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Finite-time stabilization of discrete-time conewise linear systems

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ABSTRACT

The finite-time stabilizing control design problem for discrete-time conewise linear systems is tackled in this paper. Such a class of systems consists of the union of ordinary linear time-invariant subsystems, whose dynamics are defined in prescribed conical regions, constituting a conical partition of the state space. By imposing some cone-copositivity properties to a suitable piecewise quadratic function, two sufficient conditions are preliminarily derived concerning the system's finite-time stability. By building on them, novel results are then presented for the system's finite-time stabilization through a piecewise linear output feedback controller. Such results are based on the solution of feasibility problems involving sets of Linear Matrix Inequalities (LMIs). A numerical example illustrates the effectiveness of the proposed approach.

1. Introduction

Conewise linear systems (CLSs) are collections of ordinary linear subsystems whose dynamics are constrained to cones belonging to conical partitions of the state space [1–3]. At a higher level, they can be seen as a class of switched linear systems with a state-dependent switching rule, and, as such, equivalent to the interconnection of linear systems and finite automata [4,5]. Even though they seem a simple generalization of classical linear systems, their stability analysis turns out to be "surprisingly difficult to characterize" [1]. The main reason is due to their intrinsic hybrid nature, which anyway justifies the major interest in their study, corroborated by significant applications in multi-modal systems [6]. Indeed, continuous time CLSs (discrete-time CLSs are their sampled version) forms a class of Lipschitz piecewise linear systems subject to state-triggered mode switching [7]. Some relevant examples can be found in [8], such as bimodal piecewise linear systems with continuous vector fields, and, in the context of linear complementarity systems, linear cone complementarity systems. More examples, from various areas of engineering as well as operations research, are available in [9,10]. A big effort has been spent recently on their classical Lyapunov stability analysis and stabilizing control design, see, e.g., [1,2,11]. Further approaches can be mentioned if we consider the more general category of switched systems, both in the continuous-time (see, e.g., [5,12,13]) and the discrete-time context (see, e.g., [2,14,15]). On the contrary, a few results are available about their finite-time stability (FTS) as well as finite-time stabilization, which require a separate discussion.

Actually, there exist two main distinct definitions of FTS in the literature. On one side, a system is said to be finite-time stable if all the state trajectories converge to the origin in finite time, when starting from a given initial domain [16–18]. Of course, such a definition needs system asymptotic Lyapunov stability as a pre-requisite. In this paper, we refer to an alternative FTS notion as originally proposed in [19,20] and then re-introduced more recently in, e.g., [21–23]. Roughly speaking, we call a system finite-time stable if, given a bound on the initial condition, its state remains within a specified region over a prefixed time interval. Also, if a static output (state) feedback controller can be found that makes a closed-loop system finite-time stable, we say that system to be

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finite-time stabilizable through an output (state) feedback control. Being directly related to the system's transient behavior, such concepts have a different application field and meaning with respect to Lyapunov stability and stabilization [23].

A renewed and growing interest in recent years about this topic is testified by several papers for various categories of systems, see, among others, [24–29]. However, a limited literature exists dealing with the FTS analysis of switched systems with trajectory-dependent switching signals specifically, and even fewer works have been dedicated to the related finite-time stabilization problem [30,31]. In [30] the problem of both finite-time stabilization and boundedness is addressed for switched continuous-time systems subject to state-dependent switching. The regions where the system modes are activated are assumed to be represented through quadratic functions. In [31] the FTS and stabilization via both state and output feedback for discrete-time linear systems with state-dependent disturbances are discussed. Even though the FTS concept is the same we adopt in this paper, the system description differs. Indeed, in [31] the initial and the trajectory domains are assumed to be polyhedral, possibly unbounded and the former can be not necessarily a subset of the latter. Moreover, the system state matrix is unique, whereas the disturbance signals belong to a time-varying polyhedral set.

This paper focuses on the FTS and stabilization issues for discrete-time conewise linear systems. After recalling some preliminary notions about cones, matrix cone-copositiveness, and piecewise quadratic domains in Section 2.1, the system mathematical model considered is reported in Section 2.2, together with the solution concept and the formal FTS definition adopted. The initial and the trajectory domains are assumed to be piecewise quadratic regions, which generalize the classical ellipsoidal domains. In the same section, two sufficient conditions to prove the FTS of discrete-time conewise linear systems are provided based on the use of a suitable piecewise quadratic function. The problem of finite-time stabilization via a piecewise linear static output feedback is discussed in Section 3, where some novel theoretical results are presented that allow avoiding the recourse to Bilinear Matrix Inequalities (BMI) or to overly conservative constraints relaxations. To show the effectiveness of the approach, an example is then illustrated in Section 4. Finally, Section 5 concludes the paper.

2. Preliminaries

2.1. Notation and notions

Given an index set $\mathcal{I} = \{1, \dots, p\}$, a polyhedral conical partition of $\mathbb{R}^n$ is a collection of polyhedral cones $\mathcal{C} = \{\mathcal{C}_i\}_{i\in\mathcal{I}}$ satisfying $\bigcup_{i\in\mathcal{I}} \mathcal{C}_i = \mathbb{R}^n$ and $\mathrm{int}\{\mathcal{C}_i\}\cap\mathrm{int}\{\mathcal{C}_j\} = \emptyset$, for all $i \neq j$, where $\mathrm{int}(X)$ denotes the (relevant) interior of a set X. Each cone of the partition is closed and can be represented through its $\mathcal{V}$-representation, i.e. as the conical hull of a finite number of rays (or generators), as follows

$$\mathcal{C}_i = \{x : x = E_i\theta, \theta \in \mathbb{R}^m_+\}, \tag{1}$$

where $E_i \in \mathbb{R}^{n\times m}$ is the so-called ray matrix of the cone, whose columns are the cone extremal rays. An extremal ray of a polyhedral cone is any nonzero vector of the cone that cannot be expressed as a positive linear combination of two other nonzero vectors in the cone. It is important to note that the extremal rays are defined up to a positive scalar multiple, i.e., for a given polyhedral cone, there are different ray matrices that can generate the same cone. For full-dimensional cones, it must be $m \geq n$. In case $m = n$ the cone is named simplicial and is characterized by an invertible ray matrix. If $m < n$ the cone is said to be degenerate. Without loss of generality, we assume that the conical partition enjoys the face-to-face property, i.e. the intersection between two cones of the partition is either the origin or a whole common face, which is still a polyhedral (although degenerate) cone, whose ray matrix is made of the common rays of the two cones.

