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ORIGINAL RESEARCH PAPER

Approximating the divisor functions

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Abstract
We exploit the properties of a sequence of functions that approximate the divisor functions and combine them with an analytical formula of a delta-like sequence to give a new proof of a theorem of Grönwall on the asymptotic of the divisor functions.

Keywords Divisor functions · Grönwall's theorem · Delta-like sequences

Mathematics Subject Classification 11N37 · 42A10 · 40A25

1 Introduction and statement of the main results

Given a complex number α, for any $m \in \mathbb{N} = \{1, 2, \ldots\}$ the sum of positive divisors function $\sigma_\alpha(m)$ is defined as the sum of the αth powers of the positive divisors of m [1]. It is usually expressed as

$$\sigma_\alpha(m) := \sum_{k|m} k^\alpha,$$

where $k|m$ means that $k \in \mathbb{N}$ is a divisor of m. The functions so defined are called *divisor functions*. They play an important role in number theory, namely within the study of divisibility properties of integers and the distribution of prime numbers. Such functions appear in connection with the Riemann zeta function

$$\zeta(s) := \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad (\mathrm{Re}\, s > 1).$$

Indeed, given any $\alpha \in \mathbb{R}$, it is plain that

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$$\sigma_\alpha(m) = \sum_{k|m}\left(\frac{m}{k}\right)^\alpha \leq m^\alpha \sum_{k=1}^{m}\frac{1}{k^\alpha}.$$

In particular, for $\alpha > 1$ this yields

$$G_\alpha(m) := \frac{\sigma_\alpha(m)}{m^\alpha} \leq \zeta(\alpha), \quad \forall m \in \mathbb{N}.$$

In 1913, Grönwall [2] showed in a quite elementary way that if $\alpha > 1$, then

$$\limsup_{m\to\infty} G_\alpha(m) = \zeta(\alpha). \tag{1}$$

In fact, in view of the previous inequality, immediately he derives (1) after proving that

$$\lim_{m\to\infty} G_\alpha(P_m) = \zeta(\alpha),$$

where

$$P_m := \prod_{i=1}^{m} p_i^m$$

is the mth power of the product of the first m prime numbers $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_m$.

By arguing in an alternative way, in Sect. 4 we prove the following result, which still yields Grönwall's equation (1).

Theorem 1.1 *For every real number $\alpha > 1$ one has*

$$\lim_{m\to\infty} G_\alpha(m!) = \lim_{m\to\infty} G_\alpha([1,2,\ldots,m]) = \zeta(\alpha),$$

where $[1,2,\ldots,m]$ is the least common multiple of $1,2,\ldots,m$.

The proof of this theorem is arguably more involved than Grönwall's one. However, a novelty of our approach lies in the fact that, unlike Grönwall, we do not use the multiplicativity of σ_α, i.e., $\sigma_\alpha(mn) = \sigma_\alpha(m)\sigma_\alpha(n)$ for any coprime m, n [1], but we exploit the properties of a sequence of C^∞ functions whose pointwise limit is σ_α (see Sect. 2).

Our main tool is the following theorem, which, to the best of our knowledge, is new in the literature.

Theorem 1.2 *Let $\eta_1, \eta_2 \in \mathbb{R}\backslash\mathbb{Z}$ or $\eta_1, \eta_2 \in \mathbb{Z}$, with $\eta_1 \leq \eta_2$. For every $f \in L^1[\eta_1, \eta_2]$ which is continuous in a neighborhood of the integers, we have*

$$\lim_n \sqrt{n\pi}\int_{\eta_1}^{\eta_2} \cos^{2n}(\pi x) f(x)\mathrm{d}x = \sum_{m=\lfloor\eta_1\rfloor}^{\lfloor\eta_2\rfloor} f(m) - \frac{f(\eta_1)\chi_{\mathbb{Z}}(\eta_1) + f(\eta_2)\chi_{\mathbb{Z}}(\eta_2)}{2}, \tag{2}$$

where $\lfloor\eta\rfloor$ denotes the integer part of $\eta \in \mathbb{R}$ and $\chi_{\mathbb{Z}}$ is the characteristic function of $\mathbb{Z}$.

Our hope is also that this alternative approach might provide new insights and lead to some new result in the future, such as Ingham type formulas for convolution sums involving σ_α (see [3]).

