-1)^{(n-3)/2+1}\sigma(C)\cdot 2.$$

The q basic subgraphs with a 3-type cycle totally contribute

$$(2q^+ - 2q^-)(-1)^{(n-3)/2+1}$$

to a_0. The sum of $(2p^+ - 2p^-)(-1)^{(n-1)/2+1}$ and $(2q^+ - 2q^-)(-1)^{(n-3)/2+1}$ is $\pm\, 2(p^- p^+ + q^+ - q^-)$, according to $n = 4s+1$ or $n = 4t+3$. This sum is not divisible by 4 because $p^- p^+ + q^+ - q^-$ is always odd as a consequence of the fact that $p + q = p^- + p^+ + q^+ + q^-$ is odd.

Consider a basic subgraph H of order n with $k \geq 3$ cycles. The corresponding term in a_0 contributed by such a subgraph has the form

$$(-1)^{p(H)}2^k \prod_{j=1}^{k} \sigma(C_j)$$

which is divisible by 4, where $p(H)$ is the number of components of H and $C_1, \ldots, C_k$ are the cycles in H. Hence, a_0 is not divisible by 4 and thus $a_0 \neq 0$, which proves that Σ is nonsingular. The proof of (i) is completed.

Now we prove part (ii). Let C be a pendant cycle with a vertex x shared by other cycles. Thus $\Sigma - C$ is nonsingular (by 3.1 and 3.4). Now we observe that $A(\Sigma)$ has diag $(A(C-x), A(\Sigma - C))$ as a principal submatrix, where diag $(A(C-x), A(\Sigma - C))$ denotes a 2×2 block diagonal matrix with $A(C-x)$ and $A(\Sigma - C)$ as diagonal entries. As $A(C-x)$ and $A(\Sigma - C)$ are both nonsingular, the rank of $A(\Sigma)$ is at least $n-1$, which implies that $\eta(\Sigma) \leq 1$. (see [26], Theorem 1.2, part (ii) for unsigned graphs).

Part (iii) directly follows from Lemma3.6. □

4. Particular configurations

In this section we handle the following particular configurations: 1) the wedge of signed cycles, 2) cycle-spliced signed graphs with at most 3 cycles.

4.1. Wedge of signed cycles

Let Σ be the coalescence of two cycles C_o (odd) and C_e (even). If C_e is balanced of 0-type or unbalanced of 2-type, then $\eta(\Sigma) = 1$ whatever C_o is. If C_e is balanced of 2-type or unbalanced of 0-type, then $\eta(\Sigma) = 0$ whatever C_o is. This follows from Corollary 2.8. More generally, we can compute the nullity of a wedge of n signed cycles of any type, all having a vertex v in common.

Theorem 4.1. *Let W be a wedge of n cycles all having a vertex v in common. Let h be the number of odd cycles in W, k be the number of even cycles, balanced of 0-type or unbalanced of 2-type, and l be the number of even cycles, balanced of 2-type or unbalanced of 0-type. Let W_0 be the wedge of the h odd cycles. Then*

$$\eta(W) = \begin{cases} \eta(W_0) + k & \text{if} \quad l = 0 \\ k + l - 1 & \text{if} \quad l \geq 1. \end{cases}$$

Proof. By Corollary 2.8, if C_e is balanced of 0-type or unbalanced of 2-type, then $1 = \eta(C_e - v) = \eta(C_e) - 1 = 2 - 1$ and $\eta(W) = \eta(C_e - v) + \eta(W - C_e + v) = 1 + \eta(W - C_e + v)$. If C_e is a balanced of 2-type or unbalanced of 0-type cycle, then $1 = \eta(C_e - v) = \eta(C_e) + 1 = 0 + 1 =$ and $\eta(W) = \eta(W - v) - 1 = \eta(W - C_e) = \eta(W_0 - v) + k + l - 1 = 0 + k + l - 1 = k + l - 1$. □

Example 4.2. *In Fig. 4.1 the wedge W of 5 signed cycles is depicted. The signed graph W_0 consists of the subgraph of W induced by the odd cycles*

C_1, C_4, and C_5. The nullity $\eta(W_0)$ is determined by Theorem 1.8(i), and it is $\eta(W_0) = 0$. The even cycles C_2 and C_3 are, respectively, balanced of 2-type and unbalanced of 0-type, hence it is $k = 0$ and $l = 2$. According to Theorem 4.1, it is $\eta(W) = 1$.