A symmetric matrix $M \in \mathbb{R}^{n\times n}$ is said to be cone-copositive with respect to a cone $\mathcal{C}_i$ of the partition if it is $x^TMx \geq 0$ for any $x \in \mathcal{C}_i$, and the notation is $M \geq_{\mathcal{C}_i} 0$. If equality only holds for $x = 0$, then M is said strictly cone-copositive and is denoted by $M >_{\mathcal{C}_i} 0$. In the particular case $\mathcal{C}_i = \mathbb{R}^n_+$ (i.e., the nonnegative orthant), a (strictly) cone-copositive matrix is called (strictly) copositive.

A piecewise quadratic function (PQF) defined over a conical partition $\mathcal{C}$ is a function of the type

$$\mathcal{F}_{\mathcal{C}}(x) = x^TF_ix, \quad x \in \mathcal{C}_i, \quad i \in \mathcal{I}, \tag{2}$$

where $F_i \in \mathbb{R}^{n\times n}$ are symmetric positive definite matrices in the cone $\mathcal{C}_i$. To enforce the continuity of (2) over $\mathcal{C}$, the condition

$$x^TF_ix = x^TF_jx, \quad x \in \mathcal{C}_i \cap \mathcal{C}_j, \tag{3}$$

has to be satisfied for all $i, j \in \mathcal{I}$, $i \neq j$, such that $\mathcal{C}_i \cap \mathcal{C}_j \setminus \{0\} \neq \emptyset$.[1] Being the common face between two cones of the partition still a polyhedral cone, condition (3) can be equivalently expressed by

$$E_{ij}^T(F_i - F_j)E_{ij} = 0, \tag{4}$$

where E_{ij} is the ray matrix composed of the common extremal rays between $\mathcal{C}_i$ and $\mathcal{C}_j$, see Lemma 8 in [3].

We can define a piecewise quadratic domain (PQD) over the conical partition $\mathcal{C}$ of a continuous PQF $\mathcal{F}_{\mathcal{C}}(x)$ as

$$\mathcal{X}_{\mathcal{F}_{\mathcal{C}}} = \{x : \mathcal{F}_{\mathcal{C}}(x) \leq 1, x \in \mathcal{C}_i, i \in \mathcal{I}\}, \tag{5}$$

i.e., as a compact set delimited by the unitary level curve of a PQF whose matrices satisfy the continuity conditions (4).

The set defined in (5) is a generalization of ellipsoidal domains, which is obtained when $F_i = F > 0$, for all $i \in \mathcal{I}$.

[1] Obviously, being the cones of the partition also pointed, they all share the origin as common point, where the function (2) is continuous by definition.

2.2. Finite-time stability of conewise systems

Let us consider the discrete-time conewise linear system

$$x(k+1) = A_i x(k), \quad x(0) = x_0, \; x(k) \in \mathcal{C}_i, \tag{6}$$

where $x \in \mathbb{R}^n$, $A_i \in \mathbb{R}^{n\times n}$, $i \in \mathcal{I}$, defined over a polyhedral conical partition $\mathcal{C} = \{\mathcal{C}_i\}_{i\in\mathcal{I}}$ of the whole state space. The solution to (6) from a given $x(0) = x_0$, say $x(k; x_0)$, always exists, but it can be not unique, in general, in the sense that the states sequence can be not uniquely determined from the initial condition. Indeed, on the common boundaries of the cones, the dynamics are ambiguous, and the state evolution depends on which rule is adopted to decide which dynamics to apply. If well-posedness is required, a continuity condition of the right-hand side of (6) on the partition boundaries can be imposed [32], or the conical partition $\mathcal{C}$ can be assumed to be strict [33], i.e. with pairwise disjoint and, hence, not necessarily closed cones, as similarly done in [2,12].

In general terms, we say system (6) to be finite-time stable with respect to given initial and trajectory domains if starting from any state in the initial domain, all the possible trajectories evolve within the trajectory domain during a prescribed interval of time.[2] More formally, we adapt the FTS concept in [23] to the discrete-time conewise linear systems class by choosing the initial and the trajectory domains as PQDs, as shown below.

Definition 1. Given a positive integer N and two PQDs, say $\mathcal{X}_{\mathcal{R}_\mathcal{C}}$ and $\mathcal{X}_{\Gamma_\mathcal{C}}$, defined over the conical partition $\mathcal{C}$ as

$$\mathcal{X}_{\mathcal{R}_\mathcal{C}} = \{x : \mathcal{R}_\mathcal{C}(x) \le 1, \, x \in \mathcal{C}_i, \, i \in \mathcal{I}\}, \tag{7}$$

$$\mathcal{X}_{\Gamma_\mathcal{C}} = \{x : \Gamma_\mathcal{C}(x) \le 1, \, x \in \mathcal{C}_i, \, i \in \mathcal{I}\}, \tag{8}$$

where $\mathcal{R}_\mathcal{C}(x)$ and $\Gamma_\mathcal{C}(x)$ are piecewise quadratic functions defined as

$$\mathcal{R}_\mathcal{C}(x) = x^T R_i x, \quad x \in \mathcal{C}_i, \quad i \in \mathcal{I}, \tag{9}$$