1.1 Notation

For brevity, we write $\lim_n$ instead of $\lim_{n\in\mathbb{N}} \atop n\to\infty$, and the symbol $\{\cdot\}_{n\in\mathbb{N}}$ for a sequence is abbreviated as $\{\cdot\}_n$. The power $(\cos(y))^\beta$ is written as $\cos^\beta(y)$, and the same we do for the sine function.

Given a real number $x \geq 1$, in sums like $\sum_{k=1}^{x}$ we mean that $k \in \mathbb{N}$ with $k \leq \lfloor x \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of $x \in \mathbb{R}$, i.e., the greatest integer such that $x - \lfloor x \rfloor \geq 0$.

The characteristic function of a set $A \subseteq \mathbb{R}$ is denoted by χ_A.

2 Approximation of the divisor function

In this section a sequence of functions that approximate σ_α is defined. To this end, we consider σ_α for real $\alpha > 0$, and we extend such a function to $[0,\infty)$ by letting $\sigma_\alpha(x) = 0$ when $x \in [0,\infty)\backslash\mathbb{N}$.

For any given real numbers $\alpha > 0$ and $M > 1$, let us consider the sequence $\{\mathcal{C}_{\alpha,n}\}_n \subset C^\infty[0,M]$ defined as

$$\mathcal{C}_{\alpha,n}(M;x) := \sum_{k=0}^{M} f_n(k), \tag{3}$$

where

$$f_n(t) = f_{\alpha,n}(x;t) := \begin{cases} 0 & \text{if } t = 0 \text{ or } x = 0, \\ t^\alpha \cos^{2n}\left(\dfrac{\pi x}{t}\right) & \text{otherwise.} \end{cases}$$

The main property of the functions (3) is given by the following proposition.

Proposition 2.1 *For any given real numbers $\alpha > 0$, $M > 1$ and $x \in [0,M]$, the sequence $n \to \mathcal{C}_{\alpha,n}(M;x)$ is non-increasing, and $\lim_n \mathcal{C}_{\alpha,n}(M;x) = \sigma_\alpha(x)$.*

Proof If $x = 0$, then this is obviously true. Let $x \in (0,M]$ and $k \in (0,M] \cap \mathbb{N}$ be fixed. If $x \notin \mathbb{N}$ or if $x \in \mathbb{N}$ is not divisible by k, then $|\cos(\pi x/k)| < 1$. In this case, the sequence $n \to \cos^{2n}(\pi x/k)$ is decreasing and $\lim_n \cos^{2n}(\pi x/k) = 0$. On the other hand, $\cos^{2n}(\pi x/k) = 1$ for all $n \in \mathbb{N}$ if and only if $x \in \mathbb{N}$ and k divides x. Consequently,

$$\lim_n \mathcal{C}_{\alpha,n}(M;x) = \begin{cases} \sum_{k|x} k^\alpha & \text{if } x \in (0,M] \cap \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

as claimed. $\square$

3 Delta-like sequences and proof of Theorem 1.2

In the literature (see e.g. [4, 5]) a sequence of non-negative functions $\{g_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^m)$ is known to be a *delta-like sequence* if

$$\lim_n \int_{\mathbb{R}^m} g_n(x)f(x)\mathrm{d}x = f(0) \tag{4}$$

for all $f \in C_0^\infty(\mathbb{R}^m)$. If, in addition, the functions g_n are smooth, they are also called *mollifiers*, or *approximations of the identity.*

A well-known set of delta-like sequences is made of non-negative functions $g_n \in L^1(\mathbb{R}^m)$ satisfying the properties (a) and (b) below.

(a) For every $\epsilon > 0$, there exists $\delta > 0$ such that $\int_{|x|>\delta} g_n(x)\mathrm{d}x < \epsilon$.
(b) $\int_{\mathbb{R}^m} g_n(x)\mathrm{d}x = 1$.

Here and in what follows, we have let $|x| = \sqrt{x_1^2 + \cdots + x_m^2}$.