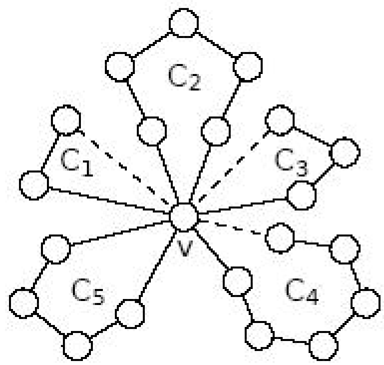

Figure 4.1: The wedge of 5 signed cycles.

4.2. Cycle-spliced signed graphs with at most 3 cycles

Using some results contained in this paper, we deduce the nullity of all cycle-spliced signed graphs Σ with $c(\Sigma) \leq 3$. Recall that $\eta(\Sigma) \neq C(\Sigma)$ by Theorem 1.3 in [10].

Let us denote by C_i^+ a positive i-type cycle and by C_i^- a negative i-type cycle, for $i = 0, 1, 2, 3$.

Case $c(\Sigma) = 1$.

By Lemma 2.6, for $\Sigma = C$, $\eta(\Sigma) = 2$ if C is a C_0^+ or a C_2^- cycle. Σ in nonsingular otherwise.

Case $c(\Sigma) = 2$.

Our graphs are a coalescence of two cycles at a vertex v, $\Sigma = C \vee_v C'$.

$\eta(\Sigma) = 3$ if $\eta(C) = \eta(C') = 2$.

$\eta(\Sigma) = 1$ if C and C' have one of the following properties:
a) C and C' are both even cycles and at least one of them has nullity 0.
b) C is odd and C' is even with $\eta(C') = 2$.

c) C and C' are both odd cycles, of the same type (both of 1-type or both of 3-type) but with opposite signs, or of different types (C of 1-type and C' of 3-type) but with the same sign.

$\eta(\Sigma) = 0$ if C and C' have one of the following properties:
a) C is odd and C' is even with $\eta(C') = 0$.
b) C and C' are both odd cycles, of the same type (both of 1-type or both of 3-type) and with the same sign, or of different types (C of 1-type and C' of 3-type) and with different signs.

Case $c(\Sigma) = 3$.

Let $\Sigma_u = \bigvee_u C^j = \bigvee_{j=1}^3 C^j$ be a wedge obtained from 3 signed cycles, C^1, C^2, C^3, by identifying the unique common vertex u, and let $\Sigma_{u,v} = C^1C^2C^3$ be a sequence of 3 cycles, C^1, C^2, C^3, where u is the unique common vertex to C^1 and C^2 and v is the unique common vertex to C^2 and C^3.

$\eta(\Sigma) = 4$ if and only if $\eta(C^j) = 2$ for $j = 1, 2, 3$, that is C^j is a C_0^+ or a C_2^- cycle, by Theorem 1.4.

If all cycles have even length and $\Sigma = \Sigma_u$, then
$\eta(\Sigma_u) = 2$ if and only if at least one among the C^j's has nullity 0, that is at least one among the C^j's is a C_0^- or a C_2^+ cycle, by Proposition 2.9, part (*ii*).

For $\Sigma = \Sigma_u$ with at least one odd cycle, we use Proposition 4.1 and get:

$\eta(\Sigma_u) = 2$ if one cycle, say C^1, is odd and the others, C^2 and C^3 have both nullity 2, or, C^1 and C^2 are odd, $\eta(C^3) = 2$ and $\eta(W_0) = \eta(C^1 \vee_u C^2) = 1$ (see the $c(\Sigma) = 2$ case above).