$$\Gamma_\mathcal{C}(x) = x^T \Gamma_i x, \quad x \in \mathcal{C}_i, \quad i \in \mathcal{I}, \tag{10}$$

with R_i, Γ_i symmetric positive definite matrices in the cone $\mathcal{C}_i$, satisfying $E_{ij}^T(R_i - R_j)E_{ij} = 0$, $E_{ij}^T(\Gamma_i - \Gamma_j)E_{ij} = 0$, respectively, $\forall i, j \in \mathcal{I}$ such that $\mathcal{C}_i \cap \mathcal{C}_j \setminus \{0\} \neq \emptyset$, the conewise linear discrete-time system (6) is said to be finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$ if

$$x_0 \in \mathcal{X}_{\mathcal{R}_\mathcal{C}} \implies x(k; x_0) \in \mathcal{X}_{\Gamma_\mathcal{C}}, \forall k \in \{0, \dots, N\}, \tag{11}$$

for any trajectory $x(k; x_0)$ starting from x_0.

Note that the inclusion of continuity conditions in the above definition is a common assumption in the literature of FTS, see, e.g., [23], and it is essential to prevent ambiguity regarding the classification of boundary points within the initial or trajectory domains.

A sufficient condition for the FTS of (6) in the sense of Definition 1 is given by the following result.

Theorem 2. *Given a positive integer N, two PQDs, $\mathcal{X}_{\mathcal{R}_\mathcal{C}}$ and $\mathcal{X}_{\Gamma_\mathcal{C}}$, defined over a conical partition $\mathcal{C}$ of $\mathbb{R}^n$ as in (7), (8), and a real scalar $\gamma \ge 1$, the conewise system (6) is finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$ if there exist a piecewise quadratic function*

$$\mathcal{P}_\mathcal{C}(x) = x^T P_i x, \quad x \in \mathcal{C}_i, \, i \in \mathcal{I}, \tag{12}$$

i.e., symmetric positive definite matrices P_i, and a positive real number λ, verifying the conditions

$$\gamma P_i - A_i^T P_j A_i >_{\mathcal{C}_i} 0, \tag{13a}$$

$$P_i - \Gamma_i \ge_{\mathcal{C}_i} 0, \tag{13b}$$

$$\lambda R_i - P_i \ge_{\mathcal{C}_i} 0, \tag{13c}$$

$$1 - \gamma^N \lambda \ge 0, \tag{13d}$$

for all $i, j \in \mathcal{I}$.

Proof. Let us consider the PQF $\mathcal{P}_\mathcal{C}(x)$ in (12) and an $x_0 \in \mathcal{X}_{\mathcal{R}_\mathcal{C}}$, i.e., such that $\mathcal{R}_\mathcal{C}(x_0) \le 1$. Note that, thanks to the continuity of (9), the condition $\mathcal{R}_\mathcal{C}(x_0) \le 1$ is well-defined also for initial points chosen on the boundaries of the cones. From (13b), (13c) it is

$$\Gamma_\mathcal{C}(x(k)) \le \mathcal{P}_\mathcal{C}(x(k)), \tag{14a}$$

$$\mathcal{P}_\mathcal{C}(x(k)) \le \lambda \mathcal{R}_\mathcal{C}(x(k)), \tag{14b}$$

[2] By extending the concepts proposed in [2, Sec.II], we could further distinguish between a *strong* and a *weak* finite-time stability, depending on whether such property is required for *any* trajectory starting from x_0 or for *at least one*. In this paper, we refer to a *strong* finite-time stability definition.

for any trajectory $x(k;x_0)=x(k)$,[3] $k\in\{0,\dots,N\}$.

Without any loss of generality, assume that $x(k+1)\in\mathcal{C}_j$ and $x(k)\in\mathcal{C}_i$. Condition (13a) then implies that

$$\mathcal{P}_{\mathcal{C}}(x(k+1))<\gamma\mathcal{P}_{\mathcal{C}}(x(k)). \tag{15}$$

By iterating (15), it is

$$\mathcal{P}_{\mathcal{C}}(x(k))<\gamma^k\mathcal{P}_{\mathcal{C}}(x_0),\ \forall k\in\{1,\dots,N\}. \tag{16}$$

Being $\gamma\geq 1$ and $x_0\in\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$, we can write from (14b)

$$\gamma^k\mathcal{P}_{\mathcal{C}}(x_0)\leq\gamma^N\lambda\mathcal{R}_{\mathcal{C}}(x_0)\leq\gamma^N\lambda. \tag{17}$$

From (14a), (16) and (17), we have

$$x(k)^T\Gamma_i x(k)\leq x(k)^T P_i x(k)<\gamma^k x_0^T P_i x_0\leq\gamma^N\lambda,\ \forall i\in\mathcal{I}, \tag{18}$$

for any $x_0\in\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ and for all $k\in\{1,\dots,N\}$. Hence, from (13d) it implies

$$x(k)^T\Gamma_i x(k)<\gamma^N\lambda\leq 1,\ \forall i\in\mathcal{I},\ \forall k\in\{1,\dots,N\}. \tag{19}$$

From (14a), (14b), it is also $x_0^T\Gamma_i x_0\leq x_0^T P_i x_0\leq 1$, $\forall i\in\mathcal{I}$, i.e., we can conclude that $\Gamma_{\mathcal{C}}(x(k))\leq 1$, for all $k\in\{0,\dots,N\}$. Note that, thanks to the continuity of (10), condition $\Gamma_{\mathcal{C}}(x(k))\leq 1$ is well-defined and unambiguously implies $x(k)\in\mathcal{X}_{\Gamma_{\mathcal{C}}}$ also for points $x(k)$ located on the boundaries of the cones. As a result, we proved that if $x_0\in\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ then $x(k)\in\mathcal{X}_{\Gamma_{\mathcal{C}}}$, $\forall k\in\{0,\dots,N\}$, for all possible solutions, that is to say, the system (6) is finite-time stable with respect to $(0,N,\mathcal{X}_{\mathcal{R}_{\mathcal{C}}},\mathcal{X}_{\Gamma_{\mathcal{C}}})$. □

Before introducing some relevant comments on the above finite time stability result, it is worth noticing that the simultaneous scaling of the initial and trajectory domains do not affect the feasibility of the LMI set (13), being the solution to the LMIs (i.e., the set of matrices that satisfy the inequalities) involved not unique and defined up to a multiplying constant. Such a consideration holds for all the theoretical results presented in this paper. Moreover, as regards the scalar parameters, while λ is an optimization variable, γ is heuristically determined in order to avoid BMIs.