We prove the following

Proposition 3.1 *If* $\{g_n\}_{n\in\mathbb{N}} \subset L^1(\mathbb{R}^m)$, *where each* g_n *is non-negative, satisfies the previous property* (a), *and*

(b′) $\lim_n \int_{\mathbb{R}^m} g_n(x)\mathrm{d}x = 1$,

then (4) *holds for every* $f \in L^1(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$ *which is continuous in a neighborhood of* $x = 0$.

Proof From (b′) it follows that for every fixed $0 < \epsilon < 1$ there exists $N > 0$ such that $1 - \epsilon < \int_{\mathbb{R}^m} g_n(x)\mathrm{d}x < 1 + \epsilon$ whenever $n \geq N$. The continuity of f at $x = 0$ yields $|f(x) - f(0)| < \epsilon$ if $|x| < \delta$ for some $\delta > 0$.

Thus, by using (a) we see that if $n \geq N$, then

$$\begin{aligned}\int_{\mathbb{R}^m} g_n(x)\big(f(x) - f(0)\big)\mathrm{d}x &= \int_{|x|<\delta} g_n(x)\big(f(x) - f(0)\big)\mathrm{d}x + \int_{|x|>\delta} g_n(x)\big(f(x) \\ &\quad - f(0)\big)\mathrm{d}x \\ &< \epsilon \int_{|x|<\delta} g_n(x)\mathrm{d}x + 2\|f\|_\infty \int_{|x|>\delta} g_n(x)\mathrm{d}x \\ &< \epsilon(1+\epsilon) + 2\|f\|_\infty \epsilon.\end{aligned}$$

We also have that

$$\begin{aligned}\int_{\mathbb{R}^m} g_n(x)\big(f(x) - f(0)\big)\mathrm{d}x &\geq -\int_{|x|<\delta} g_n(x)|f(x) - f(0)|\mathrm{d}x - \int_{|x|>\delta} g_n(x)|f(x) \\ &\quad - f(0)|\mathrm{d}x \\ &> -\epsilon \int_{|x|<\delta} g_n(x)\mathrm{d}x - 2\|f\|_\infty \int_{|x|>\delta} g_n(x)\mathrm{d}x \\ &> -\epsilon(1+\epsilon) - 2\|f\|_\infty \epsilon.\end{aligned}$$

Since ϵ is arbitrary, it follows that

$$0=\lim_n \int_{\mathbb{R}} g_n(x)(f(x)-f(0))\mathrm{d}x=\lim_n \int_{\mathbb{R}} g_n(x)f(x)\mathrm{d}x-f(0)\lim_n \int_{\mathbb{R}} g_n(x)\mathrm{d}x$$
$$=\lim_n \int_{\mathbb{R}} g_n(x)f(x)\mathrm{d}x-f(0),$$

which yields (4). □

3.1 Proof of Theorem 1.2

We need the following

Lemma 3.2 *Let* $\varphi_n(x):=\sqrt{n\pi}\cos^{2n}(\pi x)$ *and* $\eta\in(0,1)$.
If $f\in L^1(-\eta,1-\eta)$ *is continuous at* $x=0$, *then*

$$\lim_n \int_{-\eta}^{1-\eta} f(x)\varphi_n(x)\mathrm{d}x=f(0). \tag{5}$$

If $f\in L^1[0,1]$ *is continuous at* $x\in\{0,1\}$, *then*

$$\lim_n \int_0^1 f(x)\varphi_n(x)\mathrm{d}x=\frac{f(0)+f(1)}{2}. \tag{6}$$

Proof We first show that the function $\varphi_n\chi_{(-\eta,1-\eta)}(x)$ satisfies the properties (a) and (b′) in Proposition 3.1.

Without loss of generality we assume $\eta=1/2$ and for simplicity the function $\varphi_n\chi_{(-1/2,1/2)}$ will still be denoted by φ_n.

In order to prove (a), we show that the sequence $\{\varphi_n\}_n$ converges uniformly to zero in any compact subset of $\mathbb{R}\setminus\{0\}$, i.e., $\lim_n \sup_{|x|>\delta}\varphi_n(x)=0$ for every $\delta>0$. Since the φ_n are even and decreasing when $x>0$, it suffices to show that $\lim_n \varphi_n(\delta)=0$ for every $0<\delta<1/2$. By the elementary inequality $\sin(t)\geq 2t/\pi$ for all $t\in[0,\ \pi/2]$, we have

$$\lim_n \varphi_n(\delta)=\lim_n \sqrt{n\pi}\cos^{2n}(\pi\delta)$$
$$=\lim_n \sqrt{n\pi}\left(1-\sin^2(\pi\delta)\right)^n\leq\lim_n \sqrt{n\pi}(1-4\delta^2)^n=0.$$

Hence, (a) is proved.