$\eta(\Sigma_u) = 1$ if one cycle, say C^1, is odd and the others, C^2 and C^3, are both even and at least one of them has nullity 0, or, C^1 and C^2 are both odd, with $\eta(W_0) = \eta(C^1 \vee_u C^2) = 0$ (see the $c(\Sigma) = 2$ case above) and $\eta(C^3) = 2$.

$\eta(\Sigma_u) = 0$ if C^1 and C^2 are both odd and C^3 is even of nullity 0.

Now we assume $\Sigma = \Sigma_{u,v}$.

Case 1: all cycles are even and at least one cycle has nullity 0.

$\eta(\Sigma_{u,v}) = 2$ if it belongs to one of the following groups:
a) exactly one cycle has nullity 0 an the other two cycles have nullity 2 (1.6 and 2.8, part (i)).
b) Two cycles have nullity 0 and the other, with nullity 2, is a pendant cycle or contains the two cut vertices, u and v, such that the distance $d(u, v)$ is even.
c) All cycles have nullity 0 and $d(u, v)$ is even.

$\eta(\Sigma_{u,v}) = 0$ if at most one cycle has nullity 2 and $d(u, v)$ is odd.

Case 2: at least one cycle has odd length.

We use this notation:

e_0 means an even cycle of nullity 0,

e_2 means a cycle of nullity 2 (it is necessarily even),

o means an odd cycle. A sequence of three of the above symbols represents a graph $\Sigma_{u,v}$ made by cycles of the marked type, considered in the indicated order. For example, e_2e_2o is a graph $\Sigma_{u,v}$ made by the sequence of two cycles of nullity 2 (with the vertex u in common) and an odd cycle attached to the second e_2 through the vertex v.

Case 2a. Exactly one cycle is odd.

$\eta(\Sigma_{u,v}) = 2$ in the cases: e_2e_2o, e_2oe_2.

$\eta(\Sigma_{u,v}) = 1$ for the following configurations: e_0e_0o and $d(u, v)$ is even, e_2e_0o, e_0e_2o and $d(u, v)$ is even, e_0oe_0, e_0oe_2.

$\eta(\Sigma_{u,v}) = 0$ in the cases e_0e_0o with $d(u, v)$ odd, and e_0e_2o with $d(u, v)$ odd.

Case 2b. Exactly two cycles are odd. The only even cycle can be pendant or not.

Case 2b.1. The even cycle is pendant.

The following results can be obtained by Corollary 2.8 applied to the even pendant cycle.

$\eta(\Sigma_{u,v}) = 2$ if $\Sigma_{u,v} = ooe_2$ and the two odd cycles have the same type and different signs or different types and the same sign ($\eta(oo) = 1$).

$\eta(\Sigma_{u,v}) = 1$ if $\Sigma_{u,v} = ooe_2$ and the two odd cycles have the same type and same sign or different types and different signs ($\eta(oo) = 0$).

$\eta(\Sigma_{u,v}) = 0$ if $\Sigma_{u,v} = ooe_0$ whatever the two odd cycles are.

Case 2b.2. The even cycle is not pendant.

The even cycle contains the two cut vertices u and v.

$\eta(oe_0o) = 1$ if $d(u,v)$ is odd and the two odd cycles have same type and same sign or different types and different signs.

$\eta(oe_0o) = 0$ in all the other cases (when $d(u,v)$ is even, $\eta(oe_0o) = 0$ can be deduced by Lemma 2.3 in [25] applied to the components of $\Sigma - \{v\}$).

$\eta(oe_2o) = 2$ if $d(u,v)$ is even and the two odd cycles have the same type and different signs or different types and same sign (this comes from Lemma 4.3 (ii) of [10]).

$\eta(oe_2o) = 1$ if $d(u,v)$ is even and the two odd cycles have the same type and the same signs or different types and different signs.

$\eta(oe_2o) = 0$ in all other cases, that is if $d(u,v)$ is odd, whatever the two odd cycles are (from Lemma 2.4 in [26] applied to the components of $\Sigma - \{v\}$).