Remark 3. In (13a) all the combinations among the cones are considered since the state can jump to non-adjacent cones. Such an approach can be overly conservative since it would be sufficient to include only the pairs of cones involved in a one-step trajectory evolution. To actually determine all the cone transitions allowed, a reachability analysis has to be performed, which is known to be a linear programming problem [34]. In the framework of this paper, a simplified approach can be employed. Indeed, the image of each polyhedral cone of the partition under the linear map represented by the matrix of the associated linear dynamics is still a polyhedral cone, whose ray matrix is obtained by multiplying the system matrix with the ray matrix of the original cone. This way the image cone can be determined and the number of combinations to consider (i.e., the number of LMIs) can be reduced if such a cone is contained in one of the cones of the partition.

Remark 4. Conditions (13c), (13d) can be generalized to the case of different parameters $\lambda_i>0$ for each cone $\mathcal{C}_i$ of the partition. Indeed, having a unique parameter λ for all the cones corresponds to considering $\lambda=\max_{i\in\mathcal{I}}\lambda_i$, which is always well defined being the conic partition finite.

Remark 5. Suppose the condition (13a) holds with $\gamma=1$. In that case, the PQF (12) is strictly decreasing along the system trajectories, which, via standard Lyapunov arguments, implies that the system (6) is also asymptotically stable [11].

Remark 6. In Theorem 2, the PQF (12) does not need to be continuous across the boundaries of the cones to prove the system finite-time stability, i.e. the matrices P_i are not required to satisfy the continuity condition (4).

The FTS conditions in Theorem 2 are not operative, since they require the sign checking for all the points of each cone of the partition. The next theorem provides a sufficient condition based on the feasibility of a set of LMIs, i.e. of a convex problem [35,36], that can be numerically solved in an efficient way [37].

Theorem 7. *Given a positive integer N, two PQDs, $\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ and $\mathcal{X}_{\Gamma_{\mathcal{C}}}$, defined over a conical partition $\mathcal{C}$ of $\mathbb{R}^n$, and a real scalar $\gamma\geq 1$, the conewise system (6) is finite-time stable with respect to $(0,N,\mathcal{X}_{\mathcal{R}_{\mathcal{C}}},\mathcal{X}_{\Gamma_{\mathcal{C}}})$ if there exist symmetric positive definite matrices P_i, symmetric entrywise positive matrices T_i, symmetric entrywise nonnegative matrices U_i, W_i, of appropriate dimensions, for $i\in\mathcal{I}$, and a positive real number λ, such that the set of LMIs*

$$E_i^T(\gamma P_i-A_i^T P_j A_i)E_i-T_i>0, \tag{20a}$$

$$E_i^T(P_i-\Gamma_i)E_i-U_i\geq 0, \tag{20b}$$

[3] For notation simplicity, we omit the dependency on the initial condition.

$$E_i^T(\lambda R_i - P_i)E_i - W_i \geq 0, \tag{20c}$$

$$1 - \gamma^N \lambda \geq 0, \tag{20d}$$

is feasible, for all $i, j \in \mathcal{I}$.

Proof. Let us consider the function $\mathcal{P}_{\mathcal{C}}(x)$ defined in (12), together with the PQDs $\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ and $\mathcal{X}_{\Gamma_{\mathcal{C}}}$ defined in (7) and (8), respectively. We know from Theorem 2 that conditions (13a)–(13d) imply the FTS of the discrete-time conewise linear system (6) with respect to the PQDs $\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ and $\mathcal{X}_{\Gamma_{\mathcal{C}}}$. Furthermore, any (strict) cone-copositive condition on a convex polyhedral cone, as in (13a), (13b), (13c) can be transformed into a (strict) copositive condition by using the cone ray matrix [3]. More specifically, it is

$$\gamma P_i - A_i^T P_j A_i >_{\mathcal{C}_i} 0 \iff E_i^T(\gamma P_i - A_i^T P_j A_i)E_i >_{\mathbb{R}_+^n} 0, \tag{21a}$$

$$P_i - \Gamma_i \geq_{\mathcal{C}_i} 0 \iff E_i^T(P_i - \Gamma_i)E_i \geq_{\mathbb{R}_+^n} 0, \tag{21b}$$

$$\lambda R_i - P_i \geq_{\mathcal{C}_i} 0 \iff E_i^T(\lambda R_i - P_i)E_i \geq_{\mathbb{R}_+^n} 0. \tag{21c}$$

Now conditions (20a), (20b), (20c), with T_i entrywise positive matrices and U_i, W_i entrywise nonnegative matrices, imply the corresponding (strict) copositive conditions in (21a), (21b), (21c) (see Lemma 7 in [38]), and, hence, (13a), (13b), (13c). Finally, if also (20d) is satisfied, by virtue of Theorem 2 we can then conclude that the system (6) is finite-time stable in the sense of Definition 1. □

3. Finite-time stabilization via static output feedback

The FTS notion can be generalized to the framework of finite-time stabilization via static output feedback control by considering the discrete-time conewise linear system in a closed-loop with a static controller and setting the related FTS problem. This topic is sufficiently general, considering that any dynamic output feedback controller of an order less or equal to the system's order can be returned to the static output feedback case [39]. More formally, we tackle the following problem.