We now prove (b′). To this end, first let us recall the beta function [6, Formulae 8.380 2, 8.384 1]

$$\beta(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}:=2\int_0^{\frac{\pi}{2}}\sin^{2a-1}(t)\cos^{2b-1}(t)\mathrm{d}t,\quad \mathrm{Re}\ a,\mathrm{Re}\ b\in(0,+\infty),$$

where $\Gamma(z):=\int_0^\infty e^{-t}t^{z-1}\mathrm{d}t$, $\mathrm{Re}\ z>0$, is the well-known gamma function [6, Formula 8.310 1]. Since $\Gamma(1/2)=\sqrt{\pi}$ [6, Formula 8.338 2], we can write

$$\begin{aligned}\int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_n(x)\mathrm{d}x &= 2\sqrt{n\pi}\int_0^{\frac{1}{2}} \cos^{2n}(\pi x)\mathrm{d}x \\ &= 2\sqrt{\frac{n}{\pi}}\int_0^{\frac{\pi}{2}} \cos^{2n}(t)\mathrm{d}t \\ &= \sqrt{\frac{n}{\pi}}\beta\left(\frac{1}{2}, n+\frac{1}{2}\right) \\ &= \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}\sqrt{n}.\end{aligned}$$

Hence, (b′) follows after observing that by Stirling's approximation formula we get [6, Formulae 8.339 1, 2]

$$\lim_n \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\sqrt{n} = \sqrt{\pi}\lim_n \frac{(2n-1)!!}{n!2^n}\sqrt{n} = 1.$$

Now, since $f \in L^1[-\frac{1}{2}, \frac{1}{2}]$ is continuous at $x = 0$, from Proposition 3.1 we conclude that

$$\lim_n \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi_n(x)f(x)\mathrm{d}x = \lim_n \sqrt{n\pi}\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos^{2n}(\pi x)f(x)\mathrm{d}x = f(0). \tag{7}$$

Thus, (5) is proved.

It remains to prove (6).

Since $f \in L^1[0,1]$ is continuous at $x \in \{0,1\}$, the functions $x \to f(|x|)$ and $x \to f(1-|x|)$ are continuous at $x = 0$. Further, $\varphi_n(x) = \varphi_n(1-x)$ and $\varphi_n(x) = \varphi_n(-x)$. Thus, we can write

$$\begin{aligned}\int_0^1 f(x)\varphi_n(x)\mathrm{d}x &= \int_0^{\frac{1}{2}} f(x)\varphi_n(x)\mathrm{d}x + \int_{\frac{1}{2}}^1 f(x)\varphi_n(x)\mathrm{d}x \\ &= \int_0^{\frac{1}{2}} f(x)\varphi_n(x)\mathrm{d}x + \int_0^{\frac{1}{2}} f(1-x)\varphi_n(x)\mathrm{d}x \\ &= \int_{-\frac{1}{2}}^0 f(-x)\varphi_n(x)\mathrm{d}x + \int_{-\frac{1}{2}}^0 f(1+x)\varphi_n(x)\mathrm{d}x.\end{aligned}$$

From the above chain of identities it follows that

$$2\int_0^1 f(x)\varphi_n(x)\mathrm{d}x = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(|x|)\varphi_n(x)\mathrm{d}x + \int_{-\frac{1}{2}}^{\frac{1}{2}} f(1-|x|)\varphi_n(x)\mathrm{d}x.$$

Hence, (7) yields $2\lim_n \int_0^1 f(x)\varphi_n(x)\mathrm{d}x = f(0) + f(1)$, as required.