The nullities $\eta(\Sigma_{u,v}) = 1$ or, $\eta(\Sigma_{u,v}) = 0$ when $\Sigma_{u,v} = oe_0o$ and $d(u,v)$ is odd, can be deduced by a direct computation of the coefficient a_0 in the characteristic polynomial $\Phi_{\Sigma_{u,v}}(x)$ (see Lemma 3.1), together with Lemma 4.9 in [10].

$\eta(\Sigma) = 0$ for any $\Sigma = \Sigma_u$ or $\Sigma = \Sigma_{u,v}$ having only odd cycles, by Theorem 1.8, part (i).

Summary for $c(\Sigma) = 3$.

The above results for $c(\Sigma) = 3$ can be summarized as follows:

$\eta(\Sigma) = 4$ if and only if $\eta(C^j) = 2$ for $j = 1, 2, 3$.

$\eta(\Sigma) = 2$ if and only if Σ is of one the following forms:

1) $\Sigma = \Sigma_u$ and

1.1 all cycles are even and at least one of them with nullity 0.

1.2 One cycle is odd and the other two cycle have nullity 2.

1.3 Two cycles are odd, of same type and different signs or of different types and same sign, and the other has nullity 2.

2) $\Sigma = \Sigma_{u,v}$ and

2.1 all cycles are even, one has nullity 0 and the other two cycles have nullity 2.

2.2 all cycles are even, two cycles have nullity 0, the other has nullity 2 and it is a pendand cycle or it contains the two cut vertices u and v such that $d(u.v)$ is even.

2.3 all cycles are even, have nullity 0 and $d(u.v)$ is even.

2.4 one cycle is odd and the other two cycles have nullity 2.

2.5 two cycle are odd, of the same type and different signs or of different types and the same sign, and the even cycle of nullity 2 is pendant.

2.6 two cycle are odd, of the same type and different signs or of different types and the same sign, the even cycle is not pendant and of nullity 2, and $d(u, v)$ is even.

$\eta(\Sigma) = 1$ if and only if Σ is of one the following forms:
1) $\Sigma = \Sigma_u$ and

1.1 one cycle is odd and the other two cycles are both even, at least one with nullity 0.

1.2 One cycle has nullity 2 and the other two cycles are both odd, of the same type and the same sign or of different types and different signs.
2) $\Sigma = \Sigma_{u,v}$ and

2.1 one cycle is odd, the other two cycles are even, at least one has nullity 0, and $d(u, v)$ is even when the odd cycle and the even one with nullity 0 are both pendant.

2.2 two cycles are odd, of the same type and the same sign or of different types and different signs, the remainig cycle is pendant of nullity 2 or it is even, not pendant, such that $d(u, v)$ is even.

$\eta(\Sigma) = 0$ if and only if Σ is of one the following forms:
1) $\Sigma = \Sigma_u$ and at most one cycle is even and of nullity 0.
2) $\Sigma = \Sigma_{u,v}$ and

2.1 all cycles are even, at most one of them has nullity 2 and $d(u, v)$ is odd.

2.2 one cycle is odd and pendant, the other two cycles are even, at most one of them has nullity 2 and $d(u, v)$ is odd.

2.3 two cycles are odd and the even cycle is pendant of nullity 0.

2.4 two cycles are odd, the even cycle of nullity 0 is not pendant, $d(u, v)$ is even, or, $d(u, v)$ is odd and the odd cycles have different type and same sign or same types and different signs.

2.5 two cycles are odd, the even cycle of nullity 2 is not pendant and $d(u, v)$ is odd.

2.6 all cycles are odd.

The general case when Σ has both even cycles and odd cycles is left to be resolved. It is worth studying because a cycle-spliced graph is a special cactus graph: the problem of singularity of cactus graphs seems interesting and deserves to be explored. We end by posing the following problem with possible application to Chemical Graph Theory:

Problem 4.3. *Determine the nullity of cycle-spliced (signed) graphs with exactly two pendant cycles.*

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