Problem 8. Consider the controlled discrete-time conewise linear system, having the representation

$$x(k+1) = A_i x(k) + B_i u(k),\ x(0) = x_0,\ x(k) \in \mathcal{C}_i, \tag{22a}$$

$$y(k) = C_i x(k), \tag{22b}$$

defined over a polyhedral conical partition $\mathcal{C} = \{\mathcal{C}_i\}_{i\in\mathcal{I}}$ of the whole state space, where $u(k) \in \mathbb{R}^r$ is the feedback control input and $y(k) \in \mathbb{R}^q$ is the output. Given a positive integer N and two PQDs $\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}, \mathcal{X}_{\Gamma_{\mathcal{C}}}$, the finite-time stabilization problem via (piecewise linear) static output feedback for the system (22) consists in finding feedback gain matrices $K_i \in \mathbb{R}^{r\times q}$ such that the controller $u(k) = K_i C_i x(k)$, when $x(k) \in \mathcal{C}_i$, $i \in \mathcal{I}$, makes the closed-loop system

$$x(k+1) = (A_i + B_i K_i C_i)x(k),\ x(0) = x_0,\ x(k) \in \mathcal{C}_i, \tag{23a}$$

$$y(k) = C_i x(k), \tag{23b}$$

finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_{\mathcal{C}}}, \mathcal{X}_{\Gamma_{\mathcal{C}}})$.

Note that the static output controller includes as a special case the state feedback controller when all the output matrices C_i are identity matrices I.

To solve Problem 8, the results of Theorem 7 can be in principle applied to the system (23), by replacing in (20a) the matrices A_i with the closed-loop system matrices $A_i + B_i K_i C_i$. However, the LMI conditions (20a) turn into the inequalities

$$E_i^T(\gamma P_i - (A_i + B_i K_i C_i)^T P_j (A_i + B_i K_i C_i))E_i - T_i > 0, \tag{24}$$

for all $i, j \in \mathcal{I}$, which are not jointly convex in the variables P_j, K_i. One possible approach consists in resorting to a Bilinear Matrix Inequality (BMI) formulation, which is known to be nonconvex and not to provide any guarantee to find an optimal (or even a feasible) solution, in general, [5]. In case of state feedback, a usual method can be applied, which requires the relaxation of all the original cone-copositive conditions in Theorem 2, rewritten for the closed-loop system (23) with $C_i = I$, to the whole state space and the use of Schur complements [12,35]. However, such an approach is conservative and, in addition, cannot be extended to the case of output feedback.

In the next subsections we will show that, under the hypothesis of input matrices B_i of full column rank, or of output matrices C_i of full row rank, convexity can be recovered, even in the static output feedback case, without resorting to a relaxation to the whole state space. In particular, in order to obtain analogous LMI conditions, we will start from the results of Theorem 7 (which stem from Theorem 2) and combine them with a suitable generalization and adaptation of the ideas behind the so-called P-problem and Q-problem [40]. These ideas are well-established in the literature for the standard (Lyapunov) asymptotic stabilization problem and specifically for a single LTI system. Their extension to systems with multiple dynamics and their integration into the framework of output feedback finite-time stabilization of conewise linear systems represent the main contribution of this section.

3.1. Finite-time stabilization with B_i full column rank

Suppose that the matrices B_i in the closed-loop model (23) are full column rank. We can extend the FTS result of Theorem 7 to solve Problem 8 as detailed below.

Theorem 9. *Consider the closed-loop discrete-time conewise linear system (23) with B_i full column rank. Choose a positive integer N, two PQDs, $\mathcal{X}_{\mathcal{R}_{\mathcal{C}}}$ and $\mathcal{X}_{\Gamma_{\mathcal{C}}}$, and a real scalar $\gamma \geq 1$. If there exist symmetric positive definite matrices P_i, symmetric entrywise positive matrices T_i, symmetric entrywise nonnegative matrices U_i, W_i, of appropriate dimensions, and a positive real number λ, such that set of LMIs*

$$\begin{pmatrix} E_i^T \gamma P_i E_i - T_i & E_i^T (A_i^T P_j + C_i^T G_i^T B_i^T) \\ (P_j A_i + B_i G_i C_i) E_i & P_j \end{pmatrix} > 0, \tag{25a}$$

$$E_i^T (P_i - \Gamma_i) E_i - U_i \geq 0, \tag{25b}$$

$$E_i^T (\lambda R_i - P_i) E_i - W_i \geq 0, \tag{25c}$$

$$1 - \gamma^N \lambda \geq 0, \tag{25d}$$

is satisfied for all $i, j \in \mathcal{I}$, then the control law $u(k) = K_i C_i x(k)$, $x(k) \in \mathcal{C}_i$, with $K_i = F_i^{-1} G_i$ and F_i such that $B_i F_i = P_j B_i$, makes the controlled conewise system (23) finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_{\mathcal{C}}}, \mathcal{X}_{\Gamma_{\mathcal{C}}})$.