Proof of Theorem 1.2 We can assume that $[\eta_1, \eta_2] = [a-\eta, b-\eta]$, where $a \le b$ are integers and $0 \le \eta < 1$. This is obvious if $\eta_1, \eta_2 \in \mathbb{Z}$. Therefore, let us assume that $\eta_1, \eta_2 \notin \mathbb{Z}$, so that $\eta := \lfloor \eta_1 + 1 \rfloor - \eta_1 \in (0,1)$. If we let $a = \lfloor \eta_1 + 1 \rfloor$ and $b =$

$\lfloor \eta_2 + 2 \rfloor$, then $\eta_1 = a - \eta$ and $b - \eta = \lfloor \eta_2 + 2 \rfloor - \eta > \eta_2 + 1 - \eta > \eta_2$, i.e., $\eta_2 = b - \eta - \eta'$ for some $\eta' > 0$. Consequently, we can extend $f \in L^1[\eta_1, \eta_2] = L^1[a - \eta, b - \eta - \eta']$ to $\tilde{f} \in L^1[a - \eta, b - \eta]$ by letting $\tilde{f} \equiv 0$ in $[b - \eta - \eta',\ b - \eta]$. Note that $\tilde{f}$ is still continuous in a neighborhood of the integers. Further, the sum on the right-hand side of (2) does not change if f is replaced by $\tilde{f}$. Thus, without loss of generality, we can assume that f is defined in the interval $[a - \eta,\ b - \eta]$ and write

$$\begin{aligned} \lim_n \sqrt{n\pi} \int_{a-\eta}^{b-\eta} \cos^{2n}(\pi x) f(x) \mathrm{d}x &= \sum_{m=a}^{b-1} \lim_n \sqrt{n\pi} \int_{m-\eta}^{m+1-\eta} \cos^{2n}(\pi x) f(x) \mathrm{d}x \\ &= \sum_{m=a}^{b-1} \lim_n \int_{-\eta}^{1-\eta} \varphi_n(x) f(m+x) \mathrm{d}x, \end{aligned} \tag{8}$$

where φ_n has been introduced in Lemma 3.2. Indeed, by applying this lemma we have

$$\lim_n \int_{-\eta}^{1-\eta} \varphi_n(x) f(m+x) = \begin{cases} f(m) & \text{if } \eta > 0, \\ \dfrac{f(m) + f(m+1)}{2} & \text{if } \eta = 0. \end{cases}$$

Consequently, (8) becomes (2). $\square$

4 Proof of Theorem 1.1

For $x, n \in \mathbb{N}$, with $x > 1$, let us take $M = 2x$ for the approximants of $\sigma_\alpha(x)$ defined in (3), namely

$$\mathcal{C}_{\alpha,n}(2x; x) = \sum_{k=0}^{2x} k^\alpha \cos^{2n}\left(\frac{\pi x}{k}\right).$$

Recall that the function $f_n(t) := t^\alpha \cos^{2n}(\pi x/t)$ extends to a continuous function in $[0, \infty)$, with $f_n(0) = f_n(2x) = 0$. Further, it satisfies the hypotheses of Theorem 1.2 in $[0,\ 2x]$. Thus,

$$\begin{aligned} \mathcal{C}_{\alpha,n}(2x; x) &= \lim_N \sqrt{N\pi} \int_0^{2x} f_n(t) \cos^{2N}(\pi t) \mathrm{d}t + \frac{f_n(0) + f_n(2x)}{2} \\ &= \lim_N \sqrt{N\pi} \int_0^{2x} f_n(t) \cos^{2N}(\pi t) \mathrm{d}t. \end{aligned}$$

By the change of variables $t \to xt$ one has

$$\mathcal{C}_{\alpha,n}(2x; x) = x \lim_N \sqrt{N\pi}\, F_{n,N}(x),$$

where

$$F_{n,N}(x) := \int_0^2 f_n(xt)\cos^{2N}(\pi xt)\,\mathrm{d}t,$$

with

$$f_n(xt) = \begin{cases} 0 & \text{if } t = 0, \\ (xt)^\alpha \cos^{2n}\left(\frac{\pi}{t}\right) & \text{otherwise.} \end{cases}$$

Thus, Theorem 2.1 yields

$$\sigma_\alpha(x) = x \lim_n \lim_N \sqrt{N\pi}\, F_{n,N}(x), \quad \forall x \in \mathbb{N}. \tag{9}$$