Proof. The proof resumes the arguments we have introduced above. Let us consider a static output feedback control law for the controlled conewise linear system (22) of the type $u(k) = K_i C_i x(k)$, for $x(k) \in \mathcal{C}_i$, and a piecewise quadratic function $\mathcal{P}_{\mathcal{C}}(x) = x^T P_i x$, $x \in \mathcal{C}_i$, $i \in \mathcal{I}$. Being B_i of full column rank, it is useful to apply (an adaptation of) the feasibility problem known as P-problem in [40, Sec. III]. Indeed, given the matrices A_i, B_i, C_i, with B_i of full column rank for all $i \in \mathcal{I}$, and the scalar $\gamma \geq 1$, we can deduce from [40] that if there exist symmetric positive definite matrices P_i, symmetric entrywise positive matrices T_i, and matrices G_i, of appropriate dimensions, that solve the set of conditions

$$\begin{pmatrix} E_i^T \gamma P_i E_i - T_i & E_i^T (A_i^T P_j + C_i^T G_i^T B_i^T) \\ (P_j A_i + B_i G_i C_i) E_i & P_j \end{pmatrix} > 0, \tag{26}$$

for all $i, j \in \mathcal{I}$, then the inequalities (24) are satisfied with $K_i = F_i^{-1} G_i$, being F_i a solution of $B_i F_i = P_j B_i$.[4] As a result, if (25a) hold then there exist matrices F_i such that $B_i F_i = P_j B_i$, and, moreover, for $K_i = F_i^{-1} G_i$ conditions (24) are satisfied. If also (25b)–(25d) hold, by following the same steps of the proofs of Theorems 2 and 7, we can conclude that the feedback control law $u(k) = K_i C_i x(k)$, for $x(k) \in \mathcal{C}_i$, with $K_i = F_i^{-1} G_i$, $i \in \mathcal{I}$, renders the closed loop system (23) finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_{\mathcal{C}}}, \mathcal{X}_{\Gamma_{\mathcal{C}}})$. □

3.2. Finite-time stabilization with C_i full row rank

If the input matrices B_i are not full column rank, under the hypothesis that the output matrices C_i are full row rank and the cones $\mathcal{C}_i$ of the partition are full-dimensional and simplicial, it is still possible to recast the finite-time stabilization conditions as a set of LMIs. The latter assumption is sufficiently general, by considering that any full-dimensional convex polyhedral cone can be subdivided into a finite number of simplicial cones [41], and any cone-copositive condition on it can then be equivalently replaced by a finite set of sign conditions on the simplicial cones of its partition.

Let us consider the cone-copositive conditions $\gamma P_i - A_i^T P_j A_i >_{\mathcal{C}_i} 0$, $i, j \in \mathcal{I}$ preliminarily, and assume that the cones $\mathcal{C}_i$ are simplicial, i.e. the ray matrices E_i are square and invertible, for all $i \in \mathcal{I}$. Such inequalities are cone-constrained and can be equivalently rewritten in terms of the matrix $Q_i = P_i^{-1}$ by pre and post multiplying by Q_i:

$$\gamma P_i - A_i^T P_j A_i >_{\mathcal{C}_i} 0 \iff \gamma Q_i - Q_i A_i^T P_j A_i Q_i \geq_{\tilde{\mathcal{C}}_i} 0. \tag{27}$$

Indeed, we are substituting the strict cone-copositive condition $x^T(\gamma P_i - A_i^T P_j A_i)x > 0$, $x \in \mathcal{C}_i$, with the equivalent $y^T(\gamma Q_i - Q_i A_i^T P_j A_i Q_i)y > 0$, $y \in \tilde{\mathcal{C}}_i$, where $y = P_i x \in \tilde{\mathcal{C}}_i$, i.e. $\tilde{\mathcal{C}}_i$ is still a simplicial cone, whose ray matrix is $P_i E_i$. We are interested in the situation where these two cones coincide. The next Lemma provides an answer to this issue.

Lemma 10. *The two simplicial cones $\mathcal{C}_i$ and $\tilde{\mathcal{C}}_i$ coincide if the matrix $E_i^{-1} P_i E_i$ is a nonnegative monomial matrix.*[5]

Proof. For each point $x^* \in \mathcal{C}_i$ there exists a $\theta^* \in \mathbb{R}_+^n$ such that $x^* = E_i \theta^*$, which is $\theta^* = E_i^{-1} x^* \geq 0$ by construction. Such a point belongs also to $\tilde{\mathcal{C}}_i$, because there always exists a nonnegative $\tilde{\theta} = E_i^{-1} Q_i E_i \theta^* \geq 0$ (being $E_i^{-1} Q_i E_i$ a nonnegative matrix,[6]) such that it is also $x^* = P_i E_i \tilde{\theta}$, i.e. $\mathcal{C}_i \subseteq \tilde{\mathcal{C}}_i$. Vice versa, for each $y^* \in \tilde{\mathcal{C}}_i$, there exists a $\theta^* \in \mathbb{R}_+^n$ such that $y^* = P_i E_i \theta^*$, which is $\theta^* = E_i^{-1} Q_i y^* \geq 0$ by construction. This point belongs alto to $\mathcal{C}_i$, because there always exists a nonnegative $\tilde{\theta} = E_i^{-1} P_i E_i \theta^* \geq 0$ (being $E_i^{-1} P_i E_i$ a nonnegative matrix), such that it is also $y^* = E_i \tilde{\theta}$, i.e. $\tilde{\mathcal{C}}_i \subseteq \mathcal{C}_i$. As a result, it is $\tilde{\mathcal{C}}_i \equiv \mathcal{C}_i$. □

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[4] Since B_i is full column rank, F_i always exists and is invertible [40].

[5] A nonnegative *monomial* matrix is a nonnegative square matrix with exactly one element in each row and column which is not 0, i.e. it is the permutation of a positive diagonal matrix.

[6] The inverse of a nonnegative matrix is still nonnegative *iff* it is monomial [42].

If Lemma 10 holds $\forall i \in \mathcal{I}$, then, analogously to (21a), it is

$$\gamma Q_i - Q_i A_i^T P_j A_i Q_i >_{\mathcal{C}_i} 0 \iff E_i^T(\gamma Q_i - Q_i A_i^T P_j A_i Q_i)E_i >_{\mathbb{R}_+^n} 0. \tag{28}$$

Conditions (28), in turn, are implied by the inequalities:

$$\gamma E_i^T Q_i E_i - E_i^T Q_i A_i^T P_j A_i Q_i E_i - V_i > 0,\ \forall i, j \in \mathcal{I}, \tag{29}$$

where V_i are symmetric entrywise positive matrices to be determined [38]. Thanks to the Schur complements formula, conditions (29) is implied by:

$$\begin{pmatrix} E_i^T \gamma Q_i E_i - V_i & E_i^T Q_i A_i^T \\ A_i Q_i E_i & Q_j \end{pmatrix} > 0,\ \forall i, j \in \mathcal{I}. \tag{30}$$

If we now replace A_i with the closed-loop state matrices $A_i + B_i K_i C_i$, conditions (30) become

$$\begin{pmatrix} E_i^T \gamma Q_i E_i - V_i & E_i^T Q_i (A_i + B_i K_i C_i)^T \\ (A_i + B_i K_i C_i) Q_i E_i & Q_j \end{pmatrix} > 0,\ \forall i, j \in \mathcal{I}, \tag{31}$$

which are nonconvex in Q_i, K_i. The following result shows how an alternative sufficient condition for the static output stabilizability of system (23) can be derived by turning (31) into LMIs.

Theorem 11. *Consider the closed-loop discrete-time conewise linear system (23) and suppose all the cones of the partition $\mathcal{C}_i$ be simplicial and all the output matrices C_i be full row rank. Choose a positive integer N, two PQDs, $\mathcal{X}_{\mathcal{R}_\mathcal{C}}$ and $\mathcal{X}_{\Gamma_\mathcal{C}}$, and a positive real scalar $\gamma \geq 1$. If there exist symmetric positive definite matrices Q_i, with $E_i^{-1} Q_i E_i$ nonnegative monomial matrices, symmetric entrywise positive matrices V_i, symmetric entrywise nonnegative matrices U_i, W_i, matrices G_i, of appropriate dimensions, and a positive real number λ, such that the set of LMIs*

$$\begin{pmatrix} E_i^T \gamma Q_i E_i - V_i & E_i^T (Q_i A_i^T + C_i^T G_i^T B_i^T) \\ (A_i Q_i + B_i G_i C_i) E_i & Q_j \end{pmatrix} > 0, \tag{32a}$$

$$\begin{pmatrix} E_i^T Q_i E_i - U_i & E_i^T Q_i \\ Q_i E_i & \Gamma_i^{-1} \end{pmatrix} \geq 0, \tag{32b}$$

$$\begin{pmatrix} E_i^T \lambda R_i E_i - W_i & E_i^T \\ E_i & Q_i \end{pmatrix} \geq 0, \tag{32c}$$

$$1 - \gamma^N \lambda \geq 0, \tag{32d}$$

is satisfied for all $i, j \in \mathcal{I}$, then the control law $u(k) = K_i C_i x(k)$, $x(k) \in \mathcal{C}_i$, with $K_i = G_i F_i^{-1}$ and F_i such that $F_i C_i = C_i Q_j$, makes the controlled conewise system (23) finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$.

Proof. The finite-time stabilization conditions obtained by applying Theorem 2 to the controlled conewise system (23) are expressed in terms of matrices P_i. Such conditions can be expressed in terms of $Q_i = P_i^{-1}$ when the cones $\mathcal{C}_i$ of the partition are simplicial and $E_i^{-1} Q_i E_i$ are monomial matrices. We know from the so-called Q-problem, adapted from [40, Sec.III] that given the matrices A_i, B_i, C_i, with C_i of full row rank for all $i \in \mathcal{I}$, and the scalar $\gamma \geq 1$, if there exist symmetric positive definite matrices Q_i, symmetric entrywise positive matrices V_i, and matrices G_i, of appropriate dimensions, that solve the set of conditions

$$\begin{pmatrix} E_i^T \gamma Q_i E_i - V_i & E_i^T (Q_i A_i^T + C_i^T G_i^T B_i^T) \\ (A_i Q_i + B_i G_i C_i) E_i & Q_j \end{pmatrix} > 0,\ \forall i, j \in \mathcal{I}. \tag{33}$$

for all $i, j \in \mathcal{I}$, then the inequalities (31) are satisfied with $K_i = G_i F_i^{-1}$, being F_i a solution of $F_i C_i = C_i Q_j$.[7] As a result, if (32a) hold and there exist matrices F_i such that $F_i C_i = C_i Q_j$, then for $K_i = G_i^{-1} F_i$ conditions (31) are satisfied for all $i, j \in \mathcal{I}$. In turn, we have shown that such inequalities, in the case of simplicial cones and $E_i^{-1} Q_i E_i$ monomial matrices, imply

$$\gamma P_i - (A_i + B_i K_i)^T P_j (A_i + B_i K_i) >_{\mathcal{C}_i} 0 \tag{34}$$

for all $i, j \in \mathcal{I}$, which corresponds to (13a) extended to (23).

Similarly, from (32b), (32c) and applying the Schur complements, we have

$$\begin{aligned} E_i^T (Q_i - Q_i \Gamma_i Q_i) E_i - U_i \geq 0 &\implies Q_i - Q_i \Gamma_i Q_i \geq_{\mathcal{C}_i} 0 \\ &\iff P_i - \Gamma_i \geq_{\mathcal{C}_i} 0, \end{aligned} \tag{35}$$

and

$$E_i^T (\lambda R_i - Q_i^{-1}) E_i - W_i \geq 0 \implies \lambda R_i - P_i \geq_{\mathcal{C}_i} 0, \tag{36}$$

respectively. If also (32d) is satisfied, by using the same arguments of Theorem 2 we can state that the controller $u(k) = K_i C_i x(k)$, $x(k) \in \mathcal{C}_i$, $i \in \mathcal{I}$, makes the closed-loop system (23) finite-time stable with respect to $(0, N, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$. □

[7] Since C_i is full row rank, F_i always exists and is invertible [40].