Since it is plain that $f_n(xt) \geq 0$ for all $t \in [0,2]$, we can take a sufficiently large integer m, set $m' = 4(m!)$, and write

$$\begin{aligned} F_{n,N}(x) &\geq \sum_{k=1}^{m} \int_{\frac{1}{k}-\frac{1}{m'}}^{\frac{1}{k}+\frac{1}{m'}} f_n(xt)\cos^{2N}(\pi xt)\,\mathrm{d}t \\ &\geq x^\alpha \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{m'}\right)^\alpha \int_{\frac{1}{k}-\frac{1}{m'}}^{\frac{1}{k}+\frac{1}{m'}} \cos^{2n}\left(\frac{\pi}{t}\right)\cos^{2N}(\pi xt)\,\mathrm{d}t \\ &= x^{\alpha-1} \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{m'}\right)^\alpha \int_{\frac{x}{k}-\frac{x}{m'}}^{\frac{x}{k}+\frac{x}{m'}} \cos^{2n}\left(\frac{\pi x}{t}\right)\cos^{2N}(\pi t)\,\mathrm{d}t, \end{aligned}$$

after the change of variable $xt \to t$. Now, let us take $x = m!$ so that $x/k = m!/k \in \mathbb{N}$ and $\frac{x}{k} \pm \frac{x}{m'} = \frac{m!}{k} \pm \frac{1}{4} \notin \mathbb{Z}$ for all $k \in \{1, \ldots, m\}$. Thus, the previous inequality becomes

$$F_{n,N}(m!) \geq (m!)^{\alpha-1} \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{m'}\right)^\alpha \int_{\frac{m!}{k}-\frac{1}{4}}^{\frac{m!}{k}+\frac{1}{4}} \cos^{2n}\left(\frac{\pi m!}{t}\right)\cos^{2N}(\pi t)\,\mathrm{d}t.$$

Again, we apply Theorem 1.2 by taking $f(t) = \cos^{2n}(\pi m!/t)$ to see that

$$\begin{aligned} \lim_N \sqrt{N\pi} \int_{\frac{m!}{k}-\frac{1}{4}}^{\frac{m!}{k}+\frac{1}{4}} \cos^{2n}\left(\frac{\pi m!}{t}\right)\cos^{2N}(\pi t)\,\mathrm{d}t &= f\left(\frac{m!}{k}\right) + f\left(\frac{m!}{k} - 1\right) \\ &= \cos^{2n}(k\pi) + \cos^{2n}\left(k\pi \frac{m!}{m!-k}\right). \end{aligned}$$

Since for any given $k \in \{1, \ldots, m\}$ one has $\cos^{2n}(k\pi) = 1$ and

$$\lim_n \cos^{2n}\left(k\pi \frac{m!}{m!-k}\right) = \lim_n \cos^{2n}\left(\pi \frac{k}{m!/k - 1}\right) = 0,$$

we infer

$$\lim_n \lim_N \sqrt{N\pi} F_{n,N}(m!) \geq (m!)^{\alpha-1} \sum_{k=1}^{m} \left(\frac{1}{k} - \frac{1}{4(m!)} \right)^{\alpha},$$

which, together with (9), implies that

$$G_\alpha(m!) = \frac{\sigma_\alpha(m!)}{(m!)^\alpha} \geq \sum_{k=1}^{m} \frac{1}{k^\alpha} + \Delta_\alpha(m),$$

where

$$\Delta_\alpha(m) := \sum_{k=1}^{m} \left(\left(\frac{1}{k} - \frac{1}{4(m!)} \right)^{\alpha} - \frac{1}{k^\alpha} \right).$$

Since by the mean value theorem we see that

$$|\Delta_\alpha(m)| \leq \frac{\alpha/4}{m!} \sum_{k=1}^{m} \frac{1}{k^{\alpha-1}} \leq \frac{\alpha/4}{(m-1)!},$$

we deduce

$$\lim_m G_\alpha(m!) = \zeta(\alpha),$$

because, as already mentioned in Sect. 1, if $\alpha > 1$, then

$$G_\alpha(m) \leq \zeta(\alpha), \quad \forall m \in \mathbb{N}.$$

It is plain that in order to prove the same limit for $G_\alpha([1, 2, \ldots, m])$ it suffices to take $x = [1, 2, \ldots, m]$ and proceed in a completely analogous way.

Theorem 1.1 is completely proved.

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Conflict of interest The authors declare that they do not have any competing interests.

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