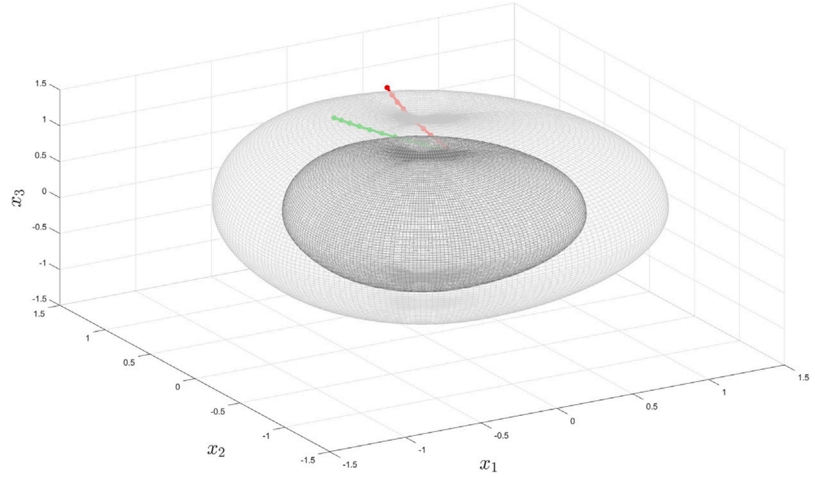

Fig. 1. State evolution in $N = 10$ steps from $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$ for the system (22) defined by (37)–(39), in open-loop (red line) and in closed-loop (green line) when applying the piecewise linear output feedback controller defined by (41). The boundary of the initial domain and the trajectory domain are highlighted with a dark gray surface and a light gray surface, respectively. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 12. It could happen that the matrices B_i are not full column rank only for some indices $i \in \mathcal{I}$, whilst the matrices C_i are full row rank for the same indices. In these mixed cases, one could merge the conditions of both Theorems 9 and 11, differentiating according to the appropriate indices.

4. Illustrative example

In this section, we present an example of a controlled discrete-time conewise linear system as in (22), derived from the zero-order-hold (ZOH) time-discretization of a continuous-time conewise system adapted from Example 2 in [43]. More specifically, it is

$$A_1 = A_2 = A_3 = A_4 = e^{\begin{pmatrix} 0 & -1 & \delta(1+\delta^2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T_s}, \tag{37a}$$

$$A_5 = A_6 = A_7 = A_8 = e^{\begin{pmatrix} 0 & -1 & -\delta(1+\delta^2) \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} T_s}, \tag{37b}$$

with $T_s = 0.1$, $\delta = 0.5$, and

$$B_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}, \quad i \in \mathcal{I} = \{1, \dots, 8\}. \tag{38}$$

The state space $\mathbb{R}^3$ is partitioned into the 8 orthants, i.e. into 8 polyhedral cones $\mathcal{C}_i$ whose ray matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{39a}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, E_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, E_7 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, E_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{39b}$$

Choose $N = 10$, the PQDs $\mathcal{X}_{\mathcal{R}_\mathcal{C}}$, $\mathcal{X}_{\Gamma_\mathcal{C}}$ related to the PQFs $\mathcal{R}_\mathcal{C}$, $\Gamma_\mathcal{C}$ defined by the matrices[8]

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_3 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{40a}$$

$$R_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, R_6 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, R_7 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, R_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{40b}$$

$$\Gamma_i = \frac{4}{9} R_i, \text{ for all } i \in \mathcal{I}, \tag{40c}$$

and $\gamma = 1.1$. In Fig. 1 the dark gray surface represents the boundary of the initial domain $\mathcal{X}_{\mathcal{R}_\mathcal{C}}$, while the light gray surface is the boundary of the trajectory domain $\mathcal{X}_{\Gamma_\mathcal{C}}$. From the initial state $x_0 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$, the open-loop system trajectory, i.e. for $u = 0$, starting from it is denoted by the red circle markers (the red line is added to help following the actual trajectory). It is clear that the uncontrolled system is not finite-time stable with respect to $(0, 10, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$ (see Fig. 1).

[8] Note that continuity conditions (4) hold.

By using Theorem 9 we can conclude that the system considered is finite-time stabilizable with respect to $(0, 10, \mathcal{X}_{\mathcal{R}_\mathcal{C}}, \mathcal{X}_{\Gamma_\mathcal{C}})$, via the output feedback $u(k) = K_i C_i x(k)$, $x(k) \in \mathcal{C}_i$, $i \in \mathcal{I}$, having the controller gains

$$K_1 = -0.0375,\ K_2 = -0.0375,\ K_3 = -0.0392,\ K_4 = -0.0392, \tag{41a}$$

$$K_5 = 0.0281,\ K_6 = 0.0302,\ K_7 = 0.0300,\ K_8 = 0.0292. \tag{41b}$$

In Fig. 1 the green circle markers denote the evolution in $N = 10$ steps of the closed-loop system (23) from the same initial state when the output feedback controller with the gains given by (41) is applied.

5. Conclusion

Two operative finite-time stabilization results for discrete-time conewise linear systems via static output feedback have been provided in this paper, depending on the rank of the input or output system matrices, respectively. Without recurring to approximate methods (BMIs) or to more conservative assumptions (relaxation to the whole state space), the controller design conditions require the solution of a feasibility problem for a set of LMIs, i.e. of a convex and numerically amenable problem. The stabilization theorems formulation exploited the FTS outcomes preliminarily derived in the first part of the paper, which adopts matrix cone-copositivity arguments involving piecewise quadratic functions. Such functions are appropriate if the initial and the trajectories domains can be modeled as piecewise quadratic domains, as is the case for classical ellipsoidal regions.

CRediT authorship contribution statement

Roberto Ambrosino: Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Software. **Raffaele Iervolino:** Conceptualization, Methodology, Writing – original draft, Writing – review & editing, Software.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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